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THIRD PART

On the nature of continuous lines delimiting the different domains of convergence.

70. While studying the uniform convergence to a limit point (like the irregular convergence to a periodic cycle), it may happen to find many limit points with uniform convergence simultaneously. Each one of them is related to an immediate domain of convergence, whose interior is separated from any exterior point by a continuous line intersecting any simple line that links an interior point of the same domain with any exterior point. One may try to get deeper inside the question by exploring the nature of these continuous lines. In the example $z_1 = \frac{-z^3+3z}{2}$, one has to deal with continuous lines showing a very complicated structure. Anyway, one may show, in some very general cases, that the complexity of such continuous lines may turn into an agglomeration of continuous lines with a much simpler nature, for example a Jordan line: closed continuous curves without multiple points.

I did not achieve in making a deep analysis of the characteristic properties of all kinds of the continuous lines arising from the iteration of rational fractions; anyway, under very general hypotheses, I achieved in showing that they may be reduced to Jordan lines. It makes a clearer knowledge on the structure of those continuous lines and it explains why difficulties arise during the delimitation of the domains of convergence.

For this purpose, one sees that the examples of iterations, which have been treated up to now (in particular the fractions in the fundamental circle) are very simple examples (*read* § 20, 22).
I start from a simple example to allow the reader to understand the method I follow.

I always take some rational fractions having (I know *a priori*) two distinct limit points with uniform convergence: so that I have been *a priori* assured that the set E′ includes a continuous line splitting the plane in regions; the present research aims at the discovering of the nature of these continuous lines.

71. *First example*. – I will take again the case of $z_1 = \frac{z+z^2}{2}$, noticed by Fatou in the Note of October 15th 1906, in *Comptes Rendus*.
We know, as we examined in the pages before, that this is the case of two limit points (0 and ∞) with uniform convergence.
The plane is split in two simply connected regions R_0 and R_∞ including respectively 0 and ∞ and sharing a continuous line E′ as the common boundary (*read* § 35).

E′ is the common limit of the preimages of two circles: the first circle C surrounding the origin $|z| = \frac{1}{2}$, and the second circle Γ surrounding the point at infinity $|z| = 4$ ([1]).

The critical points of the inverse of $\varphi(z)$ are $z = -\frac{1}{8}$ and $z = \infty$.

Therefore if z describes C once, then the set of its two preimages describes a close algebraic curve C_{-1}, surrounding C.

If z describes C twice, then each one of the preimages describes C_{-1} once; one may say the same about the preimages C_{-2}, C_{-3}, … converging to E′. In this case I say that the convergence of C_{-i} to E′ is *uniform*.
Each one of the curves C_{-i} is included in its preimage; firstly during the generation of these curves, whatever is the positive number given before, it is possible to determine a big enough index N so that, whatever the both indexes *n and p are* > N, then the *difference of the two curves* C_{-n} and C_{-p} is always < ε. One may call the difference of the two curves *the maximum* ([2]) *of the shortest distance from a point of one curve to a point of the other curve*, *when the considered point describes the curve it is lying over*.
The difference of two curves L and L′ is < ε : it means that the curve L′ is being entirely in the band which is analysed by a circle with ray ε and whose centre describes L.

72. To show this, I remark the relation

$$\frac{dz_1}{dz} = \varphi'(z) = \frac{1}{2} + z$$

where $|\varphi'(z)| > 1$ in every exterior point of the circle γ with centre $\left(-\frac{1}{2}\right)$ and ray 1.

Let us write then

$$z_1 = \frac{z+z^2}{2} = \frac{1}{2}\left(z+\frac{1}{2}\right)^2 - \frac{1}{8},$$

and we will see that when z describes the previous circle, $\left|z+\frac{1}{2}=1\right|$, then

$$\left|z_1+\frac{1}{8}\right| = \frac{1}{2}\left|z+\frac{1}{2}\right|^2 = \frac{1}{2} \;;$$

therefore z_1 describes the circle γ_1, with centre $\left(-\frac{1}{8}\right)$ and ray $\left(\frac{1}{2}\right)$, there is a circle γ_1, completely in the interior of the previous circle and including the critical point $\left(-\frac{1}{8}\right)$ in its interior and the point of convergence 0. It is enough to affirm that γ is completely inside R_0 because the images of γ have 0 as their limit point.

([1]) There is no possible misunderstanding.
([2]) It is a well determined number; it is positive when the two curves, as the case for C_{-n} and C_{-p}, are continuous curves without any common point.

For defining R_0, one may start from γ, as well as C, for approaching to the boundary E′ since it is the limit of the preimages $\gamma_{-1}, \gamma_{-2}, \dots, \gamma_{-i}, \dots$.

The γ_{-i} enjoy the same properties with their respective C_{-i}. If z describes a γ_{-i} once, then its image z_1 describes $\gamma_{-(i-1)}$ twice, and so on.

No need to insist on: one sees that the area, including O and bounded by any C_{-i}, is (for a big enough P) inside all curves γ_{-p} with an index $p > P$.

Inversely any γ_{-i} is inside all C_{-p} with a big enough index, for a big enough P.

The important property is that if z describes an area (C_{-i}) or an area (γ_{-i}), then *its two preimages* describe ($C_{-(i+1)}$) or an area ($\gamma_{-(i+1)}$) enclosing the area (C_{-i}) or an area (γ_{-i}) in its interior.

While I was researching for the immediate domain of convergence to a limit point with uniform convergence, I have insisted for a long time on the essential property allowing to define the immediate domain as the limit of the domains (C_{-i}) or (γ_{-i}), while i increases indefinitely.

73. Since in every exterior point of γ one gets $|\varphi'(z)| > 1$, the result is that, for a big enough index I, since all the C_{-i} ($i \geq$ I) surround γ, one gets the relation $|\varphi'(z)| > M > 1$ in the exterior of these C_{-i}.

For the same reason, since $\psi(z)$ is the inverse function of $\varphi(z)$, one gets, for a big enough index I,

$$|\psi'(z)| < N < I ,$$

where z is outside C_{-I}.

It suffices to choose the index I so that C_{-I} surrounds γ, because if z is outside C_{-I}, then each one of the preimages is outside $C_{-(I+1)}$; then one gets

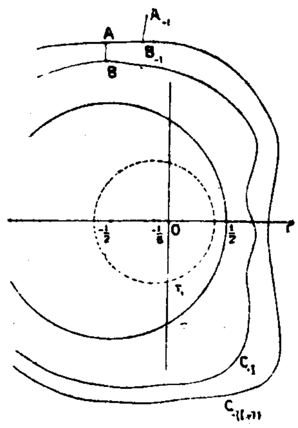
$$\psi'(z) = \frac{1}{\varphi'(z_{-1})}$$

Therefore, being z_{-1} outside γ,

$$|\psi'(z)| = \frac{I}{|\varphi'(z_{-1})|} < \frac{I}{M} = N < I .$$

Let us call δ the difference between C_{-I} and $C_{-(I+1)}$; this means that the shortest distance from any point of $C_{-(I+1)}$ is $\leq \delta$.

Let us imagine any distance, taken among the shortest ones AB.



The points A and B are inversely mapped to the preimages lying on C_{-I} and $C_{-(I+1)}$, respectively. Let B_{-1} be a preimage of B; as z describes BA, the determination of $\psi(z)$, whose affixe is B_{-1} and when z is in B, has a certain point A_{-1} as affixe, when z reaches to A. A_{-1} is a preimage of A and, when z describes BA, $\psi(z)$ describes a curve joining $B_{-1}A_{-1}$. Since, on BA, one gets $\varphi'(z) \leq N < 1$, then

$$\text{the longer arc } A_{-I}B_{-I} \leq N \times \text{ segment } AB \leq N\delta \text{ ,}$$

since $\overline{AB} \leq \delta$.
Therefore it is clear that the *difference of* $C_{-(I+1)}$ *and* $C_{-(I+2)}$ *is* $\leq N\delta$, since the shortest distance from A_{-I} [which is any point of $C_{-(I+2)}$, on the condition to properly choose A on $C_{-(I+2)}$] to $C_{-(I+1)}$, since that difference is surely $\leq$ arc$A_{-I}B_{-I}$, it is $\leq N\delta$, whatever is the point A_{-I} of $C_{-(I+2)}$.

74. One shows that the difference between $C_{-(I+p)}$ and $C_{-(I+p+1)}$ is $\leq N^p\delta$, since $N < 1$.
Therefore one sees that the difference between $C_{-(I+p)}$ and $C_{-(I+p+1)}$ describes a geometric progression. It is evident that *the difference between* C_{-n} *and* $C_{-(n+p)}$ *is less than the sum of the differences within* C_{-n} *and* $C_{-(n+1)}$, $C_{-(n+1)}$ and $C_{-(n+2)}$, …, $C_{-(n+p+1)}$, and $C_{-(n+p)}$.
Then since:

1. one may approach, from any point A of C_{-n}, to a certain point of $C_{-(n+p)}$ by a path consisting of rectilinear segments AA_1, A_1A_2, …, $A_{p-1}A_p$;
2. A_i is on $C_{-(n+i)}$ and since A_iA_{i+1} is the shortest distance of A_i to $C_{-(n+i+1)}$;
3. each one of the segments is less, whatever is the starting A, than the difference within the two curves describing their extremities;

in conclusion, if one considers the convergent sequence

$$N\delta + N^2\delta + \ldots + N^p\delta \ldots ,$$

then the difference between C_{-n} and $C_{-(n+n')}$ (given $n \geq n_0$, with a big enough n_0 and whatever is n') is less than *the rest of the previous sequence*, taken by starting from a certain rank p; *the rest* is arbitrarily small if p is big enough.
It has been proved that C_{-i} converges uniformly to their limit: E′.
Immediately, the result is that, as i diverge to infinity, C_{-i} converges to a *continuous curve*, that's to say to a set of points E′ described by the equations

$$x = f(t), \qquad y = \varphi(t) \qquad (0 \leq t \leq 1)$$

since both f and φ are *continuous functions of t* in the range (0,1).

