

CHAPTER

17

Gravitational Waves

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A modern physicist is a quantum theorist on Monday, Wednesday, and Friday, and a student of gravitational relativity theory on Tuesday, Thursday, and Saturday. On Sunday the physicist is neither, but is praying to his God that someone, preferably himself, will find the reconciliation between these two views.

—Norbert Wiener

I ask you to look both ways. For the road to a knowledge of the stars leads through the atom; and important knowledge of the atom has been reached through the stars.”

—Arthur Eddington

1.3 ■ INTRODUCTION

Gravity wave: a tidal force that propagates through spacetime.

General relativity differs from Newtonian gravity in several important ways. One way is in the behavior of light and matter in strong gravitational fields, especially near black holes. The black hole was predicted by Michell and Laplace on the basis of Newtonian gravity more than a century before Schwarzschild discovered his famous metric. However, the event horizon, singularity, and no-hair theorems are all consequences of general relativity that could not have been predicted from Newtonian physics.

Gravitational radiation is another phenomenon that has no counterpart in Newtonian physics. According to Newton, the gravitational interaction propagates instantaneously: When the Earth moves around the Sun, the Earth’s gravitational field changes all at once throughout space, according to Newton.

When Einstein formulated special relativity and recognized its requirement that no information can travel faster than the speed of light, he

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Newton: Gravity propagates instantaneously.

Einstein: No signal propagates faster than light.

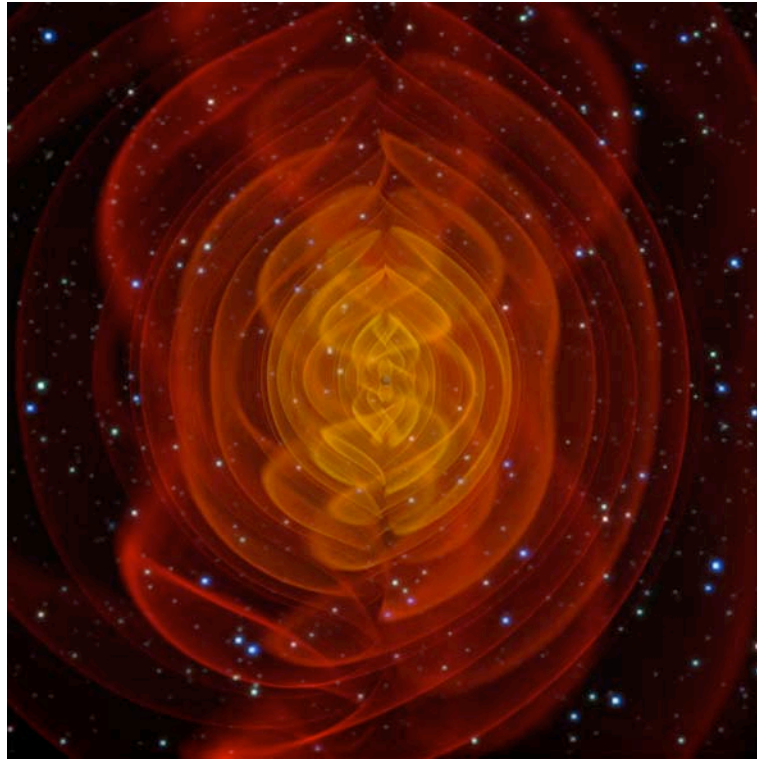


FIGURE 1 Computed emission of gravity waves. The tiny dot at the center of this image is two black holes churning spacetime as they combine into one. The swirling patterns represent distortions of spacetime that propagate outward as gravity waves. Close to the coalescing black holes, the gravity waves—essentially nothing but traveling tidal forces—are lethal. In contrast, we expect that gravity waves that could be detected on Earth are extremely small.

29 realized that Newtonian gravity would have to be modified. Not only would
 30 static gravitational fields differ from the Newtonian prediction in the vicinity
 31 of compact masses, but also time-varying fields would have to propagate. He
 32 showed that these fields would move with the speed of light, so gravity could
 33 not be used to send information faster than the speed of light, which would
 34 have destroyed the fundamental basis of all relativity.

35 Einstein had a conceptual prototype for gravity waves: electromagnetic
 36 radiation. James Clerk Maxwell predicted electromagnetic radiation in 1873
 37 and Heinrich Hertz demonstrated it experimentally in 1888. (Einstein was
 38 born in 1879.) When he grew up, Einstein quickly realized that a general
 39 relativity theory based on curved spacetime would not look like Maxwell's
 40 electromagnetic theory. After his theory was completed, Einstein and others
 41 were able to compute how gravitational fields propagate.

42 Gravity waves are essentially tidal forces that vary with time and position;
 43 that is all they are. As a gravity wave passes over you, you are alternately
 44 stretched and compressed in ways that depend on the particular form of the

Gravity waves like
 electromagnetic
 waves?

2 Gravity wave metric

3

Gravity wave on Earth:
An extremely small
traveling tidal force.

45 wave. In principle there is no limit to the size of gravity waves. Figure 1
46 pictures the calculated result of two black holes emitting gravity waves as they
47 combine into one. In the vicinity of the coalescence, gravity-wave-induced tidal
48 forces would be dangerous to life.

49 We predict that gravity waves from various sources are continually
50 sweeping over us on Earth's surface. Sections 3 and 7 describe some of these
51 sources. Basically we hope to observe these waves by detecting changes in
52 separation between two test masses suspended near to one another—changes
53 in separation caused by the traveling tidal force that constitutes a gravity
54 wave. We expect this change in separation to be *extremely* small for gravity
55 waves detectable on Earth.

56 Current gravity wave detectors on Earth are interferometers in which light
57 is reflected back and forth between free test masses along two perpendicular
58 directions, and the time difference measured between round-trip times in the
59 two directions. The “free” test masses are hung from wires that are in turn
60 supported with elaborate shock-absorbers so as to minimize the vibrations due
61 to passing trucks and even waves crashing on a distant shore. But the
62 back-and-forth pendulum-like motions of these test masses are free enough to
63 permit measurement of their change in separation due to tidal effects resulting
64 from a passing gravity wave, caused by some gigantic distant gravitational
65 event, for example the coalescence of two black holes modeled in Figure 1.

66 Does the change in separation induced by gravity waves affect everything,
67 for example a meter stick or the concrete slab on which a gravity wave
68 detector rests? Answer: Only by an amount that is entirely negligible. The
69 structure of meter sticks and concrete slabs is determined by electromagnetic
70 forces mediated by quantum mechanics. The two ends of a meter stick are not
71 freely-floating test masses. The tidal force of a passing gravity wave is much
72 weaker than the internal forces that maintain the shape of solids. The meter
73 stick—or the concrete slab underlying the vacuum chamber and detectors of a
74 gravitational-wave observatory—is stiff enough to be negligibly affected by a
75 passing gravity wave.

2.6 ■ GRAVITY WAVE METRIC

77 *Tiny but significant departure from the inertial metric*

Gravity wave
metric

78 Our analysis uses a particular gravity wave: a certain kind of plane wave
79 arriving from a very distant source and moving in the z -direction. This wave
80 (and almost all of the gravity waves we discuss in this chapter) represents a
81 very small perturbation of flat spacetime. Here is the timelike metric for such
82 a particular wave that propagates along the z -axis.

$$d\tau^2 = dt^2 - (1 + h)dx^2 - (1 - h)dy^2 - dz^2 \quad (h \ll 1) \quad (1)$$

83 In this metric h is a dimensionless function of time and space. *Numerically, h*
84 *is a fractional deviation from the flat-spacetime coefficient of dx^2 or dy^2 in the*
85 *metric.* Another name for fractional deviation of length is **strain**, so h is also

