Journal of the History of the Behavioral Sciences, Vol. 45(1), 56–65 Winter 2009
Published online in Wiley Interscience (www.interscience.wiley.com). DOI 10.1002/jhbs.20349

EARLY ALTERNATIVE DERIVATIONS OF FECHNER'S LAW

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Historians of psychology, notably Boring, fostered Fechner's idea that Weber's law is the indispensable basis for the derivation of the logarithmic psychophysical law. However, it is shown here that Bernoulli in 1738 and Thurstone in 1931 derived the logarithmic law using principles other than Weber's law and that Fechner and Thurstone based their derivations on the principles originally employed by Bernoulli. It is concluded that awareness of researchers about Bernoulli's and Thurstone's derivations could expand the directions of research on the form of the psychophysical law. © 2009 Wiley Periodicals, Inc.

In 1860, in his two-volume book *Elemente der Psychophysik*, Gustav Theodor Fechner (1801–1887) based his derivation of the logarithmic psychophysical law on Weber's law. The idea that Weber's law is indispensable for this derivation has persisted since then. The Swiss mathematician and physicist Daniel Bernoulli (1700–1782) in 1738 and the great American psychologist Luis Leon Thurstone (1887–1955) in 1931 derived the logarithmic law by alternative principles, that is, without using Weber's law. Failure of historians to appreciate these alternative derivations contributed to the current idea that Weber's law is the foundation rather than an implication of the logarithmic law. To help rectify this situation, in the following we discuss comparatively and in chronological order the alternative derivations of the logarithmic law. We emphasize that our analysis regards only historical facts and their implications, without entering the theoretical discussion of which derivation is to be preferred over the others.

Bernoulli's Derivation

Bernoulli (1738)[1] based his derivation on the following general principles, which we have isolated and named Bernoulli's Principles 1, 2, and 3.

1. Bernoulli's 1738 paper *Specimen theoriae novae de mensura sortis* is known especially by economists for the first formulation of the expected utility model and of the utility function, widely used in decision theory (Schoemaker, 1982).

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Principle 1. Bernoulli (1738, p. 176) distinguished the objective value of all goods possessed by an individual, defined by the total price of these goods, from the subjective value of all these goods, that is, the utility or total satisfaction provided by all goods.[2] He denoted these objective and subjective values by x and y, respectively. We denote any increment of x by Δx and the corresponding increment of y by Δy. Bernoulli remarked the common-sense fact that the increment Δy caused by Δx is lower for the rich than for the poor. From this fact follows the general principle that in a single individual the increment Δy caused by the same Δx decreases as x increases.[3]

Principle 2. Bernoulli (1738, p. 181) assumed that Δy is directly proportional to Δx. Obviously, Δy is some unknown increasing function of Δx. Thus, Bernoulli assumed that this function was the simplest possible. We shall see that this principle of direct proportionality was indispensable also for Fechner's derivation.

Principle 3. Bernoulli (1738, p. 181) assumed that Δy is inversely proportional to x. He derived this principle of inverse proportionality using the following reasoning. Consider two persons, *A* and *B*, with *A* having a fortune of 100,000 ducats, producing an income of 5,000 ducats, and with *B* having a fortune of 100,000 semi-ducats, producing an income of 5,000 semi-ducats. Clearly, one ducat for *A* is the same as one semi-ducat for *B*. Now suppose that *A* and *B* receive one ducat (Δx) each. It is evident that the subjective value of this ducat for *B* (Δy_B) is twice as much that for *A* (Δy_A). This means that Δy_A and Δy_B are inversely proportional to the corresponding fortunes (Bernoulli, 1738, p. 179).[4]

Bernoulli wanted to deduce the function relating y to x, that is, what we today call the psychophysical law in psychology or the utility function in economics. He did this with the help of a drawing. Figure 1 reproduces the aspects of this drawing that are relevant for our discussion.[5] Principle 1 dictates that the psychophysical law must be negatively accelerated. Figure 1 shows this general downward concavity of the function. Bernoulli determined the precise shape of the psychophysical law as follows. In his drawing, he graphically represented the idea that individuals have a minimum value, α, of x necessary to produce a value of y higher than 0.[6] Today, we call this minimum value the threshold. In Figure 1, the threshold is the abscissa of the point of intersection of the graph of the psychophysical law with the horizontal axis. Finally, Bernoulli observed that x normally increases by infinitesimal increments, dx, causing infinitesimal increments, dy, in y (Bernoulli, 1738, p. 177). By combining Principles 2 and 3, Bernoulli wrote the differential equation

$$dy = b\frac{dx}{x}, \tag{1}$$

with b constant. By integrating both sides of Equation 1 from α to x, he obtained

2. Pierre Simon Laplace (1820, p. 441) called these objective and subjective values *physical* and *moral fortunes*, respectively.