75. By showing a theoretical generation of the functions $\psi(t)$ and $\varphi(t)$, then I will deduce other properties of E′. Starting from a certain rank I, one says that all C_{-i} ($i >$ I) are in the region satisfying the following inequality $|\varphi'(z)| \geq M > 1$, as a consequence that in a region $|\psi'(z)| \leq N < 1$ holds, since $\psi(z)$ is the inverse function of $\varphi(z)$.
Let us consider the annulus between the two curves C_{-I} and $C_{-(I+1)}$. The two limit curves are two simple algebraic curves consisting of *one only closed analytic arc*, since the starting C (none of C_{-i}) does not intersect any critical point of ψ; then none of C_{-i} includes any edged point. One may generate a conformal map of the annulus (C_{-I},$C_{-(I+1)}$) over an annulus (OO′) which is enclosed within two concentric circles. It suffices that the ratio between the rays of the circles O and O′ is a proper number.

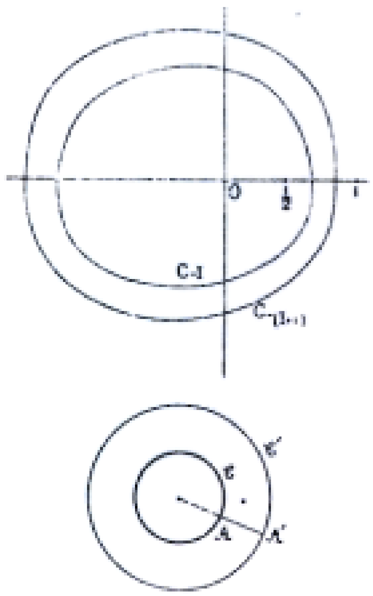


Fig. 23

The analytic function F(z) maps an interior point F(z) of the annulus ($C_{-I}C_{-(I+1)}$) to an interior point z of the annulus (OO′): F(z) is still analytic on O and O′, since both C_{-I} and $C_{-(I+1)}$ consist of one only analytic arc ([1]).

([1]) Read PICARD, *Traité d'Analyse*, t. II (2nd edition), p. 301

The rays like AA', linking a point of O to a point of O', are mapped to some analytic curves linking a point A of C_{-I} to a point B of $C_{-(I+1)}$.

A is mapped to A, and B is mapped to A', due to the function F(z).

The arc of the analytic curve AB intersects orthogonally both the curves C_{-I} and $C_{-(I+1)}$, since the ray AA' intersects orthogonally O and O' and since the angles are preserved by a conformal map, defined by F(z).

So all rays of the annulus OO' *are mapped to orthogonal trajectories of* C_{-I} *and* $C_{-(I+1)}$.

These trajectories map each point A of C_{-I} to only one point of $C_{-(I+1)}$ lying on the orthogonal trajectory of AB.
Two distinct points of C_{-I} are mapped to two distinct points of $C_{-(I+1)}$ and reciprocally.

Two orthogonal trajectories, coming out from two distinct points of C_{-I}, do not share any common points; one and only one trajectory intersects every point of C_{-I}.

76. As z describes C_{-I}, let us consider one of its preimages $z_{-1} = \psi(z)$.

It describes the area $C_{-(I+1)}$; if z describes the orthogonal trajectory AB, then z_{-1} describes a curve $A_{-1}B_{-1}$, orthogonal to $C_{-(I+1)}$ like AB is to C_{-I}; in B_{-1}, the curve $A_{-1}B_{-1}$ is orthogonal to $C_{-(I+2)}$ like AB is in B to $C_{-(I+1)}$.

When z lies in all positions in C_{-I}, then every preimage z_{-1} of z generates a trajectory that intersects orthogonally both $C_{-(I+1)}$ and $C_{-(I+2)}$; if the orthogonal trajectory AB, coming out from z, intersects *twice in succession* the annulus (C_{-I}, $C_{-(I+1)}$), then the trajectory $A_{-1}B_{-1}$, preimage of AB [which is described by any determination of $\psi(z)$, when z describes AB] intersects once the annulus ($C_{-(I+1)}$,$C_{-(I+2)}$).
If z describes twice in succession C_{-I}, then each one of its two preimages is mapped to any of the two determinations of $\psi(z)$, described by $C_{-(I+1)}$ once and in a continuous way, always going in the same direction.

77. Every orthogonal trajectory AB of the annulus (C_{-I},$C_{-(I+1)}$) admits two preimages that are the orthogonal trajectories of the annulus ($C_{-(I+1)}C_{-(I+2)}$).
Therefore this last annulus is intersected by orthogonal trajectories for $C_{-(I+1)}$, and $C_{-(I+2)}$, so that any arbitrary pair of these trajectories cannot share any common point or they may be the same.
These new trajectories extend those of the annulus (C_{-I},$C_{-(I+1)}$) *with the continuity of the tangent line*, but *without the need of the analytic continuation* ([1]).

([1]) There is no reason to believe that the trajectories of ($C_{-(I+1)}$,$C_{-(I+2)}$), extending any pair of determined trajectories of (C_{-I},$C_{-(I+1)}$), are the preimages of these last ones.

In fact the trajectory BD of ($C_{-(I+1)}$,$C_{-(I+2)}$), intersecting the point B, has, for the tangent line in B, *the normal line with the curve* $C_{-(I+1)}$; but there is an *analytic arc*, preimage of a trajectory of $C_{-I}C_{-(I+1)}$, *coming out from* B_1, which is *the image of* B; there are no *a priori reasons* so that the preimage BD of the trajectory B_1D_1 of ($C_{-I}C_{-(I+1)}$) is the *analytic continuation* of the trajectory AB of (C_{-I}, $C_{-(I+1)}$).

The annulus ($C_{-I}C_{-(I+1)}$) ([1]) is therefore intersected by trajectories, orthogonal to C_{-I}, $C_{-(I+1)}$, $C_{-(I+2)}$: each one of these trajectories consists of two analytic arcs joining on $C_{-(I+1)}$, in respect of the continuity for the tangent line.

Two trajectories do not share any common point; if it does not happen, then it means they coincide.

This process may keep on going indefinitely because the region within C and Γ does not include any critical points of $\psi(z)$.
So one defines these trajectories *orthogonal to all* C_{-I} with index $i \geq$ I.

Each one of these trajectories consists of an infinity of arcs (the analytic portions within C_{-i} and $C_{-(i+1)}$) where every portion is analytic so that they join each other, in respect of the continuity of the tangent line.

Two trajectories may coincide or they may share no common interior point in R_0 ([2]).

78. One sees that, with the help of orthogonal trajectories, a biunivocal and continuous correspondance may be fixed between the points of any C_{-i} (with index $i >$ I) and the points with C_{-I} ([3]).

After corresponding a parameter (ranging from 0 to 1) to every point of C_{-I} (for example, one chooses on C_{-I} an origin ω and a direction to follow; so, if l is the length of C_{-I}, then every point A of C_{-I} corresponds to the parameter $t = \frac{s}{l}$, since s is the arc of C_{-I}, described in the positive direction, for moving from ω to A; since t ranges from 0 to 1, then A describes once C_{-I} in the positive direction from ω to ω), one sees that the coordinates x_i, y_i (of a point describing C_{-i}) are continuous functions of the parameter t.

$$x_i = f_i(t)\ ,\ y_i = g_i(t) \qquad\qquad (0 \leq t \leq 1\ ;\ i = \mathrm{I}, \mathrm{I}+1, \ldots, \infty),$$

([1]) If one makes the conformal map of the annulus (C_{-I},$C_{-(I+1)}$) over an annulus (OO″), consisting of two concentric circles, then there are no reasons for the found trajectories to be mapped to the rays (that orthogonally intersect the circles O and O″), because, in this representation, *the curve* $C_{-(I+1)}$ won't be mmaped in general to a circle, concentric with OO″.

([2]) *A priori*, nothing can stop two distinct trajectories, coming out from two distinct points of C_{-I}, to converge to *the same point* of E′; but we will see, further, that this is impossible.

([3]) One can fix a correspondence with the points lying on the same orthogonal trajectory.

and simultaneously, one will get

$$f_i(a) = f_i(b)$$
$$g_i(a) = g_i(b)$$

if $a = 0$, $b = 1$, or if $a = b$ (1).
All the points of C_{-i} ($i = I + 1, I + 2, \ldots$), lying on the same orthogonal trajectory, correspond to the same value of t.

79. I will show that these $f_i(t)$ converge uniformly to a limit $f(t)$ so as $\varphi_i(t)$.
While A, with the parameter t, describes C_{-I}, then the arc of the trajectory, lying between C_{-I} and $C_{-(I+1)}$, has a maximum Δ and a minimum, both positive, (the maximum Δ plays an analogous role like the *difference* between C_{-I} and $C_{-(I+1)}$).
It is clear that, since $|\psi'(z)| < N$ in all the region where there are these C_{-I} with the index $i \geq I$, every portion of the trajectory, which lies within $C_{-(I+1)}$ and $C_{-(I+2)}$, is $< N\Delta$, since these portions are described by $\psi(z)$, while z describes these portions which are lying between C_{-I} and $C_{-(I+1)}$.
In the same way, one realizes that every portion of the trajectory, lying between $C_{-(I+p)}$ and $C_{-(I+p+1)}$, is $< N^p\Delta$. One deduces immediately that

$$|f_{I+p}(t) - f_{I+p+1}(t)| \leq N^p\Delta$$
$$|g_{I+p}(t) - g_{I+p+1}(t)| \leq N^p\Delta$$
$$(p = 0, 1, \ldots, \infty)$$

whatever t is, because evidently $[x_{i+1}(t) - x_i(t)]$, the difference of the abscissas of two extreme points of an arc of a trajectory, lying between C_{-i} and $C_{-(i+1)}$, is (in absolute value) less than the length of that arc, since it is $\leq N\Delta$ and according to what one has just seen.

