2.2 Weyl spinors

The simplest non-trivial representations are $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$. The corresponding spinors are ψ_R and ψ_L (right- and left-handed Weyl spinors), respectively. By definition, the action of $\vec{J}_{R,L}$ on ψ_R is given by

$$\vec{J}_R\psi_R = \tfrac{1}{2}\vec{\sigma}\psi_R \ , \ J_L\psi_R = 0 \tag{2.2.1}$$

This leads to the action of rotations and boosts, respectively,

$$\vec{L}\psi_R = (\vec{J}_R + \vec{J}_L)\psi_R = \tfrac{1}{2}\vec{\sigma}\psi_R \ , \ \vec{M}\psi_R = -i(\vec{J}_R - \vec{J}_L)\psi_R = -\tfrac{i}{2}\vec{\sigma}\psi_R \tag{2.2.2}$$

Thus, under a finite Lorentz transformation,

$$\psi_R \to U_R\psi_R \ , \ U_R = e^{-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}++\frac{1}{2}\vec{v}\cdot\vec{\sigma}} \tag{2.2.3}$$

where we used eq. (2.1.17) - see also (2.1.8).
Working similarly, we find the effect of a general Lorentz transformation on ψ_L,

$$\psi_L \to U_L\psi_L \ , \ U_L = e^{-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}-\frac{1}{2}\vec{v}\cdot\vec{\sigma}} \tag{2.2.4}$$

It is useful to construct quantities which transform nicely, such as vectors and scalars. These will represent physical quantities. One such quantity is

$$V^\mu = \psi_R^\dagger\sigma^\mu\psi_R \ , \ \sigma^\mu = (I,\vec{\sigma}) \tag{2.2.5}$$

Let us show that V^μ is a four-vector. To this end, we consider infinitesimal rotations and boosts. Under a rotation, it is clear from (2.2.3) that $V^0 = \psi_R^\dagger\psi_R$ doesn't change. The spatial components of V^μ transform as

$$\begin{aligned}\vec{V} &\to \psi_R^\dagger(1+\tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma}+o(\theta^2))\vec{\sigma}(1-\tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma}+o(\theta^2))\psi_R \\ &= \vec{V}+\tfrac{i}{2}\theta_i\psi_R^\dagger[\sigma^i,\vec{\sigma}]\psi_R+o(\theta^2)\end{aligned} \tag{2.2.6}$$

Using (2.1.7), we deduce

$$\delta\vec{V} = \vec{\theta}\times\vec{V} \tag{2.2.7}$$

showing that $\vec{V}$ is a vector in three dimensions (*cf.* eq. (2.1.11)). Notice that this property was a direct consequence of the fact that $\vec{\sigma}$ transforms as a vector, which is the content of eq. (2.1.7).
Under an infinitesimal boost,

$$V^0 \to \psi_R^\dagger(1+\tfrac{1}{2}\vec{v}\cdot\vec{\sigma}+o(v^2))^2\psi_R = V^0+\vec{v}\cdot\vec{V}+o(v^2) \tag{2.2.8}$$

and

$$\begin{aligned}\vec{V} &\to \psi_R^\dagger(1+\tfrac{1}{2}\vec{v}\cdot\vec{\sigma}+o(v^2))\vec{\sigma}(1+\tfrac{1}{2}\vec{v}\cdot\vec{\sigma}+o(v^2))\psi_R \\ &= \vec{V}+\vec{v}V^0+o(v^2)\end{aligned} \tag{2.2.9}$$

where we used $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$. Thus, V^μ transforms correctly under boosts as well.[1] It follows that V^μ is a four-vector.
To construct a Lagrangian density, we need to turn ψ_R into a *field* and then find a scalar. Then a Lorentz transformation will not only act on the spinor indices of ψ_R but also on its argument. It is easy to show that $\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$ is a scalar. For the Lagrangian density we need a *real* quantity, so we define

$$\mathcal{L} = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R \tag{2.2.10}$$

By treating ψ_R and $\psi_R^\dagger$ as independent variablem the field equation is simple,

$$\sigma^\mu \partial_\mu \psi_R = 0 \tag{2.2.11}$$

This is the Weyl equation. Notice that

$$\det \sigma^\mu \partial_\mu = \begin{vmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{vmatrix} = \partial_\mu \partial^\mu \tag{2.2.12}$$

