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The Impact of Daily and Seasonal Cycles In Solar-Generated Steam On Oil Recovery

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Abstract

Steam-injection projects located in areas with abundant sunshine can exploit solar energy to generate steam, instead of burning gas. Using mirrors, sunlight can be concentrated and used to heat water flowing inside a pipe to steam-temperature at high pressure. This solar-generated steam is only available during the day, causing a day-night cycle in the steam-rate, and is also affected by seasonal variations in solar energy, causing a summer-winter cycle in the steam-rate.

Using both analytical modelling and thermal reservoir simulation, we investigate the impact of the daily and seasonal cycles in steam rate on oil recovery. We compare the oil recovery from solar-generated steam and the recovery resulting from a continuous and constant-rate steam-injection, as obtained by burning gas, for instance.

Our analytical model for the periodic heating of matrix-blocks in a fractured reservoir shows that for fractured reservoirs with a typical fracture-spacing larger than one metre, the impact of daily cycles in steam-injection can be ignored.

Using thermal reservoir simulation, we also compare oil recovery from two representative models of realistic reservoirs: a fractured reservoir (recovery mechanism: thermally-enhanced gas-oil gravity drainage) and a non-fractured reservoir (recovery mechanism: steam-drive), for both steam-injection profiles: constant-rate and cyclic-rate. Our simulations show that the seasonal cycles in the solar-generated steam rate are reflected as seasonal cycles in the oil-rate.

The simulation results indicate that for the same cumulative amount of steam injected (during the same time-span), the oil recovery from solar-generated steam-injection and that from constant-rate steam-injection are essentially the same, both for the fractured reservoir and for the non-fractured reservoir. Therefore, from a subsurface oil-recovery point of view, solar-generated steam provides a viable alternative to constant-rate steam-injection.

Introduction

The objective of this paper is to investigate if the oil recovery from a typical steam-injection EOR project would be negatively impacted if steam would be injected at a cyclical rate rather than at a constant rate. In particular, we want to investigate the feasibility to generate steam from solar energy instead of burning gas. We want to point out that in this paper we focus on some of the subsurface recovery aspects only, and not on surface and facility aspects of the steam generation itself. In a nutshell, the idea is to use mirrors to concentrate sunlight into a pipe through which water is flowing, and to heat the water in this way to steam-temperature at a high pressure. Pressures of up to 60 bar were reported to have been achieved in the European Direct Steam Generation research project in Spain [1]. Steam generated in this way is affected by two natural cycles: a day-night cycle (solar-steam being available only during the day) and a seasonal cycle (due to the summer-winter cycle in solar irradiance). The paper is organised as follows: we first develop an analytical model for the response of a matrix-block the surface of which is subjected to a periodically changing temperature. This analytical model is conceived to investigate the behaviour of a matrix-block in a fractured reservoir that is subject to temperature fluctuations that could arise due to changes in the steam-injection rate. This model allows us to estimate to which extent, for a given fracture-spacing, daily or seasonal variations in steam-injection rate manifest themselves in variations of the temperature of the matrix-block itself. This model will show us that for fractured reservoirs with a fracture spacing of several metres, daily fluctuations in applied temperature can for practical purposes be neglected, but that seasonal fluctuations are important and are reflected in the average temperature of the matrix-blocks.

Subsequently we present simulation results comparing solar-cyclic steam-injection and constant-rate steam-injection, for a fractured reservoir and for a non-fractured reservoir. These results show that the seasonal variation in the steam-injection rate is reflected as an oscillation in the oil-rate. However, our simulations show that if the same amount of cumulative steam is injected into the reservoir, during the same time-span, then solar-cyclic steam-injection and constant-rate steam-injection give the same oil recovery. In other words, the simulations indicate that solar-generated steam-injection does not negatively impact the oil recovery. Thus from a sub-surface point of view, solar-generated steam seems a viable alternative to constant-rate steam-injection.

Theory

To get a feeling for the temperature response of a matrix-block in a fractured reservoir that is subjected to cyclic steam-injection, we consider a homogeneous one-dimensional matrix-block of size L that is subjected to a periodically changing temperature at its matrix-fracture boundaries (located at $x = \pm L/2$). Then the temperature evolution inside the matrix-block which initially has a uniform temperature T_i, is given by

$$\begin{cases} \partial_t T = \alpha \partial_x^2 T, \\ T(x,t) = T_i, \ \ x = \pm L/2, \ \ t \le 0, \\ T(-L/2,t) = T(L/2,t) = T_s(t), \ \ t > 0, \end{cases} \qquad \text{(Eq. 1)}$$

where $T_s(t)$ denotes the prescribed temperature applied at the matrix-fracture interfaces at $x = \pm L/2$. We assume that from $t = 0$ onwards, the temperature at the boundaries of the matrix-block is raised by an amount ΔT and that simultaneously an oscillating temperature is applied at the sides of the matrix-block, i.e.

$$T_s(t) = \Delta T + A_T \sin(\omega t).$$

The resulting temperature profile inside the matrix-block, $T(x,t)$, and the volume-averaged matrix-block temperature, $\overline{T}(t)$, are derived in the Appendix. The volume-averaged matrix-block temperature is given by:

$$\overline{T}(t_d) = T_i + \overline{T}_{step} + \overline{T}_{periodic,\, transient} + \overline{T}_{periodic,\, steady-state}, \qquad \text{(Eq. 2)}$$

where

$$\overline{T}_{step} = \Delta T \left[1 - 8 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \pi^2} \exp(-(2n+1)^2 \pi^2 t_d) \right]$$

$$\overline{T}_{periodic,\ transient} = 8 A_T \sum_{n=0}^{\infty} \frac{\omega_d}{\omega_d^2 + ((2n+1)^2 \pi^2)^2} \exp(-(2n+1)^2 \pi^2 t_d) \qquad \text{(Eq. 3)}$$

$$\overline{T}_{periodic,\ steady-state} = 8 A_T \sum_{n=0}^{\infty} \frac{\sin(\omega_d t_d - \varphi_n)}{\sqrt{\omega_d^2 + ((2n+1)^2 \pi^2)^2}}.$$

In the abvove expressions we expressed time in units of the characteristic response time of the matrix-block, and measured in these new units, the time-values become "dimensionless" and are denoted with a subscript d (the symbols in the above formulas are defined in the Appendix).

