Gravitation & Cosmology, Vol. 13 (2007), No. 3 (51), pp. 178–184

BI-QUATERNION SQUARE ROOTS, ROTATIONAL RELATIVITY, AND DUAL SPACE-TIME INTERVALS

Alexander Yefremov

Institute of Gravitation and Cosmology, Peoples' Friendship University of Russia

Received 2 September 2007

Diagonal quadratic forms of any dimension, built on a sign-indefinite metric, are represented as squares of six-dimensional (6D) bi-quaternion (BQ) vectors having a definable norm. In particular, the line element of 4D Minkowski space-time is written as a square of BQ-vector whose spatial and temporal parts are mutually orthogonal. Lorentz transformations of BQ-vector components with simultaneous SO(3,C) transformations of the quaternion frame yield a correlation between matrix representations of these groups, distinguishing the SO(1,2) subgroup of mixed space-time rotations. The admitted variability of the subgroup parameters leads to a BQ-vector formulation of relativity theory, comprising all features and effects of Special Relativity with an additional ability to describe motion of arbitrary non-inertial frames. Abandoning the requirement of BQ-vector norm existence leads to an unconventional 6D model of relativity, such that the imaginary part of a space-time interval is observed on the light cone.

Бикватернионные "корни квадратные", "вращательная" теория относительности и дуальные пространственно-временные интервалы
Александр Ефремов

Показано, что диагональные квадратичные формы с индефинитной метрикой произвольной размерности представимы как квадраты 6-мерных бикватернионных (BQ) векторов с определяемой нормой. В частности, линейный элемент четырехмерного пространства-времени Минковского записан как квадрат BQ-вектора, пространственная и временная части которого взаимно ортогональны. Одновременное применение преобразований Лоренца (к компонентам вектора) и SO(3,C)-вращений (к кватернионной триаде) дает соотношение компонент матриц этих групп и выделяет подгруппу SO(1,2) смешанных пространственно-временных поворотов. Параметры подгруппы могут быть переменными величинами, это позволяет сформулировать BQ-векторную версию теории относительности, которая содержит все основные черты и эффекты специальной теории относительности, но, кроме того, включает возможность описывать произвольные движения неинерциальных систем отсчета. Отказ от требования существования нормы BQ-вектора приводит к нестандартной формулировке 6-мерной теории относительности, мнимая часть линейного элемента которой наблюдается на световом конусе.

1. Introduction

Quaternion (or Rotational) relativity theory (the Q-model of relativity) is already discussed not only as a curious mathematical event [1] but also as a helpful tool able to explain and predict real physical phenomena such as the Pioneer anomaly or the observable displacements of planets [2–4]. Nonetheless, the Q-model is yet hardly noticeable in the shade of the incontestable authority of Einstein's relativity which, by the beginning of the 21st century, has become absolutely "conventional". Another, less psychological but more technical reason for a cautious attitude toward the Q-model can be found in somewhat vague formality of its appearance in the process of a "complex continuation" of quaternion frame firstly attributed to a non-inertial (rotating) observer in Newtonian mechanics [5]. The aim of this work is to again attract attention to the Q-model, demonstrating that, on the one hand, it may also arise in a "square-root procedure", well familiar and acceptable in theoretical physics, and hence, on the other hand, it can be regarded as a certain mathematical basement for construction of "conventional" theories, Einstein's special and general relativities certainly included.

In Sec. 2, a short generic study of BQ-square roots of diagonal bilinear forms is suggested, including necessary details from the theory of quaternion multiplication invariance. In Sec. 3, the basic element of the Q-model, the SO(1,2)-form invariant BQ-vector, is produced in the process of a direct derivation from Einstein's space-time interval, and simple relations between the components of matrices of involved groups are straightforwardly deduced, accompanied with clear examples. Sec. 4 contains examples of producing "unconventional" relativities on the quaternion basis. A short discussion in Sec. 5 concludes the study.

