A new proof of the Solvable Signalizer Functor Theorem

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Autumn 2007

1 Introduction

A new proof of the following theorem of Glauberman [5] is presented.

The Solvable Signalizer Functor Theorem. *Let G be a finite group, A an elementary abelian r-subgroup of G with rank $m(A) \geq 3$ and θ a solvable A-signalizer functor on G. Then θ is solvably complete.*

For the benefit of the uninitiated, in §3 there is an exposition of elementary signalizer functor theory - including the basic definitions.

Signalizer functors were invented by Gorenstein as a tool for use in the Classification of Finite Simple Groups, see [8] for a discussion. The first solvable signalizer functor theorem was established by Gorenstein who considered the case $r = 2, m(A) \geq 5$. Goldschmidt [6], [7] improved this work, dealing with the cases r odd, $m(A) \geq 4$ and $r = 2, m(A) \geq 3$. Glauberman was the first to prove the definitive Solvable Signalizer Functor Theorem. A proof similar in outline to Glauberman's appears in the book by Kurzweil and Stellmacher [9].

Bender [3] gives a remarkably short proof in the case $r = 2$. His argument is quite different from Glauberman's. The ingredients are:

- An idea of Glauberman enabling effective use of induction.
- Bender's Maximal Subgroup Theorem.
- Glauberman's ZJ-Theorem.
- A fixed point theorem.

In attempting to generalize Bender's proof to arbitrary r, two difficulties arise: the ZJ-Theorem cannot be applied to all solvable groups of even order; and the fixed point theorem is not valid when r is a Fermat prime.

Aschbacher, in the first edition of his book *Finite Group Theory*, gives a proof of the Solvable Signalizer Functor Theorem along these lines. He uses the less powerful Glauberman Failure of Factorization Theorem as a substitute for the ZJ-Theorem and develops techniques for dealing with Fermat primes.

Unfortunately the difficulties are such that the resulting proof is much more complex than Bender's. Indeed, so much so, that in the second edition of *Finite Group Theory*, Aschbacher abandons the general case and presents a proof only for $r = 2$.

The proof presented here follows Bender's in outline. A recent result of the author [4] is a more suitable substitute for the ZJ-Theorem and we use Aschbacher's idea for dealing with Fermat primes. The resulting argument is similar to Aschbacher's but with several layers of complexity removed. Since we have to use [4], there is no overall reduction in length. However, the resulting conceptual simplification and the applicability of [4] to other problems make the effort worthwhile.

2 Preliminaries

All groups considered in this paper are finite.

Definition 2.1. *Let A and G be groups. Then* ***A* acts coprimely on *G*** *if we are given a homomorphism $\theta : A \longrightarrow \mathrm{Aut}(G)$; the orders of A and G are coprime; and at least one of A or G is solvable.*

Definition 2.2. *Suppose that A acts coprimely on G and let p be a prime. Then:*

(a) $\mathrm{Syl}_p(G; A)$ *is the set of maximal, under inclusion, A-invariant p-subgroups of G.*

(b) $O_p(G; A) = \bigcap \mathrm{Syl}_p(G; A)$.

Coprime Action. *Suppose A acts coprimely on the group G.*

(a) Let p be a prime. Then $\mathrm{Syl}_p(G; A) \subseteq \mathrm{Syl}_p(G)$; $C_G(A)$ acts transitively by conjugation on $\mathrm{Syl}_p(G; A)$; and every $AC_G(A)$-invariant p-subgroup of G is contained in $O_p(G; A)$.

(b) Suppose G is solvable and π is a set of primes. Then G possesses a unique maximal $AC_G(A)$-invariant π-subgroup.

(c) Suppose K is an A-invariant normal subgroup of G. Set $\overline{G} = G/K$. Then $C_{\overline{G}}(A) = \overline{C_G(A)}$.

(d) $G = C_G(A)[G, A]$ and $[G, A] = [G, A, A]$.

(e) Suppose X and Y are A-invariant subgroups of G with $G = XY$. Then $C_G(A) = C_X(A)C_Y(A)$.

(f) Suppose G is solvable and $[F(G), A] = 1$. Then $[G, A] = 1$.

(g) Suppose G is solvable, p is a prime and $[P, A] = 1$ for some $P \in \mathrm{Syl}_p(G; A)$. Then $[G, A] \leq O_{p'}(G)$.

(h) Suppose A is elementary abelian and noncyclic. Then

$$\begin{aligned} G &= \langle\, C_G(B) \mid B \in \mathrm{Hyp}(A)\,\rangle \\ &= \langle\, C_G(a) \mid a \in A^{\#}\,\rangle. \end{aligned}$$

Let $T \leq A$. Then

$$\begin{aligned} [G,T] &= \langle\, [C_G(B),T] \mid B \in \mathrm{Hyp}(A)\,\rangle \\ &= \langle\, [C_G(a),T] \mid a \in A^{\#}\,\rangle. \end{aligned}$$

Proof. (a),(b),(c). See [9, p.185-187]. (d) is [9, p.184] and (e) is [9, p.188].