The first term in Eq. 2 is just the initial temperature; the second term in Eq. 2 gives the transient response due to the step-wise increase in the temperature from T_i to $T_i + \Delta T$ at $t = 0$; the third term in Eq. 2 gives the transient response due to the externally applied sinusoidal temperature; the fourth term in Eq. 2 gives the "steady-state" periodic response to the applied sinusoidal temperature. Note that in Eq. 3 ΔT and A_T just appear as simple overall multiplication factors, but that, roughly speaking, the individual terms in the periodic part of the solution are roughly attenuated with a factor $1/\omega_d = T/2\pi\tau$, and in addition, that in the steady-state part a phase-shift occurs.

It is worthwhile to compare the above solution with that following a step-change in the applied temperature boundary conditions only; in that case the volume-weighted matrix-block temperature is given by

$$\overline{T}(t_d) = T_i + \overline{T}_{step}. \qquad \text{(Eq. 4)}$$

It is intuitively clear that if the period of the applied temperature oscillation (T) is much smaller than the characteristic thermal response time of the matrix-block (τ), then the matrix-block does not have sufficient time to adapt to the applied oscillatory part of the temperature boundary conditions. In the limit where $\tau/T \to \infty$ (i.e. $\omega_d \to \infty$), the impact of the oscillatory part of the applied boundary temperature becomes negligible and in that case the average matrix-block temperature can effectively be replaced by that due to a simple step-change in temperature, Eq. 4. It is only for the cases where τ/T is of the order of 1 or smaller, that the matrix-block has sufficient time to adapt to the externally applied temperature and that the periodic temperature variation is reflected in the response of the matrix-block.

To get a feeling for the impact of daily and seasonal cycles in steam-injection rate on matrix-blocks, we apply the above theory to matrix-blocks of different sizes (notice that the size of the matrix-block is equal to the fracture spacing): 1 and 10 m, and using a typical value of the thermal diffusivity of 1e-6 m2/s. For illustrative purposes we choose $\Delta T = 200\ C$ and $A_T = 25\ C$. As shown in Figure 1, applied daily temperature fluctuations are reflected in the average temperature of blocks of 1 m, but they are virtually absent in blocks of 10 m. In fact, the ratio of the amplitude of the average temperature induced inside the matrix-block and the amplitude of the applied surface temperature: $A_T/\Delta T$, which we refer to as the amplitude attenuation factor, is a function of T/τ only, and is shown in Figure 2.

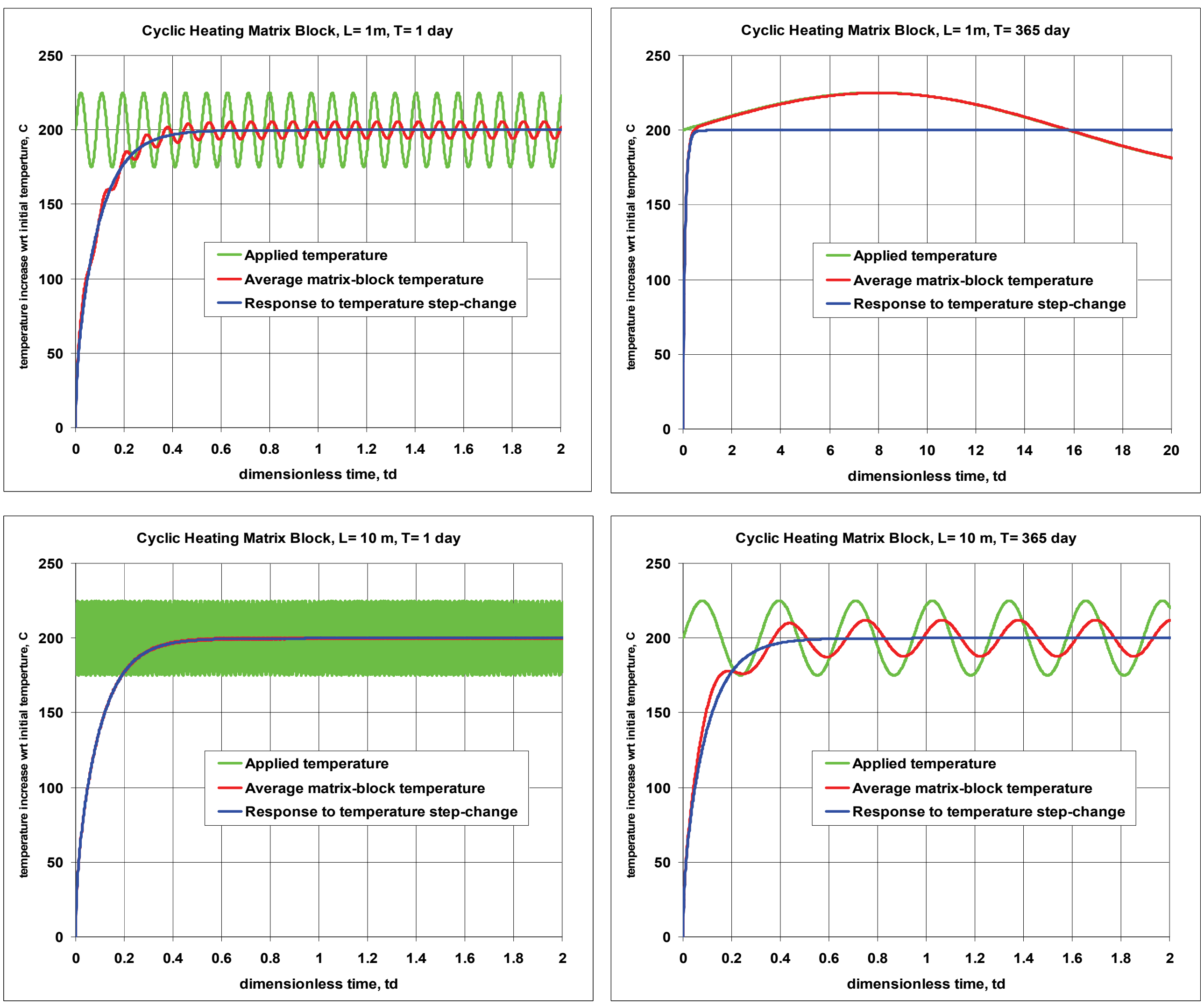


Figure 1: Applied oscillating surface temperature (green curves) and induced temperature (red).