Therefore the set of $f_i(t)$ converge *uniformly*, for $i \to \infty$ to a continuous limit function $f(t)$. In the same way, the sequence of $g_i(t)$ converges *uniformly* to a continuous limit function $g(t)$. The sum is a convergent sequence

$$(\Sigma) \qquad \Delta + N\Delta + N^2\Delta + \ldots + N^p\Delta + \ldots$$

[it is a geometric progression ($N < 1$)], then one deduces

$$|f_{I+p}(t) - f_{I+p+q}(t)| < \varepsilon$$
$$|g_{I+p}(t) - g_{I+p+q}(t)| < \varepsilon$$

whatever is $q > 0$, since the value of ε has been fixed previously, as soon as the value of p is so big that the rest R_p of the sequence (Σ) is $< \varepsilon$.

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(1) C_{-i} fits exactly the definition of closed and simple Jordan curve, which for short we call *the Jordan curve*.

80. Therefore, the set E′ is the set of the points

$$x = f(t), \qquad y = g(t)(0 \leq t \leq 1)$$

since both *f* and *g* are *continuous functions*, and

$$f(0) = f(1)$$
$$g(0) = g(1)$$

Every orthogonal trajectory consists of an infinity of arcs, whose the lengths are ≤ than the terms of the progression (Σ). Therefore everyone of these trajectories has a *finite length* and the point, where the trajectory intersects C_{-i}, converges uniformly to *one and only limit point*, as *i* increases indefinitely; that's to say, when the point [$f(t)$, $\varphi(t)$] corresponds to the value of *t*, characterising the considered trajectory.

81. A doubt, related to the particular chosen representation for C_{-i}, may arise from the spirit of this lecture; the same doubt may exist for a particular representation of the limit of C_{-i}. Anyway, it is easy to waste. If P is any point of E′, then it is, as we already saw before, the limit of the points of C_{-i}.

On every C_{-i}, one may choose a point P_i so that the set of P_i includes its limit point in P itself. For example, one gets P_i as the base of the shortest distance $\overline{PP_i}$, dropped down from P on C_{-i}, then

$$\overline{PP}_{i+1} < \overline{PP}_i < \overline{PP}_{i-1}$$

because $C_{-(i+1)}$ surrounds C_{-i}; $\overline{PP_i}$ is a function of *i*, decreasing constantly to zero as *i* increases indefinitely. Evidently, due to this choice of P_i, the set of P_i includes P as the only limit point. Let t_1 be the value of the parameter *t* corresponding to the point P_i; then the coordinates of P_i are the pair ($f_i(t_i)$; $g_i(t_i)$).
The set of t_i is in general an infinite countable set, ranging within 0 and 1.
It has at least a limit point τ (if an infinity of t_i coincides, then one takes, as τ, one of the values coinciding with an infinity of t_i: it occurs to remove the objection for the case where there will be a finite number of distinct t_i, due to the previous coincidences), that's to say: one may find in all cases an infinity of increasing indexes

$$i_1, i_2, \ldots, i_{p1}, \ldots .$$

so that, whatever ε is, one gets $| \tau - t_{Ip} | < \varepsilon$, as soon as $p > P$, when P is big enough.
It is clear that, due to the uniform convergence of $f_i(t)$ to $f(t)$ and of $g_i(t)$ to $g(t)$, the sequence

$$f_{i1}(t_{i1}), f_{i2}(t_{i2}), \ldots, f_{ip}(t_{ip}), \ldots$$

converges to $f(\tau)$ and

$$g_{i1}(t_{i1}), g_{i2}(t_{i2}), \ldots, g_{ip}(t_{ip}), \ldots$$

converges to $g(\tau)$. Now, the first sequence is the sequence of the abscissas of the sequence

$$P_{i1}, P_{i2}, \ldots, P_{ip}, \ldots$$

converging to P and only P; the second sequence is the sequence of ordinates of the same original sequence. The result is that P (which is an arbitrary point of E′) has coordinates, coming out from the functions $f(t)$ and $g(t)$ for the value τ of the parameter. Therefore *the continuous curve*

$$x = f(t), \qquad y = \varphi(t) \qquad (0 \leq t \leq 1)$$

is the *same as the set* E′.

82. Nothing of above assures us *a priori* that the set of t_i includes more than only one limit point τ; therefore nothing can assure *a priori* that E′ is a *simple* closed continuous curve, that's to say where every point of the curve corresponds only to one value t of the parameter (except for the values 0 and 1: both of them give the same value).
If every point P of E′ is mapped at least to a value τ of the parameter t, then at least an orthogonal trajectory of the considered system (where P is an extreme point) shall intersect the point generated by the value τ of the parameter in the region C_{-1}.
This orthogonal trajectory, going to P, is a *simple line* [that's to say a continuous curve that can be mapped biunivocally and continuously to a segment of a straight line (*read* Preliminaries, §2 and 4)], whose every point, but P, is inside R_0.
It means that *every point* P *of* E, *since it is a boundary point of* R_0, *is therefore accessible from the interior of* R_0, *following one of the previous orthogonal trajectories*.

83. Now, with the help of new considerations, included in the theory of continuous boundaries of a domain, I will show that E′ is a SIMPLE closed *Jordan line*.

In fact, every point P of E′ is a *simple* point of the boundary of R_0.
The opposite hypothesis would induce me to think that one may find a simple closed line L or a closed Jordan curve, coming out from P and getting back to P ([1]), whose all points but P are inside R_0, so that that curve defines an interior region ([2]) *including at least a boundary point* Q *of* R_0 ([3]), because one cannot reduce that Jordan curve L to an only point P by a continuous movement without intersecting again the point Q of E, distinct from P.

([1]) Since P is accessible from the interior of R_0, no contradictions arise.
([2]) To be intended as the *interior* of a simple closed Jordan curve.
([3]) All authors do not give a definition like this for a multiple point of the boundary (*read*, for example, CARATHÉODORY, *Math. Ann.,* t. LXXIII, p. 362 and following, §44,45,46,48 and on Th. XXIV); but in the case where the considered boundary point is *accessible*, then those definitions come back to the one I gave above.

An already explained method [*read* §50] proves that Q cannot be the limit for its preimages of an arbitrary point of R_∞ without contradiction, because there may be some interior points of R_∞ in the interior of the line L; these points cannot be linked to points, which are both inside R_∞ and sufficiently far from the plane (for example, $| z | > 4$) so that they are outside L; this linkage cannot work, since there is no simple line intersecting L again. The line L cannot exit from R_∞ (because the points of L are inside R_0) without intersecting P, that's to say without intersecting an interior point of R_∞: this contradicts that R_∞ consists of one piece and it is simply connected.

Since every point P of E′ is *accessible* by R_0 and since it is *a simple point of the boundary of* R_0, *this boundary cannot be anything else but a Jordan curve* (*read* Th. XXIV, p.366, of the Memoir of CARATHÉODORY, cited before).

84. It is not so necessary to apply that theorem to verify that E′ is *a Jordan curve*.

In fact, every point of E′ is accessible from the interior of R_0. But all reasonings made for R_0 can be applied even to R_∞ and to the approximation of E′ with the help of the curves Γ_{-i} ; since in the region lying between C_{-I} and Γ, which is the same region wherein all Γ_{-i} are traced, one gets $| \psi'(z) | < N < 1$.

This method is used to verify the uniform convergence of Γ_{-i} to E′ in the same way it is used to verify the convergence of C_{-i} to E′.

Then, *every point of* E′ *is accessible from the interior of* R_0. A continuous closed curve E′, where every point is accessible from its interior (R_0) and from its exterior (R_∞), is *a Jordan curve*: *that's to say a continuous closed curve whose points can be mapped biunivocally and continuously to the points of a circumference* (*read* SCHŒNFLIES, *Die Lehre von den Punktmannigfaltigkeiten*, 2nd Part, Chapter V, § 11 and 12).

85. The simple closed line E′, the common boundary of R_0 and R_∞, encloses the roots points of the equations

$$z = \varphi_n(z) \qquad (n = 1, 2, ..., r) ,$$

where $| \varphi'_n(z) | > 1$ (points of E), everywhere dense in itself. In particular, that simple closed line intersects the point $z = 1 = \varphi(1)$, $\varphi'(1) = \frac{3}{2}$.

Every point of E′ has all its images and all its preimages in E′. That's to say E′ is invariant by the simple rational substitution $z_1 = \varphi(z)$ at by its inverses. Fatou, with the help of the theory of functional equations, showed in his Note (in *Comptes Rendus*) that the curve is never analytic.
It is easy to see that in any point of E it has not usually a determined tangent line.

86. Firstly, let us remark that, due to the following relation

$$z_1 = \frac{z+z^2}{2} = \frac{1}{2}\left(z+\frac{1}{2}\right)^2 - \frac{1}{8}\ ,$$

for any arbitrary point z_1 of E′, there are two preimages z, symmetric in respect of the point $-\frac{1}{2}$ and lying on E′.

That's to say, the curve E′ *admits the point* $-\frac{1}{2}$ *as the centre of the symmetry*; it is clear that it admits the real axis Ox as the symmetry axis, since C and C_{-i} are symmetric in respect of that axis. Moreover, a pair of points belonging to E is symmetric in respect of that axis, since $z = \varphi_n(z)$ is an equation with real coefficients: the points of E, which are everywhere dense on the simple Jordan curve E′, can determine the symmetric curve in respect of Ox, like the set E.

