# Partial Wave Analysis for Dummies

**Jim Napolitano**

***RPI and Cornell***

## Outline

- **The Basic Quantum Mechanics**
- **Partial Wave Analysis in Textbooks**
- **An Example for CLEO-c: $J/\psi \to \gamma\pi\pi, \gamma KK$**
- **Extensions, Complications, Technical Details**
- **Advanced Example: $b_1(1235) \to \omega\pi$**
- **Advanced Example: $\gamma p \to p\pi^+\pi^-$**

**CLEO Lunch Seminar**
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# Disclaimers

**Jim Napolitano is one of the "dummies"!**

**The term "Partial Wave Analysis" is poorly defined and over used. Always keep your own physics in mind, and use what is relevant for your problem.**

**Other terms used include "Amplitude Analysis" and "Partial Wave Decomposition". These terms have different meanings for different people!**

**Important distinction for many analyses:**
**Energy-Independent versus Energy-Dependent.**

**You are getting my own slant on all of this!**

# The Basic Quantum Mechanics

For a complete set of states $|\ell\rangle$, write an arbitrary state

$$|\alpha\rangle = \sum_\ell c_\ell |\ell\rangle$$

For orthonormal $|\ell\rangle$, $\langle \ell' | \ell \rangle = \delta_{\ell\ell'}$, so

$$\langle \ell | \alpha \rangle = c_\ell$$

Therefore, we can write

$$|\alpha\rangle = \mathcal{I} |\alpha\rangle$$

where

$$\mathcal{I} = \sum_\ell |\ell\rangle\langle \ell|$$

This is called "inserting a complete set of states" and is the essence of "Partial Wave Analysis"

# Partial Wave Analysis in Textbooks

**Perhaps you learned that Partial Wave Analysis meant**

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

**where**

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)e^{i\delta_\ell}\sin\delta_\ell$$

**This is a special case, namely elastic scattering of a nonrelativistic spinless particle from a static, central potential.**

**Nevertheless, it is obtained by inserting a complete set of states in the scattering matrix.**

## The Details: Sakurai "Modern QM" §7.6

$$
\begin{aligned}
f(\vec{k}', \vec{k}) &\propto \langle \vec{k}' | T | \vec{k} \rangle \\
&\propto \sum_{\ell' m'} \sum_{\ell m} \langle \vec{k}' | \ell' m' \rangle \langle \ell' m' T | \ell m \rangle \langle \ell m | \vec{k} \rangle \\
&\propto \sum_{\ell m} T_\ell(E) Y_\ell^m(\hat{k}') Y_\ell^{m\star}(\hat{k}) \\
&\propto \sum_{\ell m} T_\ell(E)(2\ell + 1) P_\ell(\cos\theta)\delta_{m0} \\
&\propto \sum_{\ell} T_\ell(E)(2\ell + 1) P_\ell(\cos\theta)
\end{aligned}
$$

**The rest is a parameterization of $T_\ell(E)$ in terms of "phase shifts" $\delta_\ell$ that makes use of the behavior of the scattered wave at distances far from the potential.**

**This is mostly irrelevant for today's discussion!**

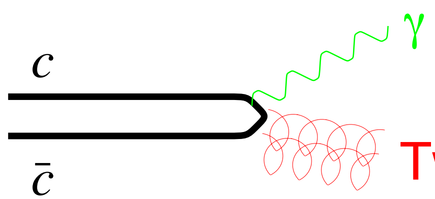
# An Example for CLEO-c: $J/\psi \to \gamma\pi\pi, \gamma KK$

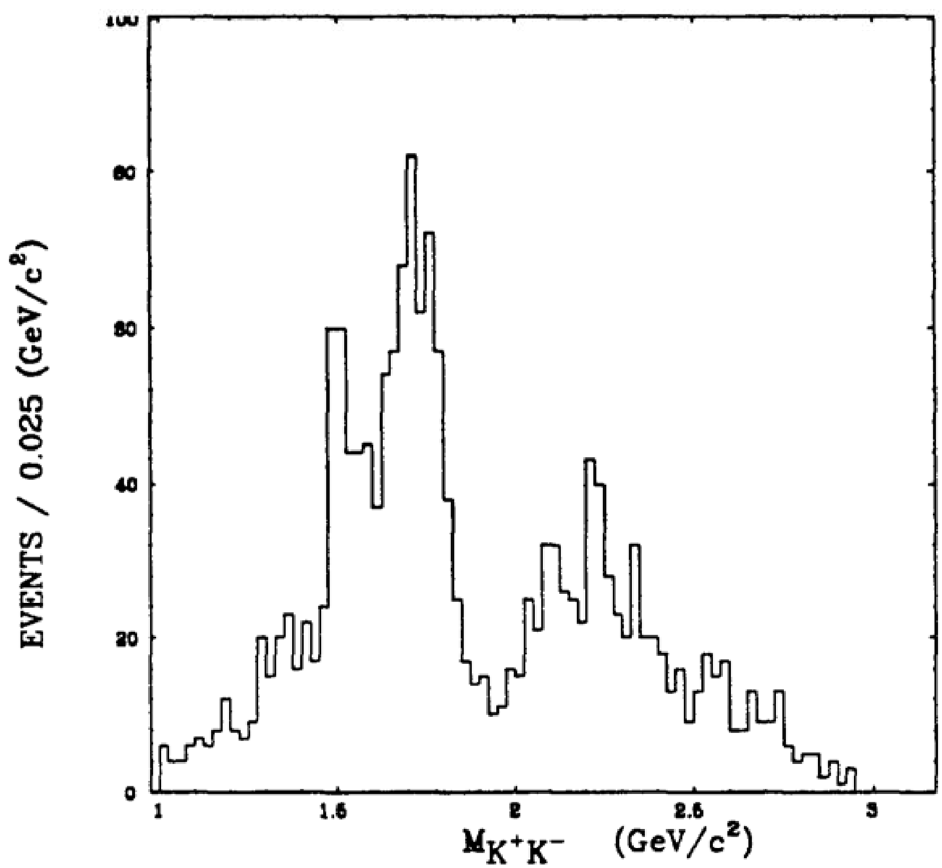
Data: Mark III at SPEAR

Recall:
The search for glueballs

$$J/\psi \to \gamma\ X$$
$$X \to K^+K^-$$

**$J/\psi$ Radiative Decay:** $J/\psi \to \gamma G$




Need to know the quantum numbers of the glueball candidate $G$

**Illustrates most crucial concepts of Partial Wave Analysis.**

## Step One: Some References

**To really learn this, you need to review basic formalisms of quantum mechanics, especially the theory of angular momentum and rotational symmetry. In addition to a good quantum mechanics textbook I recommend the following, all available electronically from** SPIRES**:**