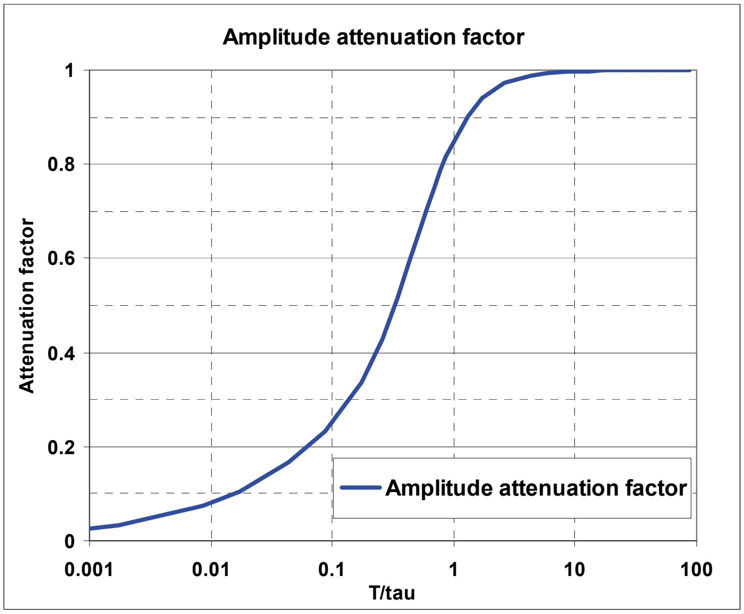


Figure 2: Temperature amplitude attenuation factor as a function of the applied dimensionless period of the temperature $T_d = T/\tau$ (applicable to all fracture spacings).

For matrix-blocks with a fracture spacing larger than 1 m, for daily fluctuations $T/\tau \leq 0.09$ (where a thermal diffusivity $\alpha = 10^{-6}$ m2/s is assumed), and from the results shown in Figure 2, we can conclude that for matrix-blocks with fracture-spacings of 1 m or larger, *daily* temperature variations are not significant as the temperature attenuation factor is smaller than 0.2. In those cases, from an engineering point of view the temperature response can be approximated by the temperature response to a simple step-change in temperature. The yearly (seasonal) variations, however, are significant for all fracture spacings considered. For a fracture spacing of 1 m or smaller, the average matrix-block temperature can follow the applied surface temperature almost perfectly (once the initial transient phenomena have subsided, i.e. after approximately $0.5t_d$, see Figure 1). We want to emphasise that the applied temperature amplitude of 25 C was chosen arbitarily and for illustrative purposes only. In fact we do not expect variations in steam injection rate to lead to a +/- 25 deg C temperature variation at the fracture-matrix interface, but rather smaller temperature fluctuations. The amplitude attenuation factor shown in Figure 2, however, is generally applicable, independent of the applied temperature amplitude and the fracture spacing under consideration. In other words, for a given fracture spacing and applied temperature fluctuation, the results shown in Figure 2 can be used to calculate the resulting average matrix-block temperature.

Simulations and results

The objective of this section is to investigate whether the daily and seasonal cycles in the solar steam-injection rate would negatively impact the oil recovery when compared to the recovery obtained under a constant steam-injection rate. We consider both a fractured and a non-fractured reservoir. To make an apple-for-apple comparison between the solar-cyclic and the constant-rate simulations, the steam-injection rate in the continuous (constant-rate) steam-injection simulation is taken equal to the average of the solar-cyclic rate. Notice that this requires the reservoir to have sufficient injectivity such that the peaks in the cyclic steam-rate can actually be injected into the reservoir. Prior to start of steam-injection, the dominant recovery mechanism in the fractured reservoir is gas-oil gravity drainage (GOGD) and following steam-injection the dominant recovery mechanisms becomes thermally-enhanced GOGD. Similarly, prior to start of steam-injection, the dominant recovery mechanism in the non-fractured reservoir is primary depletion, and following steam-injection, the dominant recovery mechanism becomes steam-drive.

The key input into the simulations is the assumed solar steam rate profile. In our simulations the seasonal variation in the solar-generated steam-rate is modelled by an oscillatory function that has an amplitude that is 25% of the yearly average steam-rate, as shown in Figure 3. For simplicity, we assume that only during 10 out of the 24 hours a day solar steam is available (at a rate that does not change during a particular day). Notice that this implies that the actual steam-rate injected during the 10 hours that steam-injection occurs, is higher by a factor 2.4 (24 h/10 h) than the seasonal steam-rate shown in Figure 3.