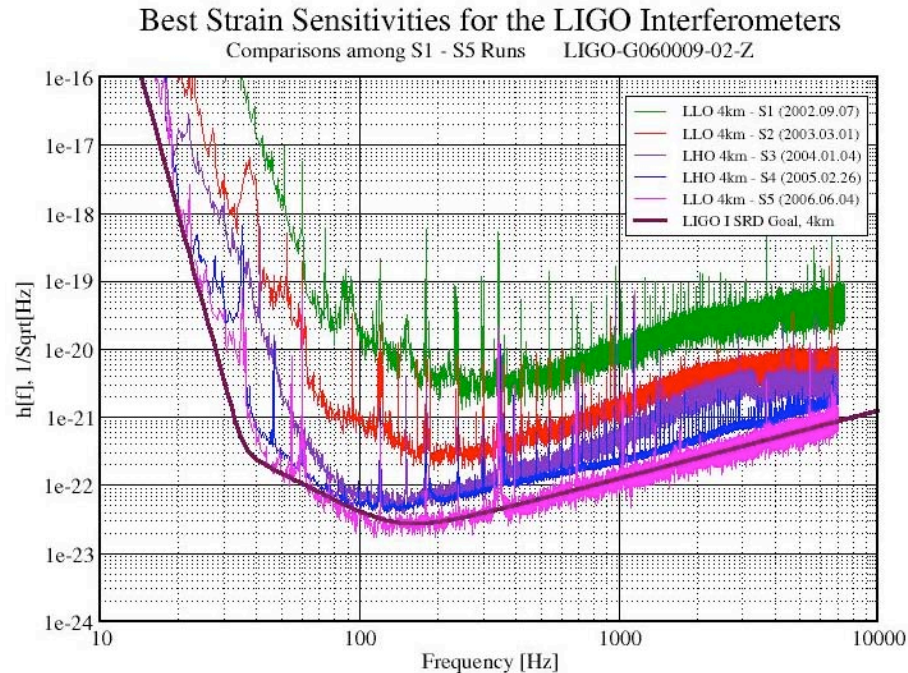


FIGURE 2 Progressive improvements in sensitivity of LIGO interferometers. On the vertical axis $1e-19$, for example, means a fractional change in separation of 10^{-19} between test masses. The bottom solid line is the current goal. Spikes occur at frequencies of electrical or acoustical noise. To be detectable, gravity wave signals must cause greater displacement than what is represented by these noise curves. As of mid-2009, gravity wave signals have yet to be detected. FIND MOST RECENT UPDATE THAT SHOWS EVOLUTION OF SENSITIVITY: SCOTT HUGHES?

h = gravity
wave strain

86 called the **gravity wave strain**. The wave leading to (1) is a transverse wave,
87 since h describes a perturbation of space only in the x and y directions
88 transverse to the z -direction of propagation. The strain h varies with both
89 position and time. Its maximum value is very much less than one. Let two free
90 test masses be at rest a distance D apart in the x or y direction. When a
91 z -directed gravity wave passes over them, the change in their separation,
92 called the **displacement**, equals hD , which follows directly from the
93 definition of h as a “fractional deviation.”

94 FOLLOWING IS NUMERICALLY INCONSISTENT. GET LATEST
95 LIGO PARAMETERS AND SENSITIVITIES. One can use Einstein’s field
96 equations to make predictions about the magnitude of the function h in
97 equation (1) for various kinds of astronomical phenomena. Currently, gravity
98 wave detectors use laser interferometry and go by the full name **Laser**
99 **Interferometer Gravitational Wave Observatory** or **LIGO** for short.
100 The first-generation LIGO, called Initial LIGO, was able to detect waves with
101 (approximately) $h > 10^{-19}$ for frequencies within a range of about 100 hertz.

LIGO gravity
wave detector

2 Gravity wave metric

5

LIGO sensitivity

(Abbreviation: Hz. Recall that one hertz is one cycle per second.) The second-generation LIGO, called Advanced LIGO, is about 10 times more sensitive; it is planned to be operational around 2014. Advanced LIGO can also be tuned in frequency to achieve higher sensitivity in frequency bands of interest.

Figure 2 compares the gradually-improving sensitivities of LIGO over time. The displacement sensitivity is expressed in the units of meter/(hertz)^{1/2} because the amount of noise limiting the measurement grows with the frequency range being sampled. Note that the instruments are designed to be most sensitive near 150 hertz. This frequency is determined by the different kinds of noise faced by experimenters: Quantum noise limits the sensitivity at high frequencies, while seismic noise is the largest problem at low frequencies. If the range of sampled frequencies—*bandwidth*—is 100 hertz, then the best sensitivity is about $10^{-22} \times 100^{1/2} = 10^{-21}$. This means that along a length of 4 kilometers = 4×10^3 meters, the change in length is approximately $10^{-21} \times 4 \times 10^3 = 4 \times 10^{-18}$ meters, which is approximately one thousand times smaller than a proton, or a hundred million times smaller than a single atom!

?

Hold on! Your gravity wave detector sits on Earth's surface, but equation (1) says nothing about curved spacetime described, for example, by the Schwarzschild metric. The expression $2M/r$ measures departure from flatness in the Schwarzschild metric. At Earth's surface, $2M/r \approx 1.4 \times 10^{-9}$, which is 10^{13} —ten million million!—times greater than the corresponding gravity wave factor $h \sim 10^{-22}$. Why doesn't the quantity $2M/r$ —which is much larger than h —appear in (1)?

!

First, the factor $2M/r$ is essentially constant over the size of LIGO, so we can ignore it. Second—and more important—the LIGO detector is “tuned” to detect a time-varying gravity wave of frequency near 150 hertz. LIGO is totally insensitive to the small *static* curvature introduced by the factor $2M/r$ at Earth's surface. For this reason, we simply omit static curvature factors from equation (1), effectively describing gravity waves “in free space” as well as for the predicted $h \ll 1$.

Einstein's equations become a wave equation.

In free space and for small values of h , Einstein's field equations actually reduce to a wave equation for h . For the most general case, this wave has the form $h = h(x, y, z, t)$. When x, y, z , and t are all in geometric units (for example meters), this wave equation takes the form:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{\partial^2 h}{\partial t^2} \quad (\text{free space and } h \ll 1) \quad (2)$$

For simplicity, think of a wave moving along the z -axis. The most general solution to the wave equation under these circumstances is

$$h = h_{+z}(z - t) + h_{-z}(z + t) \quad (3)$$

Assume gravity wave moves in $+z$ direction.

The expression $h_{+z}(z - t)$ means a function h of the variable $z - t$ and *not* some constant h times the quantity $(z - t)$. The function $h_{+z}(z - t)$ describes

6

Chapter 17 Gravitational Waves

141 a wave moving in the positive z -direction and the function $h_{-z}(z+t)$ describes
 142 a wave moving in the negative z -direction. In this chapter we deal only with a
 143 gravity wave propagating in the positive z -direction and hereafter use

$$h \equiv h(z-t) \equiv h_{+z}(z-t) \quad (\text{wave moves in } +z \text{ direction}) \quad (4)$$

144 The argument $z-t$ means that h is a function of *only* the combined variable
 145 $z-t$. Indeed, h can be *any function whatsoever* of the variable $(z-t)$. The
 146 form of this variable tells us that, whatever the profile of the gravity wave at
 147 any given time; as time passes, that profile displaces itself in the positive
 148 z -direction with the speed of light (one in our units) .

LIGO sensitive
75 to 500 hertz

149 Figure 2 shows that the LIGO gravity wave detector has maximum
 150 sensitivity to gravity waves of frequencies between 75 and 500 hertz, with a
 151 peak sensitivity at around 150 hertz. Even at 500 hertz, the wavelength of the
 152 gravity wave is very much longer than the overall 4-kilometer dimensions of
 153 the LIGO detector. Therefore *we can assume in the following that at any given*
 154 *time the value of h is spatially uniform over the entire LIGO detector.*

QUERY 1. Uniform h ?

155 Using numerical values, verify the claim in the preceding paragraph that h is effectively uniform over
 156 the LIGO detector. 158

Draw global
map coordinates
on rubber sheet.

160 It is important to understand the meaning of the coordinates in metric
 161 (1). These are global *map* coordinates; global coordinates are always fictions
 162 that we choose to reveal aspects of a spacetime we cannot visualize. For $h \neq 0$,
 163 these global coordinates are invariably distorted. Think of the three mutually
 164 perpendicular planes formed by pairs of space coordinates (x, y) , (y, z) , and
 165 (z, x) . Draw a grid of lines on a rubber sheet lying in each corresponding
 166 plane. The gravity wave distorts the rubber sheet as it passes through it.