[1]e-mail: a.yefremov@rudn.ru

2. BQ-square roots

Complex quadratic forms

Let there exist a quadratic form with possibly complex components x^A in a manifold of arbitrary dimension N,

$$s^2 = g_{AB}x^A x^B,$$

where g_{AB} is a metric ($A, B, C... = 1, 2, ..., N$, and the summing convention is valid for all dimensions). In the standard procedure of Lamé lift of the components to the locally flat tangent space, there emerge new (possibly also complex) components

$$X^{(C)} = g^{(C)}_A x^A.$$

Contraction of the tangent space indices of two Lamé matrices returns the manifold metric

$$g^{(C)}_A g_{(C)B} = g_{AB},$$

while contraction in the manifold indices yields the N-dimensional Kronecker symbol, the diagonal flat metric

$$g_{(B)A} g^A_{(C)} = \delta_{BC}.$$

Thus

$$s^2 = g_{AB}x^A x^B = \delta_{CD} g^{(C)}_A x^A g^{(D)}_B x^B = \delta_{CD} X^{(C)} X^{(D)},$$

or

$$s^2 = {X^{(1)}}^2 + {X^{(2)}}^2 + {X^{(3)}}^2 + ... + {X^{(N)}}^2.$$

Assume that some components $X^{(C)}$ are real while others are pure imaginary, e.g.,

$$\begin{aligned} {X^{(1)}}^2 + ... + {X^{(M)}}^2 &= a^2, \\ {X^{(M+1)}}^2 + ... + {X^{(N)}}^2 &= -b^2; \end{aligned}$$

(a, b are real numbers), this makes the metric forming s^2 indefinite:[2]

$$s^2 = a^2 - b^2. \tag{1}$$

Thus the interval measured in a multidimensional manifold (with an indefinite metric tensor) is reduced to the form of squared length of a triangle cathetus from the Pythagoras relation.

Quaternions and bi-quaternions

Let us now briefly turn to quaternions and bi-quaternions. A quaternion number[3]

$$Q = u + a_k \mathbf{q}_k$$

comprises real coefficients u, a_k attached to the real unit 1 (not shown) and to the imaginary vector triad $\mathbf{q}_k$, respectively, the units obeying the following multiplication rule:

$$1\,\mathbf{q}_k = \mathbf{q}_k\,1, \qquad \mathbf{q}_j\mathbf{q}_k = -\delta_{jk} + \varepsilon_{jkn}\mathbf{q}_n. \tag{2}$$

It is easily shown (see e.g. [7]) that Eqs. (2) preserve their form under transformations of the triad $\mathbf{q}_k$ by matrices either of the SO(3,C) group or of the SL(2C) group, while the real unit is not transformed. Below only $SO(3,C)$-vector rotations are applied[4]

$$\mathbf{q}_{k'} = O_{k'n}\mathbf{q}_n$$

where $O_{k'n} \in$ SO(3,C) is an arbitrary 3×3 matrix (with generally complex components) satisfying the orthogonality conditions

$$O_{k'n}O_{m'n} = \delta_{k'm'}, \qquad O_{k'n}O_{k'p} = \delta_{np}.$$

All quaternions constitute an algebra with the set of operations similar to those of complex numbers: addition, subtraction, multiplication, division and Q-conjugation

$$\bar{Q} \equiv u - a_k\mathbf{q}_k.$$

This so-called third exclusive algebra is the last (in dimension) to support associative multiplication, but the property of commutation in multiplication is lost.