It follows that a Weyl spinor also satisfies the *massless* Klein-Gordon equation

$$\partial_\mu \partial^\mu \psi_R = 0 \tag{2.2.13}$$

which admits plane-wave solutions

$$\psi_R = u\, e^{-ip\cdot x} \tag{2.2.14}$$

where u is a spinor and $p^2 = 0$ (massless particle). Suppose the energy is positive. We are free to choose axes so that the momentum is along the z-axis. Then $p^\mu = (p, 0, 0, p)$. Plugging into the Weyl equation, we obtain

$$p_\mu \sigma^\mu u = p(I - \sigma^3)u = 0 \tag{2.2.15}$$

Therefore, $\sigma^3 u = u$, whose solution is $u = \binom{1}{0}$. It follows that the spin is along the momentum, i.e., the helicity is positive, i.e., this is a right-handed particle, which justifies our use of the subscript R in ψ_R.
The above discussion may be repeated for the left-handed spinors ψ_L. Under Lorentz tranformations,

$$\vec{L}\psi_L = \tfrac{1}{2}\vec{\sigma}\psi_L \ \ , \ \ \vec{M}\psi_L = \tfrac{i}{2}\vec{\sigma}\psi_L \tag{2.2.16}$$

It follows that

$$V^\mu = \psi_L^\dagger \bar{\sigma}^\mu \psi_L \ \ , \ \ \bar{\sigma}^\mu = (I, -\vec{\sigma}) \tag{2.2.17}$$

is a vector. The Lagrangian density (a scalar) is

$$L = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L \tag{2.2.18}$$

[1] Note that the factor $\gamma = (1 - v^2)^{-1/2}$ is missing because $\gamma = 1 + o(v^2)$.

leading to the field equation

$$\bar{\sigma}^\mu \partial_\mu \psi_L = 0 \tag{2.2.19}$$

The solutions also solve the massless Klein-Gordon equation.[2] A plane-wave solution $\psi_L = u\, e^{-ip\cdot x}$ with momentum along the z-axis satisfies the Weyl equation if $u = \binom{0}{1}$. Thus, ψ_L is a left-handed spinor (of negative helicity).
Finally, two important observations:

- The Lagrangian density for a Weyl spinor is linear in the derivatives, unlike in the Klein-Gordon case, where it is quadratic. This is because there is no Lorentz-invariant operator linear in the derivatives that acts non-trivially on a scalar field.
- ψ_R and ψ_L are related to each other by parity ($P : \psi_R \leftrightarrow \psi_L$). This follows from the action of P on Lorentz generators, $P : \vec{L} \to \vec{L}$, $\vec{M} \to -\vec{M}$ and so, on account of (2.1.21), $P : \vec{J}_R \leftrightarrow \vec{J}_L$.

2.3 The Dirac equation

2.3.1 The equation

Let us combine ψ_R and ψ_L by adding their respective Lagrangian densities,

$$\mathcal{L} = i\psi_R^\dagger \sigma \cdot \partial \psi_R + i\psi_L^\dagger \bar{\sigma} \cdot \partial \psi_L \tag{2.3.1}$$

We may add a mixed quadratic term, $\psi_R^\dagger \psi_L$, which is Lorentz-invariant (this follows easily from (2.2.3) and (2.2.4)).[3] Actually, only the real part contributes to the Lagrangian. This is implemented by adding the complex conjugate of $\psi_R^\dagger \psi_L$. We obtain the more general Lagrangian density

$$\mathcal{L} = i\psi_R^\dagger \sigma \cdot \partial \psi_R + i\psi_L^\dagger \bar{\sigma} \cdot \partial \psi_L - m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R) \tag{2.3.2}$$

where m is an arbitrary constant whose physical significance is yet to be determined. The field equations are easily deduced,

$$\begin{aligned} i(\partial_0 + \vec{\sigma} \cdot \vec{\nabla})\psi_R - m\psi_L &= 0 \\ i(\partial_0 - \vec{\sigma} \cdot \vec{\nabla})\psi_L - m\psi_R &= 0 \end{aligned} \tag{2.3.3}$$

They reduce to the respective Weyl equations (2.2.11) and (2.2.19) in the limit $m = 0$. They can be collectively written in terms of a four-component spinor as

$$i\gamma^\mu \partial_\mu \psi = m\psi \ , \ \ \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \ , \ \ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{2.3.4}$$

[2] This is a consequence of $\det \sigma \cdot \partial = \partial^2$; notice also that $(\sigma \cdot \partial)(\bar{\sigma} \cdot \partial) = \partial^2$.

[3] Note that neither $\psi_R^\dagger \psi_R$ nor $\psi_L^\dagger \psi_L$ is Lorentz-invariant which is why we did not consider adding them to the Lagrangian density in our discussion of Weyl spinors.