167 Glue map clocks to the intersections of these grid lines on the rubber sheet
 168 so that they move as the rubber sheet distorts. A gravitational wave moving in
 169 the $+z$ direction (Figure 5) passes through a rubber sheet lying in the xy
 170 plane, so that the grid ruled on the rubber sheet stretches and contracts in
 171 different directions within the plane of the sheet (Figures 3 and 4). The map
 172 clocks, glued at intersections of map coordinate grid lines, ride along with the
 173 grid as the sheet distorts, so that the map coordinates of any clock does not
 174 change.

Map time t
read on clocks
glued to the
rubber sheet.

175 Think of two ticks on a single map clock. Between ticks the map
 176 coordinates of the clock do not change: $dx = dy = dz = 0$. Therefore metric (1)
 177 tells us that the wristwatch time $d\tau$ between two ticks is also map time dt
 178 between ticks. Therefore map time t corresponds to the time measured on the
 179 clocks glued to the rubber sheet, even when the strain h varies at their
 180 locations.

2 Gravity wave metric

7

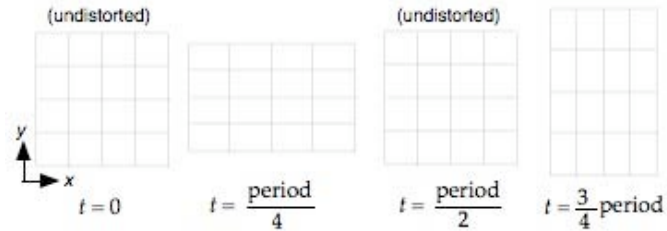


FIGURE 3 Change in shape (greatly exaggerated!) of the map coordinate grid at four times as a periodic wave passes through in the z -direction (perpendicular to the page). NOTE carefully!: The x -axis is stretched while the y -axis is compressed and vice versa. The areas of the panels remain the same.

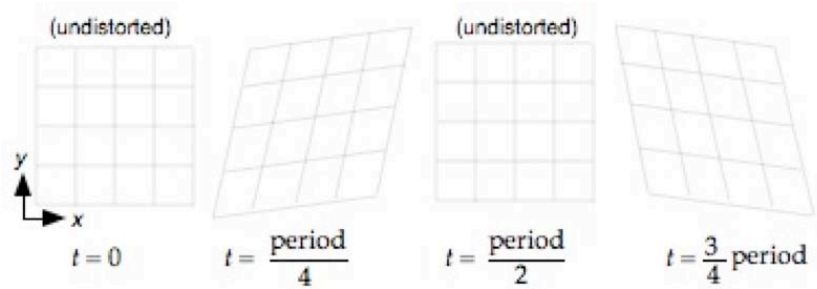


FIGURE 4 Effects of a periodic gravity wave with polarization “orthogonal” to that of Figure 3 on the map grid in the xy plane. Note that the axes of compression and expansion are at 45 degrees from the x and y axes. All grids stay in the xy plane as they distort. As in Figure 3, the areas of the panels are all the same.

181 Figure 3 represents the map time variation of the space distortion of the
 182 rubber sheet at a given location due to a particular polarization of the gravity
 183 wave. Although gravity waves are transverse like electromagnetic waves, the
 184 polarization forms of gravity waves are different from those of electromagnetic
 185 waves. Figure 4 shows the distortion caused by the wave “orthogonal” to that
 186 shown in Figure 3.

3. ■ SOURCES OF GRAVITY WAVES

188 *Many sources; only one with clear prediction*

189 Sources of gravity waves include collapsing stars, exploding stars, stars in orbit
 190 around one another, and the big bang. Neither electromagnetic waves nor
 191 gravity waves result from a spherically symmetric distribution of charge (for
 192 electromagnetic waves) or matter (for gravitational waves), even when that
 193 spherical distribution pulses symmetrically in and out. Therefore, *symmetric*
 194 collapses or explosions emit no waves, either electromagnetic or gravitational.
 195 The most efficient source of electromagnetic radiation is oscillating pairs of
 196 electric charges of opposite sign, the result technically called **dipole**
 197 **radiation**. But mass has only one “polarity;” there is no gravity dipole
 198 radiation from masses that oscillate back and forth along a line. Emission of
 199 gravity waves requires *asymmetric* movement or oscillation; the technical name
 200 for the simplest result is **quadrupole radiation**. Happily, most collapses and
 201 explosions are asymmetric; even the motion in a binary system is sufficiently
 202 asymmetric to emit gravitational waves.

203 We study here gravity waves emitted by a binary system consisting of two
 204 neutron stars—or a neutron star and a black hole—orbiting about one another
 205 (Section 6). All such pairs that we have detected are too far away to see
 206 directly; at least one neutron star needs to be a **pulsar** that emits a steady
 207 stream of pulses that we can receive at a great distance. Pulsars turn out to be
 208 extremely stable clocks. As the two objects orbit, they also emit gravity waves
 209 that cause the binary system to lose energy, so that the orbiting objects
 210 gradually spiral in toward one another. These orbits are fairly well described
 211 by Newtonian mechanics until about one millisecond before the two objects
 212 coalesce.

213 Emitted gravity waves are nearly periodic during the Newtonian phase of
 214 orbital motion. As a result, these particular gravity waves are easy to predict
 215 and therefore easy to search for. When the two objects coalesce, they emit a
 216 burst of gravity waves (Figure 11). After coalescence the resulting structure
 217 vibrates (“rings down”), emitting more gravity waves as it settles into its final
 218 state as a black hole. Initial LIGO has already completed its efforts and would
 219 have been sensitive enough (Figure 2) to detect binary neutron star systems
 220 coalescing at a distance of about 26 million light years. Unfortunately, no such
 221 coalescences were detected during more than one year of observation.
 222 Advanced LIGO extends the detection radius to 200 Megaparsecs ≈ 650
 223 million light years. The volume of space increases approximately as the cube of
 224 the distance, so the improved sensitivity will vastly increase the number of
 225 galaxies that can be “seen” by LIGO from about one thousand to millions,
 226 increasing the odds of success thousands of times.

Binary system
emits gravity
waves . . .

. . . whose
amplitude is
predictable.

QUERY 2. Increased volume of detection

Use numerical values given in the preceding paragraph to calculate to two significant figures the increased “odds of success” of Advanced LIGO compared with Initial LIGO.

4 Motion of Light in Map Coordinates

9

From other sources:
hard to predict.

231

232 Binary coalescence is the only source for which we can currently make a
233 clear prediction of the signal (and therefore of the detection distance limit).
234 Other conceivable sources include supernovae and the collapse of a massive
235 star to form a black hole—the event that triggers so called **gamma-ray**
236 **bursts**. But we have only speculations about how far away any of these can be
237 and still be detectable by either Initial LIGO or Advanced LIGO.

238

DETECTORS DO NOT AFFECT GRAVITY WAVES

239

240 We are used to the fact that metal structures can distort or reduce the
241 amplitude of electromagnetic waves passing across them. Even the
242 presence of a receiving antenna can distort an electromagnetic wave in
243 its vicinity. The same is not true of gravity waves, whose generation or
244 modification requires massive moving structures. Gravity wave detectors
245 have negligible effect on the waves that they are designed to detect.

246

247 Indeed, it is the smallness of the influence that gravity waves have on
mechanical structures that makes gravity waves so difficult to detect.

247

QUERY 3. Electromagnetic waves vs. gravity waves. Discussion.

What property of electromagnetic waves makes their interaction with conductors so huge compared with the interaction of gravity waves with matter of any kind?

251

4.1 MOTION OF LIGHT IN MAP COORDINATES

253 *Light reflected back and forth between mirrored test masses.*

254

255 The LIGO detector is an *interferometer* that employs mirrors mounted on
“test masses” suspended at rest at the ends of an L-shaped vacuum cavity.

256

257 The length of each leg of the L is 4 kilometers. Detection of the gravity wave is
accomplished by measuring the relative round-trip *time delay* between light
258 sent down one leg of the detector and light sent down the other, perpendicular
259 leg.

260

261 Suppose that a gravity wave of the polarization illustrated in Figure 3
moves in the z -direction as shown in Figure 5 and that one leg of the detector
262 lies along the x -direction and the other leg along the y -direction. In order to
263 analyze the operation of LIGO, we need to know (a) how light propagates
264 along the x and y legs of the interferometer and (b) how the test masses at the
265 ends of the legs move when the z -directed gravity wave passes over them. In
266 the present section we analyze the motion of light in map coordinates; Section
267 5 begins the description of the motion of test masses in map coordinates.