After all, E′ admits Ox and another straight line (intersecting the point $-\frac{1}{2}$ and parallel to the axis Oy) as axes of symmetry. Ox intersects E′ in two points, the point 1 and the point –2: the segment (-2,1) is in the interior of R_0, the two half-lines (1,+∞) and (-2,-∞) are in the interior of R_∞. Immediately it is clear that the segment ($-\frac{1}{2}$,1) of the real axis is in the interior of R_0, since all its points but $z = 1$ have 0 as the only limit point of their images; the segment (-2,$-\frac{1}{2}$), symmetric of the previous segment in respect of the point $z = -\frac{1}{2}$ (which is the centre of E′) is in the interior of R_0.

The half-line (1,+∞) is in the interior of R_∞, such as its symmetric half-line (-2,-∞), in respect of $z = -\frac{1}{2}$. E′ includes only the points $z = 1$ and $z = -2$ on Ox; therefore E includes only the point $z = 1$ on Ox.

87. Let us consider a point ζ of E, an imaginary root of $z = \varphi_n(z)$ and not lying on Ox. Let us define $\zeta_1, \zeta_2, \ldots, \zeta_{n-1}$ the points of E: they generate a periodic cycle of rank n with ζ ([1]). One gets

$$\varphi'_n(z) = \varphi'(\zeta)\varphi'(\zeta_1) \ldots \varphi'(\zeta_{n-1})$$

Now,

$$\varphi'(z) = z+\frac{1}{2}\ .$$

The arguments of $\zeta+\frac{1}{2}$, $\zeta_1+\frac{1}{2}$, …, $\zeta_{n-1}+\frac{1}{2}$ are the angles generating with Ox the segments going from the point $-\frac{1}{2}$ to the points $\zeta, \zeta_1, \zeta_2, \ldots \zeta_n$.

([1]) If ζ is imaginary, then the images ζ_i of the group are in general imaginary too.

These arguments do not vanish if ζ is any arbitrary imaginary point of E.
The sum of these arguments does not enjoy any particular property; just it is not *constantly null* for the points of E.
In general, $\varphi'_n(\zeta)$ *is not real* in any arbitrary point of E.
Doubts may arise for the periodic cycles with rank of an even number, as the groups with rank 2, where the points may be $\left(\zeta = \frac{-3-i\sqrt{15}}{2}, \zeta_1 = \frac{-3+i\sqrt{15}}{2}\right)$; it may happen that $\zeta, \zeta_1, \zeta_2, \ldots, \zeta_n$ are symmetric two by two in respect of the real axis [it occurs when $z = \varphi_n(z)$ has real coefficients]: in this case $\varphi'_n(\zeta)$ is real. But this does not surely happen for the groups having an odd number of periodic points: roots of this group cannot be symmetric two by two.

It may be supposed that any arbitrary point ζ (indexed at n) of E, but exceptional points, retrieves an imaginary value of $\varphi'_n(\zeta)$. In the general case, it may happen that the argument of $\varphi'_n(\zeta)$ is incommensurable with 2π.

Let us consider such a point of E, so that $\zeta = \varphi_n(\zeta)$ so that $\text{ARG}(\varphi'_n(\zeta)) \neq 0$ or π.
Since the critical points of the inverse of $\varphi_n(z)$ are the same as the inverse of $\varphi(z)$ [that's to say $-\frac{1}{8}$ and ∞] and the images until the rank n have only 0 and ∞ as limit points (n goes to infinity), then one may surround ζ by a circle, where the inverse function of $\varphi_n(z)$ has an holomorphic branch $\psi_n(z)$ that becomes equal to ζ for $z = \zeta$. One will get $\psi'_n(\zeta) = \frac{1}{\varphi'_n(\zeta)}$. Therefore $|\psi'_n(\zeta)| < 1$ and $\text{ARG}(\psi'_n(\zeta)) \neq 0$ or π, or the argument of $\psi'_n(\zeta)$ is incommensurable with 2π, in general. One may surround ζ with a circle C with ray ρ, wherein the relation $|z - \zeta| < \rho$ involves

$$|\psi_n(z) - \zeta| < H|z - \zeta| \qquad (0 < H < 1).$$

That's to say, the points $\psi_n(z), \psi_{2n}(z), \ldots, \psi_{pn}(z)$, which are preimages of rank n, $2n$, $3n$ … of the point z, converge to ζ, since $|z - \zeta| < \rho$.
But, it is clear that, given a small enough $|z - \zeta|$, one gets

$$\text{ARG}[\psi_n(z) - \zeta] - \text{ARG}(z - \zeta) = \text{considerably near to } \text{ARG}[\psi'_n(\zeta)]$$
$$\text{ARG}[\psi_{pn}(z) - \zeta] - \text{ARG}(z - \zeta) = \text{considerably near to } \text{ARG}[\psi'_{pn}(\zeta)] = p\text{ARG}[\psi'_n(\zeta)] .$$

It means that the points $\psi_n(z)$, $\psi_n(z)$ … converge to ζ and the rays linking ζ to two consecutives points (in this sequence) generate *an angle converging to* $\text{ARG}[\psi'_n(\zeta)]$, when the considered points converge to ζ as their index increases to ∞.

If $\text{ARG}[\psi'_n(\zeta)] \neq 0$ or π, then $\text{ARG}[\psi'_{pn}(\zeta) - \zeta]$ does not converge to a fixed limit, as p increases indefinitely, then $\psi'_{pn}(z)$ converges to ζ.

If ARG[$\psi'_n(\zeta)$] is incommensurable with 2π, then ARG[$\psi'_{pn}(\zeta)$ - ζ] is *near to any chosen value* ranging within 0 and 2π, for a big enough value of p. Let us remark immediately that if s is chosen on E′, then all the $\psi'_{pn}(z)$ are on E′: that's to say, E′ *cannot include in* ζ *a determined tangent line*; in the same way, if ARG[$\psi'_n(\zeta)$] is incommensurable with 2π, then E′ has in a neighbourhood of ζ some points which are near to every straight line coming out from ζ.

ζ is analogous to those points of loxodromic substitutions lying on the boundary of the domain of existence for a kleinian function [*read* POINCARÉ, *Mèmoire sur les groupes kleinéens* (*Acta, t.* III, 1883, p. 77) and SCHŒNFLIES, *Die Lehre* …, Chapter V, § 14, already cited]. The curve E′ surrounds one of that points ζ in the same way as a double logarithmic spiral with only a pole (1).
But I won't say anything more: the previous works about the nature of a Jordan lines have already and sufficiently explained the behaviour of such curves in a neighbourhood of one of their points.

88. But, since I wrote about a comparison between the points ζ of E lying on E′ and since the double points of the substitution of a kleinian group admit a fundamental curve of the same kind as the curves, notified by Poincaré, then I will point at a theoretic generation of E′, compared with the generation of the fundamental curve of a kleinian group, starting by the fundamental domain of that same group.

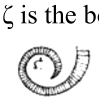
In the last case, Poincarè starts from a simply connected fundamental domain D, bounded by some circles C_1, C_2,… C_n, so that C_1 touches C_2, C_2 touches C_3, … C_{n-1} touches C_n, and finally C_n touches C_1; all contacts are exterior the circles.

After adding to D the domains coming out from D by symmetries related to the sides of the domain D and after adding to the already resulting domain its symmetrics in respect of its sides … indefinitely, then the *locus* of the vertexes of the domains, the derived set of the set of these vertexes, is the same curve as the one notified by Poincaré.

89. I recall that, in the beginning of this Memoir, I pointed my attention to the comparison between the generation of the preimages of a point z_1 in the iteration of the relation $z_1 = \varphi(z)$ and the generation of the homologous points from a point z_1 in an automorphic group as preimages (1) of a point z_1, since the preimage z (of z_1) is defined by the following relation

$$f(z, z_1) = [z_1 - S_1(z)] \dots [z_1 - S_n(z)] = 0.$$

(1) There is no contradiction. Between two non intersecting logarithmic spirals (with the same pole ζ) one may observe a simply connected area for which ζ is the boundary point (*look at the figure below*).



The $S_n(z)$ are the fundamental substitutions of the group (*read* § 17).
The fundamental domain D of the kleinian group includes, whatever is the interior point z in the domain of existence, an homologous point only in the interior of D (two points only when the homologous point is on the boundary of D).

90. I will determine a fundamental domain V for the iterations of $z_1 = \frac{z+z^2}{2}$, that's to say an interior domain D of R_0, so that if one choices any arbitrary interior point z of R_0, then z has, in the set of its preimages, *at most a preimage of the same rank* ([2]) that is inside D (except for those points lying on the contour of D); starting from a certain index i, there exists, in D, *a image of any rank* z_i, $z_{(i+1)}$, … .

I can make an easier exposition: let us firstly come back and fix the substitution

$$z = -\frac{1}{2} + Z$$

then $z_1 = \varphi(z) = \frac{z+z^2}{2}$ could be rewritten as

$$Z_1 = \frac{4Z^2+3}{8}$$

so that the limit points with uniform convergence are both ∞ and $\frac{1}{2}$. Every point Z_1 of the plane has two symmetric preimages in respect of the new origin $Z = 0$. Therefore, it has one and only preimage *in the upper half-plane* $\mathrm{Im}(z) > 0$, defined D_{-1}.

Let us remark immediately that $Z_1 = \frac{3}{8}$ is a critical point for the function $Z = \Phi(Z_1)$, the inverse of $Z_1 = \frac{4Z^2+3}{8}$. If Z_1 describes the real axis from $-\infty$ to $+\infty$, then the half-line $\left(+\infty, \frac{3}{8}\right)$ is mapped, for Z, to the half-line $(-\infty, 0)$, since Z is the preimage of Z_1 lying in the upper half-plane when Z_1 is any point in the plane; when Z_1 describes $\left(\frac{3}{8}, -\infty\right)$, Z describes *the positive part of the imaginary axis* O*y*; while Z_1 describes the upper half-plane $\mathrm{Im}(Z_1) > 0$, the point, belonging to the preimages of Z_1 and lying on the upper plane, describes the first quadrant XOY [$\mathrm{Re}(Z) > 0$, $\mathrm{Im}(Z) > 0$], defined D_{-2}.