- **L. P. Chen, "An amplitude analysis of the $K\bar{K}$ system ($M \leq 2$ GeV/$c^2$) produced in $J/\psi$ radiative decay," SLAC-0386**
- **J. D. Richman, "An Experimenter's Guide To The Helicity Formalism," CALT-68-1148**
- **B. S. Zou and D. V. Bugg, "Covariant tensor formalism for partial wave analyses of $\psi$ decay to mesons," Eur. Phys. J. A 16, 537 (2003)**

# Step Two: Formal Beginning

For the reaction $e^+e^- \to J/\psi \to \gamma X \to \gamma m\bar{m}$, write

$$dN = |\mathcal{M}(e^+e^- \to \gamma m\bar{m})|^2 dM_X d\Omega_X d\Omega_m^\star$$

where $M_X$, $\Omega_X$, and $\Omega_m^\star$ specify the kinematics.

$\mathcal{M}$ is a matrix element of some "decay operator" $U$:

$$\mathcal{M} = \langle f|U|i\rangle = \langle f|U_A U_B \cdots |i\rangle$$

We make the assumption that

$$\mathcal{M}(e^+e^- \to \gamma m\bar{m}) = \sum_X \mathcal{M}(J/\psi \to \gamma X) \times \mathcal{M}(X \to m\bar{m})$$

*(??Are we inserting a complete set* $|X\rangle\langle X|$*??)*

Our goal is to learn about $U$. We do this by parameterizing $\mathcal{M}$, and fitting the parameters to our data. These parameters are coefficients of "partial waves".

# Step Three: The Helicity Basis

**Consider the two-body decay $\alpha \rightarrow 1 + 2$ where particle $\alpha$ has angular momentum quantum numbers $J$ and $M$. (Of course, we say $\alpha$ has "spin $J$".)**

**The helicity operator $h \equiv \vec{S} \cdot \vec{p}/|\vec{p}|$ (eigenvalues $\lambda$) is invariant under rotations, so it commutes with the total angular momentum operator $\vec{J}$.**

**Therefore, we can form basis states $|jm\lambda_1\lambda_2\rangle$ that are simultaneous eigenstates of operators $\vec{J}^2$, $J_z$, $h_1$, and $h_2$.**

**We will be inserting this complete set of states.**

**The tricky part is to properly rotate the "axis of quantization". For this, we use Wigner $\mathcal{D}$-functions.**

$$\mathcal{M} = \langle f|U|i\rangle = \langle 1+2|U|\alpha\rangle = \langle \theta, \phi, \lambda_1, \lambda_2|U|JM\rangle$$

**Now insert $|jm\lambda_1'\lambda_2'\rangle$ where the $|jm\rangle$ quantization axis is the same as for $|JM\rangle$. Use the rotational symmetry of $U$.**

$$\begin{aligned}
\mathcal{M} &= \sum_{jm\lambda_1'\lambda_2'} \langle \theta, \phi, \lambda_1, \lambda_2|jm\lambda_1'\lambda_2'\rangle\langle jm\lambda_1'\lambda_2'|U|JM\rangle \\
&= \sum_{\lambda_1'\lambda_2'} \langle \theta, \phi, \lambda_1, \lambda_2|JM\lambda_1'\lambda_2'\rangle\langle JM\lambda_1'\lambda_2'|U|JM\rangle \\
&= \sum_{\lambda_1'\lambda_2'} \langle \theta, \phi, \lambda_1, \lambda_2|JM\lambda_1'\lambda_2'\rangle A^{(J)}_{\lambda_1'\lambda_2'}
\end{aligned}$$

**We have now parameterized the interaction $U$ by $A^{(J)}_{\lambda_1'\lambda_2'}$ for the decay of a particle of spin $J$ into daughter particles with helicities $\lambda_1'$ and $\lambda_2'$. The angular distribution of this decay is governed by the factor(s) $\langle \theta, \phi, \lambda_1, \lambda_2|JM\lambda_1'\lambda_2'\rangle$ which are independent of the dynamics. However,...**

$$\langle \theta, \phi, \lambda_1, \lambda_2 | J M \lambda_1' \lambda_2' \rangle = \delta_{\lambda_1, \lambda_1'} \delta_{\lambda_2, \lambda_2'} \sqrt{\frac{2J+1}{4\pi}} \mathcal{D}^{J\star}_{M, \lambda_1 - \lambda_2}(\phi, \theta, 0)$$

See Richman for details. Recall that $\mathcal{D}$ functions are matrix elements of the rotation operator $\mathcal{R}(\alpha, \beta, \gamma)$ for Euler angles $\alpha, \beta, \gamma$, i.e.

$$\mathcal{D}^{j}_{m'm}(\alpha, \beta, \gamma) = \langle j m' | \mathcal{R}(\alpha, \beta, \gamma) | j m \rangle$$

There is an arbitrary choice of phase because $\phi$ is undefined at $\theta = 0$. Our choice of $\gamma = 0$ defines the phase convention. In any case, this leaves us with. . .

$$\begin{aligned} \mathcal{M} &= \sum_{\lambda_1' \lambda_2'} \langle \theta, \phi, \lambda_1, \lambda_2 | J M \lambda_1' \lambda_2' \rangle A^{(J)}_{\lambda_1' \lambda_2'} \\ &= A^{(J)}_{\lambda_1 \lambda_2} \sqrt{\frac{2J+1}{4\pi}} \mathcal{D}^{J\star}_{M, \lambda_1 - \lambda_2}(\phi, \theta, 0) \end{aligned}$$