268

269 With what map speed does light move in the x -direction in the presence of
a gravity wave implied by metric (1)? To answer this question, set $dy = dz = 0$
270 in that equation, yielding

LIGO is an
interferometer.

Motion of light in
map coordinates.

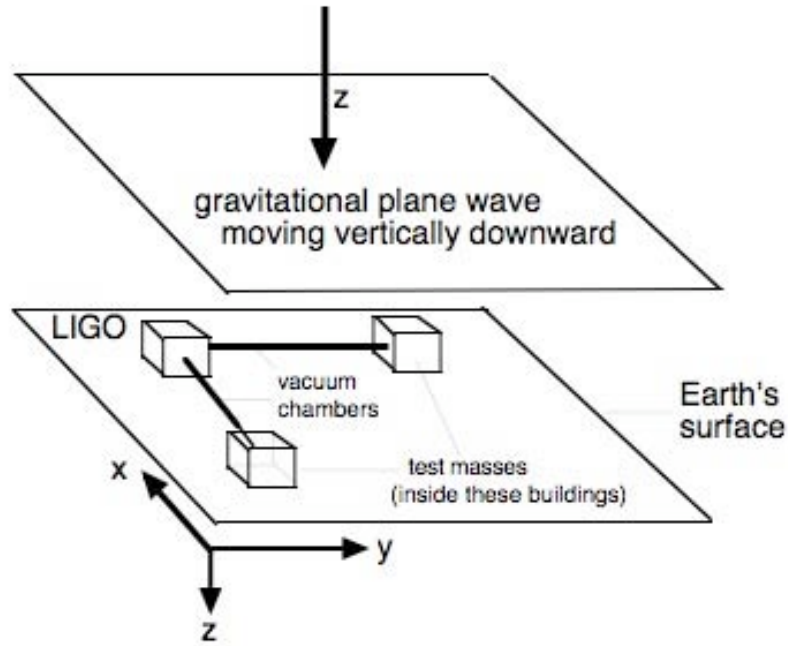


FIGURE 5 Perspective drawing of the relative orientation of legs of the LIGO interferometer lying in the x and y directions on the surface of Earth and the z -direction of the incident gravity wave descending vertically. [Illustrator: Rotate lower plate and contents CCW 90 degrees, so corner box is above the origin of the coordinate system. Same for Figure 10.]

$$d\tau^2 = dt^2 - (1 + h)dx^2 \quad (5)$$

As always, the proper time is zero between two adjacent events on the worldline of a light pulse. Set $d\tau = 0$ to find the speed of light in the x -direction.

$$\frac{dx}{dt} = \pm(1 + h)^{-1/2} \quad (\text{light moving in } x \text{ direction}) \quad (6)$$

The plus and minus signs correspond to a pulse traveling in the positive or negative x -direction, respectively—that is, in the plane of LIGO in Figure 5. Remember that the magnitude of h is very much smaller than one, so we use the approximation inside the front cover. To first order:

$$(1 + \epsilon)^n \approx 1 + n\epsilon \quad |\epsilon| \ll 1 \text{ and } |n\epsilon| \ll 1 \quad (7)$$

Apply this approximation to (6) to obtain

5 Motion of Ligo Test Masses in Map Coordinates

11

Gravity wave
modifies map
speed of light.

$$\frac{dx}{dt} \approx \pm(1 - \frac{h}{2}) \quad (\text{light moving in } x \text{ direction}) \quad (8)$$

In words, the map speed of light is changed (slightly!) by the presence of our gravity wave. Since h is a function of time as well as position, the map speed of light in the x -direction is not constant, but varies as the wave passes through. (Should we worry that the speed in (8) does not have the standard value one? No! This is a *map speed*—a mythical beast—measured directly by no one.)

By similar arguments, the map speeds of light in the y and z directions for the wave described by the metric (1) are:

$$\frac{dy}{dt} \approx \pm(1 + \frac{h}{2}) \quad (\text{light moving in } y \text{ direction}) \quad (9)$$

$$\frac{dz}{dt} = \pm 1 \quad (\text{light moving in } z \text{ direction}) \quad (10)$$

5.1 MOTION OF LIGO TEST MASSES IN MAP COORDINATES

“Obey the Principle of Maximal Aging!”

Consider two test masses with mirrors suspended at opposite ends of the x -leg of the detector. The signal of the interferometer due to the motion of light along this leg will be influenced only by the x -motion of the test masses due to the gravity wave. In this case the metric is the same as (5).

How does a test mass move as the gravity wave passes over it? As always, we answer this question with the Principle of Maximal Aging, maximizing the wristwatch time of the test mass across two adjoining segments of its worldline between fixed end-events. In what follows we verify the surprising result anticipated in Section 2 above, namely that a test mass initially at rest in map coordinates rides with the expanding and contracting map coordinates drawn on the distorting rubber sheet, so this test mass does not move with respect to map coordinates as a gravity wave passes over it. This result comes from showing that an out-and-back jog in the vertical worldline in map coordinates leads to smaller aging and therefore does not occur for a free test mass.

Figure 6 pictures this idealized case: an incremental linear deviation from a vertical worldline from origin 0 to the event at $t = 2t_0$. Along Segment A the displacement x increases linearly with time: $x = v_0 t$, where the speed v_0 is a constant. Along segment B the displacement returns to zero at the same constant rate. The strain h has average values \bar{h}_A and \bar{h}_B along segments A and B respectively. We use the Principle of Maximal Aging to find the value of the speed v_0 that maximizes the wristwatch time along this worldline. We will find that $v_0 = 0$. In other words, the free test mass initially at rest in map coordinates stays at rest in map coordinates; it does not deviate from the vertical worldline in Figure 6. Now for the details.

Write the metric (5) in approximate form for one of the segments:

Idealized case:
Linear jogs
out and back.

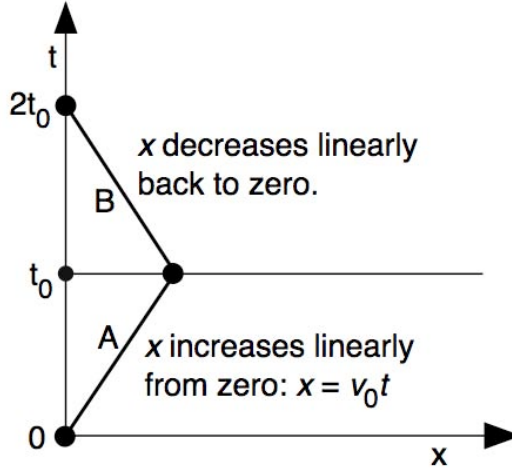


FIGURE 6 Trial worldline for a test mass; incremental departure from vertical line of a particle at rest. Segments A and B are very short.

$$\Delta\tau^2 \approx \Delta t^2 - (1 + \bar{h})\Delta x^2 \quad (11)$$

where \bar{h} is an average value of the strain h across that segment. Now we apply (11) first to Segment A in Figure 6, then to Segment B. We are going to take derivatives of these expressions, which will look awkward applied to Δ symbols. Therefore we temporarily ignore the Δ symbols in (12) and let τ stand for $\Delta\tau$, t for Δt , and x for Δx , holding in mind that these symbols actually represent increments, so equations in which they appear are approximations. With these substitutions, equation (11) becomes, for the two adjoining worldline segments:

$$\begin{aligned} \tau_A &\approx \left[t_0^2 - (1 + \bar{h}_A)(v_0 t_0)^2 \right]^{1/2} && \text{Segment A} \\ \tau_B &\approx \left[t_0^2 - (1 + \bar{h}_B)(v_0 t_0)^2 \right]^{1/2} && \text{Segment B} \end{aligned} \quad (12)$$

so that the total wristwatch time along the bent worldline from $t = 0$ to $t = 2t_0$ is the sum of the right sides of equations (12).

We want to know what value of v_0 (the out-and-back speed of the test mass) will lead to a maximal value of the total wristwatch time. To find this, take the derivative with respect to v_0 of the sum of individual proper times and set the result equal to zero.

$$\frac{d\tau_A}{dv_0} + \frac{d\tau_B}{dv_0} \approx -\frac{(1 + \bar{h}_A)v_0 t_0^2}{\tau_A} - \frac{(1 + \bar{h}_B)v_0 t_0^2}{\tau_B} = 0 \quad (13)$$

5 Motion of Ligo Test Masses in Map Coordinates

13

Initially at rest
in map coordinates?
Then stays at rest
in map coordinates.