Unlike quaternions, the coefficients of a bi-quaternion are assumed to be complex numbers:

$$B = (u + iw) + (b_k + ia_k)\,\mathbf{q}_k$$

(u, w, a_k, b_k are real numbers), which obviously destroys the algebraic division together with the algebra itself for this set due to the fact that the norm of a BQ number is in general undefinable:

$$\begin{aligned} |B|^2 \equiv B\bar{B} &= [(u + iw) + (b_k + ia_k)\,\mathbf{q}_k] \\ &\quad\times [(u + iw) - (b_n + ia_n)\,\mathbf{q}_n] \\ &= u^2 - w^2 + 2i\,uw + b_k b_k - a_k a_k + 2i\,a_k b_k. \end{aligned} \tag{3}$$

Eq. (3) shows that there is a possibility to single out a set of BQ-vectors of the type

$$\mathbf{s} = (b_k + ia_k)\,\mathbf{q}_k, \qquad a_k b_k = 0 \tag{4}$$

having definite norms; indeed, the norm squared of the BQ-vector (4) is

$$\begin{aligned} |s|^2 = \mathbf{s}\bar{\mathbf{s}} = -\mathbf{s}^2 &= -(b_k + ia_k)\,\mathbf{q}_k\,(b_n + ia_n)\,\mathbf{q}_n \\ &= b_k b_n - a_k a_n = b^2 - a^2, \end{aligned} \tag{5}$$

a^2 and b^2 being squared lengths of the vector a_k and b_k. The operation of division in this set is of course not saved since the BQ-vector norm (5) may vanish, but the determined norm, as is seen from Eq. (5), up to the sign coincides with the square of the BQ-vector itself, which is necessary for defining bi-quaternion square roots.

---

[2] In fact, we could introduce an indefinite metric tensor from the very beginning, but involving imaginary components will be helpful below in physical applications.

[3] Small Latin indices are 3-dimensional, δ_{jk}, ε_{jkn} are the 3D Kronecker and Levi-Civita symbols, respectively.

[4] The spinor SL(2C) description of Q-frame rotation is multivariant since SL(2C) is known to cover SO(3,C) twice, and that is why the simpler vector group SO(3,C) is used here while the very interesting spinor approach deserves a closer study and a detailed presentation.

BQ-square root theorem

Comparing Eqs. (1) and (5), one arrives at the key expression

$$s^2 = \mathbf{s}\mathbf{s}. \tag{6}$$

Here, $\mathbf{s}$ is a BQ-vector from Eq. (4) composed of two 3D-vectors $\mathbf{a}$, $\mathbf{b}$, mutually imaginary and orthogonal; so this object geometrically represents an element of 3D complex vector space (or 6D real vector space). Thus Eqs. (1), (5), (6) prove the following

BQ-Square Root Theorem. *Any quadratic form (or interval measured in a manifold) with indefinite metric (when represented in the tangent space) is a square of a BQ-vector of the type (4).*

In other words, such a form admits a six-dimensional bi-quaternion square root.

Simplifications and examples of square roots in minimal dimensions

Now define two Q-vectors (the first one is in fact a BQ-vector with a definable norm, or a Q-vector in imaginary space):

$$\mathbf{a} = ia_k\mathbf{q}_k = ia\, e_k\mathbf{q}_k,$$
$$\mathbf{b} = b_k\mathbf{q}_k = bn_k\mathbf{q}_k,$$

where $\mathbf{q}_k$ is an arbitrary quaternion triad, the vectors $e_k \equiv a_k/a$ and $n_k \equiv b_k/b$ are unit ($e_ke_k = 1$, $n_kn_k = 1$) and mutually orthogonal ($e_kn_k = 0$). Then the BQ-vector (4) is written as

$$\mathbf{s} = \mathbf{a} + \mathbf{b} = (iae_k + bn_k)\mathbf{q}_k. \tag{7}$$

It is easy to verify that two unit and orthogonal vectors e_k, n_k in fact constitute two rows (or two columns) of an orthogonal matrix $O_{k'n} \in$ SO(3,C), so that, e.g., the following expressions are valid:

$$\mathbf{q}_{1'} = e_k\mathbf{q}_k, \qquad \mathbf{q}_{2'} = n_k\mathbf{q}_k,$$

hence the BQ-vector (7) acquires the simpler form

$$\mathbf{s} = ia\mathbf{q}_1 + b\mathbf{q}_2. \tag{8}$$

It is worth noting that Eq. (8) gives an example of BQ-square roots from intervals of manifolds of minimal dimensions. It is shown in the book [6] (pp. 160–169) that the left-hand side of BQ-vectors of the type (8) turns out to be itself a Q-vector in imaginary space,