(f). We have $[G,A] \leq C_G(F(G))$. Since G is solvable, $C_G(F(G)) \leq F(G)$. Thus $[G,A,A] = 1$. Apply (c).

(g). Set $\overline{G} = G/O_{p'}(G)$, so $F(\overline{G}) = O_p(\overline{G})$. Then $[O_p(\overline{G}),A] \leq [\overline{P},A] = 1$. Apply (f).

(h) is [9, p.193]. Recall that if A is an elementary abelian r-group then $\mathrm{Hyp}(A)$ is the set of subgroups of A with index r. □

$P \times Q$-Lemma ([9, p.187]). *Suppose $P \times Q$ acts on the group V where P and V are p-groups and Q is a p'-group. If $[C_V(P),Q] = 1$ then $[V,Q] = 1$.*

Goldschmidt Lemma ([9, p.190]). *Let P be a p-subgroup of the solvable group G. Then $O_{p'}(N(P)) \leq O_{p'}(G)$.*

Definition 2.3. *A group M has* **characteristic p**, *where p is a prime p, if $C_M(O_p(M)) \leq O_p(M)$.*

Remark. This is equivalent to $F^*(M) = F(M) = O_p(M)$; and if M is solvable to $O_{p'}(M) = 1$.

Definition 2.4. *A* **weak primitive pair** *for a group G is a pair (M_1, M_2) of distinct subgroups of G that satisfy*

- *whenever $\{\,i,j\,\} = \{\,1,2\,\}$ and $1 \neq K \operatorname{char} M_i$ with $K \leq M_1 \cap M_2$ then $N_{M_j}(K) = M_1 \cap M_2$.*

Moreover, (M_1, M_2) has **characteristic p** *if in addition:*

- *for each i, M_i has characteristic p and $O_p(M_1)O_p(M_2) \leq M_i$.*

Definition 2.5. *If M and H are solvable subgroups of the group G we define*

$$M \rightsquigarrow H$$

to mean there exists $X \leq F(M)$ with $XC_{F(M)}(X) \leq H$.

Bender's Maximal Subgroup Theorem. *Suppose that (M,H) is a weak primitive pair of solvable groups for the group G and that $M \rightsquigarrow H$. Then:*

(a) $O_q(H) \cap M = 1$ for all $q \notin \pi(F(M))$.

(b) *If $H \rightsquigarrow M$ or $O_q(H) = 1$ for all $q \notin \pi(F(M))$ then there is a prime p such that M and H have characteristic p.*

We use the following as a substitute for the ZJ-Theorem.

Theorem 2.6 ([4, Corollary C]). *Let G be a group. The following configuration is impossible.*

- *(M_1, M_2) is a weak primitive pair of characteristic p for G.*
- *M_1 and M_2 are solvable.*
- *For each i there is a group A_i that acts coprimely on M_i and $O_p(M_1)O_p(M_2) \leq O_p(M_i\,; A_i)$.*

3 Signalizer Functors

This section is aimed at the uninitiated. It develops elementary signalizer functor theory. Henceforth, G is a group, r is a prime and A a noncyclic elementary abelian r-subgroup of G.

Definition 3.1. *An **A-signalizer functor on G** is a mapping θ which for each $a \in A^{\#}$ assigns an A-invariant r'-subgroup*

$$\theta(C(a))$$

of $C(a)$ and satisfies

$$\theta(C(a)) \cap C(b) \leq \theta(C(b))$$

*for all $a, b \in A^{\#}$. Moreover, **θ is solvable** if $\theta(C(a))$ is solvable for all $a \in A^{\#}$.*

A trivial way of constructing signalizer functors is as follows: let M be an A-invariant r'-subgroup of G and define $\theta(C(a)) = C(a) \cap M$. The aim of signalizer functor theory is to show that every signalizer functor arises in this way. Since there are many other ways of constructing signalizer functors, see for example [8], we have a useful tool for constructing subgroups of G.

Definition 3.2. *An A-signalizer functor θ on G is **complete** if there is an A-invariant r'-subgroup M os G such that*

$$\theta(C(a)) = C(a) \cap M$$

*for all $a \in A^{\#}$. If M is solvable then we say that θ is **solvably complete**.*

Remarks. Since A is noncyclic it follows from Coprime Action(h) that M is uniquely determined. If θ is (solvably) complete then the subgroups $\theta(C(a))$ generate a proper (solvable) r'-subgroup of G.

Henceforth θ is an A-signalizer functor on G.

Definition 3.3. *A* ***θ*-subgroup** *of G is an A-invariant r'-subgroup H of G with the property*

$$C(a) \cap H \leq \theta(C(a))$$

for all $a \in A^{\#}$.

By the definition of signalizer functor, the subgroups $\theta(C(a))$ are θ-subgroups.