327 In middle expression of (13), all quantities are fixed except for v_0 . The only
328 way that (13) can be satisfied is if $v_0 = 0$. *The test mass initially at rest does*
329 *not change its map x -coordinate as the gravity wave passes over.*

330 Our result seems rather specialized in two senses: First, it treats only the
331 vertical worldline traced out by a test mass at rest. Second, it deals only with
332 a very short segment of the worldline, along which \bar{h} is considered to be nearly
333 constant. Concerning the second point, you can think of (13) as a tiny
334 out-and-back “jog” *anywhere* on a much longer vertical worldline. Then our
335 result implies that *any* jog in the vertical worldline does not lead to an
336 increased value of the wristwatch time, even if h varies a lot over a longer
337 stretch of the worldline.

Not at rest in map
coordinates? Maybe
kink in map worldline.

338 The first specialization, the vertical worldline, is important: The gravity
339 wave does not cause a kink in a *vertical* map worldline. The same is typically
340 *not* true for a particle that is moving in map coordinates before the gravity
341 wave arrives. (We say “typically” because the kink may not appear for some
342 directions of motion of the test mass and for some polarization forms and
343 directions of propagation of the gravity wave.) In this more general case, a
344 kink in the worldline corresponds to a change of velocity. In other words, a
345 passing gravity wave can change the map velocity of a moving particle just as
346 if it were a velocity-dependent force. If the particle velocity is zero, then the
347 force is zero: a particle at rest in map coordinates remains at rest.

348

QUERY 4. Disproof of relativity? (optional)

349 “AHA!” declares Kristin Burgess. “Now I can disprove relativity once and for all. If the test mass
350 *moves*, a passing gravity wave can cause a kink in the worldline of the test mass as observed in the
351 local inertial Earth frame. No kink appears in its worldline if the test mass is at rest. But if a worldline
352 has a kink in it as observed in one inertial frame, it will have a kink in it as observed in all overlapping
353 relatively-moving inertial frames. An observer in any such frame can detect this kink. So the *absence* of
354 a kink tells me *and every other inertial observer* that the test mass is ‘at rest’? We have found a way to
355 determine absolute rest using a local experiment. Goodbye relativity!” Is Kristin right? (A detailed
356 answer is beyond the scope of this book, but you can use some relevant generalizations drawn from
357 what we already know to think about this paradox. As an analogy from flat-spacetime
358 electromagnetism, think of a charged particle at rest in a purely magnetic field: The particle
359 experiences no magnetic force. In contrast, when the same charged particle moves in the same frame, it
360 may experience a magnetic force for some directions of motion.)

362

At rest in map
coordinates?
Still can move
in Earth coordinates.

363 In this book we make every measurement in a local inertial frame, not
364 using differences in global map coordinates. So of what possible use is our
365 result that a particle at rest in global coordinates does not move in those
366 coordinates when a gravity wave passes over it? Answer: Just because
367 something is at rest in map coordinates does not mean that it is at rest in
368 local inertial Earth coordinates. In the following section we find that a gravity
369 wave *does* move a test mass as observed in the Earth coordinates.
370 LIGO—attached to the Earth—can detect gravity waves!

6. ■ DETECTION OF A GRAVITY WAVE BY LIGO

372 *Make measurement in the local Earth frame.*

373 Suppose that the gravity wave that satisfies metric (1) passes over the LIGO
 374 detector oriented as in Figure 5. We know how the test masses at the two ends
 375 of the legs of the detector respond to the gravity wave: they remain at rest in
 376 map coordinates (Section 5). We know how light propagates along both legs:
 377 as the gravity wave passes through, the map speed of light varies slightly from
 378 the value one, as given by equations (8) through (10) in Section 4.

Earth frame
tied to LIGO slab

379 The trouble with map coordinates is that they are arbitrary and need not
 380 correspond to what an observer measures. Recall that we require all
 381 measurements to take place in a local inertial frame. So think of a local
 382 reference frame anchored to the concrete slab on which LIGO rests. As
 383 explained in the Introduction (Section 1), the gravity wave has essentially no
 384 effect on this slab. Call the coordinates in the resulting local coordinate
 385 system **Earth coordinates**. Earth coordinates are analogous to shell
 386 coordinates for the Schwarzschild black hole; useful only locally but yielding
 387 the numbers that predict results of measurements. The metric for the local
 388 inertial frame then has the form:

$$\Delta\tau^2 \approx \Delta t_{\text{Earth}}^2 - \Delta x_{\text{Earth}}^2 - \Delta y_{\text{Earth}}^2 - \Delta z_{\text{Earth}}^2 \quad (14)$$

389 Compare this with the approximate version of (1):

$$\Delta\tau^2 \approx \Delta t^2 - (1+h)\Delta x^2 - (1-h)\Delta y^2 - \Delta z^2 \quad (h \ll 1) \quad (15)$$

390 Legally, in order to make the coefficients in (15) constants we should use
 391 the symbol \bar{h} , with a bar over the h , to indicate the average value of the
 392 gravity wave amplitude over the detector. However, in Query 1 you showed
 393 that for the frequencies at which LIGO is sensitive, the wavelength is very
 394 much greater than the dimensions of the detector, so the amplitude h of the
 395 gravity wave is effectively uniform across the LIGO detector. Therefore it is
 396 not necessary to take an average, and we use the symbol h without a
 397 superscript bar.

Earth frame
coordinate
differences

398 Comparing (14) with (15) yields:

$$\Delta t_{\text{Earth}} = \Delta t \quad (16)$$

$$\Delta x_{\text{Earth}} = (1+h)^{1/2}\Delta x \approx (1+\frac{h}{2})\Delta x \quad h \ll 1$$

$$\Delta y_{\text{Earth}} = (1-h)^{1/2}\Delta y \approx (1-\frac{h}{2})\Delta y \quad h \ll 1$$

$$\Delta z_{\text{Earth}} = \Delta z$$

399 where we use approximation (7). Notice, first, that Earth time lapse Δt_{Earth}
 400 between two events is identical to their map time lapse Δt and the z
 401 component of their space separation in Earth coordinates, Δz_{Earth} , is identical
 402 to the z component of their separation in map coordinates, Δz .

6 Detection of a gravity wave by LIGO

15

Test masses move
in Earth coordinates.

Light speed = 1
in Earth frame.

Different Earth
times along
different legs

Time difference
after N round trips.

Now for the differences! Let Δx be the map x -coordinate separation between the pair of mirrors in the x -leg of the LIGO interferometer and Δy be the map separation between the corresponding pair of mirrors in the y -leg. As the z -directed wave passes through the LIGO detector, the test masses at rest at the ends of the legs stay at rest in map coordinates, as Section 5 showed. Therefore the value of Δx remains the same during this passage, as does the value of Δy . But the presence of the time-varying strain $h(t)$ in (16) tells us that these test masses move when observed in Earth coordinates. More: when the distance between test masses increases (say) along the Earth x -axis, it decreases along the perpendicular Earth y -axis; and vice versa. Perfect for detection of a gravity wave by an interferometer!

Earth metric (14) is that of an inertial frame in which the speed of light has the value one in whatever direction it moves. With light we have the opposite weirdness to that of the motion of test masses initially at rest: In map coordinates light moves at speeds different from one in the presence of this gravity wave—equations (8) through (10)—but in Earth coordinates light moves with speed one. This is reminiscent of the corresponding case near a Schwarzschild black hole: Light moves at speeds different from one in Schwarzschild map coordinates but at speed one in shell coordinates.

In summary the situation is this: As the gravity wave passes over the LIGO detector, the speed of light propagating down the two legs of the detector has the usual value one as measured by the Earth observer. However, for the Earth observer the separations between the test masses along the x -leg and the y -leg change: one increases while the other decreases, as given by equations (16). The result is a difference in the round-trip times of light along the two legs. It is this difference that LIGO is designed to measure and thereby to detect the gravity wave.