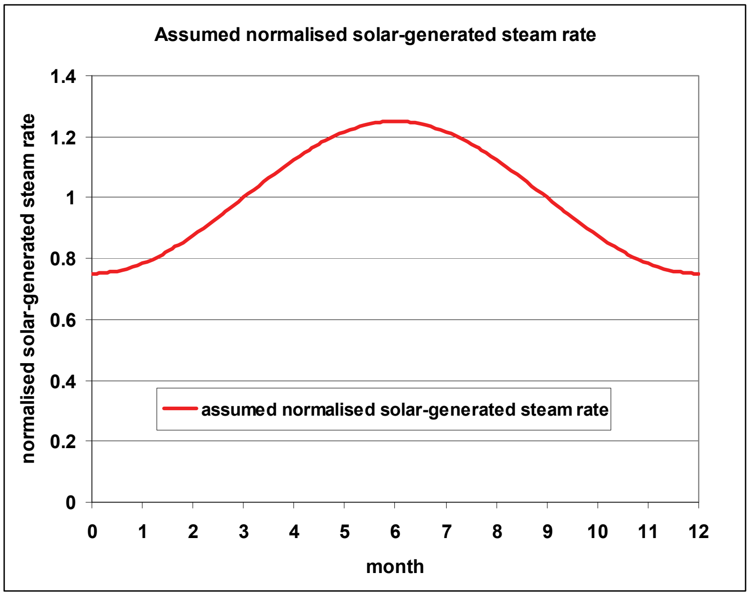


Figure 3: Assumed solar steam-rate (seasonal variation).

Impact of yearly cyles in solar-generated steam in a fractured reservoir

The characteristics of the fractured reservoir we considered are as follows: matrix permeability of the order of 10 mD, matrix porosity of the order of 30%, a fracture spacing of 1 m, and an initial oil viscosity of 220 cP. At steam-temperature the oil viscosity is assumed to be 2 cP. We assume that at the start of steam-injection the field has been in production for 21 years; during those 21 years, the field has been producing under isothermal gas-oil gravity drainage. The reservoir has a dome-like structure, with a gas-cap of about 40 m. In the simulations, steam-injection into the gas-cap starts at t=21 years. The resulting simulated oil-rate for the cyclic steam-injection and the constant steam-injection are shown in Figure 4. The simulations show that the steam-injection is very beneficial: the oil-rate increases almost by a factor of 5 within less than 5 years following start of steam-injection. The oscillation in the solar-generated steam is clearly reflected in the predicted oil-rate; the amplitude in the oil-rate is much smaller than the amplitude in the applied steam-injection. The same data are shown in Figure 5, where we have normalised the cyclic-solar oil- and steam-rates by dividing them by the corresponding rates due to the constant-rate steam-injection, i.e. $q_{steam,cyclic}(t)/q_{steam,continuous}(t)$ and $q_{oil,cyclic}(t)/q_{oil,continuous}(t)$. These results show that a $\pm 25\%$ relative change in the steam-injection rate is reflected as a $\pm 7\%$ relative change in the resulting oil-rate. Notice also that initially the peak in the oil-rate occurs about 2 months after the corresponding peak in steam-rate, and that this delay later reduces to about 1 month. The cumulative oil recovery for the solar-cyclic rate and that for the constant-rate steam-injection are compared in Figure 6. The results in Figure 6 reconfirm the observations made above: as long as the cumulative amount of steam injected under the solar-cyclic steam-injection is equal to the cumulative amount of steam injected under constant-rate steam-injection (and injected during the same time-span), then the cumulative oil recovery is essentially the same. In both cases, the cumulative amount of energy injected into the reservoir, available for heating of the rock, is the same, and gives rise to the same cumulative recovery.

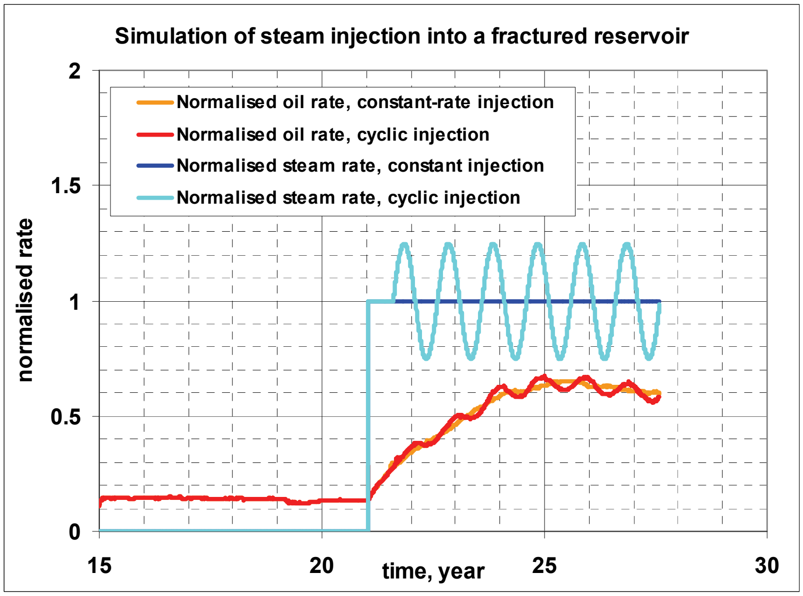


Figure 4: Comparison of the simulated oil rates for solar-cyclic and constant-rate steam-injection for the fractured reservoir considered. Shown in blue are the applied steam-injection rates.

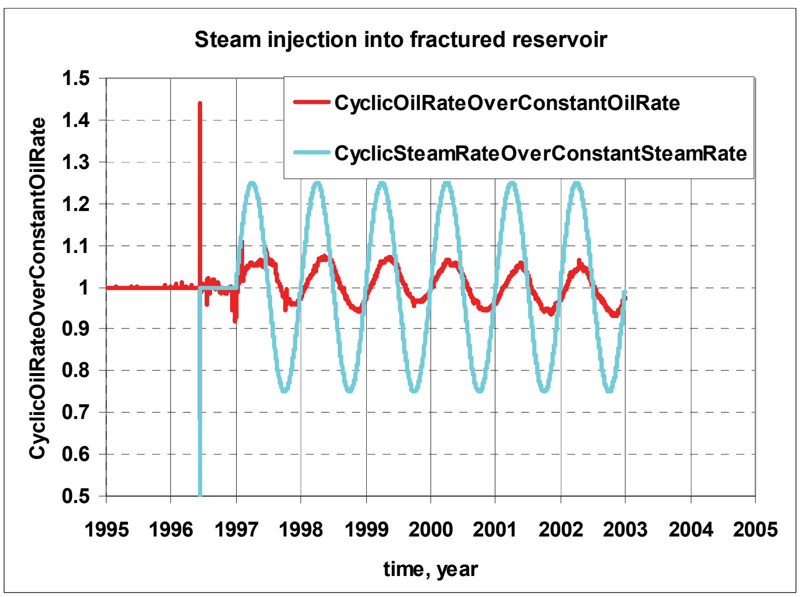


Figure 5: Normalised solar-cyclic steam-injection rates ($q_{steam,cyclic}(t)/q_{steam,continuous}$) and oil-rates ($q_{oil,cyclic}(t)/q_{oil,continuous}$) obtained by dividing the momentary solar-rates by the corresponding momentary continuous-rate results.