Lemma 3.4. *Let H be a θ-subgroup of G. Then:*

(a) Every A-invariant subgroup of H is a θ-subgroup.

(b) If B is a noncyclic subgroup of A then

$$H = \langle\, \theta(C(b)) \cap H \mid b \in B^{\#} \,\rangle.$$

(c) If K is also a θ-subgroup and $HK = KH$ then HK is a θ-subgroup.

Proof. (a) is trivial. (b) and (c) follow from Coprime Action(h) and (e) respectively. □

The θ-subgroups of G are partially ordered by inclusion. Hence we may refer to the **maximal *θ*-subgroups of *G***. Trivially:

Lemma 3.5. *The following are equivalent:*

(a) θ is complete.

(b) G possesses a unique maximal θ-subgroup.

Moreover, the subgroup M in the definition of complete is the unique maximal θ-subgroup.

Using Coprime Action(h) we have:

Lemma 3.6. *The following are equivalent:*

(a) θ is (solvably) complete.

(b) For every noncyclic $B \leq A$, $\langle\, \theta(C(b)) \mid b \in B^{\#} \,\rangle$ is a (solvable) θ-subgroup.

(c) There exists a noncyclic $B \leq A$ such that $\langle\, \theta(C(b)) \mid b \in B^{\#} \,\rangle$ is a (solvable) θ-subgroup.

Given H with $A \leq H \leq G$ we can consider the θ-subgroups of G that are contained in H. If there is a unique maximal such subgroup then we denote it by

$$\theta(H)$$

and say that ***θ*(*H*) is defined**. Thus θ is complete if and only if $\theta(G)$ is defined. A moments thought reveals that if $a \in A^{\#}$ then every θ-subgroup of $C(a)$ is contained in $\theta(C(a))$. Thus there is no ambiguity in the notation $\theta(C(a))$.

Alternatively, we can define an A-signalizer functor θ_H on H by

$$\theta_H(C(a)) = \theta(C(a)) \cap H.$$

Then $\theta(H)$ is defined if and only if θ_H is complete.

The following are left for the reader:

Lemma 3.7. *Suppose $A \le N \le G$ and $\theta(N)$ is defined. Then*

$$M \cap N = M \cap \theta(N)$$

for every θ-subgroup M.

Lemma 3.8. *Suppose $1 \neq B \le A$. Then*

(a) *$\theta(C(B))$ is defined and $\theta(C(B)) = \bigcap_{b \in B^{\#}} \theta(C(b))$.*

(b) *If $b \in B^{\#}$ then $\theta(C(b)) \cap C(B) = \theta(C(B))$.*

(c) *If M is a θ-subgroup then $C_M(B) = M \cap \theta(C(B))$.*

Lemma 3.9. *Let $a \in A^{\#}$. Then*

$$\theta(C(a)) = \langle\, \theta(C(B)) \mid a \in B \in \mathrm{Hyp}(A) \,\rangle.$$

The subgroup $\theta(C(A))$ plays a prominent role in signalizer functor theory.

Next we consider quotients.

Lemma 3.10. *Suppose $N \trianglelefteq G$ is a (solvable) θ-subgroup. Set $\overline{G} = G/N$ and define $\overline{\theta}$ by*

$$\overline{\theta}(C(\overline{a})) = \overline{\theta(C(a))}$$

for each $a \in A^{\#}$. Then

(a) *$\overline{\theta}$ is a $\overline{A}$-signalizer functor on $\overline{G}$.*

(b) *Suppose $N \le M \le G$. Then M is a θ-subgroup if and only if $\overline{M}$ is a $\overline{\theta}$-subgroup.*

(c) *θ is (solvably) complete if and only if $\overline{\theta}$ is (solvably) complete.*

Definition 3.11. *Let p be a prime. A subgroup of G is a* **(p, θ)-subgroup** *if it is both a p-subgroup and a θ-subgroup. The set of maximal (p, θ)-subgroups is denoted by*

$$\mathrm{Syl}_p(G; \theta).$$

Transitivity Theorem ([9, p.369]). *Let p be a prime and assume that $m(A) \ge 3$. Then $\theta(C(A))$ acts transitively by conjugation on $\mathrm{Syl}_p(G; \theta)$.*

Finally we define

$$\pi(\theta) = \bigcup_{a \in A^{\#}} \pi(\theta(C(a)))$$

where for any group X the set of prime factors of $|X|$ is denoted by $\pi(X)$.