# Step Four: Apply Formalism to our Sequential Decay

$\mathcal{M}(J/\psi \to \gamma X)$

For $e^+e^- \to J/\psi$, have $J = 1$ and $M = \pm 1$
Since photons are massless, $\lambda_\gamma = \pm 1$
For spin $J_X$, $\lambda_X = 0, \pm 1, \ldots, \pm J_X$
Need $C_{J/\psi} = C_\gamma C_X$ so must have $C_X = +1$

$\mathcal{M}(X \to m\bar{m})$

For pseudoscalar $m, \bar{m}$, need $\lambda_m = 0$
Have $J(m\bar{m}) = \ell$ and $P = C = (-1)^\ell$
$\Rightarrow$ Only $J_X = 0, 2, \ldots$ allowed

**Putting it together:**

$$\mathcal{M}^{\lambda_\gamma}_{J_X,\lambda_X} = a^{\lambda_\gamma}_{J_X,\lambda_X}\sqrt{\frac{2J_X+1}{4\pi}}\mathcal{D}^{1^\star}_{\lambda_\psi,\lambda_X-\lambda_\gamma}(\Omega_X)\mathcal{D}^{J_X^\star}_{\lambda_X,0}(\Omega^\star_m)$$

$$dN = \frac{3}{16\pi}\sum_{\lambda_\psi,\lambda_\gamma}\left|\sum_{J_X,\lambda_X}\mathcal{M}^{\lambda_\gamma}_{J_X,\lambda_X}\right|^2 dM_X d\Omega_X d\Omega^\star_m$$

**where** $a^{\lambda_\gamma}_{J_X,\lambda_X} \propto A^{(1)}_{\lambda_X,\lambda_\gamma}A^{(J_X)}_{0,0}$

**Parity conservation:** $a^{-\lambda_\gamma}_{J_X,-\lambda_X} = a^{\lambda_\gamma}_{J_X,\lambda_X}$

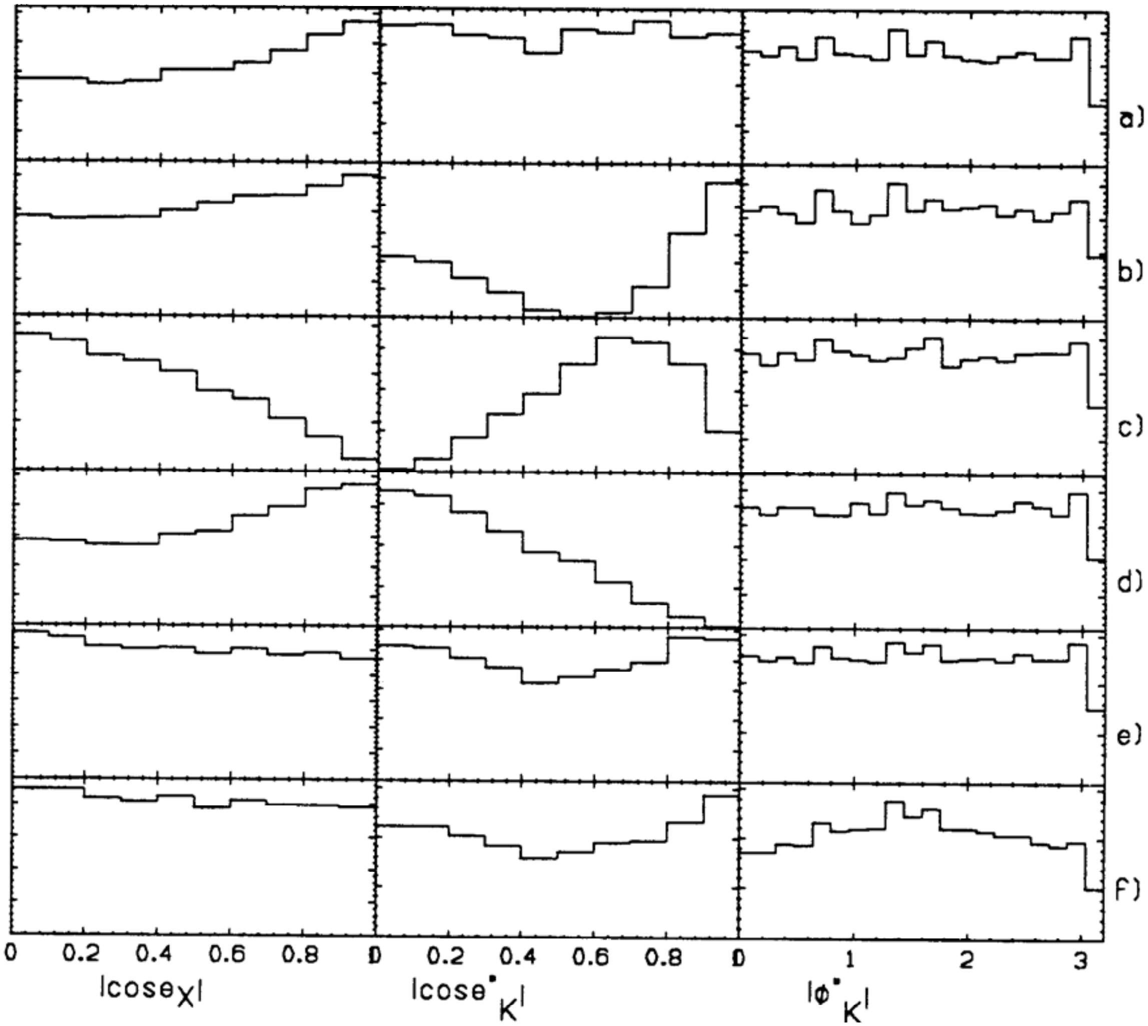
**So, the "joint" angular distribution is described by the *complex* parameters**

$$\begin{array}{lll} a_{0,0} & & \\ a_{2,0} & a_{2,1} & a_{2,2} \\ \ldots & & \end{array}$$