$$\mathbf{s} \equiv is\,\mathbf{q}_{1'} = ia\mathbf{q}_1 + b\mathbf{q}_2.$$

This means that the manifold of minimal dimension whose interval can be presented as a BQ-vector squared is the 1D real line (directed along $\mathbf{q}_{1'}$),

$$\mathbf{s}^2 = (is\,\mathbf{q}_{1'})(is\,\mathbf{q}_{1'}) = s^2,$$

but it is also always associated with the 2D-complex plane (formed by $\mathbf{q}_1$ and q_2). A relation between the two intervals is simply Eq. (1).

It may be useful to give an example of a "complete" 3D space interval,

$$s^2 = a^2 - c^2 - d^2.$$

Its obvious BQ-square root is

$$\mathbf{s} = ia\mathbf{q}_1 + b\mathbf{q}_2 + \ddot{e}\mathbf{q}_3,$$

but the opportunity to rotate the Q-triads also allows one to reduce it to the 2D case in the simple form

$$\mathbf{s} = ia\mathbf{q}_1 + \sqrt{c^2+d^2}\left(\frac{b}{\sqrt{c^2+d^2}}\mathbf{q}_2 + \frac{c}{\sqrt{c^2+d^2}}\mathbf{q}_3\right)$$
$$= ia\mathbf{q}'_1 + \sqrt{c^2+d^2}\,\mathbf{q}'_2,$$

which is in fact a latent representative of 6D-space:

$$\mathbf{s} = ia\mathbf{q}_{1'} + \sqrt{c^2+d^2}\,\mathbf{q}_{2'}$$
$$\equiv ia\mathbf{q}_{1'} + b\,\mathbf{q}_{2'} = (ia_{k''} + b_{k''})\,\mathbf{q}_{k''}.$$

The example of famous 4D space is thoroughly analyzed in the next section.

3. Q-model of relativity as a BQ-square root of the SR space-time interval

Square root of the 4D vector norm in Minkowski space-time

Let there be a quadratic norm of an arbitrary 4D-vector defined in special relativity (SR)[5]

$$A^2 = \eta_{\alpha'\beta'}a^{\alpha'}a^{\beta'}$$
$$= \eta_{\alpha'\beta'}L^{\alpha'}_{\lambda}a^{\lambda}L^{\beta'}_{\rho}a^{\rho} = \eta_{\lambda\rho}a^{\lambda}a^{\beta} = A'^2, \tag{9}$$

naturally, invariant under Lorenz transformations of the vector components,

$$a^{\alpha'} = L^{\alpha'}_{\lambda}a^{\lambda}, \tag{10a}$$
$$\eta_{\alpha'\beta'}L^{\alpha'}_{\lambda}L^{\beta'}_{\rho} = \eta_{\lambda\rho} = \eta_{\lambda'\rho'}. \tag{10b}$$

Let us write Eq. (9) in a developed form:

$$A^2 = a_0^2 - a_ka_k = a_{0'}^2 - a_{k'}a_{k'} = A'^2. \tag{10}$$

According to the *BQ-Square Root Theorem* and Eq. (4), these expressions can always be put in the form of squares of BQ-vectors[6]:

$$\mathbf{A} = (ie_ka_0 + a_k)\mathbf{q}_k, \qquad e_ke_k = 1,$$
$$\mathbf{A}' = (ie_{k'}a_0 + a_{k'})\mathbf{q}_{k'}, \qquad e_{k'}e_{k'} = 1,$$

with the respective orthogonality conditions

$$e_ka_k = 0, \qquad e_{k'}a_{k'} = 0, \tag{11}$$

[5]It can be, in particular, the space-time interval of special relativity. All Greek indices are 4-dimensional, Latin indices are 3-dimensial, $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowski 4D metric.