3. This principle was stated by the Swiss mathematician Gabriel Cramer (1704–1752) in 1728 (Bernoulli, 1738, pp. 190–191) and by the French naturalist Georges Louis-Leclerc, Count of Buffon (1707–1788) in 1730 (Buffon, 1777, p. 76).

4. Laplace presented Bernoulli's derivation of the logarithmic law on page 441 of his renowned 1820 book on the theory of probabilities. In two sentences on page 190 of this book, he synthetically described Principles 1–3. On page 441, he recalled these principles subsuming them all under the single term "principle."

5. Figure 1 is essentially identical to a figure used by Thurstone (1931) to present his own derivation of the logarithmic law. A faithful reproduction of Bernoulli's original drawing—that is, Figure 5 in Table 7 in the volume containing his article—may be found in the English translation of his article published in the journal *Econometrica* (1954, 22, pp. 23–36).

6. Bernoulli did not expand on this idea.

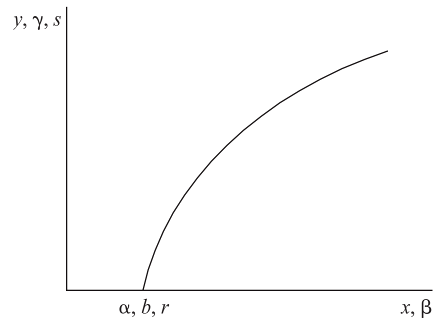

FIGURE 1.
Illustration of the threshold (α, Bernoulli; *b*, Fechner; *r*, Thurstone) and downward concavity of the psychophysical law. The law relates utility (*y*, Bernoulli; *s*, Thurstone) or sensation magnitude (γ, Fechner) to total amount of possessed goods or commodities (*x*, Bernoulli and Thurstone) or to stimulus magnitude (β, Fechner), respectively.

$$y = b \log \frac{x}{\alpha}, \tag{2}$$

which states that the precise shape of the psychophysical law is logarithmic.

FECHNER'S DERIVATION

Fechner (1860, vol. 2, pp. 10–13)[7] based his derivation on Weber's law and on Bernoulli's Principle 2, which he called "mathematical auxiliary principle" (*mathematisches Hülfsprincip*).

Weber's law. Consider a sensation magnitude γ determined by a stimulus magnitude β. Fechner (1860, vol. 2, p. 9) used the symbol $d\gamma$ to denote a just noticeable sensation increment, from γ to $\gamma + d\gamma$, and the symbol $d\beta$ to denote the corresponding stimulus increment, from β to $\beta + d\beta$. Fechner (1860, vol. 1, p. 65) attributed to the German physiologist Ernst Heinrich Weber (1795–1878) the empirical finding (Weber, 1834) that $d\gamma$ remains constant when the relative stimulus increment $d\beta/\beta$ remains constant, and named this finding Weber's law (*Weber'sches Gesetz*). Fechner (1860, vol. 2, p. 10) underlined that Weber's law was empirical.

Mathematical auxiliary principle. Fechner (1860, vol. 2, p. 10) assumed that $d\gamma$ remains directly proportional to $d\beta$ as long as $d\gamma$ and $d\beta$ remain very small. This is Bernoulli's Principle 2, limited to the cases when $d\gamma$ and $d\beta$ are very small. Shortly, we shall see why this principle was fundamental for Fechner's derivation.