**Our task is now to fit these parameters to the data.**

# Predicted angular distributions

## The results are far from obvious!




$a_{0,0} \neq 0$

$a_{2,0} \neq 0$

$a_{2,1} \neq 0$

$a_{2,2} \neq 0$

b+c+d

$a_{2,0} = a_{2,1} = a_{2,2}$

## Interferences are important!

## Step Five: Fit to the Data Independently in Bins of $\Delta M_X$

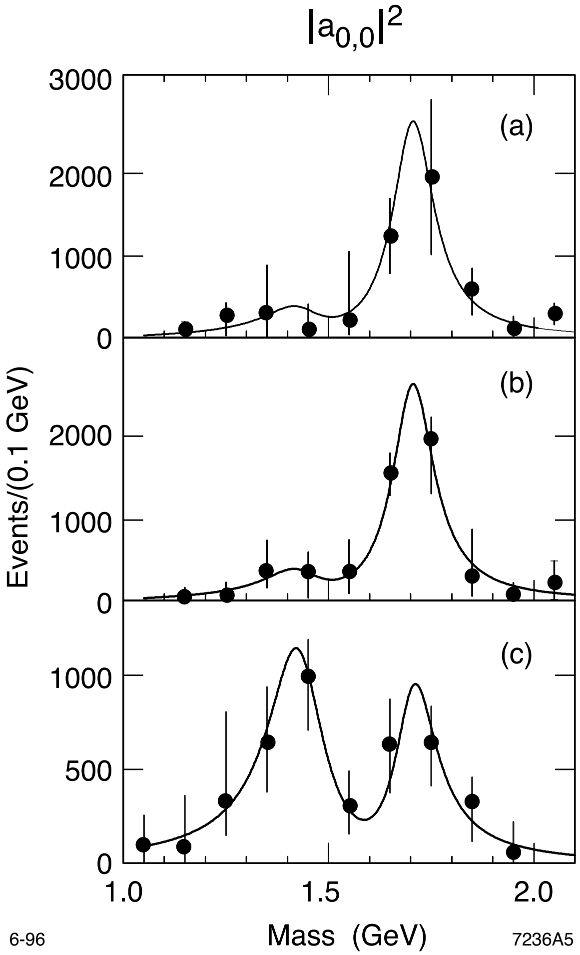
**Mark III Technique:** Formulate the "accepted moments" and fit these to the data by minimizing $\chi^2$. Then, extract amplitudes $a_{J,\lambda}$ by solving nonlinear algebraic equations.

**Unbinned Extended Maximum Likelihood:** Fit for $a_{J,\lambda}$ directly, including the Poisson fluctation probability in the number of events. Includes the acceptance specifically for each term in the expansion.

Important: The "acceptance" (geometric, plus analysis cuts) is determined with a "phase space" Monte Carlo simulation. This simulation contains no "physics", but populates each element of $\Delta\Omega_X, \Delta\Omega_m^\star$ evenly.

# The Result: Evidence for $f_0(1380)$, $f_0(1710)$, $f_2(1270)$, $f_2(1525)$

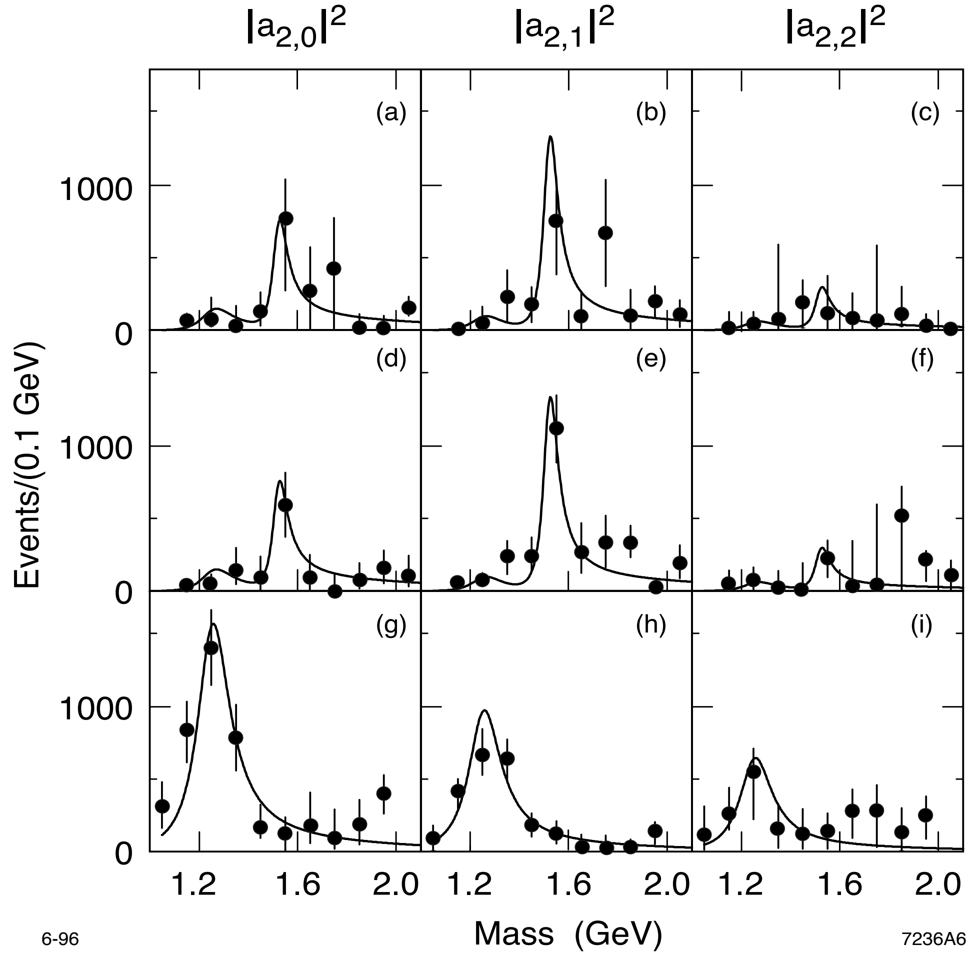
## Recall $J/\psi \rightarrow \gamma X$, where $X \rightarrow \ldots$