What will be the value of this difference in round-trip times between light propagation along the two legs? Let D be the length of each leg in the absence of the gravity wave. The round-trip time is twice this length divided by the speed of light, which has the value one in Earth coordinates. From equations (16) we find that the difference in round-trip times between light propagated along the two legs is

$$\Delta t_{\text{Earth}} = 2D \left(\frac{h}{2} + \frac{h}{2} \right) = 2Dh \quad (\text{one round trip of light}) \quad (17)$$

Using the latest interferometer techniques, LIGO reflects the light back and forth down each leg approximately $N = 140$ times. That is, light executes approximately 140 round trips, which multiplies the detected time shift, increasing the sensitivity of the detector by the same factor. Equation (17) becomes

$$\Delta t_{\text{Earth}} = 2NDh \quad (N \text{ round trips of light}) \quad (18)$$

Quantities N and h have no units, so the unit of time in (18) is the same as the unit of D , for example meters.

443

QUERY 5. LIGO fast enough?

Do the 140 round trips of light take place in a time small compared with one period of the gravity wave being detected? (If it does not, then LIGO detection is not fast enough to track the change in gravity strain.)

447

448

449

QUERY 6. Application to LIGO.

Each leg of the LIGO interferometer is of length $D = 4$ kilometers. Assume that the laser emits light of wavelength 1000 nanometer $= 10^{-6}$ meter (infrared light from a NdYAG laser). Suppose that we want LIGO to reach a sensitivity of $h = 10^{-22}$. For $N = 140$, find the corresponding value of Δt_{Earth} . Express your answer as a decimal fraction of the period T of the laser light used in the experiment. (For background see http://www.ligo.caltech.edu/LIGO_web/about/)

456

457

QUERY 7. Faster derivation?

In this book we insist that global map coordinates are arbitrary human choices, so that we cannot depend on map coordinate differences to be directly measurable quantities. However, the value of h in (1) is so small that the metric differs only slightly from an inertial metric. This once, therefore, we treat map coordinates as directly measurable and ask you to redo the derivation of equations (17) and (18) using only map coordinates.

Remember that test masses initially at rest in map coordinates do not change their coordinates as the gravity wave passes over them (Section 4), but the gravity wave alters the map speeds of light, and differently in the x -direction, equation (8), than in the y -direction, equation (9). Assume that each leg of the interferometer has the length D_{map} in map coordinates.

- A. Find an expression for the difference Δt in map time between the two legs for one round trip of the light.
- B. How great do you expect the difference to be between times Δt and Δt_{Earth} and the difference between distances D (in Earth coordinates) and D_{map} ? Taken together, will these differences be great enough so that the result of your prediction and that of equation (18) could be distinguished experimentally?

474

475

QUERY 8. Different directions of propagation of the gravity wave

Thus far we have assumed that the gravitational plane wave of the polarization described by equation (1) descends vertically onto the LIGO detector, as shown in Figure 5. Of course the observers cannot prearrange in what direction an incident gravity wave will move. Suppose that the wave described by (1) propagates along the direction of, say, the y -leg of the interferometer, while the x -direction lies along the other leg, as before. What is the equation that replaces (18) in this case?

7 Binary System as a Source of Gravity Waves

17

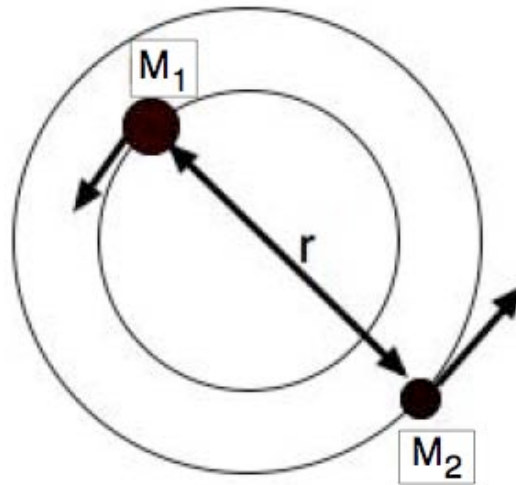


FIGURE 7 A binary system with each object in a circular path.

QUERY 9. LIGO fails to detect a gravity wave?

Think of various directions of propagation of the gravity wave pictured in Figure 3, together with different directions of x and y in equation (1) with respect to the LIGO detector. Give the name **orientation** to a given set of directions x and y —the transverse directions in (1)—plus z (the direction of propagation) in (1) relative to the LIGO detector. How many orientations are there for which LIGO will detect *no signal whatever*, even when its sensitivity is 10 times better than that needed to detect the wave arriving in the orientation shown in Figure 5? Are there zero such orientations? one? two? three? some other number less than 10? an infinite number?

7 ■ BINARY SYSTEM AS A SOURCE OF GRAVITY WAVES

Proof of the existence of gravity waves?

Unequal masses,
each in circular
orbit.

Energy of the system.

Now we consider in more detail gravity waves generated by a binary system consisting of two neutron stars, each in circular orbit around their center of mass. The binary system is the only known example of a stellar system for which we can explicitly calculate the emitted gravity waves. Suppose that the stars of the binary system have masses M_1 and M_2 and are assumed to orbit at a constant distance r apart, as shown in Figure 7.

The basic parameters of the orbit are adequately computed using Newtonian mechanics, according to which the energy of the system in conventional units is given by the expression:

$$E_{\text{conv}} = -\frac{GM_{1,\text{kg}}M_{2,\text{kg}}}{2r} \quad (\text{Newton}) \quad (19)$$

Rate of energy loss . . . As these neutron stars orbit, they generate gravity waves. General relativity predicts the rate at which the orbital energy is lost to this radiation. In conventional units, this rate is:

$$\frac{dE_{\text{conv}}}{dt_{\text{conv}}} = -\frac{32G^4}{5c^5r^5} (M_{1,\text{kg}}M_{2,\text{kg}})^2 (M_{1,\text{kg}} + M_{2,\text{kg}}) \quad (\text{Newtonian circular orbits}) \quad (20)$$

. . . derived from Einstein's equations. Equation (20) assumes that the two stars are separated by much more than their Schwarzschild radii and that they are moving at nonrelativistic speeds. Deriving equation (20) involves a lengthy and difficult calculation starting from Einstein's field equations. The same is true of the derivation of the metric (1) for a gravity wave. These are two of only three equations in this chapter that we simply quote from a more advanced treatment of general relativity.

QUERY 10. Energy and rate of energy loss in geometric units

Convert equations (19) and (20) to geometric units to be consistent with our notation and to get rid of the constants G and c . Use the sloppy professional shortcut, "Let $G = c = 1$."

A. Show that (19) and (20) become:

$$E = -\frac{M_1M_2}{2r} \quad (\text{Newton: geometric units, } E \text{ in units of length}) \quad (21)$$

$$\frac{dE}{dt} = -\frac{32}{5r^5} (M_1M_2)^2 (M_1 + M_2) \quad (\text{geometric units, } E \text{ in units of length}) \quad (22)$$

B. Verify that in both of these equations E has the unit of length.

C. Suppose you are given the value of E in meters. Show how you would convert this value first to kilograms and then to joules.

QUERY 11. Rate of change of radius

Derive an expression for the rate at which the radius changes as a result of this energy loss. Show that the result is:

$$\frac{dr}{dt} = -\frac{64}{5r^3} M_1M_2 (M_1 + M_2) \quad (\text{Newton: circular orbits}) \quad (23)$$

8 Binary Pulsar PSR1913+16

19

8.7 ■ BINARY PULSAR PSR1913+16528 *Proof of gravity waves?*Hulse and Taylor
discover binary.

529 On July 2, 1974 Russell A. Hulse was carrying out observations at the world's
 530 largest radio telescope at Arecibo, Puerto Rico. Hulse—a graduate student
 531 working under the direction of Joseph H. Taylor, then at the University of
 532 Massachusetts, Amherst—detected signals from a pulsar later named
 533 PSR1913+16. (PSR stands for “pulsar” and the numbers denote its celestial
 534 coordinates.) Here is an account of the discovery, excerpted from the Nobel
 535 Foundation website (which also has wonderful illustrations)
 536 <http://www.nobel.se/physics/laureates/1993/illpres/discovery.html>
 537 (Copyright ©2001 The Nobel Foundation)

THE DISCOVERY OF THE BINARY PULSAR

538 During 1974 Joseph Taylor and Russell Hulse were searching for new
 539 pulsars with the Arecibo telescope. They discovered 40, one of which was
 540 to be very important.
 541

542 When Hulse was observing the new pulsar, which has been named
 543 PSR1913+16, he found that the pulses arrived sometimes more often
 544 and sometimes less. The simplest interpretation was that the pulsar was
 545 orbiting another star very closely and at high velocity: Here one “pulsar
 546 year” is only about eight hours.