Fechner expressed Weber's law in words rather than a formula. However, one can express this law by the formula

$$d\gamma \bullet \frac{d\beta}{\beta} \tag{3}$$

7. For a theoretical critique of Fechner's derivation, see Luce and Edwards (1958). Regarding Fechner's ideas and their impact on the culture of the time, see Heidelberger (2004).

with the symbol "•" meaning "is constant when there is constancy of." In Expression 3, Fechner replaced the symbol "•" with that of equality. That is, he turned Weber's law into the equation

$$d\gamma = k\frac{d\beta}{\beta}, \tag{4}$$

with k a constant representing the different units of measurement for γ and β.

We can now see why the mathematical auxiliary principle was indispensable to Fechner. In Expression 3, the symbol $d\gamma$ means the just noticeable sensation increment, not any sensation increment (Weber, 1834, 1846). By definition, for each β there is only one just noticeable sensation increment. Thus, Expression 3 means that for each β there is one single value of $d\gamma$ and one single value of $d\beta$. That is, with β fixed, Expression 3 assumes no variation of $d\gamma$ with $d\beta$. Instead, when β is fixed, Equation 4 states that $d\gamma$ varies with $d\beta$, in contradiction to Expression 3. To remedy this contradiction, Fechner was forced to employ the mathematical auxiliary principle—that is, Bernoulli's Principle 2. He limited the application of this principle to the case where $d\gamma$ and $d\beta$ are very small.

Fechner (1860, vol. 2, p. 10) remarked that Equation 4 did not allow one to calculate measures of sensation magnitude. For this calculation to be possible, one needs to first interpret $d\gamma$ and $d\beta$ as differentials and then integrate both sides of Equation 4 (Fechner, 1860, vol. 2, p. 33). When this is done, after some simplification one obtains

$$\gamma = k\log\frac{\beta}{b} \tag{5}$$

where b is the absolute threshold, that is, the minimum value of β necessary to produce a sensation. Today, as in Fechner's time, Equation 5 is called Fechner's law.[8]

Fechner's Comment On Bernoulli

What did Fechner have to say about Bernoulli's derivation of the logarithmic law? He repeatedly claimed throughout his 1860 book that Weber's law was the foundation of the logarithmic law. However, in the first volume of this book, Fechner recognized that Bernoulli's derivation of the logarithmic law was founded on principles other than Weber's law. He expressed this clearly on page 237 by citing two quotations from Bernoulli (1738), the first exemplifying Principle 1 and the second illustrating Principle 3, and then by admitting that "On this [Principles 1 and 3] he [Bernoulli] founds at page 181 the differential and at page 182[9] the logarithmic formula, which later[10] we more generally base on Weber's law."

We may note that, curiously, Fechner quoted Bernoulli's Principles 1 and 3 but not Bernoulli's Principle 2. We have seen that one cannot derive Fechner's law from Weber's law without employing the mathematical auxiliary principle, that is, without employing Bernoulli's Principle 2.

8. For example, referring to Equation 5, James Ward (1876, p. 453) said that "This is what is generally spoken of as Fechner's Law." Fechner named Equation 4 the "fundamental formula" (*Fundamentalformel*). Since Equation 5 allows one to calculate γ from β, when k receives an arbitrary value and b is determined empirically, he named this equation the "measurement formula" (*Massformel*).

9. Here Fechner made a mistake: The differential and the logarithmic formulas are both on page 181.

10. "Later" means on pages 10–13 in the second volume of the 1860 book, where Fechner derived his fundamental and measurement formulas corresponding to the Bernoulli "differential" (Equation 1) and "logarithmic" formulas (Equation 2), respectively.

How did Fechner accommodate his claim that Weber's law was the foundation of the logarithmic law with his own admission that Bernoulli used alternative principles to derive the logarithmic law? In the quotation just given, Fechner asserted that he derived the logarithmic law more generally than Bernoulli did. Later, Fechner (1860, vol. 2, pp. 550–551) explained what he meant by "more generally." Essentially, he believed that his derivation was more general because it applied to all sensations, while Bernoulli's derivation applied only to the special case of utility. However, Fechner (1860, 1877, 1882, 1887) failed to provide any compelling reason why the principles employed in Bernoulli's derivation should not be extendable to sensations.

Thurstone's Derivation

Thurstone (1931)[11] based his derivation[12] on the following five postulates, which he named 1, 2, 3, 4, and 5.