[6]The unit vector e_k is freely chosen in a plane orthogonal to the vector a_k, the spatial part of a^{α}; analogously e'_k is chosen in a plane orthogonal to the vector a'_k.

so that

$$\begin{aligned}\mathbf{A}\,\mathbf{A} &\equiv (ie_k a_0 + a_k)\mathbf{q}_k (ie_n a_0 + a_n)\mathbf{q}_n \\ &= (ie_k e_n a_0^2 + ie_n a_k + ie_k a_n + a_k a_n)\,(-\delta_{kn}) \\ &= a_0^2 - a_k a_k,\end{aligned}$$

$$\begin{aligned}\mathbf{A}'\mathbf{A}' &= (ie_{k'} a_0 + a_{k'})\mathbf{q}_{k'} (ie_{n'} a_0 + a_{n'})\mathbf{q}_{n'} \\ &= (ie_{k'} e_{n'} a_0^2 + ie_{n'} a_{k'} + ie_{k'} a_{n'} + a_{k'} a_{n'})(-\delta_{kn}) \\ &= a_{0'}^2 - a_{k'} a_{k'},\end{aligned} \qquad (12)$$

or

$$A^2 = \mathbf{A}\mathbf{A} = a_0^2 - a_k a_k = a_{0'}^2 - a_{k'} a_{k'} = \mathbf{A}'\mathbf{A}'. \qquad (13)$$

Thus if the squared norm (9) of the vector is invariant as a number (length) under Lorenz transformations, then the formula for a square root following from Eq. (13)

$$\mathbf{A} = (ie_k a_0 + a_k)\mathbf{q}_k = (ie_{k'} a_{0'} + a_{k'})\mathbf{q}_{k'} = \mathbf{A}' \qquad (14)$$

shows that a BQ-vector of the type $\mathbf{A}$ must be form-invariant under some transformation (generalized rotation) of the Q-triad

$$\mathbf{q}_{k'} = O_{k'\,n}\,\mathbf{q}_n. \qquad (15)$$

Connection between the components of the Lorentz group matrix and the SO(3,C) matrix

Before searching for a relation between the components of the matrices O and L, it is convenient to represent the vectors a_k and $a_{k'}$ in a form automatically satisfying the ortogonality conditions (11). It is obvious that the vector e_k is orthogonal to a 2D plane having the projector metric

$$\begin{aligned}&b_{kn} \equiv \delta_{kn} - e_k e_n \Rightarrow \\ &\quad e_n b_{kn} = e_n(\delta_{kn} - e_k e_n) = e_k - e_k = 0.\end{aligned} \qquad (16)$$

If the 3D vector a_k is explicitly attached to this 2D plane,

$$a_k = a_n b_{nk}, \qquad (17)$$

then the condition (4) is automatically satisfied:

$$e_k a_k = e_k a_n b_{kn} = 0.$$

The same is to be made with a vector having primed indices $a_{k'}$:

$$a_{k'} = a_{n'} b_{k'n'} = a_{k'}(\delta_{kn} - e_{k'} e_{n'}). \qquad (18)$$

Inserting the expressions (17) and (18) into Eq. (14) yields:

$$\begin{aligned}\mathbf{A} &= (ie_k a_0 + a_n b_{nk})\mathbf{q}_k \\ &= (ie_{p'} a_{0'} + a_{m'} b_{m'p'})\mathbf{q}_{p'} = \mathbf{A}'.\end{aligned} \qquad (19)$$

Eq. (19) allows one to express the components of the matrix O via those of the matrix L. The result is achieved if, in Eq. (19), one writes the components $a^{\alpha'}$ in terms of the components a^{λ} as in Eq. (10) while simultaneously rotating the Q-triad as in Eq. (15):

$$\begin{aligned}&(ie_k a_0 + a_n b_{nk})\mathbf{q}_k \\ &\quad = (ie_{p'} L_{0'\lambda} a^{\lambda} + L_{m'\lambda} a^{\lambda} b_{m'p'})\,O_{p'\,k}\,\mathbf{q}_k.\end{aligned} \qquad (20)$$