([1]) Since z_1 is said to be the image of z: it is evident that the set of the *images* of a point is the same as the set of the *preimages*.
([2]) In the case where the homologous points, of any z_1 in a automorphism group, are defined as the images or as the preimages of z_1 by the relation $f(z, z_1) = 0$, then there is an infinity of preimages falling into D, but *they are confused in the same point z*, only one of them with different rank. These are all the properties of D that I gave to D too.

Therefore every point of the plane has a preimage of rank 1 in the upper half-plane D_{-1} and a second preimage, of the previous point, that is symmetric in respect of 0; between the preimages of rank 2: the first one of the two preimages, coming out from the first preimage of rank 1, lies in the first quadrant, the second one lies in the third quadrant; the preimages, coming out from the second preimage of rank 1, lie in the second and in the fourth quadrant respectively [since, in respect of the real axis, two symmetric points are mapped to symmetric preimages in respect of the same axis (since $Z_1 = \frac{4Z^2+3}{8}$ has real coefficient), therefore any point of the lower half-plane is mapped to a preimage in the fourth and to another opposite preimage in the second quadrant].
Every point of the plane has one and only one preimage of rank 2 in the first quadrant D_{-2}. Let us keep on going with the same process.

Every point Z of the upper half-plane has only one preimage Z_{-1} in the first quadrant; every point Z_{-1} of the first quadrant has only one preimage Z_{-2} in a part of the first quadrant, described by the preimage Z_{-1} of rank 1 of Z, as Z describes the first quadrant. Then Z describes:

1. The real half-axis $\left(+\infty, \frac{3}{8}\right)$, when Z_{-1} describes $(+\infty,0)$;
2. The segment $\left(\frac{3}{8},0\right)$, when Z_{-1} describes a segment OO_{-1} of the imaginary axis OY (O_{-1} is the preimage of O for the affixe $\frac{i\sqrt{3}}{2}$);
3. The imaginary half-axis, when z_{-1} describes a half-branch of hyperbola.

It is easy to obtain the equation of this hyperbola: $Z = X+iY$, mapped to to $Z_1 = X_1+iY_1$, so that

$$X_1 = \frac{4(X^2-Y^2)+3}{8}$$
$$Y_1 = XY$$

The requested hyperbola H can be obtained by fixing $X_1 = 0$; therefore it is described by the following equation

$$X^2 - Y^2 + \frac{3}{4} = 0 \ .$$

This is an equilateral hyperbola. When Z describes the first quadrant, one of its preimages (lying over OX) describes an area OO_{-1}, bounded by OX, and the branch H including the bisector of XOY. I define this area D_{-3}.
Therefore every point of the plane has only one preimage of rank 3 in D_{-3}, as it has only one preimage of rank 2 in D_{-2}.
Any arbitrary point of the plane has only one preimage of rank 1 in D_{-1} only one preimage of rank 2 in D_{-2} and only one preimage of rank 3 in D_{-3}.

But an arbitrary point of the plane has two preimages of rank 2 in D_{-1}, two preimages of rank 3 in D_{-2}, four preimages of rank 3 in D_{-3}, etc.
Therefore, if we would seek the fundamental domain D so that it has *only and at most one preimage of any rank in* D for every interior point of R_0, we couldn't take for D nor D_{-1}, nor D_{-2}, … the process may keep on going indefinitely.

Every interior point Z of D_{-3} has only one preimage in D_{-4}, inside D_{-3} and bounded by:

1. The axis OX;
2. The segment OO_{-1} of OY;
3. The arc $O_{-1}O_{-2}$ of the branch of hyperbola H bounding D_{-3} (since O_{-2} is the preimage of O_{-1} on that branch).
4. One infinite branch H_{-1} of an algebraic curve, preimage of rank 1 of H and coming out from O_{-2} orthogonally and traced in the interior of D_{-3}, having an asymptotic direction that generates an angle of $\frac{\pi}{8}$ with OX; the asymptotic direction of H is $\left(+\frac{\pi}{4}\right)$ with OX.

In the same way, every interior point of D_{-1} has only one preimage in an area D_{-5}, preimage in D_{-4} and in the interior of D_{-4} and bounded by OX; OO_{-1}, the arc of H going from O_{-1} to O_{-2}, the arc of H going from O_{-2} to O_{-3} (the preimage of O_{-2}); finally an infinite branch H_{-2} of the algebraic curve, preimage of H_{-1} and starting orthogonally with H_{-1} from O_{-3} and traced in D_{-4}; its asymptotic direction is $\frac{\pi}{16}$.

91. Let us pursue this process indefinitely: each one of the areas D_{-1}, D_{-2}, … is included in the previous one; if Z describes D_{-i}, then it has only one preimage of rank 1 in $D_{-(i+1)}$; $D_{-(i+1)}$ is the preimage of D_{-i}.

The area D_{-i} does not converge to zero as *i* increases indefinitely.

In fact, let us examine the neighbourhood of $z = \frac{1}{2}$, which is a limit point with uniform convergence; for example, let us examine *the part* (O_1), *lying over* OX, *of the area bounded by small enough circle* ([1]), *centered at* $z = \frac{1}{2}$: there is an area lying in D_{-1}, D_{-2}.

If it is in D_{-i}, so it is in $D_{-(i+1)}$ too.
In fact, if Z describes the area (O_1), then its preimage of rank 1, lying over OX, describes an area (O), including the area (O_1) in its interior ([2]); the boundaries of these two areas do not share anything else but a segment of the real axis.
Now (O) is included in $D_{-(i+1)}$, since $D_{-(i+1)}$ is described by the preimage of Z lying in the upper half-plane, when Z describes D_{-i}.

Therefore $D_{-(i+1)}$, including (O), includes (O_1) that is interior in (O).
All the D_{-i} include (O_1).
So there is not any ambiguity on the *limit area* Δ *of* D_{-i} *for* $i \to \infty$.
This area includes the area (O_1). It is bounded by an infinity of algebraic lines :

1. The real axis OX.
2. The segment OO_{-1} of OY.
3. The arc $O_{-1}O_{-2}$ of the branch H;
4. The arc $O_{-2}O_{-3}$ of the branch H_{-1}, being the preimage of H;
5. The arc $O_{-3}O_{-4}$ of the branch H_{-2}, being preimage of H_{-1}, …

$(i+1)^0$ The arc $O_{-(i+1)}O_{-i}$ of the branch $H_{-(i-2)}$, being the preimage of $H_{-(i-3)}$ … .

Δ is a curvilinear polygon with an infinity of sides, where all angles are right; if one travels on these sides from the point $+\infty$ to O, on the real axis by describing OO_{-1}, …, then one finds the area Δ on the right and each side of the polygon is the *preimage* of rank 1 of the previous side.

It is necessary to remark that the point $\frac{3}{8}$ shall be considered as a vertex of the polygon (critical vertex), whose preimage is O; OO_{-1} is the preimage of the segment $\left(0, \frac{3}{8}\right)$ of OX, and OX is the preimage of the half-line $\left(+\frac{3}{8}, +\infty\right)$ of OX.

92. *An interior point* of Δ is inside all areas D_{-i}; therefore, the preimage of rank 1 of that point, lying over OX, is still inside all D_{-i} according to the definition of D_{-i}; therefore that preimage is in the *interior* of Δ.
Every interior point of Δ has only one preimage of rank 1 in Δ. Every interior point of Δ has its image inside Δ, since that image is inside all areas D_{-i}.

All the images of an interior point in Δ are in the interior of Δ. Let us examine the preimages of an arbitrary point of Δ. There is only one of rank 1 inside Δ. However, every *exterior point of* Δ *has no preimages of rank* 1 *inside* Δ *neither on its contour* since the image of an interior point of Δ (or on its contour) is still in the interior in Δ or on its contour;

([1]) One may consider the circle $\left|Z - \frac{1}{2}\right| \le \frac{1}{16}$, because the distance of $\frac{1}{2}$ to the near critical point $Z = \frac{3}{8}$ is $\frac{1}{8}$, we will take a circle O_1 with ray $< \frac{1}{8}$, leaving the critical point in the exterior of that circle for avoiding the difficulty relating to the chosen preimage.

([2]) If $\left|Z - \frac{1}{2}\right| \le \frac{1}{4}$, one gets $\left|Z_1 - \frac{1}{2}\right| \le \frac{3}{4}\left|Z - \frac{1}{2}\right|$, by a process employed many times in this Memoir.

in opposite with the hypotheses, it is necessary to admit that the given point is in the interior of Δ. Therefore any exterior point of Δ has not *any preimage inside* Δ.

But if an interior point in Δ has one and only one preimage of rank 1, then it has *only one preimage of any rank* inside Δ.
Therefore Δ enjoys already this property so that every point of the plane has only one preimage of any rank inside Δ, if this point is in the interior of Δ; the point has no preimages if it lies outside Δ.

93. I wrote that Δ *is all entire at the finite distance*; every interior point of Δ has, for example, a module $| Z | < 4$; in the same way every point of the contour of Δ, except if it belongs to the axis OX, is at a finite distance.