4 Fixed point theorems

Lemma 4.1 ([1, (36.2), p.193]). *Let r, p and q be distinct primes, a an element of order r faithful on a p-group P and V a faithful $GF(q)\langle a \rangle P$-module. If r is a Fermat prime and $p = 2$ assume P is abelian. Then $C_V(a) \neq 0$.*

Theorem 4.2. *Let r and p be primes, a an element of order r acting on a solvable r'-group G and P an $\langle a \rangle C(a)$-invariant subgroup of G with $P = [P, a]$. If r is a Fermat prime and $p = 2$ assume P is abelian. Then $P \leq O_p(G)$.*

Proof. Let G be a minimal counterexample. Let V be a minimal a-invariant normal subgroup of G and set $\overline{G} = G/V$. Note that V is an elementary abelian q-group for some prime q. Coprime Action(c) and the minimality of G imply $\overline{P} \leq O_p(\overline{G})$. Let N be the inverse image of $O_p(\overline{G})$ in G, so $P \leq N \trianglelefteq G$. Then $q \neq p$. Also $O_p(G) = 1$ since otherwise we could have chosen $V \leq O_p(G)$.

Choose S with $P \leq S \in \mathrm{Syl}_p(N)$. Then $N = SV$ and $C_P(V) \leq C_S(V) = O_p(N) \leq O_p(G) = 1$. The minimality of G forces $G = PV$. If $[V, a] = 1$ then $P = [P, a] \leq C_P(V) = 1$, a contradiction. Thus $[V, a] \neq 1$. Since a has prime order $r \neq p$ and since $C_P(V) = 1$ it follows that $\langle a \rangle P$ is faithful on V.

Lemma 4.1 implies $C_V(a) \neq 1$. As P is $C(a)$-invariant we have $[P, C_V(a)] \leq P \cap V = 1$. Now $G = PV$ so $C_V(a)$ is an a-invariant normal subgroup of G. The choice of V forces $V = C_V(a)$. This contradicts the previous paragraph and completes the proof. □

The following results are needed when r is a Fermat prime.

Lemma 4.3 ([1, (36.4), p.194]). *Let r and p be distinct odd primes, let a be an element of order r faithful on a p-group P, assume $\langle a \rangle P$ acts faithfully on a 2-group T, and assume V is a faithful $GF(p)\langle a \rangle PT$-module. Then $C_V(a) \neq 0$.*

Theorem 4.4. *Let a be an element of prime order r acting on the solvable r'-group G. Let $X = [X, a]$ be an $\langle a \rangle C(a)$-invariant subgroup of G. Then $O_2(O^2(X)) \leq O_2(G)$.*

Proof. This is a special case of [1, (36.5), p.194]. Let G be a minimal counterexample, set $T = O_2(O^2(X)) \neq 1$ and let V be a minimal a-invariant normal subgroup of G. Then V is an elementary abelian p-group for some prime $p \neq 2$. Arguing as in Theorem 4.2, we obtain $O_2(G) = 1, G = XV, X \cap V = 1$ and $\langle a \rangle X$ is faithful on V. Moreover $[C_V(a), X] \leq X \cap V = 1$ so $C_V(a) \trianglelefteq G$ and then $C_V(a) = 1$.

Let $q \in \pi(X) - \{2, p\}$, choose $Q \in \mathrm{Syl}_q(X; a)$ and set $H = [Q, a]V$. Using Coprime Action(e), $C_H(a) \leq [Q, a]$ so Theorem 4.2 implies $[Q, a] \leq O_q(H)$. Then $[Q, a] \leq C_X(V) = 1$ and $|X|_q = |C_X(a)|_q$. Let X_0 be an a-invariant Hall $\{2, p\}$-subgroup of X. It follows that $X = C_X(a)X_0$ whence $X = [X, a] \leq X_0$ and we conclude that X is a $\{2, p\}$-group.

Let $P \in \mathrm{Syl}_p(X; a)$. If $[P, a] = 1$ then Coprime Action(g) implies $X = [X, a] \leq O_{p'}(X) = O_2(X)$ whence $O^2(X) = 1$, a contradiction. Thus $1 \neq [P, a] \leq O^2(X)$. Now $\langle a \rangle P$ acts faithfully and irreducibly on V so $O_p(X) =$

1. Then $O_p(O^2(X)) = 1$ so as $O^2(X)$ is a solvable $\{2, p\}$-group, it follows that $\langle a \rangle [P, a]$ acts faithfully on T. The previous lemma implies $C_V(a) \neq 1$, a contradiction. □

5 The minimal counterexample

Henceforth we assume the Solvable Signalizer Theorem to be false and consider a minimal counterexample with $|G| + |\pi(\theta)|$ minimal.

Lemma 5.1. *(a) If B is a noncyclic subgroup of A then*

$$\langle \theta(C(b)) \mid b \in B^{\#} \rangle$$

is not contained in a solvable θ-subgroup of G.

(b) No nontrivial solvable θ-subgroup is normal in G.

(c) If $X \neq 1$ is a solvable θ-subgroup then $\theta(N(X))$ is defined and solvable.

(d) If M is a maximal solvable θ-subgroup and $1 \neq X \trianglelefteq M$ is A-invariant then $M = \theta(N(X))$.

(e) If M_1 and M_2 are distinct maximal solvable θ-subgroups then (M_1, M_2) is a weak primitive pair for G.