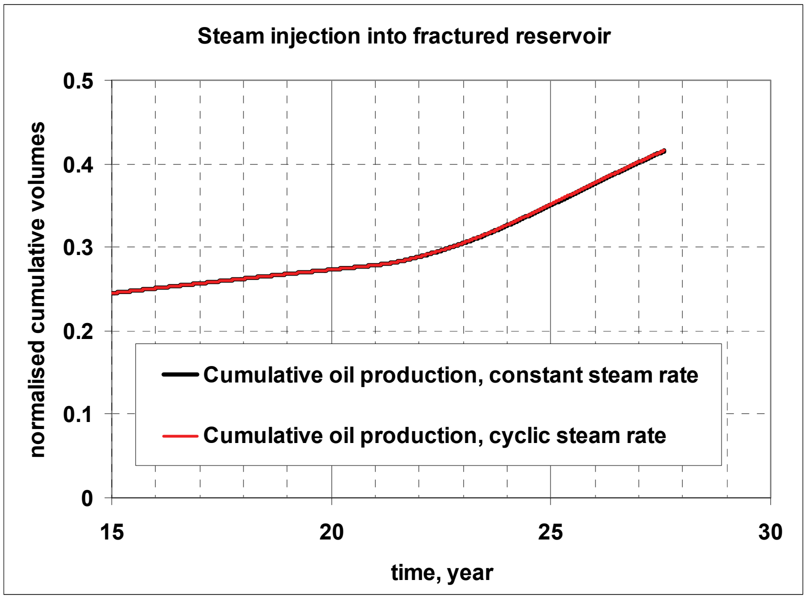


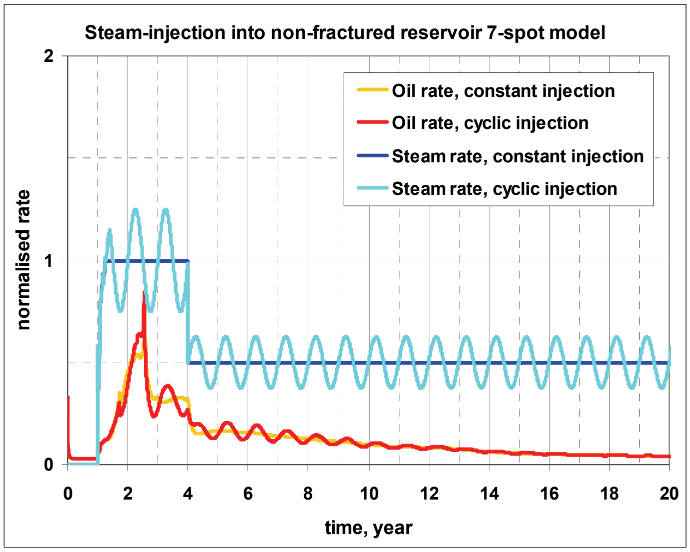
Figure 6: Comparison of the simulated cumulative oil recovery for solar-cyclic and constant-rate steam-injection for the fractured reservoir considered.

Impact of yearly cycles in solar-generated steam in a non-fractured reservoir

To investigate the impact of steam-injection into a non-fractured reservoir, we consider steam-injection into a 7-spot pattern consisting of a central steam-injector surrounded by 6 producers. In our simulations, we investigated the impact of the seasonal and daily steam-injection rates separately. We first consider the impact of the seasonal cycles (without taking into account the daily cycles). In the simulation, oil is produced during the first year under isothermal conditions. Steam-injection is started after 1 year. Subsequently steam is injected at a constant rate for three years. We would have continued this steam-injection rate in the simulations, but after 3 years steam breakthrough was observed in the producers, and to limit the steam production, the steam-injection rate was subsequently halved and maintained at that halved rate.

The simulated oil rate under the (seasonal) solar-cyclic steam-injection and the constant-rate steam-injection results are shown in Figure 7. There is an immediate response in the oil-production once steam-injection starts (following one year of normal production). The solar-cyclic oil rate oscillates around the “constant-rate” steam-injection, and the amplitude of the oscillations in the oil-rate gradually decreases.
This is also shown in a normalised way in the right side of Figure 7; this figure also shows that the delay between the peak in the oil rate and the peak in the steam-injection rate initially about 1 month, and later reduces to half a month approximately, the figure shows that the applied $\pm 25\%$ relative change in the steam-injection rate is reflected as a $\pm 25\%$ relative change in the oil rate initially, but that this reduces to $\pm 5\%$ in the oil-rate later on. This reduction in ratio of the amplitudes is interpreted to be due to the change of character of the recovery mechanism from an initial steam drive mode to subsequently a gravity drainage mode.

The cumulative steam-injection and cumulative oil-production are shown in Figure 8 and show that as long as the cumulative amount of steam injected under the solar-cyclic steam-injection is equal to the cumulative amount of steam injected under constant steam-injection (and injected during the same time-span), then the cumulative oil recovery is essentially the same.