547 By observing the shift in the pulses, Hulse and Taylor found that the
 548 stars were equally heavy, each weighing about 1.4 times as much as the
 549 Sun. Since they were not visible on any photographs either, it was
 550 concluded that the other body, somewhat unexpectedly, was also a
 551 neutron star. Seen from Earth, however, it does not show up as a pulsar.
 552

MEASURING gravity waves

553 Since the two neutron stars in PSR1913+16 are moving so fast and close
 554 together they should, according to General Relativity, emit large
 555 amounts of gravity waves. This makes them lose energy: Their orbits will
 556 therefore shrink and their orbiting period will shorten.
 557

558 **Indirect evidence:** The binary pulsar has been observed continuously
 559 since its discovery, and the orbiting period has in fact decreased.
 560 Agreement with the prediction of General Relativity is better than
 561 1/2%. This is considered to prove that gravity waves really exists. In
 562 turn, this result is currently one of our strongest supports for the
 563 validity of the General Theory of Relativity.

Data from
binary pulsar

564 The signal from the pulsar constituted a very stable clock, stable to 10
 565 significant figures. As a result, Hulse and Taylor were able to use general
 566 relativity to analyze the motion of the system in detail, verifying many general
 567 relativity predictions, some of which allowed them to determine the individual
 568 orbiting masses M_1 and M_2 (given below), which Newtonian mechanics does

not reveal. Their results show that the binary system PSR1913+16 has the following parameters:

$$M_1 = (1.442 \pm 0.003)M_{\text{Sun}} \quad (\text{pulsar}) \quad (24)$$

$$M_2 = (1.386 \pm 0.003)M_{\text{Sun}} \quad (\text{companion})$$

$$a = 2.3418 \pm 0.0001 \text{ light seconds} \quad (\text{Semi-major axis of both})$$

$$e = 0.617127 \pm 0.000003 \quad (\text{Eccentricity of both})$$

Orbital period, ≈ 7.75 hours

Rate of advance of the periastron ≈ 4.2 degrees per Earth-year

Distance from Earth ≈ 7 kiloparsecs or about 20 000 light years.
(This distance is quite uncertain.)

Each neutron star follows its own elliptical path about the center of mass. The semi-major axis of the elliptical orbit for a neutron star—label it a —is half of the major axis, the longest distance from one side of its orbit to the other. The semi-minor axis—label it b —is half of the minor axis. Then the eccentricity $e \equiv (a^2 - b^2)^{1/2}/a$. The word **periastron** refers to the point of closest approach of these “astron”omical objects (just as the word *perihelion* refers to the point of closest approach of an orbiting object to our Sun: Greek, “Helios”). Note how large the rate of this periastron advance is compared with 43 arcseconds of advance of the perihelion of the planet Mercury *per Earth-century*.

The non-zero eccentricity in equation (24) tells us that the neutron stars in PSR1913+16 are *not* in circular orbits. General relativity predicts that when a binary system has non-circular orbits it will radiate gravity waves at a greater rate than when the orbits are circular. Nevertheless, in the following QUERIES we assume for simplicity that the orbits are effectively circular, as in Figure 7. That is, we assume a binary system in which each companion is in a circular orbit with constant radial separation r equal to the major axis, twice the value of the semi-major axis given in (24). This is equivalent to setting to zero the eccentricity of each neutron star orbit.

Meaning of
periastron.

We assume
circular orbits.

QUERY 12. Shrinkage of r per orbit

For a single orbit, the separation r between the orbiting neutron stars (assumed to be in circular orbits) does not change much, but it does change a little. For one orbit, what is the approximate value of the change in this separation r ? Express your answer in millimeters. (*Hint: No integration is needed for an approximate calculation of this incremental change.*)

8 Binary Pulsar PSR1913+16

21

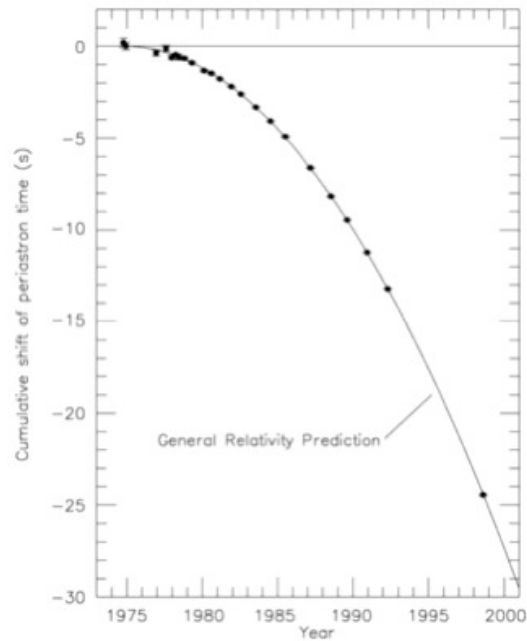


FIGURE 8 Decrease in the period in seconds (vertical axis) over the years 1975 to 1998 (horizontal axis) of binary system PSR1913+16. Agreement with the prediction of general relativity, assuming the change is due to emission of gravity waves, is now within 0.3 percent. This agreement appears to eliminate any other possible explanation for the change in orbits. From a paper (and Copyright ©2000) by J.H. Taylor and J. M. Weisberg.

601

QUERY 13. Energy radiated by idealized binary PSR1913+16

- What is the power currently being radiated in gravity waves? Express your answer as a unitless measure (energy in meters divided by time in meters) and also in watts (joules per second).
- Use equation (19) or (21) to calculate how much total energy in joules will be radiated in gravity waves from the present year to the future time when the two companions are separated by $r = 20$ kilometers (approximately the sum of their radii)? This total energy corresponds to how many kilograms of mass converted entirely to energy?
- How long a time in years will it be before the two neutrons stars in PSR1913+16 are separated by only $r = 20$ kilometers, so that coalescence is imminent? (Only in the last millisecond or so before coalescence does the Newtonian description become completely useless.)

612

613 ADD QUERY ABOUT RADIATION RATE OF SUN-MERCURY BINARY
 614 SYSTEM AND LENGTH OF TIME TO COALESCE DUE TO GRAV
 615 RADIATION.

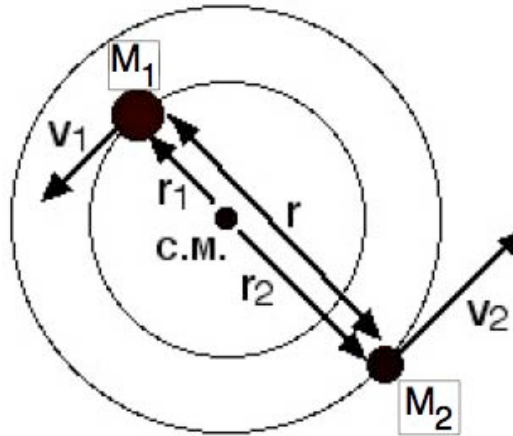


FIGURE 9 Figure 7 augmented to show the center of mass and orbit radii of individual components of PSR1913+16.

9. ■ GRAVITY WAVE AT EARTH DUE TO DISTANT BINARY SYSTEM

How far away from a binary system can we detect gravity waves?

Can LIGO on Earth's surface detect the gravity waves emitted by the distant binary system PSR1913+16 (idealized as one in which the neutron stars move in circular orbits as shown in Figure 7)? To answer this question we need to calculate the magnitude of h in the metric of equation (1).

Gravity waveform . . .

Here is the third and final result of general relativity quoted without proof in this chapter. The function $h(z, t)$ is given by the equation (in conventional units)

$$h(z, t) = -\frac{4G^2 M_1 M_2}{c^4 r z} \cos \left[\frac{2\pi f(z - ct)}{c} \right] \quad (\text{conventional units}) \quad (25)$$

where f is the frequency of the binary orbit, r is the (constant!) distance between orbiters in Figures 7 and 9, and z is the distance from source to detector. Convert (25) to geometric units by setting $G = c = 1$. Note that $h(z, t)$ is a function of z and t .