Proof. This is a consequence of the minimality of G and the elementary results from §3. □

Theorem 5.2. *Let $p \in \pi(\theta)$ and $P \in \mathrm{Syl}_p(G; \theta)$. Then $C_A(P) = 1$.*

Proof. We repeat the proof given in [3], which relies on an idea of Glauberman. Let $a \in A^{\#}$. Now A acts on the solvable r'-group $\theta(C(a))$ so Coprime Action(b) implies that $\theta(C(a))$ possesses a unique maximal $A\theta(C(a)) \cap C(A)$-invariant p'-subgroup, which we denote by

$$\psi(C(a)).$$

Lemma 3.7 implies $\theta(C(a)) \cap C(A) = \theta(C(A))$. Thus $\psi(C(a))$ is the unique maximal $A\theta(C(A))$-invariant p'-subgroup of $\theta(C(a))$. It follows that ψ is an A-signalizer functor. Indeed, if $b \in A^{\#}$ then

$$\psi(C(a)) \cap C(b) \leq \theta(C(a)) \cap C(b) \leq \theta(C(b))$$

and so the left hand side, being an $A\theta(C(A))$-invariant p'-subgroup of $\theta(C(b))$, is contained in $\psi(C(b))$.

By construction, $\pi(\psi) \subseteq \pi(\theta) - \{p\}$ so the minimality of $\pi(\theta)$ implies that ψ is solvably complete. Let

$$K = \psi(G).$$

For each $a \in A^{\#}$, $K \cap C(a) = \psi(C(a)) \leq \theta(C(a))$ so K is a solvable θ-subgroup.

Assume the theorem to be false. Choose T with $\mathbb{Z}_r \cong T \leq C_A(P)$. Since $\theta(C(A)) \leq C(T)$, the Transitivity Theorem implies that T centralizes every (p,θ)-subgroup of G. For each $a \in A^{\#}$, an A-invariant Sylow p-subgroup of $\theta(C(a))$ is a (p,θ)-subgroup so Coprime Action(g) yields

$$[\theta(C(a)), T] \leq O_{p'}(\theta(C(a))) \leq \psi(C(a)) = C_K(a) \leq \theta(C(a)).$$

Coprime Action(d) implies

$$[\theta(C(a)), T] = [C_K(a), T]. \qquad (*)$$

Let $B \in \mathrm{Hyp}(A)$ with $T < B < A$. Now B is noncyclic so Coprime Action(h) and $(*)$ imply

$$[K,T] = \langle\, [\theta(C(b)), T] \mid b \in B^{\#} \,\rangle.$$

For each $b \in B^{\#}$, $\theta(C(B))$ normalizes $\theta(C(b))$ and T. It follows that $\theta(C(B))$ normalizes $[K,T]$. Lemma 3.8 implies $\theta(C(T))$ normalizes $[K,T]$. Lemma 3.4(c) implies that

$$\theta(C(T))[K,T]$$

is a solvable θ-subgroup.

Let $a \in A^{\#}$. Using Coprime Action(d) and $(*)$ we have

$$\begin{aligned} \theta(C(a)) &= \big(\theta(C(a)) \cap C(T)\big)[\theta(C(a)), T] \\ &\leq \theta(C(T))[K,T]. \end{aligned}$$

Lemma 5.1(a) supplies a contradiction. □

Corollary 5.3. *Suppose $a \in A^{\#}$, M is a maximal solvable θ-subgroup, $\theta(C(a)) \leq M$, $p \in \pi(F(M))$ and $S \in \mathrm{Syl}_p(M;A)$. Then:*

(a) $[S,a] \neq 1$.

(b) $[F(M),a] \neq 1$.

(c) Suppose $[O(M),a] = 1 \neq O(M)$ *and* $p \neq 2$. *Then*

$$1 \neq \big[[S,a], O_2(M)\big] \leq O_2(O^2([M,a])).$$

Proof. (a). Suppose $[S,a] = 1$. Let $N = \theta(N(S))$, so $N = C_N(a)[N,a]$. Now $C_N(a) \leq \theta(C(a)) \leq M$. Also $[N,a] \leq C(S) \leq C(O_p(M)) \leq N(O_p(M))$. Then $[N,a] \leq \theta(N(O_p(M))) = M$ by Lemma 5.1. Thus $N \leq M$. Choose P with $S \leq P \in \mathrm{Syl}_p(G;\theta)$. Then $N_P(S) \leq \theta(N(P)) = N \leq M$ whence $N_P(S) = S$ and then $S = P \in \mathrm{Syl}_p(G;\theta)$. Since $[S,a] = 1$, Theorem 5.2 supplies a contradiction.

(b). Apply Coprime Action(f).