In fact, every point of the contour of Δ, not lying on OX, has a image on OO_{-1} and another one on the segment $\left(O, \frac{3}{8}\right)$ of the axis OX; then, all the points of the last segment are in the interior of R_0, as one immediately realizes.
Therefore, all the points of the contour of Δ, except the half-line $\left(\frac{3}{2}, +\infty\right)$ of OX ([1]), are in the interior of R_0.
One may conclude that Δ is entirely in R_0, since Δ is a simply connected domain and it is bounded by one only contour. Anyway this is an easy removable obstacle.
Since every interior point Z of Δ has all its images Z_n inside Δ, if θ_n refers to the argument of the image Z_n, then one would get $\tan(\theta_n) > 0$, whatever n is.
Now, if r and θ are the coordinates of Z, then $Z = r^{ei\theta}$; and if r_1 and θ_1 are the coordinates of Z_1, then $Z_1 = r_1 e^{i\theta}{}_1$.
Then one gets

$$r_1^2 = \left(\frac{4r^2-3}{8}\right)^2 + \frac{3r^2}{4}\cos^2(\theta)$$

and

$$\tan\theta_1 = \frac{8r^2\tan\theta}{4r^2+3-(4r^2-3)\tan^2\theta}\ .$$

If $| r | \geq 4$, one gets

$$4r^2 - 3 > 0\ , \qquad r_1 \geq \frac{4r^2-3}{8} > 4 \quad \text{and} \qquad 4r_1^2 - 3 > 0$$

then,

$$\tan\theta_1 > \tan\theta\ , \qquad \text{on condition that} \qquad \tan^2\theta < \frac{4r^2+3}{4r^2-3}\ ,$$

([1]) Moreover, as it would be recognized later, the point does not lie on the boundary of Δ, because it is not the limit of the interior points of Δ.

and

$$\tan\theta_1 < 0 \qquad \text{if} \qquad \tan^2\theta < \frac{4r^2+3}{4r^2-3}.$$

In the last case, if $\tan^2\theta > \frac{4r^2+3}{4r^2-3}$, $\tan\theta_1 < 0$; therefore z_1 does not lie inside D; therefore z is not inside D anymore. If $\tan^2\theta < \frac{4r^2+3}{4r^2-3}$, then one gets

$$\tan\theta_1 > \tan\theta \qquad \text{and} \qquad r_1 > r\,;$$

therefore, the bigger θ_1, the bigger r.
One keeps on going until

$$\tan^2\theta < \frac{4r^2+3}{4r^2-3}$$

as

$$\tan\theta_{i+1} > \tan\theta_i \quad (\theta_{i+1} > \theta_i\,] \qquad \text{and} \qquad r_{i+1} > r_i\,.$$

But, as r_i increases indefinitely as the index i grows, then $\frac{4r_1^2+3}{4r_1^2-3}$ decreases constantly to 1, while $\tan\theta_i$ constantly grows until, for a certain index p, one will get $\tan^2\theta_p > \frac{4r_p^2+3}{4r_p^2-3}$: since the image Z_{p+1} is out from Δ, Z cannot belong to Δ.
It happens that a certain Z_p will satisfy

$$\tan^2\theta_p > \frac{4r_p^2+3}{4r_p^2-3}$$

because the opposite hypothesis obliges the function $\tan\theta_p$ to grow: it will converge to a positive limit ≤ 1 for $p \to \infty$; but the relation

$$\tan\theta_p = \frac{8r_{p-1}^2\,\tan\theta_{p-1}}{4r_{p-1}^2+3-(4r_{p-1}^2-3)\tan^2\theta_{p-1}},$$

where r_p diverges to ∞ if p increases indefinitely, proves that it is impossible for both $\tan\theta_{p+1}$ and $\tan\theta_p$ to converge to the same limit $l \neq 0$, because one should get for l the following relation

$$l = \frac{8l}{4(1-l^2)} = \frac{2l}{1-l^2}$$

which is satisfied only by the real number $l = 0$ and the imaginary numbers $l = \pm\, i$. Therefore, it is proved that every point of the circle $|\,Z\,| = 4$ is outside Δ, since at least one of its images is outside Δ (the one image satisfying $\tan\theta_p < 0$).

Therefore *all the entire* Δ *is in the interior of* $|\,Z\,| \leq 4$. Every *interior point* of Δ has all its images in the interior of Δ, not on its contour: therefore all these images are at finite distance. Every interior point of Δ is in the interior of R_0 like every point of the contour of Δ except the half-line $\left(+\frac{3}{2},+\infty\right)$ of the real axis OX. *We agree that the half-line does not belong to the contour of* Δ, moreover there are not any points of Δ, out of the real axis in

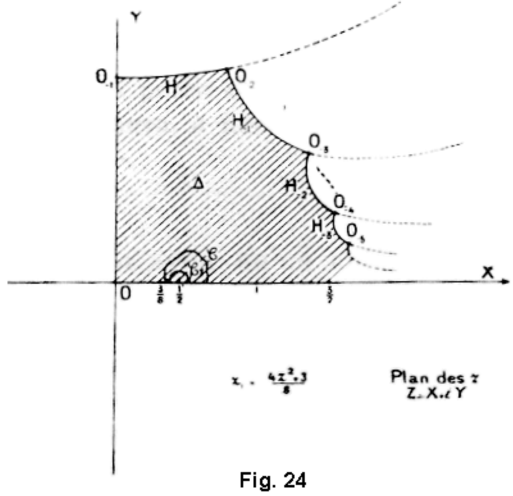


Fig. 24

the neighbourhood of every point belonging to $\left(+\frac{3}{2},+\infty\right)$ of OX. The boundary of Δ will be the segment $\left(0,\frac{3}{2}\right)$ of OX, after OO_{-1}, $O_{-1}O_{-2}$, and so on …, since all the sides are in the interior of R_0 and since they confine with a simply connected region, including the area (O_1) $\left|Z-\frac{1}{2}\right| \le \frac{1}{16}$ and lying over OX, already mentioned above.

94. Let Δ' be the symmetric domain of Δ in respect of the real axis.
Let us examine the domain D, consisting of the union of Δ and Δ' after the elimination of the common segment of the boundary $\left(O,\frac{3}{2}\right)$ that lies on the real axis. D is a simply connected domain, consisting of one piece and in the interior of R_0; it does not share anything else but the point $Z = \frac{3}{2}$ with the boundary of R_0. Since both Δ and Δ' enjoy the same properties (in fact any arbitrary pair of symmetric points, in respect of OX, has their own symmetric preimages or images, in respect of OX), then the relation $Z_1 = \frac{4Z^2+3}{8}$ fixes a *biunivocal* mapping of the interior of Δ' to itself, as the interior of Δ to itself.
The set Δ + Δ' = D includes the circle $O_1 \left|Z-\frac{1}{2}\right| < \frac{1}{16}$, surrounding the limit point $Z = \frac{1}{2}$ with uniform convergence.

The result is that every interior point of R_0, (since all its images are in the interior of O_1, as it starts from a certain rank) has, starting from a certain rank, all its images in the interior of D.

95. D enjoys the following properties:

1. Every interior point of R_0 may have at most only a preimage of any rank in the interior of D. It may happen that the point is in the interior of O or not. If the same point is in the interior of Δ, then it has a preimage of any rank inside Δ; if it is in the interior of Δ', then it has a preimage of any rank inside Δ'.

2. Every interior point of R_0, starting from a certain rank, has all its images in the interior of D (only an image of the same rank).

Every interior point of D is mapped to an interior preimage of D; it can be said that the interior of D is biunivocally mapped to itself; it is necessary to suppose, due to the critical point in $Z = \frac{3}{8}$, that D has been cut by following the segment $\left(O, +\frac{1}{2}\right)$ of OX (including $Z = \frac{3}{8}$); it is necessary to imagine that this last traced cut belongs to the boundary of D.

[The preimage of this cut is the segment $\left(-\frac{i\sqrt{3}}{2}, +\frac{i\sqrt{3}}{3}\right)$ of OY, related with the segment $\left(O, +\frac{1}{2}\right)$ of OX.]

96. Due to the 2nd property of D, one sees that R_0 can be obtained by adding both the areas Δ and Δ' to all their preimages of rank 1, after to all their preimages of rank 2, … and so on; this process keeps on going indefinitely by suppressing all the common parts of the boundaries of the new determined areas.

In this way R_0 is generated, starting from Δ and recalling the generation of the domain of existence of a kleinian group, as it has been cited above. In fact, the preimages of Δ are the set Δ and its symmetric set in respect of O; the preimages of Δ' are the set Δ' and its symmetric set in respect of O.

There are four joint areas, after the elimination of the common boundaries [the segment $\left(-\frac{3}{2}, +\frac{3}{2}\right)$ of OX, the segment $\left(-\frac{i\sqrt{3}}{2}, +\frac{i\sqrt{3}}{3}\right)$ of OY] that generate a curvilinear polygon D_{-1}, which is symmetric in respect of OX and OY and it is bounded by some arcs of algebraic curves H , H_{-1} , …, that confine Δ: the polygon whose all angles are right.

D_{-1} is the polygon, described by the two preimages of Z, when Z describes D.

D_{-1} consists of D and its symmetric in respect of OY.

Now, two symmetric (in respect of OY) points are mapped to some preimages being symmetric two by two in respect of the equilateral hyperbola H, which is the preimage of the axis OY (as the same definition of symmetry in respect of an analytic curve, according to Schwarz). Therefore, when Z describes D_{-1}, then its two preimages describe the polygon D_{-1}, which shall be necessarily joint to its image set in respect of the hyperbola H. This image set consists of *two curvilinear separated polygons*: the first one touches D_{-1} along the side $O_{-1}O_{-2}$ and its symmetric in respect of OY ([1]); the second polygon is the symmetric of the first one in respect of O.

The set of D_{-1} and its image in respect of H generates a curvilinear polygon D_{-2} with the same kind of those in D_{-1}.

Let us consider the same reasoning that from D_{-1}, which is generated by the set D and its symmetric in respect of OY, deduces the set D_{-2}, which is generated by the set D and its symmetric in respect of the hyperbola H, the side of D_{-1} the preimage of OY; this method proves that, when Z describes D_{-2}, then its two preimages describe a domain D_{-3}, consisting of D_{-2} and in its image in respect of the algebraic curve H_{-1}, the preimage of the hyperbola H.