The components of the same Q-triad vectors $\mathbf{q}_k$ of Eq. (20) are equalized with the right-hand side written in more detail:

$$\begin{aligned}ie_k a_0 + a_n b_{nk} &= [ie_{p'}(L_{0'0} a_0 - L_{0'k} a_k) \\ &\quad + (L_{m'0} a_0 - L_{m'k} a_k) b_{m'p'}]\,O_{p'\,k}\,.\end{aligned} \qquad (21)$$

Equalizing the coefficients of the same components of the arbitrary vector $a_{\lambda'}$ in the left- and right-hand sides of Eq. (21), one arrives at primal relations between the matrix elements:

$$ie_k = (ie_{p'} L_{0'0} + L_{m'0} b_{m'p'})\,O_{p'\,k}, \qquad (22a)$$

$$b_{nk} = (-ie_{p'} L_{0'm} - L_{m'k} b_{m'p'})\,O_{p'\,k}. \qquad (22b)$$

Multiplying Eqs. (22) by the matrix $O_{s'\,k}$ (with contraction in k) and using the matrix orthogonality relation $O_{p'k} O_{s'k} = \delta_{p's'}$ yields a more convenient relation between the matrix elements

$$ie_k O_{s'k} = ie_{s'} L_{0'0} + L_{m'0} b_{m's'}, \qquad (23a)$$

$$b_{nk} O_{s'k} = -ie_{s'} L_{0'm} - L_{m'k} b_{m's'}. \qquad (23b)$$

Eqs. (23) relate the six independent real-valued components of the matrix L (with $\det L = 1$) and the three independent complex parameters of the matrix O. Given the vectors e_k and $e_{k'}$, one immediately arrives at the sought-for explicit connection between the components. The procedure is illustrated by the following examples.

Examples of relations between Lorentz and SO(3,C) matrix elements

Any matrix of the Lorentz group, as well as that of $SO(3,C)$, can always be represented as a product of irreducible elements ("simple rotations"), so it is sufficient to find a component relation for these elements. There are two most general types of such rotations: (i) ordinary rotation (at a real angle) and (ii) hyperbolic rotation (at an imaginary angle); other cases are combinations of these two. Each simple rotation implies that at least one spatial direction is not involved in the transformation: it is allowed to perform ordinary rotations about this direction but not hyperbolic ones. Let such a direction be No. 1, i.e., $e_k = e_{k'} = \delta_{1k}$. Then from Eqs. (23) one straightforwardly finds

$$O_{\,k'm} = \begin{pmatrix} L_0^{0'} & -iL_2^{0'} & -iL_3^{0'} \\ iL_0^{2'} & L_2^{2'} & L_3^{2'} \\ iL_0^{3'} & L_2^{3'} & L_3^{3'} \end{pmatrix}. \qquad (24)$$

It is worth noting that the components of $L_{\lambda}^{\alpha'}$ with the indices $(1,\,1')$ do not enter into the matrix (24), while positions marked by the indices $(1,\,1')$ in the matrix $O_{\,k'm}$ are occupied by "time-related" Lorentz matrix components. Consider three examples.

Ordinary spatial rotation

Let the Lorentz transformation be a simple rotation of the 4D coordinate system about the spatial direction 1 by the real angle γ, then Eq. (24) establishes the following matrix relation:

$$L_{\lambda}^{\alpha'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\gamma & \sin\gamma \\ 0 & 0 & -\sin\gamma & \cos\gamma \end{pmatrix} \Rightarrow$$

$$O_{k'n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{pmatrix}. \tag{25}$$