Figure 10 schematically displays the notation of equation (25), along with relative orientations and relative magnitudes assumed in the equation. This equation makes the Newtonian assumptions that (a) the two stars are separated by a distance r much larger than their Schwarzschild radii and (b) they move at nonrelativistic speeds. Additional assumptions are:

(c) The distance z between the binary system and Earth is very much greater than a wavelength of the gravity wave. This assumption assures that the radiation at Earth constitutes the so-called “far radiation field” where it assumes the form of a plane wave given in equation (4).

(d) The binary stars are orbiting in the xy plane, so that from Earth the orbits would appear as circles if we could see them (which we cannot, because

9 gravity wave at Earth Due to Distant Binary System

23

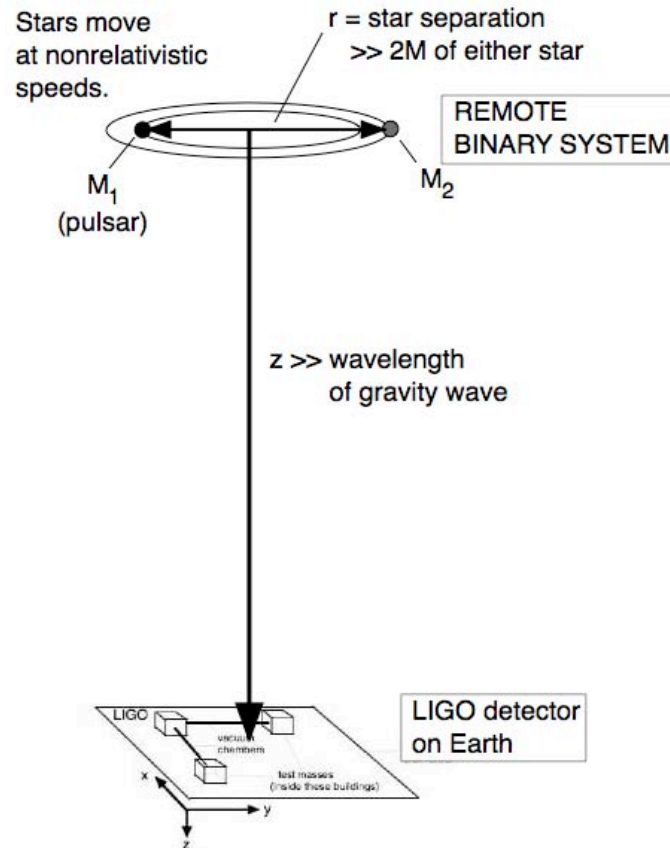


FIGURE 10 Schematic diagram, *not to scale*, showing notation and relative magnitudes for equation (25). The binary system and the LIGO detector lie in parallel planes.[Illustrator: See note in caption to Figure 5.]

they are too far away). Unfortunately this assumption is not true of the plane of the orbit of binary PSR1913+16, as we know from Doppler shifts of signals from the orbiting pulsar.

... for one case

Equation (25) describes only one linear polarization at Earth, the one generated by metric (1) and shown in Figure 3. The orthogonal polarization shown in Figure 4 is also transverse and equally strong, with components proportional to $(1 \pm h)$. The formula for the magnitude of h in that orthogonally polarized wave is identical to (25) with a sine function replacing the cosine function. We have not displayed the metric for that orthogonal polarization.

Detection requirements

In order for LIGO to detect a gravity wave, two conditions must be met: (a) the amplitude h of the gravity wave must be sufficiently large, and (b) the frequency of the wave must be in the range in which LIGO is most sensitive

(100 to 400 hertz). QUERY 13 deals with the amplitude of the wave. The frequency of gravity waves, discussed in QUERY 14, contains a surprise.

QUERY 14. Amplitude of gravity wave from PSR1913+16 at Earth

- A. Use (25) to calculate the maximum amplitude of h at Earth due to the radiation from the “idealized circular-orbit” binary system PSR1913+16. Consider this amplitude to be positive.
- B. Can either Initial LIGO or Advanced LIGO detect the gravity waves whose amplitude is given in part A?
- C. What is the maximum amplitude of h at Earth just before coalescence of PSR1913+16, when the neutron stars are separated by a distance $r = 20$ kilometers (but with orbits still described approximately by Newtonian mechanics)?
-

QUERY 15. Frequency of gravity waves emitted from PSR1913+16

- A. In order for either Initial LIGO or Advanced LIGO to detect the gravity waves whose amplitude is given in Query 13, the frequency of the gravity wave must be in the range 100 to 400 hertz. In Figure 9 the point C. M. is the stationary center of mass of the pulsar system. Using the symbols in Figure 9, fill in the steps to complete the following derivation.

$$\frac{v_1^2}{r_1} = \frac{GM_1}{r_1^2} \quad (\text{for } M_1, \text{ Newton, conventional units}) \quad (26)$$

$$\frac{v_2^2}{r_2} = \frac{GM_2}{r_2^2} \quad (\text{for } M_2, \text{ Newton, conventional units}) \quad (27)$$

$$M_1 r_1 = M_2 r_2 \quad (\text{center-of-mass condition}) \quad (28)$$

$$f_{\text{orbit}} \equiv \frac{1}{T_{\text{orbit}}} = \frac{v_1}{2\pi r_1} = \frac{v_2}{2\pi r_2} \quad (\text{common orbital frequency}) \quad (29)$$

where f_{orbit} and T_{orbit} are the frequency and period of the orbit, respectively. From these equations, show that for $r \equiv r_1 + r_2$ the frequency of the orbit is

$$f_{\text{orbit}} = \frac{1}{2\pi} \left[\frac{G(M_1 + M_2)}{r^3} \right]^{1/2} \quad (30)$$

- B. Here is a surprise: The frequency f of the gravity wave generated by this binary pair and appearing in (25) is twice the orbital frequency.

$$f_{\text{gravity wave}} = 2f_{\text{orbit}} \quad (31)$$

Why this doubling? Essentially it is because gravity waves are waves of tides. Just as there are two high tides and two low tides per day caused by the moon’s gravity acting on the Earth, there are two peaks and two troughs of gravity waves generated per binary orbit.

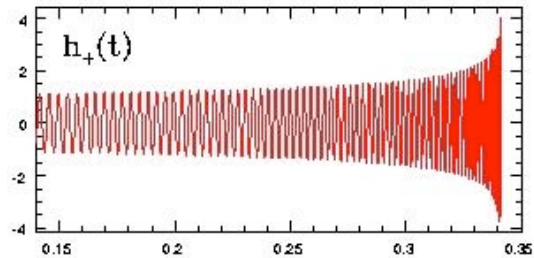


FIGURE 11

The “chirp” of one polarization of gravity wave as the two components of a binary system coalesce. (The plot for the other polarization is indistinguishable from this one.)

- C. Approximate the average of the component masses in (24) by the value $M = 1.4M_{\text{Sun}}$. Find the distance r between the binary stars when the orbital frequency is 75 hertz, so that the frequency of the gravity wave is 150 hertz. [ANS: Approximately 100 km.]
- D. Using results quoted earlier in this chapter, estimate the time for the binary system to decay from the current radial separation to the radial separation calculated in part C.
ANS: $t_2 - t_1 \approx 5(r_2^4 - r_1^4)/(256M^3)$, everything in unit meter.

“Chirp” at
coalescence

Newtonian mechanics predicts the motion of the binary system surprisingly accurately until the two components touch, a few milliseconds before they coalesce. As this happens, the gravity wave sweeps upward in both frequency and amplitude in what is called a **chirp**. Figure 11 is a predicted wave form for such a chirp.

To hear an audio simulation of the chirp, visit one of the following websites:

<http://www.lsc-group.phys.uwm.edu/~patrick/work/talks/itp/chirp.002.au>
<http://www.lsc-group.phys.uwm.edu/~patrick/work/talks/itp/chirp.002.wav>

Detection of such a waveform sweeping through the frequencies for which LIGO is sensitive would be a “smoking gun” for the coalescence of a binary source. Although LIGO cannot detect emission from PSR1913+16, we expect that many other binary systems are close to provide a detectable signal for Advanced LIGO.

10 ■ REFERENCES

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- LIGO sensitivity, Figure 2, at
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705 Chirp wave shape, Figure 11, at

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707 Websites for updates:

708 www.ligo.org

709 www.ligo.caltech.edu