Postulate 1. Thurstone defined utility as the satisfaction deriving from accumulated commodities of some kind. He denoted this satisfaction by s and "the number of items of a commodity" by x. Expressing an obvious fact, the first postulate states that s is an increasing function of x. To illustrate this fact, Thurstone used a diagram essentially identical to that reproduced in Figure 1. In this diagram, s is equivalent to Bernoulli's y.

Postulate 2. There is a lower value of x, which Thurstone denoted by r, below which x produces no utility (the threshold). In Figure 1, r is equivalent to Bernoulli's α.

Postulate 3. This postulate defines motivation as "the anticipated increment in satisfaction per unit increase in the commodity. . . . It is consistent with common sense that the motivation to acquire an additional unit of the commodity is smaller, the greater the amount already possessed" (Thurstone, 1931, p. 141). Motivation is defined by the ratio of differentials ds/dx.

Postulate 4. Motivation is finite when satisfaction is zero. This postulate states that the psychophysical law must have a shape such that motivation is not infinite at threshold, when s is zero.

Postulate 5. Motivation is inversely proportional to x. Thurstone considered this to be his most fundamental psychological postulate. He derived the logarithmic law from the mathematical expression of this postulate. In his words:

> This psychological postulate can be written more concisely in the form
>
> $$\frac{ds}{dx} = \frac{k}{x}, \tag{6}$$
>
> in which s = satisfaction, x = amount of commodity possessed, ds/dx = motivation, and k = a constant which characterizes the person and the particular commodity. Integrating, we have

11. In psychophysics, Thurstone is most famous for his law of comparative judgment, published in articles posthumously collected in the volume *The Measurement of Values* (Thurstone, 1959). For a discussion of the law of comparative judgment, see Luce (1994).

12. For a history from 1930 to 1970 of the impact in economics of Thurstone's (1931) derivation, see Moscati (2007). More recently, Thurstone's derivation has led in economics to a generalization that goes under the name of "Fechner-Thurstone utility function" (Basmann, McAleer, & Slottje, 2007; Basmann, Molina, & Slottje, 1983). Thurstone (1931) never mentioned Bernoulli. Regarding the "whole subject" of his paper, he referred the reader only to Irving Fisher's (1892) doctoral thesis, reprinted in 1925 as a book (Thurstone, 1959, p. 124). Fisher (1892, 1927) also did not mention Bernoulli.

$$\int ds = k \int \frac{dx}{x} \tag{7}$$

or

$$s = k \log x + c, \tag{8}$$

which is certainly no stranger in psychophysics. It is our old friend, Fechner's law. (Thurstone, 1931, p. 142; equations renumbered)

Simple algebra shows that Equation 8 reduces to

$$s = k \log \frac{x}{r}. \tag{9}$$

To reply to the potential reader who could object that Thurstone merely adopted Fechner's law, Thurstone (1931, p. 142) noted that Equation 8 depends strictly on the concept of motivation. Using this concept, one could conjecture other postulates in place of Postulate 5. For example, according to Thurstone (1931, p. 142), the postulate that motivation is inversely proportional to the "amount of *satisfaction* already attained from the commodity" (italics in the original) would be psychologically more plausible than Postulate 5. The mathematical expression of this other postulate is

$$\frac{ds}{dx} = \frac{k}{s}, \tag{10}$$

which, after integration and simplification, leads to the psychophysical law

$$s = \sqrt{p \cdot x + q} \tag{11}$$

with p and q constants. Thurstone dropped Equation 11 because it did not fit his data as well as the logarithmic law did. However, the example of Equation 11 is important because it stresses that Thurstone's concept of motivation is an autonomous principle since it may be used in different implementations.

Comparison of Derivations

Equations 2, 5, and 9 are the same equation in the sense that each symbol in one position of one equation has the same meaning as the symbol in the same position in the other equations. Thus, the logarithmic law can be equivalently founded on different principles or postulates: Principles 1–3 (Bernoulli), Weber's law conjoined with the mathematical auxiliary principle (Fechner), or Postulates 1–5 (Thurstone). The following comparisons show that these principles and postulates overlap partially.