Ordinary hyperbolic rotation

A Lorentz boost in the direction, e.g., No. 2 (or No. 3, but not No. 1 since the latter is not to be involved in the transformation[7]) is given by a simple rotation at an imaginary angle ψ; this converts the ordinary trigonometric functions to the hyperbolic ones. Eq. (24) yields the SO(3,C) version of the transformation:

$$L_{\lambda}^{\alpha'} = \begin{pmatrix} \cosh\psi & 0 & \sinh\psi & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\psi & 1 & \cosh\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow$$

$$O_{k'n} = \begin{pmatrix} \cosh\psi & -i\sinh\psi & 0 \\ i\sinh\psi & \cosh\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

Mixed rotation

A combination of the above two simple rotations is given by a product of the respective matrices (25) and (26):

$$L_{\lambda}^{\alpha'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\gamma & \sin\gamma \\ 0 & 0 & -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \cosh\psi & 0 & \sinh\psi & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\psi & 1 & \cosh\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cosh\psi & 0 & \sinh\psi & 0 \\ 0 & 1 & 0 & 0 \\ \cos\gamma\sinh\psi & \cos\gamma & \cos\gamma\cosh\psi & \sin\gamma \\ -\sin\gamma\sinh\psi & -\sin\gamma & -\sin\gamma\cosh\psi & \cos\gamma \end{pmatrix},$$

$$O_{k'n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \cosh\psi & -i\sinh\psi & 0 \\ i\sinh\psi & \cosh\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cosh\psi & -i\sinh\psi & 0 \\ i\cos\gamma\sinh\psi & \cos\gamma\cosh\psi & \sin\gamma \\ -i\sin\gamma\sinh\psi & -\sin\gamma\cosh\psi & \cos\gamma \end{pmatrix}.$$

It is easily verified that the resulting matrices still satisfy the component relation Eq. (24). The developed procedure allows one to similarly construct any SO(3,C) transformation specified by an arbitrary Lorentz matrix; this prompts suggestion of a quaternion model of relativity.

[7]For a boost in direction 1, another condition instead of $e_k = e_{k'} = \delta_{1k}$ should be used, e.g., $e_k = e_{k'} = \delta_{2k}$.

Q-model or rotational relativity

The main idea of the Q-model is to replace the line element of Einstein's relativity and its Lorentz invariance group with an adequate BQ-vector invariant under the rotational group represented by the matrices O of Eq. (22). Thus, instead of the quadratic form of 4D coordinates, an observer has at his or her disposal a movable Q-triad with time and distances measured along its unit vectors and dealt with exactly in the same manner as in Newtonian mechanics. There are just two differences: (i) rotations may be ordinary and hyperbolic, and (ii) the basic BQ-vector, as a genuine vector, must be built of small time and space displacements, hence being a differential with a structure similar to Eq. (7)[8]

$$d\mathbf{s} = (i\,dt\,e_k + dx_k)\mathbf{q}_k, \qquad e_k dx_k = 0. \tag{27}$$

The BQ-vector (27) preserves invariance under the rotations (15), (23) of the Q-triad,

$$d\mathbf{s} = (i\,dt\,e_k + dx_k)\mathbf{q}_k = (i\,dt\,e_{k'} + dx_{k'})\mathbf{q}_{k'},$$

and its square returns the interval of Einstein's relativity (in tangent space)

$$d\mathbf{s}d\mathbf{s} = (i\,dt\,e_k + dx_k)\mathbf{q}_k(i\,dt\,e_n + dx_n)\mathbf{q}_n = dt^2 - dr^2 = ds^2.$$