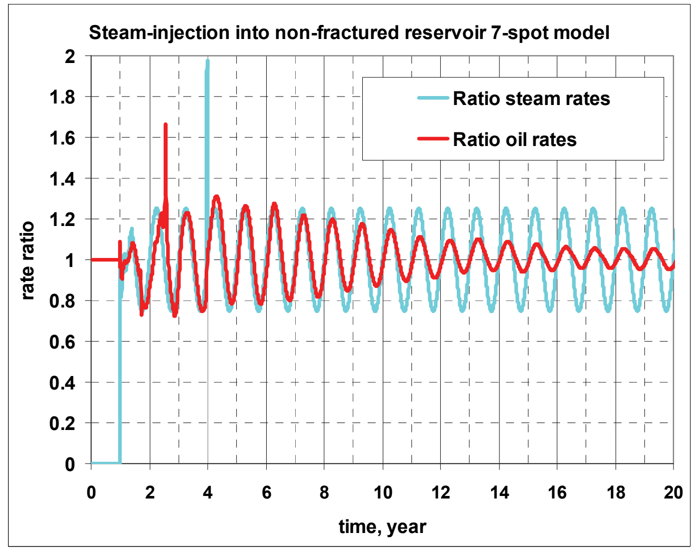


Figure 7: Simulated steam-injection and resulting oil production for solar-cyclic and constant-rate steam-injection (left). Normalised solar-cyclic oil production rate ($q_{oil,cyclic}(t)/q_{oil,continuous}$) and normalised steam rate ($q_{steam,cyclic}(t)/q_{steam,continuous}$).

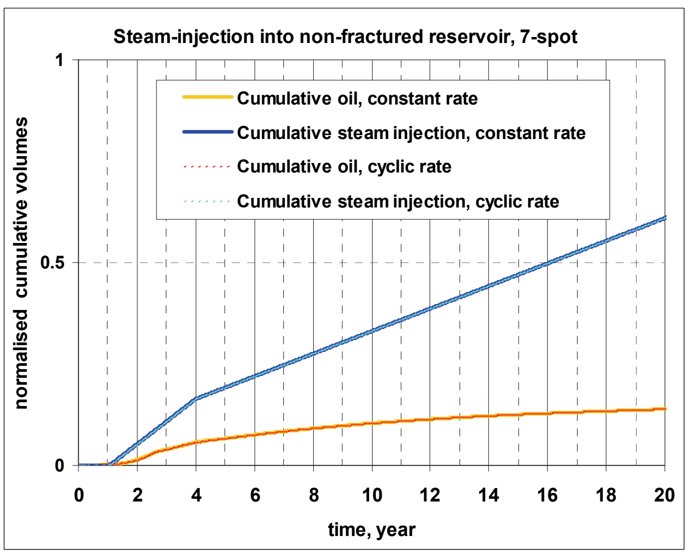


Figure 8: Comparison of simulated cumulative steam-injection and oil production for solar-cyclic and constant-rate steam injection for the non-fractured reservoir 7-spot model.

Impact of daily cycles in the steam-injection rate in a non-fractured reservoir

We assume that due to the daily cycles steam is available only during 10 hours per 24 hours. This means that in the simulations the time-step size must be much smaller than 10 hours (we used a maximum time-step size of 1 hour) and therefore we limited the simulation period to 4 years. We also want to point out that in the analysis of the impact of the daily cycles, we have not taken into account the seasonal variations in solar-cyclic steam. The steam-injection rate and the oil-rate are shown in Figure 9 (and correspond to the production/injection into the $1/12^{th}$ symmetry element). We want to point out that the solar-cyclic steam injection rate is plotted by showing symbols only and that, following a start-up period, the steam-injection rate is either zero or at it maximal value (the solar-cyclic steam-injection rate is not plotted as a dotted line because on the scale displayed in Figure 9, the whole area below the maximal rate and to the right of 1 year would be coloured blue if we would plot lines).

The oil rates for the continuous-rate and solar-cyclic steam injection are quite similar, see Figure 9 (showing simulation output for a 4 year period) and Figure 10 (which shows the solar-cyclic and continuous steam injection rates and oil rates during a period of 9 days). Note that initially both for the constant-steam injection and the solar-cyclic injection, following start of steam-injection, due to limited (initial) well injection capacity , the steam-injection needs some time, approximately 0.2 year, to build up before the maximum rates can be injected. This highlights the significance of the assumption that was already mentioned above: that for the comparison between constant-rate and solar-cyclic steam-injection, we assume that the reservoir has sufficient injectivity to inject also the peaks in the solar-cyclic steam, such that the same cumulative amount of steam can be injected into the reservoir during the same time-span.

The cumulative steam-injection and oil- and water production for the continuous and daily-cyclic steam-injection cases are shown in Figure 11. The simulation results indicate that during the simulated period of 4 years, there is little difference in oil production between daily-cyclic steam-injection and continuous steam-injection. In other words, the daily cycles in solar-generated steam do not have a negative impact on the oil recovery (compared to constant-rate steam injection) as long as the cumulative amounts of steam injected into the reservoir (during the same time-span) are the same.

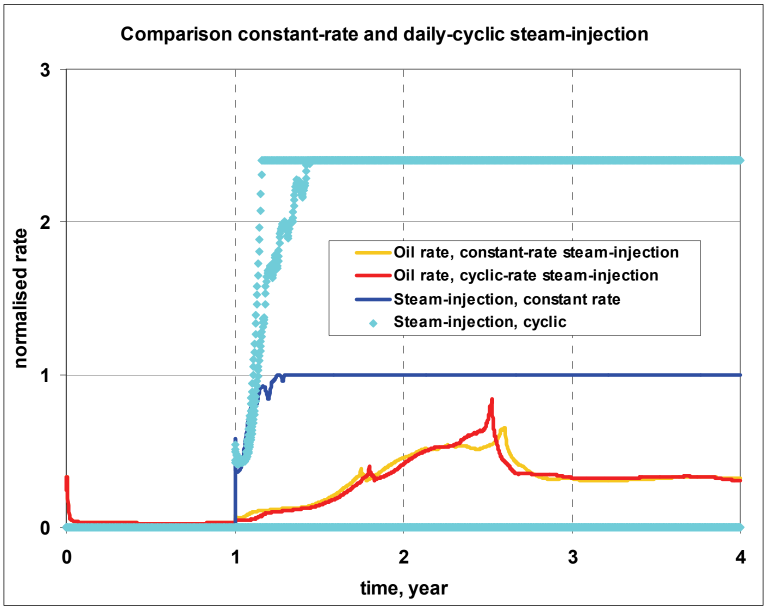


Figure 9: Simulation results showing the comparison between constant-rate steam-injection, and daily cyclic steam-injection for the non-fractured reservoir.