Bernoulli vs. Fechner. Bernoulli's Principle 1 is different from Weber's law in that it refers to Δy as any possible increment of y, while Weber's law refers only to the just noticeable increment of y (Weber, 1834). Fechner employed Bernoulli's Principle 2 (mathematical auxiliary principle). He did not employ Bernoulli's Principle 3.

Bernoulli vs. Thurstone. Thurstone did not employ Bernoulli's Principles 1 and 2. Bernoulli's Principle 3 partially matches Thurstone's Postulate 5. This match is partial in that Bernoulli applied his Principle 3 to the concept of utility increment (dy), while Thurstone applied Bernoulli's Principle 3 to the concept of motivation (ds/dx).

Bernoulli's Principles 2 and 3 are fundamentally the only ones that are needed for the derivation of the psychophysical law (Bernoulli's Principle 1 may be considered as a preparatory principle). We have just seen that Fechner and Thurstone based their derivations on Bernoulli's Principles 2 and 3, respectively. These principles are simple since they concern one single variable, that is, dx or x. Fechner and Thurstone also used complex principles concerning the ratio between two variables, that is, the ratio defining Weber's law and the ratio defining motivation, respectively. Clearly, Bernoulli's derivation is the most parsimonious conceptually, since it uses only simple principles.

Bernoulli's principles agree with Weber's law. It may, in fact, be shown that Bernoulli's principles imply Weber's law. The demonstration is as follows.

Let Equation 2 apply to sensations. One can start by considering any sensation increment $\Delta y = y(x + \Delta x) - y(x)$, with y the function defined by Equation 2, x the stimulus intensity, and Δx the stimulus increment. Thus, using Equation 2 yields

$$\Delta y = b \log \frac{x + \Delta x}{\alpha} - b \log \frac{x}{\alpha} \quad (12)$$

with b a constant and α the threshold. After some simplification, Equation 12 becomes

$$\Delta y = b \log\left(1 + \frac{\Delta x}{x}\right). \quad (13)$$

Since b is constant, it is a simple deduction from Equation 13 that

$$\Delta y \bullet \frac{\Delta x}{x} \quad (14)$$

with the symbol "•" having the meaning previously defined.

Expression 14 applies to any sensation increment Δy, including the just noticeable sensation increment. It follows that Bernoulli's Principles 1–3 imply Weber's law (Expression 3) as a special case of Expression 14.[13]

Discussion

We have shown that Weber's law can be seen as an implication of Bernoulli's principles, rather than being the basis of the logarithmic law as in Fechner's derivation. However, the idea that Weber's law is the independent basis of the logarithmic law is widespread. Historians fostered this idea. To exemplify, we use Edwin Garrigues Boring's book *A History of Experimental*

13. More recently, Stephen Warren Link (1992) proposed another derivation of the logarithmic law. As we have just seen, the previous derivations of Bernoulli, Fechner, and Thurstone are based on hypotheses about the functional relations between the increment in sensory magnitude, the corresponding increment in stimulus magnitude, and the stimulus magnitude. Link's derivation differs qualitatively from these previous derivations, since it rests on the specification of the statistical–physiological nature of the discriminative process involved in the determination of the just noticeable sensory difference. Link's derivation is interesting because it shows that the logarithmic law can be derived without using Weber's law. Briefly, the derivation is as follows. Given a standard stimulus magnitude, $\mathbf{S}_B$, and a just noticeably different comparison stimulus magnitude, $\mathbf{S}_A$, Link postulates that the sensation magnitude corresponding to $\mathbf{S}_A$ is $S = \Theta \cdot A$, with Θ a parameter representing the discriminability of $\mathbf{S}_A$ from $\mathbf{S}_B$ and with A a measure of response resistance. Using a theory of response variability based on a Poisson process, he demonstrates mathematically that $\Theta = \ln(\mathbf{S}_A/\mathbf{S}_B)$, implying that $S = A \cdot \ln(\mathbf{S}_A/\mathbf{S}_B)$. This last equation tends to Fechner's law when $\mathbf{S}_B$ tends to the threshold.

Psychology, published in 1929 and re-edited in 1950. The example from Boring is important because his book has been authoritative throughout most of the twentieth century—and still is.