$K_S K_S$

$K^+ K^-$

$\pi^+ \pi^-$




**Solid lines: An Energy (i.e Mass) Dependent Fit**

# Extensions, Complications, Technical Details

**Lots of data and many amplitudes to fit, means you need lots of disk space and lots of CPU time.**

**Mathematically ambiguous solutions. If not, maybe statistically ambiguous solutions.**

**The sum over intermediate states $X$ is really infinite. Did you truncate the the set of basis states too early?**

**What's the right way to handle multi-hadron final states?**

**Are you summing over non-orthogonal states? Is the "double counting" a big problem for your physics goals?**

# A General Use Program: PWA2000

**Authors:**
**John Cummings (RPI) and Dennis Weygand (JLab)**

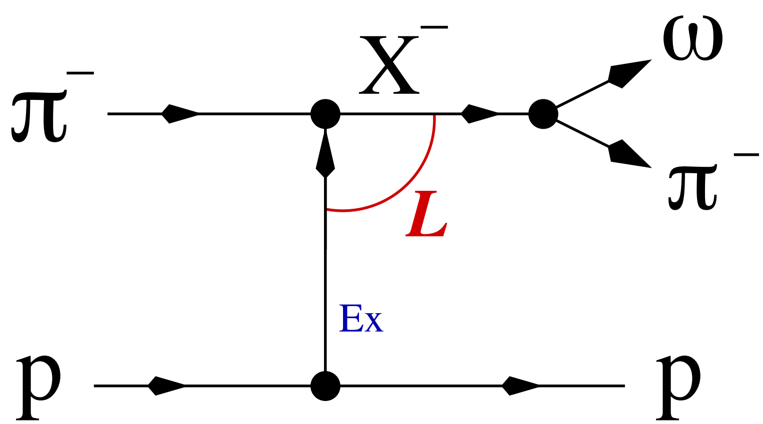
**Various features, including**

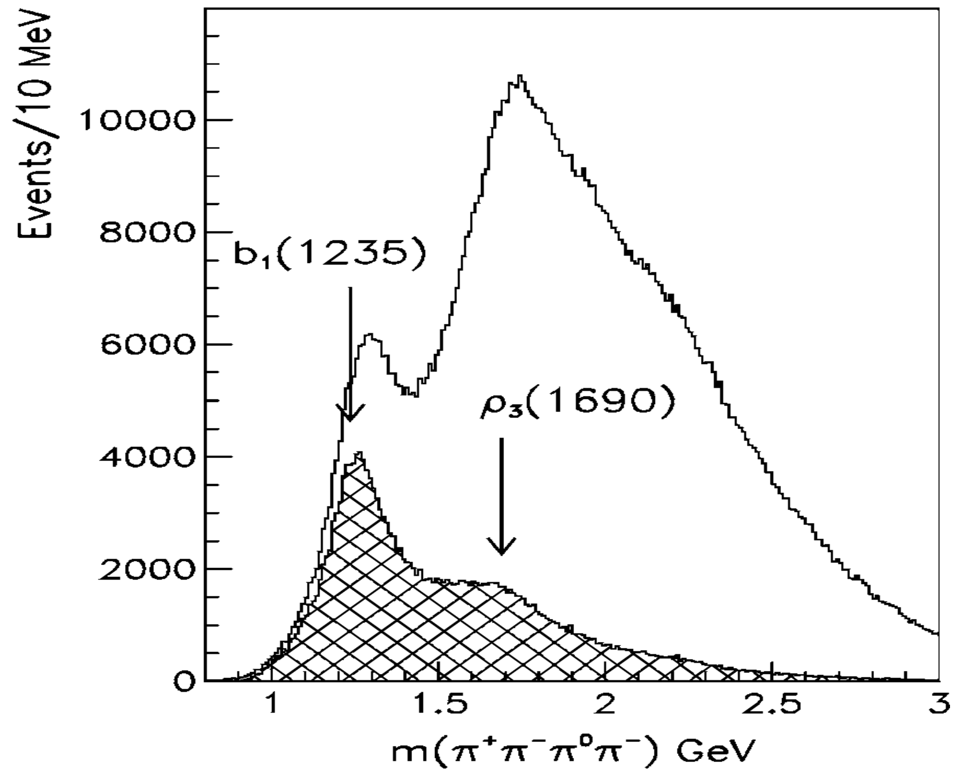
- **Extended maximum likelihood fitting**
- **Built-in options for quasi-two-body decays**
- **Acceptance incorporated into fit function**
- **Tools for display, comparison, combining amplitudes**
- **Object-oriented approach**