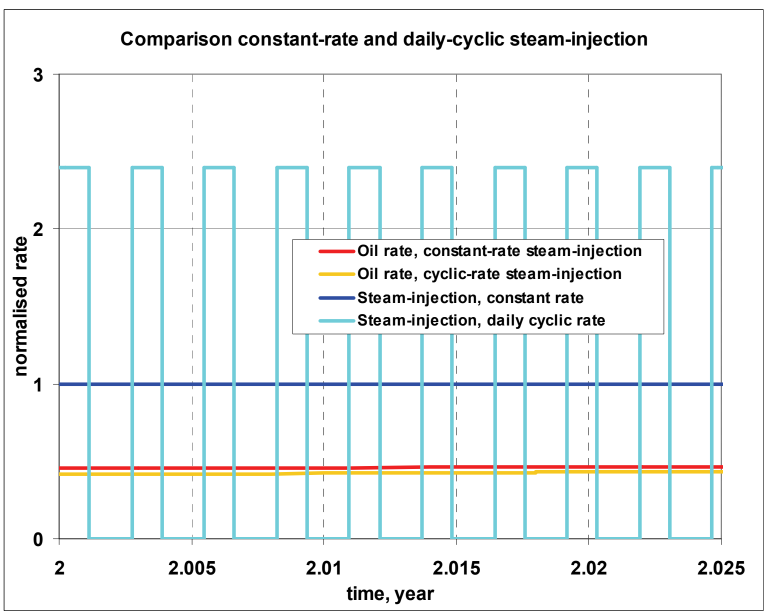


Figure 10: Simulation results comparing constant-rate steam-injection and daily cyclic steam-injection for the non-fractured reservoir (showing a period of nine days just after year 2).

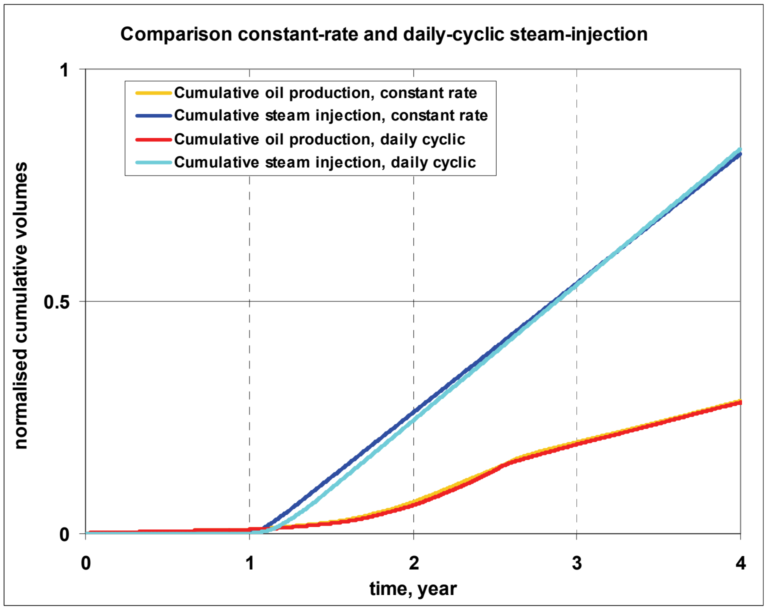


Figure 11: Simulation results showing the comparison of the cumulative oil recovery, water production and water-injection between constant-rate steam-injection, and daily cyclic steam-injection for the non-fractured reservoir.

Conclusions

1. Our analytical model of a fractured reservoir that is subjected to periodic temperature oscillations at its surface shows that if the dimensionless period of the oscillation $T_d = T/\tau \leq 0.08$, then the ratio in amplitude of the volume-averaged matrix-block temperature and the externally applied surface temperature (i.e. the temperature attenuation factor) becomes smaller than 0.2 and can for practical purposes be ignored. This means that for fractured reservoirs with a fracture-spacing of 1m or larger, applied daily oscillations in the surface temperature can be ignored.

2. Our thermal reservoir simulations for both a fractured reservoir and a non-fractured reservoir show that although an oscillation in the steam-injection rate is reflected as an oscillation in the oil-rate, solar-cyclic steam injection and constant-rate steam-injection give the same cumulative oil recovery if the same cumulative amount of steam is injected into the reservoir in the same time-span.

3. From a sub-surface recovery point of view, provided that the reservoir has sufficient injectivity and the peaks in solar-generated steam rate can be injected into the reservoir, solar-generated cyclic steam-injection is an alternative to constant-rate steam injection.

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Nomenclature

Symbol	Meaning	Units
T	Temperature; period of applied temperature oscillation	K; s
$\overline{T}$	Volume-averaged temperature	K
L	Fracture spacing	m
t	Time	s
x	Position	m
α	Rock thermal diffusivity	m^2/s
ΔT	Step in temperature	K
A_T	Amplitude of applied temperature oscillation	K
$\tau = L^2 / \alpha$	Characteristic time of thermal diffusion of matrix block of size L and thermal diffusivity α	s
t_d	Time measured in units of τ ("dimensionless time")	-
ω	Angular frequency	rad/s
ω_d	Angular frequency measured in units of τ ("dimensionless angular frequency")	rad
φ	Angle	rad

References

1. Zarza, E., Valenzuela, L., Leon, J., Hennecke, K., Eck, M., Weyers, H.-D., Eickhoff, M. : "Direct steam generation in parabolic throughs: Final results and conclusions of the DISS project", Energy, vol. 29, pp. 635-644, 2004.