(c). Since $O^2([M,a]) \trianglelefteq M$ and $[S,a] \leq O^2([M,a])$ the inclusion is clear. Suppose $\big[[S,a], O_2(M)\big] = 1$. Now $[O(M),a] = 1$ whence

$$[S,a] \leq C(O_2(M)) \cap C(O(M)) \leq C(F(M)) \leq F(M)$$

and so $[S,a] \leq O(M)$ because $p \neq 2$. Then $[S,a] = [S,a,a] \leq [O(M),a] = 1$, contrary to (a). □

6 A uniqueness theorem

Let

$\mathcal{L}$	*be the set of nontrivial $\theta(C(A))$-invariant solvable θ-subgroups, and*
$\mathcal{L}^*$	*be the set of maximal members of $\mathcal{L}$ under inclusion.*

Trivially, if $M \in \mathcal{L}^*$ then $\theta(C(A)) \leq M$ and $C_M(A) = \theta(C(A))$. Moreover, $\mathcal{L}^* \neq \emptyset$ since otherwise $\theta = 1$.

Lemma 6.1. *Suppose B is a noncyclic subgroup of A, M is a maximal solvable θ-subgroup of G and $x \in \theta(C(B)) - M$. Then (M, M^x) is a weak primitive pair.*

Proof. We have $\theta(C(B)) \cap N(M) \leq \theta(N(M)) = M$ so $M \neq M^x$. Let $1 \neq K \operatorname{char} M$ and set $N = N_{M^x}(K)$. If $b \in B^\#$ then $x \in \theta(C(B)) \leq \theta(C(b))$ so

$$C_N(b) \leq C_{M^x}(b) = \big(C_M(b)\big)^x \leq \theta(C(b))^x = \theta(C(b)).$$

Hence $C_N(b) \leq \theta(C(b)) \cap N(K) \leq \theta(N(K)) = M$. Coprime Action(h) implies $N \leq M$, so $N_{M^x}(K) \leq M$.

Let $1 \neq K \operatorname{char} M^x$ and set $y = x^{-1}$. Then $K^y \operatorname{char} M$ so applying the above, with y in place of x, yields $N_{M^y}(K^y) \leq M$. Thus $N_M(K) \leq M^x$. The proof is complete. □

Theorem 6.2. *Let p be a prime. Then $\mathcal{L}^*$ contains at most one member with characteristic p.*

Proof. Suppose $M \in \mathcal{L}^*$ has characteristic p. Let

$$P = \bigcap \mathrm{Syl}_p(G; \theta).$$

The Transitivity Theorem implies that P contains every $\theta(C(A))$-invariant (p, θ)-subgroup of G. In particular, $O_p(M) \leq P$. Set

$$Q = P \cap M.$$

Since $C_M(A) = \theta(C(A))$ it follows that $Q = O_p(M\,; A)$.

We claim that $\theta(N(Q)) \leq M$. Assume false and set $N = \theta(N(Q))$. By Coprime Action(h) there exists $B \in \mathrm{Hyp}(A)$ and $x \in C_N(B) - M$. Since N is a θ-subgroup we have $x \in \theta(C(B))$. Also B is noncyclic because $m(A) \geq 3$. Lemma 6.1 implies (M, M^x) is a weak primitive pair. Now $x \in N(Q)$ so

$$O_p(M)O_p(M^x) \leq Q = Q^x = O_p(M\,; A) = O_p(M^x\,; A^x) \leq M \cap M^x.$$

Consequently (M, M^x) has characteristic p. Note that A and A^x act coprimely on M and M^x respectively. Theorem 2.6 supplies a contradiction. We deduce that $\theta(N(Q)) \leq M$. In particular, $N_P(Q) \leq P \cap M = Q$ whence $P = Q \leq M$.

Now suppose $H \in \mathcal{L}^*$ also has characteristic p and $H \neq M$. Lemma 5.1 implies (M, H) is a weak primitive pair. By the previous paragraph,

$$O_p(M)O_p(H) \leq P = O_p(M\,; A) = O_p(H\,; A).$$

Then (M, H) has characteristic p. Theorem 2.6, with $A_1 = A_2 = A$, supplies a contradiction. □

Corollary 6.3. *Let $M, H \in \mathcal{L}^*$ with $M \rightsquigarrow H$. Suppose at least one of the following hold:*

(a) $H \rightsquigarrow M$, or

(b) $O_q(H) = 1$ for all $q \notin \pi(F(M))$.

Then $M = H$.

Proof. Apply Lemma 5.1(e), Bender's Maximal Subgroup Theorem and Theorem 6.2. □

7 The generic case

For each $a \in A^\#$ let

$$\mathcal{M}_a = \{\, M \in \mathcal{L}^* \mid \theta(C(a)) \leq M \,\}.$$

Note that $\theta(C(A)) \leq \theta(C(a))$, so as $\mathcal{L}^* \neq \emptyset$ it follows that $\mathcal{M}_a \neq \emptyset$. Choose

$$M_a \in \mathcal{M}_a$$

but if possible with $C_{O_p(M_a)}(a) = 1$ for some $p \in \pi(F(M_a))$.