**Program available now. Paper in preparation.**

# Advanced Example: $b_1(1235) \to \omega\pi$

**Two principle decay modes: $b_1 \to (\omega\pi)_S$ and $b_1 \to (\omega\pi)_D$**



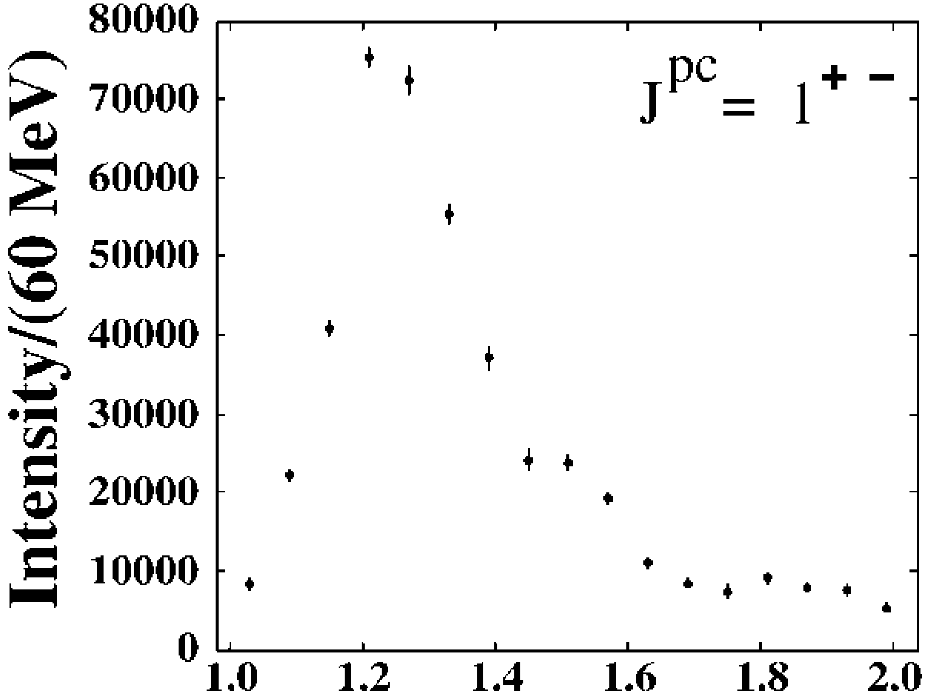



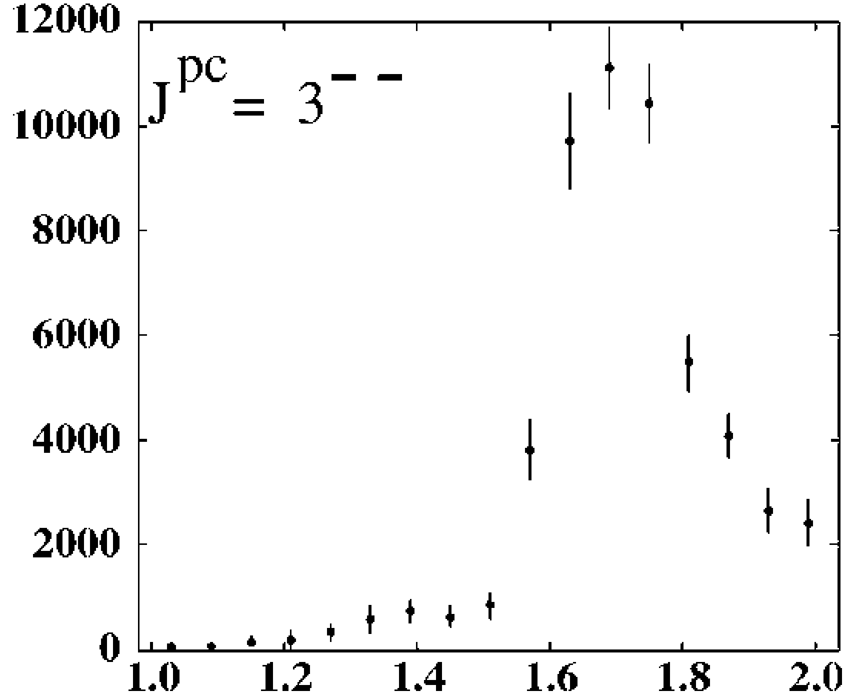

M. Nozar *et al.*, Phys. Lett. B 541, 35 (2002)

**Result of partial wave analysis:**
$|D/S| = 0.269 \pm 0.020$ and $\phi(D-S) = 10.54^\circ \pm 2.4^\circ \pm 3.9^\circ$

# $b_1(1235) \to \omega\pi$: Separate quantum numbers

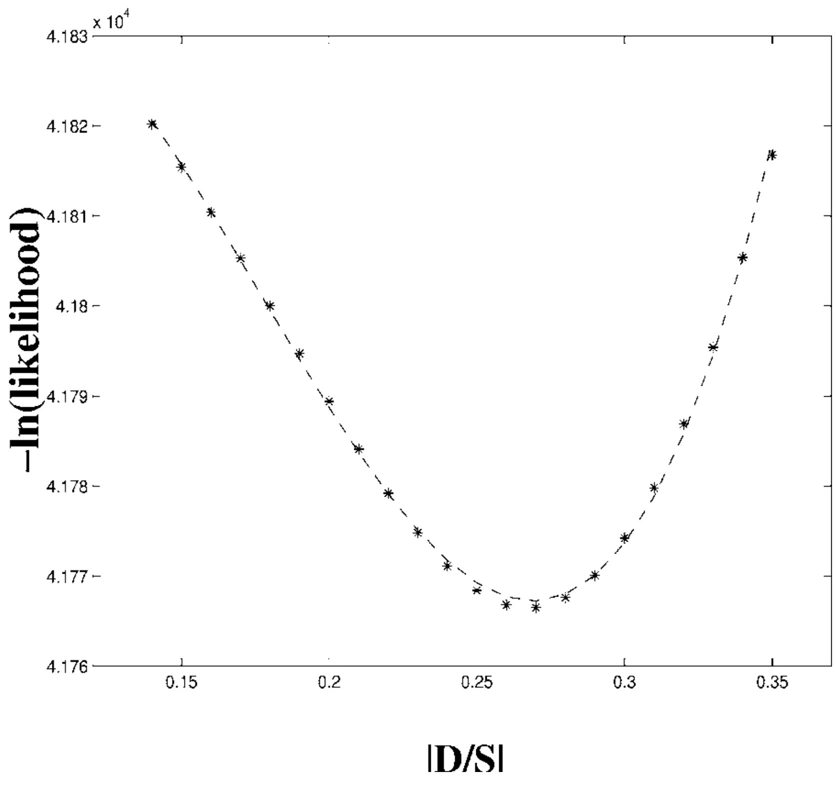
## Dominated by $b_1(1235)$ and $\rho_3(1670)$