One can foster the idea that Weber's law is the independent basis of the logarithmic law by ignoring all derivations that are alternative to Fechner's derivation. Boring (1950), and virtually every other psychologist, ignored Thurstone's derivation of the logarithmic law. Psychologists ignored or sporadically showed only vague recognition of Bernoulli's derivation.[14] Boring (1950) is the most important example of vague recognition of Bernoulli. He described Bernoulli's contribution as follows:

> Bernoulli's interest in the theory of probabilities as applied to games of chance had led to the discussion of *fortune morale* and *fortune physique*, mental and physical values which he believed (1738) to be related to each other in such a way that a change in the amount of "mental fortune" varies with the ratio that the change in the physical fortune has to the total fortune of its possessor. . . . In this way *fortune morale* and *fortune physique* became mental and physical quantities, mathematically related, quantities that correspond exactly, both in kind and relationship, to mind and body in general and to sensation and bodily energy in particular, the terms that Fechner sought to relate, in the interests of his philosophy, by way of Weber's law. (Boring, 1950, pp. 284–285)

Boring did not say how Bernoulli "mathematically related" the moral and physical fortunes. That is, he did not inform the reader that Bernoulli derived the logarithmic law on the basis of principles other than Weber's law.[15]

The present analysis shows that Weber's law is not essential for the derivation of the logarithmic law. With attention focused on Bernoulli's and Thurstone's derivations of the logarithmic law, theoretical and empirical research on the psychophysical law could take new directions. The following considerations help clarify this possibility. In terms of functions, Fechner's derivation of the psychophysical law starts from the equation

$$d\gamma = E\left(\frac{d\beta}{\beta}\right) \tag{15}$$

while Bernoulli's derivation starts from the equation

$$d\gamma = \frac{F(d\beta)}{G(\beta)}. \tag{16}$$

Throughout his 1860 book, Fechner asserted that Weber's law was approximately valid only within a limited range of stimulus values. Since then, attempts have been made to find

14. For example, Ward (1876, p. 457) attributed the merit of having found a formula identical to Fechner's logarithmic law to Laplace, ignoring Bernoulli completely; and, in his famous 1942 book *Sensation and Perception in the History of Experimental Psychology*, Boring discusses Fechner without ever mentioning Bernoulli. Cattell (1928), Gescheider (1997, p. 296), and Stevens (1975, pp. 4–5) are examples of vague recognition. Stevens (1975, p. 5) was more detailed than others, describing Bernoulli's derivation of the logarithmic law in two sentences: "Bernoulli derived his logarithmic function by first making a simple assumption. The added utility, he said, grows smaller as the number of *dollars* grows larger—a simple inverse relation" (italics in the original). Stevens's description is rather cryptic. It refers to Bernoulli's Principles 1 and 3 but omits Bernoulli's Principle 2.

15. Boring withheld this information even though Fechner (1860, vol. 1, p. 237) recognized the different basis of Bernoulli's derivation. Later, in a single sentence, Boring (1961, p. 4) asserted that Bernoulli contended that moral fortune is proportional to the logarithm of physical fortune. Without further specification, this assertion obscures the fact that Bernoulli derived the logarithmic law by reasoning from first principles.

the function E in Equation 15 that would be the most descriptive of empirical data.[16] For example, it has been proposed that

$$E\left(\frac{d\beta}{\beta}\right) = c_1 + c_2 \left(\frac{b}{\beta}\right)^n \tag{17}$$

with n, c_1, and c_2 constants (Riesz, 1928). This rather complicated equation implies a more complicated form of the logarithmic law, which predicts empirical data more precisely than Equation 5 does. Should one accept that Bernoulli's derivation is extensible to sensations, the research for more complicated forms of the logarithmic law would be separately focused on the functions F and G in Equation 16 rather than on the single function E in Equation 15. Similarly, Thurstone's concept of motivation (*ds/dx*) and the possibility that it applies to sensations still wait to be explored.

Acknowledgment

We would like to thank Norman Henry Anderson, Stephen Warren Link, Lawrence Edward Marks, and two anonymous reviewers for useful comments on an earlier version of this paper.

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16. For a list and a discussion of these attempts, see Baird (1997), Baird and Noma (1978), Krueger (1989), Murray (1993), Norwich (1993), and Norwich and Wong (1997).

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