Appendix: Temperature inside a matrix-block subjected to a periodically changing surface temperature

As a model for a matrix-block in a fractured reservoir that is subjected to periodic steam-injection, we consider a one-dimensional matrix-block that is subjected to a periodically changing temperature at its surface (matrix-fracture boundaries). We calculate the evolution of the temperature inside such a one-dimensional matrix block and will use the analytical result to analyse whether applied daily or seasonal temperature variations are important.

Consider a homogeneous one-dimensional matrix-block of size L (bounded by two fractures; the matrix-block size is equal to the fracture-spacing). We assume that initially the matrix-block has a uniform temperature T_i. From $t = 0$ onwards, the temperature at the matrix-fracture interfaces at $x = -L/2$ and $x = L/2$ is changed. Using the convolution theorem, the temperature inside the matrix is given by

$$T(x,t) - T_i = \int_0^t \frac{dT_s(t')}{dt'} u(x,t-t')dt' = \int_0^t T_s(t') \frac{\partial}{\partial t} u(x,t-t')dt', \qquad \text{(Eq. A1)}$$

where $u(x,t-t')$ denotes the temperature change at position x and at time t caused by a unit temperature increase that occurred at both of the boundaries at $x = -L/2$ and $x = L/2$ at time t' ($t \ge t'$). The temperature response inside a one-dimensional matrix-block due to a unit-step temperature change at the boundaries at $x = -L/2$ and $x = L/2$ that occurred at time t' is a well-known result and is given by

$$u(x,t-t') = 1 - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\exp\left(\frac{-(2n+1)^2\pi^2\alpha(t-t')}{L^2}\right)\cos\left((2n+1)\pi\frac{x}{L}\right). \qquad \text{(Eq. A2)}$$

Calculating $\partial_t u(x,t-t')$ from this expression and substituting the resulting expression in Eq. (xx), we obtain

$$T(x,t) - T_i = \int_0^t T_s(t') \times \left[\frac{4\alpha}{L^2}\sum_{n=0}^{\infty}(-1)^n(2n+1)\pi\exp\left(\frac{-(2n+1)^2\pi^2\alpha(t-t')}{L^2}\right)\cos\left((2n+1)\pi\frac{x}{L}\right)\right]dt'. \qquad \text{(Eq. A3)}$$

In the following, we assume that from $t = 0$ onwards, the temperature at the boundaries of the matrix-block is raised by an amount ΔT and that simultaneously, from $t = 0$ onwards an oscillating temperature is applied at the sides of the matrix-block, i.e. $T_s(t) = \Delta T + A_T \sin(\omega t)$.

It is convenient to introduce a new unit of time, the characteristic time of thermal diffusion of the block, $\tau = L^2/\alpha$, and to define a "dimensionless" time $t_d = t/\tau$ and a "dimensionless" length $x_d = x/L$. We also define a "dimensionless" angular velocity

$$\omega_d = 2\pi/(T/\tau) = 2\pi\tau/T \qquad \text{(Eq. A4)}$$

When expressed in these new units, the above integral becomes

$$T(x_d,t_d) - T_i = \int_0^{t_d}\left[\Delta T + A_T\sin(\omega_d t'_d)\right]\times\left[4\sum_{n=0}^{\infty}(-1)^n(2n+1)\pi\exp\left(-(2n+1)^2\pi^2(t_d - t'_d)\right)\cos\left((2n+1)\pi x_d\right)\right]dt'_d \qquad \text{(Eq. A5)}$$

The average temperature of the matrix-block is obtained by integration of this expression: using $\bar{f} = \frac{2}{L}\int_0^{\frac{L}{2}} f\,dx$, we find:

$$\overline{T}(t_d) - T_i = 8\int_0^{t_d} [\Delta T + A_T \sin(\omega_d\, s_d)] \times \left[\sum_{n=0}^{\infty} \exp\left(-(2n+1)^2 \pi^2 (t_d - s_d)\right)\right] ds_d \qquad \text{(Eq. A6)}$$

Upon integration we obtain:

$$\overline{T}(t_d) = \overline{T}_i + \overline{T}_{step} + \overline{T}_{periodic,\,transient} + \overline{T}_{periodic,\,steady-state},$$

where

$$\overline{T}_{step} = \Delta T\left[1 - 8\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2\pi^2} \exp(-(2n+1)^2\pi^2 t_d)\right]$$

$$\overline{T}_{periodic,\;transient} = 8A_T \sum_{n=0}^{\infty} \frac{\omega_d}{\omega_d^2 + ((2n+1)^2\pi^2)^2} \exp(-(2n+1)^2\pi^2 t_d) \qquad \text{(Eq. A7)}$$

$$\overline{T}_{periodic,\,steady-state} = 8A_T \sum_{n=0}^{\infty} \frac{\sin(\omega_d t_d - \varphi_n)}{\sqrt{\omega_d^2 + ((2n+1)^2\pi^2)^2}},$$

with

$$t_d = t/\tau = \alpha t/L^2,$$
$$\omega = 2\pi/T$$
$$\omega_d = 2\pi\tau/T \qquad \text{(Eq. A8)}$$
$$\varphi_n = \arctan(\omega_d/(2n+1)^2\pi^2)$$

The first term of this expression is just the initial temperature, the second line gives the transient response due to the step-wise increase in the temperature from T_i to $T_i + \Delta T$, the third line gives the transient response due to the externally applied sinusoidal temperature, and the fourth line gives the "steady-state" periodic response to the applied sinusoidal temperature.

The above solution should be compared with the solution when the temperature at the boundaries of the matrix-block ($-1/2 \le x_d \le 1/2$) is raised to, and maintained at, temperature $T_i + \Delta T$:

$$\overline{T}(t_d) = T_i + \overline{T}_{step} \qquad \text{(Eq. A9)}$$