The strategy is quite simple. Lemma 5.1 implies not all the M's can be equal. Using §4 and §6 we are able to force many of the M's to equal one another. The resulting tension leads to a contradiction.

Lemma 7.1. *Let B be a noncyclic subgroup of A. Then there exists $b, c \in B^\#$ such that $M_b \neq M_c$.*

Proof. This follows from Lemma 5.1(a). □

Lemma 7.2. *Let $a \in A^\#$ and $H \in \mathcal{M}_a$ with $M_a \rightsquigarrow H$. Then $M_a = H$.*

Proof. Assume false. By Corollary 6.3 there exists $q \in \pi(F(H)) - \pi(F(M_a))$. Bender's Maximal Subgroup Theorem implies $O_q(H) \cap M_a = 1$. In particular $C_{O_q(H)}(a) = 1$.

The definition of M_a implies $C_{O_p(H)}(a) = 1$ for some $p \in \pi(F(M_a))$. Let $Z = Z(O_p(M_a))$. Then $Z \leq H$ because $M_a \rightsquigarrow H$. Now $C_Z(a) = 1$ so $Z = [Z, a]$. As $C_H(a) = \theta(C(a)) = C_M(a)$ we see that Z is $\langle a \rangle C_H(a)$-invariant. Note that Z is abelian. Theorem 4.2 forces $Z \leq O_p(H)$. But then

$$O_q(H) \leq \theta(N(Z)) = M_a,$$

contradicting $O_q(H) \cap M_a = 1$. □

Lemma 7.3. *Let $a \in A^\#$ and $1 \neq X \leq F(M_a)$ such that X is $A\theta(C(a))$-invariant. Then $\theta(N(X)) \leq M_a$.*

Proof. Choose H with $\theta(N(X)) \leq H \in \mathcal{M}_a$. Then $N_{F(M_a)}(X) \leq \theta(N(X)) \leq H$ so $M_a \rightsquigarrow H$. Apply the previous lemma. □

Lemma 7.4. *Suppose $p \neq 2$, $a \in A^{\#}$ and $1 \neq P \leq C_{O_p(M_a)}(a)$ with $P \in \mathcal{L}$. Then $\theta(N(P)) \leq M_a$.*

Proof. Assume false and choose H with $\theta(N(P)) \leq H \in \mathcal{L}^*$ and $H \neq M_a$. Since $P \leq F(M_a)$ we have $M_a \rightsquigarrow H$. We claim:

$$\left.\begin{array}{l} F(H) \textit{ does not contain a nontrivial} \\ A\theta(C(a))\textit{-invariant subgroup of } F(M_a). \end{array}\right\} \qquad (*)$$

Indeed, suppose X is such a subgroup. Lemma 7.3 implies $\theta(N(X)) \leq M_a$ so $H \rightsquigarrow M_a$. Corollary 6.3 supplies a contradiction, which establishes $(*)$.

Let $q \notin \{2, p\}$. Then $O_q(M_a) \leq \theta(N(P)) \leq H$. Now $C_H(a) \leq \theta(C(a)) \leq M_a$ so $O_q(M_a)$ is $C_H(a)$-invariant. Theorem 4.2 forces $[O_q(M_a), a] \leq O_q(H)$. Then $(*)$ implies $[O_q(M_a), a] = 1$.

Let $P^* = C_{O_p(M_a)}(a) \geq P$. Then P^* is $A\theta(C(a))$-invariant. Moreover $C_{O_p(M_a)}(P^*) \leq \theta(C(P)) \leq H$ so Theorem 4.2 implies $[C_{O_p(M_a)}(P^*), a] = 1$. Applying the $P \times Q$-Lemma to the action of $\langle a \rangle \times P^*$ on $O_p(M_a)$ we deduce that $[O_p(M_a), a] = 1$.

We have shown that $[O(F(M_a)), a] = 1$ so $[O(M_a), a] = 1$ by Coprime Action(f). Now $p \neq 2$ and $1 \neq P \leq O_p(M_a)$ so $O(M_a) \neq 1$. Corollary 5.3 implies

$$O_2(O^2([M_a, a])) \neq 1.$$

Now $[O_p(M_a), a] = 1$ so $[M_a, a] \leq \theta(C(O_p(M_a))) \leq \theta(C(P)) \leq H$. Theorem 4.4 implies

$$O_2(O^2([M_a, a])) \leq O_2(H)$$

and $(*)$ supplies a contradiction. □

Theorem 7.5. *Let $a \in A^{\#}$. Then $[O(M_a), a] = 1$.*

Proof. Assume false. Coprime Action(f) implies there exists $p \neq 2$ such that $[O_p(M_a), a] \neq 1$ and then Coprime Action(h) implies there exists $B \in \mathrm{Hyp}(A)$ with

$$P := \left[C_{O_p(M_a)}(B), a\right] \neq 1.$$

Note that $P = [P, a]$ and $P \in \mathcal{L}$. We claim that:

$$P \leq O_p(L) \quad \text{whenever} \quad P \leq L \in \mathcal{L}^*. \qquad (*)$$

Indeed, $C_L(a) \leq \theta(C(a)) \leq M_a$ so $\langle P^{C_L(a)} \rangle$ is a p-group. Apply Theorem 4.2.