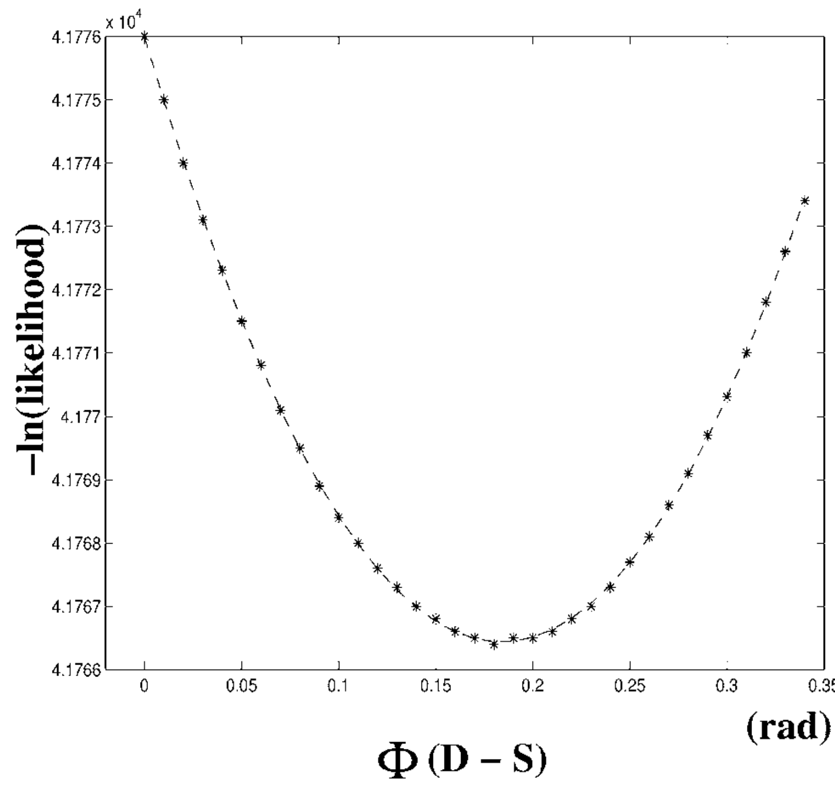


# $b_1(1235) \to \omega\pi$: Adjusting $D/S$

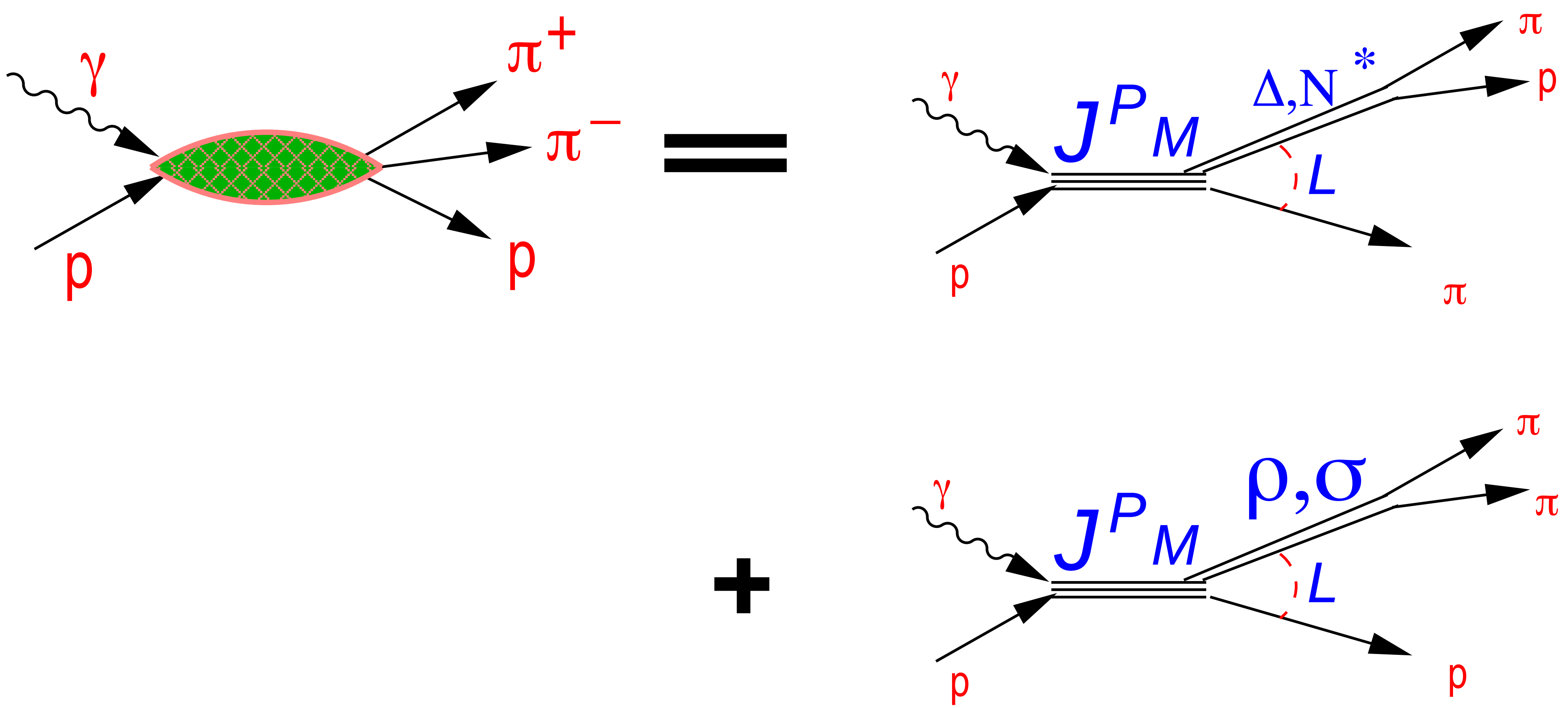
## Constrain magnitude or phase and vary the other



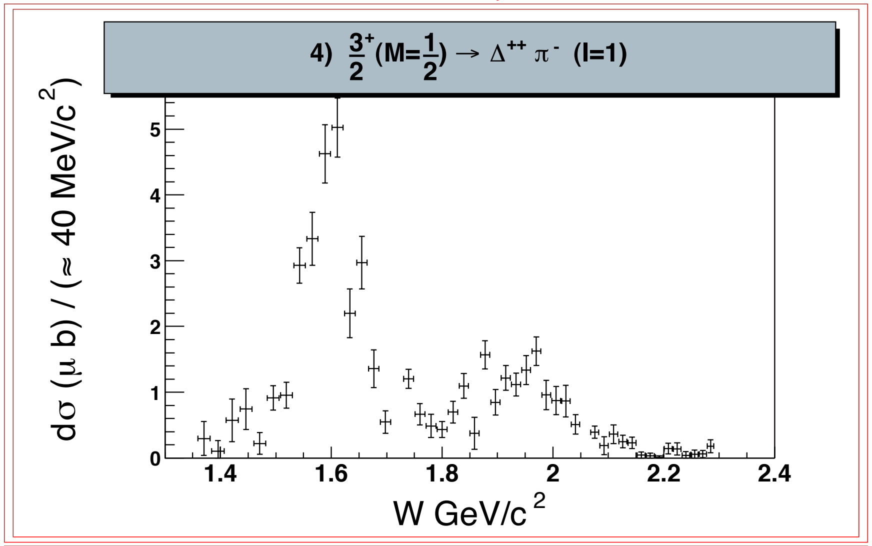
# Advanced Example: $\gamma p \rightarrow p\pi^+\pi^-$