The image of D_{-2}, in respect of H_{-1}, consists of four separated polygons, which are symmetric to each other in respect of two coordinated axes, touching respectively D_{-2} by following four arcs that are deduced from the arc $O_{-2}O_{-3}$ of H_{-1}, which is the side of Δ (*extended arc by its image, related to* H), by the related symmetry with the coordinated axis and the origin. The process can keep on going indefinitely.

$D_{-(i+1)}$ is deduced from D_{-i} by adding the image of D_{-i} in respect of the algebraic curve $H_{-(i-1)}$, which is the preimage of OY with rank i ; as Z describes D_{-i}, its two preimage describe $D_{-(i+1)}$. R_0 is evidently the limit of D_{-i}, for $i \to \infty$.

The attention is pointed to the process for generating R_0, starting from D; this process is used to generate the domain of existence of a kleinian function with a fundamental curve, starting from the fundamental domain D of the group and then adding the domain D to its images, in respect of its sides.

Nothing's shocking then, if the boundary of R_0, like the limit curve notified by Poincaré, is a closed *Jordan curve* E′ whereon the points (where the curve has no tangent lines) may generate an everywhere dense set.

([1]) The first branch of the hyperbolas H.

The points of E, everywhere dense on E′, since they are the *points coinciding with one of its preimages* ([1]), play a role aside with the double points ([2]) of a kleinian group substitutions lying on the fundamental curve of that group.
In all these last points (the double points of the loxodromic substitutions), the fundamental curve of the group cannot have a determined tangent (*read* SCHÖNFLIES, *loc. Cit.*, Chapter 4, §14). For the same reason, E′ cannot have a determined tangent line in each point ζ of E, where $\arg\varphi'_n(\zeta) \neq 0$ or π. The shown example belongs to a simple type of a category of much more general examples (appearing further) approaching to the same conclusions.
Those examples have been chosen in respect of the previous examples, exposed previously in the 3rd and 4th *Application* of the second part of this Memoir.

97. *Second example*. – Firstly, let us take in consideration the case of any fraction of second degree $z_1 = \varphi(z)$, so that it admits two limit points ζ_1 and ζ_2 with uniform convergence; let us suppose that one may surround ζ_1 and ζ_2, respectively with two curves C and Γ, so that:

1. The image of the area (C), including ζ_1, is an area (C_1), *in the interior of* (C) and including a critical point of $\psi(z)$ which is the inverse function of $\varphi(z)$;
2. The image of the area (Γ), including ζ_2, is an area (Γ_1), *in the interior of* (Γ) and including a critical point of $\psi(z)$ which is the inverse function of $\varphi(z)$; [this means that both (C) and (Γ) include a root of $\varphi'(z) = 0$];
3. In the double connected area, between C and Γ, one gets

$$|\varphi'(z)| > M > 1$$

Then it is evident that, between C and L, one gets

$$|\psi'(z)| \leq \frac{1}{M} < 1$$

The area (C) is in the interior of R_1, which is the domain of convergence to ζ_1; the area (Γ) is in the interior of R_2, which is the domain of convergence to ζ_2.
The preimages (C_{-i}) of the area (C) converge uniformly to an area R_1, since the preimage curves C_{-i} of C converge uniformly to their limit, due to $|\psi'(z)| < N$ between C and Γ: a region intersected by C_{-i}. Thus, the preimages Γ_{-i} of Γ converge uniformly to their limit in the same way as C_{-i}: a simple closed *Jordan curve* E′ splitting R_1 from R_2.
The generation of R_0 (referring first case) can be extended to this case by replacing the real axis (of the previous example) with a proper break in the plane joining the critical points of $\psi(z)$. Instead of the 3rd condition, *one may claim only the following condition*:

([1]) Because it satisfies $Z = \varphi_n(Z)$, for $n = 1, 2, \ldots, \infty$; the points of E, coinciding with their images of rank n, coincide with one of their preimages of rank n too.
([2]) The double points are, in fact, points coinciding with one of their homologues in the group.

4. *Between* C *and* Γ, one gets $|\varphi'_n(z)| > M > 1$ *for a certain value of n* ; one sees that the set C_{-n}, C_{-2n}, ..., C_{-pn}, … converges uniformly to their limit: a *simple closed Jordan curve*. One chooses the (previous) first and the second conditions as properly as possible; generally, there would be some preimages (with enough high rank) of small circles surrounding ζ_1 and ζ_2.

98. *Third example. – a.* But it is clear that a fraction with rank 2, satisfying the conditions of the previous example, [*all conclusions are still valid* (1)] may be substituted by a fraction with an arbitrary degree, having only two points where $\varphi'(z)$ vanishes (2) (that's to say whose inverse function has only two critical points) and having two limit points ζ_1 and ζ_2 with uniform convergence in respect of the first and second condition of the previous example, and both the third and the fourth condition of the second example. Then, there is a *simple closed Jordan curve* E′ splitting the domains of convergence R_1 and R_2 to ζ_1 and ζ_2 respectively.

b. Recalling the application 4° (§ 35), one may avoid the restriction for $\varphi'(z)$ vanishing in only two distinct points.
Let $z_1 = \varphi(z)$ a rational function with degree K, having two limit points, ζ_1 and ζ_2, with uniform convergence

$$[\zeta_1 = \varphi(\zeta_1),\ |\varphi'(\zeta_1)| < 1\ ;\ \zeta_2 = \varphi(\zeta_2),\ |\varphi'(\zeta_2)| < 1]$$

enjoying the following properties, analogous with the 1st and 2nd property of the 3° and 4° previous examples.

1. One may find a simple closed curve C, surrounding ζ_1 (and separating ζ_1 from ζ_2) and confining an area (C) including ζ_1, so that *the set of* K *preimages* of a interior point z of (C) describes an area (C_{-1}) including (C) in its interior, since the curve C_{-1} is a simple closed curve which is described once by every preimage of z as z describes K times the curve C.

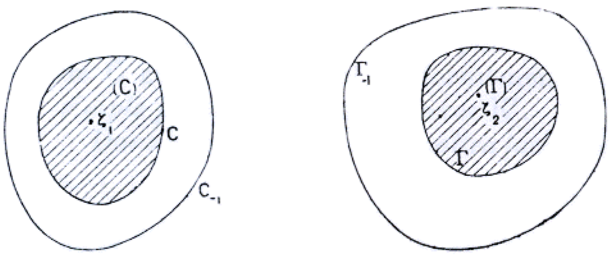


Fig. 25

(1) Except the generation of domains of convergence by symmetries.

(2) For example, a fraction of the type $z_1 = \dfrac{az^k + b}{cz^k + d}$ (*read* Application 4°, § 35).

2. One may find a simple closed curve Γ, surrounding ζ_2 and separating ζ_2 from both ζ_1 and from the curve C, since Γ is outside C so that the area (Γ), bounded by Γ and including ζ_2, has, for the set of its K preimages, a simply connected area (Γ_{-1}) including the area (Γ) and bounded by simple closed curve Γ_{-1}; this last curve Γ_{-1} is described once by each preimage of z as z describes the curve Γ for K times (*look at* the picture above).

3. Moreover, one may suppose that in the double connected area, bounded by C and L (or in the area bounded by two arbitrary preimages C_{-i} and Γ_{-j} of C and Γ), one constantly gets $|\varphi'_n(z)| > M > 1$ for a certain index n (more often, $|\varphi'(z)| > M > 1$).

Then, according to the 4th application (§ 35, case *b*), the plane is split into two regions R_1 and R_2; each one of the two domains of convergence to the points ζ_1 or ζ_2 (respectively) is separated by a continuous line E′ which is the limit of the preimages of C and Γ.
But *it is enough that in the double connected area, bounded by any two preimages* C_{-i} *and* Γ_{-j} *of* C *and* Γ, *the following relation holds*

$$|\varphi'_n(z)| > M > 1$$

for a certain index n so to ensure that the preimages of C and Γ converge uniformly to a simple closed Jordan curve coinciding with E′.

During the practice, *one easily recognizes the case where*

$$|\varphi'_n(z)| > M > 1$$

in the annulus bounded by C *and* Γ and referring to the previous general case.

99. *Fourth example*. – In particular, let us come back to the examples, notified by Fatou in his Note on May 21th 1917 in *Comptes Rendus* and recovered in this Memoir (application 4°, *b*, of the 2nd part, § 35).

Let us take a function in the fundamental circle O so that it admits two symmetric limit points ζ_1 and ζ_2 with uniform convergence, in respect of the fundamental circle O.

In the case where the fundamental circle is at a finite distance (that's to say it cannot be reduced to a straight line while ζ_1 and ζ_2 are the centre of the circle and the point at infinity, respectively): this occurs by applying an homographic mapping.
It has been seen previously that $|\varphi'(z)| > 1$ is retrieved all over the circle.

The function $\psi(z)$, the inverse of $\varphi(z)$, admits in general K - 1 simple critical points in the interior of O and other K - 1 simple critical points which are the symmetric points of the first ones in respect of O.

One sees that, since z is an arbitrary point in the interior of the circle O, then one gets

$$|\varphi(z)| < |z|;$$

according to the Schwarz's Lemma, if z is outside the circle O, then

$$|\varphi(z)| > |z|.$$

So

$$\varphi(0) = 0, \varphi(\infty) = \infty, \qquad |\varphi'(0)| < 1$$

Therefore, let us consider a circle C, centered at ζ_1 and in the interior of O, like its symmetric Γ in respect of O; C has been chosen to be *as near as possible to* O for obtaining:

1. The critical points of $\psi(z)$ are in the interior of C or outside Γ;
2. Between C and Γ, one gets

$$|\varphi'(z)| > M > 1$$

It is possible to get $|\varphi'(z)| > 1$ and never $|\varphi'(z)| = 1$.