Let $b, c \in B^{\#}$. Then $P \leq M_a \cap C(b) \leq \theta(C(b)) \leq M_b$ so $P \leq O_p(M_b)$. Lemma 7.4 implies $\theta(N(P)) \leq M_b$. Also, $P \leq O_p(M_c)$ whence $N_{F(M_c)}(P) \leq \theta(N(P)) \leq M_b$ so $M_c \rightsquigarrow M_b$. By symmetry, $M_b \rightsquigarrow M_c$ so Corollary 6.3 forces $M_b = M_c$. Lemma 7.1 supplies a contradiction. □

We remark that if $r = 2$ then each M_a has odd order and a contradiction follows from Theorem 7.5 and Corollary 5.3. Also if r is not a Fermat prime then the restriction $p \neq 2$ is not needed in Lemma 7.4 and then the proof of Theorem 7.5 yields $[F(M_a), a] = 1$, again contradicting Corollary 5.3.

8 The Fermat case

We use an idea of Aschbacher [1]. Choose $a \in A^{\#}$ with $O(M_a)$ maximal. Lemma 7.1 and Theorem 6.2 imply that not all the M's have characteristic 2. Hence $O(M_a) \neq 1$ and we may choose an odd prime p with $O_p(M_a) \neq 1$. Theorem 7.5 implies $[O_p(M_a), a] = [O(M_a), a] = 1$. By Coprime Action(h) there exists B with $\mathbb{Z}_r \times \mathbb{Z}_r \cong B < A, a \in B$ and

$$P := C_{O_p(M_a)}(B) \neq 1.$$

Lemma 8.1. *Let $b \in B^{\#}$ and suppose $O_2(M_a) \cap M_b \neq 1$. Then $M_a = M_b$.*

Proof. Let $T = O_2(M_a) \cap M_b \in \mathcal{L}$. Then $Z(O_2(M_a))O(M_a) \leq \theta(N(T))$ and as $M_a = \theta(N(Z(O_2(M_a))))$ it follows from the Goldschmidt Lemma that $O(M_a) \leq O(\theta(N(T)))$ and then that

$$O(M_a) \cap M_b \leq O(M_b). \tag{$*$}$$

In particular, $P \leq O(M_b)$.

By Theorem 7.5, $[O(M_b), b] = 1$ so $[M_b, b] \leq \theta(N(P)) \leq M_a$, by Lemma 7.4. Corollary 5.3 and Theorem 4.4 imply

$$1 \neq O_2(O^2([M_b, b])) \leq O_2(M_a).$$

Now $O_2(O^2([M_b, b])) \trianglelefteq M_b$ whence $O(M_a) \leq \theta(N(O_2(O^2([M_b, b])))) = M_b$, and then $(*)$ yields $O(M_a) \leq O(M_b)$. The choice of a forces $O(M_a) = O(M_b) \neq 1$, whence $M_a = M_b$. □

We are now in a position to derive a final contradiction. Choose $S \in \mathrm{Syl}_p(M_a\,; A)$. By Lemma 7.1 there exists $b \in B^{\#}$ with $M_b \neq M_a$. Then $O_2(M_a) \cap M_b = 1$. In particular

$$C_{O_2(M_a)}(b) \leq O_2(M_a) \cap \theta(C(b)) \leq O_2(M_a) \cap M_b = 1.$$

Set $H = [S, b]O_2(M_a)$. Then $C_H(b) \leq [S, b]$. Theorem 4.2 forces $[S, b] \leq O_p(H)$, whence

$$\big[[S, b], O_2(M_a)\big] = 1.$$

Corollary 5.3(c) implies

$$\big[[S, a], O_2(M_a)\big] \neq 1,$$

so as $a \in B \cong \mathbb{Z}_r \times \mathbb{Z}_r$ it follows that $\langle\, b \,\rangle$ is uniquely determined. We deduce that

$$M_c = M_a$$

for all $c \in B - \langle\, b \,\rangle$.

Coprime Action(h) implies

$$\begin{aligned} [O_2(M_b), b] &= \langle\, [C_{O_2(M_b)}(c), b] \mid c \in B - \langle\, b \,\rangle \,\rangle \\ &\leq M_a. \end{aligned}$$

Theorem 7.5 and Corollary 5.3(b) imply $[O_2(M_b), b] \neq 1$ and $O_2(M_a) \neq 1$. Using Lemma 7.3 we obtain

$$1 \neq N_{O_2(M_a)}([O_2(M_b), b]) \leq O_2(M_a) \cap M_b = 1.$$

This contradiction completes the proof of the Solvable Signalizer Functor Theorem.

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