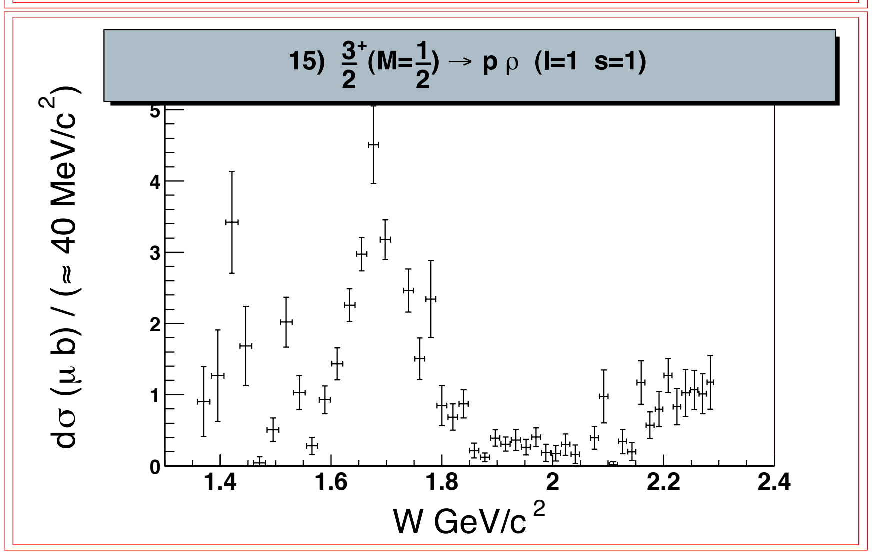
## Goal: Baryon spectroscopy



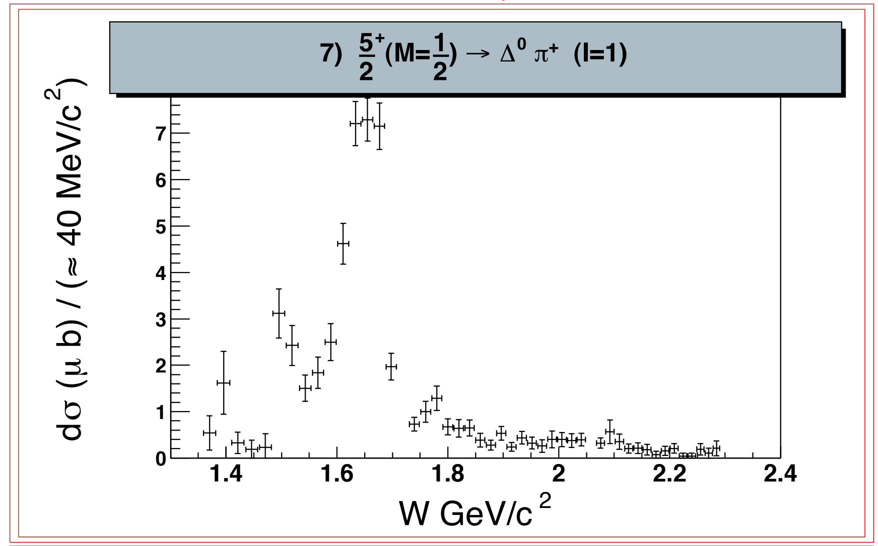
# $\gamma p \to p\pi^+\pi^-$: Some Preliminary Results

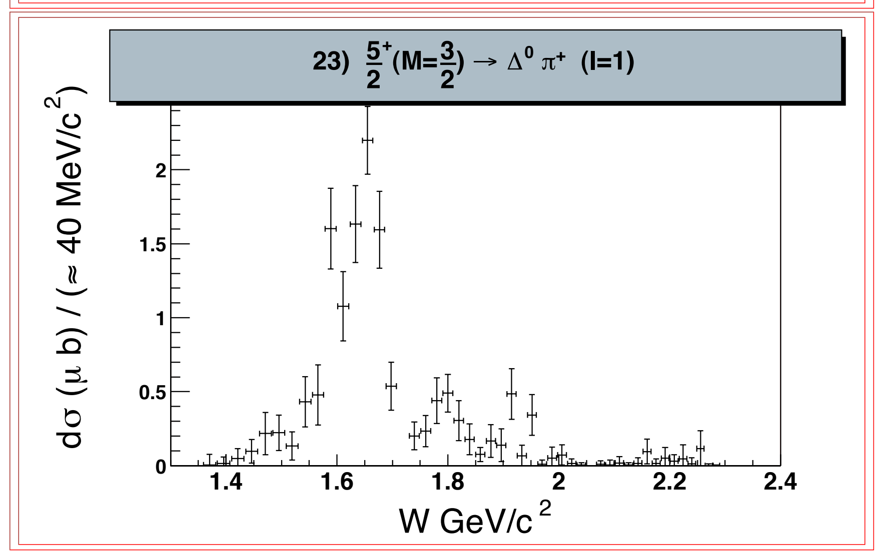
## $J^P = 3/2^+$




## $J^P = 5/2^+$



# Conclusions

**Partial Wave Analysis (whatever it is) is a useful tool for extracting physics, especially when the effects of interference are important.**

**The technique is not a panacea. Always keep your physics goals in mind, and use these to make decisions for your particular analysis.**

**Multihadron final states are a particularly thorny problem.**