Evidently if z describes C, then its K preimages describe a simple closed curve C_{-i}, surrounding C and being traced in the annulus (C,O) between C and O.
The preimages of C, C_{-1}, C_{-2}, … surround mutually, since C_{-i} in the annulus ($C_{-(i-1)}$,O), surrounding $C_{-(i-1)}$, …
Each one of the preimages of Γ, Γ_{-1}, Γ_{-2}, … includes the previous indexed curve.

Γ_{-i} is in the annulus ($\Gamma_{-(i-1)}$,O) and it surrounds O.
All previous conditions 1°, 2°, 3° are satisfied.
All C_{-i} and Γ_{-j} converge uniformly to the circle O.

100. But if the coefficients of $\varphi(z)$ range within proper restricted limits, then the rational function $\Phi(z)$ behaves in respect of the following relations:

1. $\Phi(z)$ would have two limit points with uniform convergence, near to 0 and to ∞, since both the two roots 0 and ∞ of $z = \varphi(z)$ are mapped to two near roots of the function $z = \Phi(z)$ with a *small* variation of the module of the derivative in those points. I define Z_1 and Z_2 those two limit points .

2. $\Psi(z)$, the inverse function of $\Phi(z)$ has still K - 1 critical points in the interior of C and K - 1 critical points in the exterior of Γ; it is evident since the critical points of $\Psi(z)$ are near to the critical points of $\psi(z)$.

3. The set of K preimages $\Psi(z)$ of z (as z describes C) describes a closed curve C'_{-1} (already found) surrounding C and very near to C_{-1}. Therefore C'_{-1} surrounds C and it will be traced in the annulus (C,Γ). In the same way, the set of K preimages of z,

while z describes Γ, describes a closed curve Γ'_{-1} lying in the annulus (C,Γ) surrounding C'_{-1}.

4. Finally one gets $|\,\Phi'(z)\,| > M_1 > 1$ within C and Γ like one gets $|\,\varphi'(z)\,| > M > 1$.

These four conditions are satisfied *if the variations of the coefficients of* $\varphi(z)$, *turning* $\varphi(z)$ *into* $\Phi(z)$*, are small enough*.

The set of the previous four conditions proves that the plane is split into two simply connected regions R_1 and R_2, which are the respective domains of Z_1 and Z_2; these two regions are separated by a simple closed *Jordan curve* E′ which is the common limit of the preimages of the circle C and Γ by the transformation $z_1 = \Phi(z)$.

Evidently E′ is traced in the annulus (C,Γ).
This annulus is very near to the fundamental circle O.
Therefore it is clear that: in respect of some very small changes (in absolute value) of the coefficients of a fraction $\varphi(z)$ in the fundamental circle O and of the type mentioned before (arbitrary changes), then the *inequalities*, coming out from the new coefficients*,* give birth to much more general fractions than the fractions in the fundamental circle so that one obtains some new fractions $\Phi(z)$, related to those splitting the plane into two regions separated by a simple closed *Jordan curve* near to the circle O.

Resuming, the curve (separating the domains of convergence to ζ_1 and to ζ_2) *changes continuously* in respect of its coefficients as the coefficients of $\varphi(z)$ change infinitesimally.
This is really far to be *a priori* evident.

101. Let us consider one of the new functions $\Phi(z)$, coming out from a function $\varphi(z)$ in the fundamental circle by small enough changes of the coefficients.
Then, let us examine the *Jordan curve* E′ splitting the domains ζ_1 and ζ_2; the roots of the equation

$$z = \Phi_n(z)$$

are everywhere dense on E′.
These roots satisfy the following condition:

$$|\,\Phi'_n(z)\,| > 1 \qquad\qquad (n = 1, 2, \ldots, \infty).$$

Let us examine one of these roots: there would be a point P of the set E so that $\Phi'_n(z)$ *is not real* in P. Further, some more specifications.

If one considers, together with $\Phi(z)$, the fraction in the fundamental circle $z_1 = \varphi(z)$ where $\Phi(z)$ comes from, then $\Phi(z)$ has some arbitrary coefficients acting like parameters related only to certain *inequalities* in respect of $\varphi(z)$: for example, every coefficient of $\Phi(z)$ is forced to have its affixe in the interior of a small circle, centered at the affixe of *the value of the considered coefficient in* $\varphi(z)$.

The values of $\Phi'_n(z)$ in the points of the roots for $z = \Phi_n(z)$ ($n = 1, 2, \ldots, \infty$) are some functions with the previous parameters; if $\Phi'_n(z)$ is real, then *a relation of equality holds within certain parameters*.

A better precision is attained by discovering that the value of $\Phi'_n(z)$ is a algebraic function of *coefficients-parameters* of $\Phi(z)$ in a root point of $z = \Phi_n(z)$; if $\Phi'_n(z)$ is real, then an algebraic relation (not an identity) lies within certain parameters.
If one considers all the relations for $n = 1, 2, \ldots, \infty$, then one gets an infinity of algebraic (R) relations.

Since each one of these parameters may change in a small circle of their plane, then *it is always possible to choose these parameters so that one of the previous relations* (R) *could not be satisfied*.

It is achieved, for example, by choosing arbitrarily all parameters but one; after, this last parameter is chosen so that it is different from the roots of all algebraic equations relating to this parameter, whereunto the algebraic relations (R) are reduced in these conditions: the found roots generate a *countable set*, while the set of the values, assumed by the last parameter, has the power of continuum.

Such a chosen fraction $\Phi(z)$ may be defined a *general fraction*: that's to say a fraction *whose coefficients do not satisfy any particular relation*.

On the other hand, $\Phi(z)$ can belong to the *general* type, so that *any of the values of* $\Phi'_n(z)$ *in a root point* $z = \Phi_n(z)$, *for* $n = 1, 2, \ldots, \infty$, *does not hold a commensurable argument with* 2π: it still states that these parameters shall not satisfy any of the *equality relations* of the group (R) of relations in a *countable infinity*.

For any general fraction $\Phi(z)$, the following statement holds: *in any point* P *of* E, *the curve* E′ *cannot have a determined tangent line and the points of* E *are everywhere dense on* E′.
(In particular, what we said before can be applied to the general fractions of the 2°, 3°, 4° examples, since in all these fractions, the *coefficients-parameters* are submitted only to retrieve *inequality* relations: each one of the coefficients may arbitrarily lie in a small area of the plane where they are represented by the affixe.)

In all examples, belonging to the applications 1°, 2°, 3°, 4° of the second part and remarked to be as simple as possible, one finds already some splitting curves of domains revealing to be not as simple as analytic curves.

It is a big score to affirm that these curves are *simple closed Jordan curves*.

102. *Fifth example*. – Here is a more complicated example where the boundaries of certain domains are simple closed Jordan curves and the other boundaries are closed continuous curves; but this last curves are not *simple*, since they include everywhere dense double points on that same curves.

I will deal with the fraction

$$z_1 = \frac{-z^3+3z}{2} = \varphi(z)$$

already seen in the 2nd part (§ 48).
Let us examine the domain R_1, surrounding the limit point $z = 1$: this is the limit of the preimage areas (Γ_{-1}), (Γ_{-2}), …, (Γ_{-i}), … of the area Γ $\left(|z-1| \leq \frac{1}{2}\right)$, so that these preimages surround this last area.
It is been already remarked that, inside the circle Γ and Γ', both used to define R_1 and R'_1, (as the same notation in the 2nd Part), one gets

$$\left|\frac{dz_1}{dz}\right| \geq \frac{9}{8} > 1\,;$$

in this exterior region of both Γ and Γ', where the curves Γ_{-i} are traced.
In this region one gets

$$\left|\frac{dz}{dz_1}\right| = |\psi'(z_1)| \leq \frac{8}{9}\ .$$

Therefore, one may conclude, as it has been already done for R_0 in the example $z_1 = \frac{z+z^2}{2}$ that the curves Γ_{-i}, preimages of Γ and surrounding Γ, converge uniformly to a limit curve O since it is a continuous curve and it is described by the equations

(O) $\qquad x = f(t), \qquad y = g(t)$ for $0 \leq t \leq 1$

since both f and g are two continuous functions in t.

Every boundary point of R_1 is a point of O and it is *accessible from the interior of* R_1.
[This result has been achieved with the help of the orthogonal trajectories of the annulus (Γ,Γ_{-1}), which have been extended in all R_1 like it has been done for $z_1 = \frac{z+z^2}{2}$.]
On the other hand, I have shown [*read* §50, note ([1])] that every point of the boundary of R_1 is simple for that boundary.

Therefore, one concludes that this boundary O is a *simple closed Jordan curve*.

The same conclusion can be imposed to the boundary O′ of R'_1, symmetric to O in respect of the origin; the same holds for the boundaries of all $R_1^{-(i)}$ and $R_1'^{-(i)}$ of R_1 and R'_1: the total domain of convergence to (-1) or (-1), respectively.

The boundary E′ of the domain R_∞, of the point at infinity, is a *continuous curve*

$$x = f(t), \qquad y = g(t) \qquad \text{for } 0 \le t \le 1$$

since it could be defined, starting from the circle C, $|z| = 3$, as the limit of the preimages C_{-i} of C converging *uniformly to their limit*, due to the following inequality

$$|\psi'(z_1)| \le \frac{8}{9};$$

this last inequality has been verified outside Γ and Γ′, in all the regions where all C_{-i} are traced inside ([1]).
On that boundary the origin is a *multiple point* like the preimages (everywhere dense on E′) of the origin.

Therefore E′ *is closed and continuous*, *but it is not a* «SIMPLE JORDAN POINT»: E′ consists in the joining of :

1. the Jordan curves O;
2. O′;
3. of all their preimages;
4. of all the limit points of the set of the preimages (points whose neighbourhood there are an infinity of preimages).

([1]) As in the previous example, every point of E′ is accessible from the interior of the domain R_∞ of the point at infinity.