Eq. (27) implies that the observer deals with velocities rather than with coordinates, and that a time interval is measured in 3D "imaginary" space in a direction which is always orthogonal to the velocity of the observed point (particle), the latter moving in 3D "real" space. If the particle is a reference body of the observer's frame Σ', then $dx_{k'} = 0$, and the basic BQ-vector measures the proper time-interval $d\mathbf{s} = idt'e_{k'}\mathbf{q}_{k'}$ of Σ'. Eq. (15), connecting different frames (e.g. Σ' and Σ), can be symbolically written as

$$\Sigma' = O\,\Sigma \tag{28}$$

and, in this form, it will be called the "rotational equation"; it is the main tool of the Q-model, allowing one to analyze the behaviour of any particle or frame provided the conditions of relative motion are established. Since the Q-triads (frames) $\Sigma = \{\mathbf{q}_k\}$ may comprise a set of variable parameters (angles) depending, e.g., on the observer's time, Eq. (28), in general, links non-inertial frames; this makes the Q-model a powerful tool for solving any relativistic problems.[9] In a simpler form, convenient for solving particular problems, Eq. (27) is written similarly to Eq. (8) as

$$d\mathbf{s} = idt'\mathbf{q}_{1'} = i\,dt\mathbf{q}_1 + dr\mathbf{q}_2. \tag{29}$$

[8]The fundamental velocity $c = 1$ is the same in all frames.

[9]And this distinguishes the Q-model from special relativity where a short set of "non-inertial problems" is usually regarded, necessarily involving auxiliary conditions taken from general relativity, such as Fermi-Walker transport of vectors etc. [7, 8].

Q-velocity of a particle (or of a frame) is found as a derivative of Eq. (27) or (29) with respect to the observer's time, e.g.,

$$\mathbf{v} \equiv \frac{d\mathbf{s}}{dt'} = i\mathbf{q}_{1'} = \frac{dt}{dt'}\left(i\mathbf{q}_1 + \frac{dr}{dt}\mathbf{q}_2\right), \tag{30}$$

it has a unit imaginary time component for the observer, and its square universally equals unity, as is the case for 4-velocity in Einstein's relativity:

$$\mathbf{v}^2 = (i\mathbf{q}_{1'})(i\mathbf{q}_{1'}) = 1$$
$$= \left(\frac{dt}{dt'}\right)^2 \left(i\mathbf{q}_1 + \frac{dr}{dt}\mathbf{q}_2\right)\left(i\mathbf{q}_1 + \frac{dr}{dt}\mathbf{q}_2\right)$$
$$= \left(\frac{dt}{dt'}\right)^2 \left[1 - \left(\frac{dr}{dt}\right)^2\right]$$

since the standard relations hold:

$$\frac{dt}{dt'} = \cosh\psi, \qquad \frac{dr}{dt} = \tanh\psi.$$

Q-acceleration is a result of time derivative of Eq. (30), and in the particular case of the Σ'-observer it is

$$\mathbf{a} \equiv \frac{d\mathbf{v}}{dt'} = \frac{d^2\mathbf{s}}{dt^2} = i\omega_{1'k'}\mathbf{q}_{k'} = i\omega_{1'2'}\mathbf{q}_{2'} + i\omega_{1'3'}\mathbf{q}_{3'}, \tag{31}$$

where $\omega_{m'k'}$ is a standard geometric object, connection, antisymmetric in its indices and controlling the "rotations" of a Q-triad under small changes of its parameters. In the case given by Eq. (31), the connection components describe two spatial components of the acceleration "felt" by the observer in the frame Σ', or equivalently they can be interpreted as two components of a specific force[10] acting on the Σ' reference body. Eqs. (27) and (28), together with the introduced Q-kinematic magnitudes, velocity and acceleration, defined for observed objects, comprise the main set of relations of the Q-model of relativity. This tool, mostly described in detail in the book [6], is shown to be very simple and effective in the solution of a great number of relativistic problems, including the kinematics of inertial and non-inertial frames, and quite promising in developing BQ-vector relativistic dynamics. Among the most worthy results, one would cite a complete description of the relativistic harmonic oscillator [9], a more or less successful attempt to explain the Pioneer anomaly from pure relativistic positions [4], and a prediction of an observable (from the Earth) position shift of satellites of the Solar System planets [6], a new relativistic effect deserving experimental verification.

Dual space-time interval

The main features of the Q-model are: (i) frames of reference are represented by Q-triads depending on the parameters of motion; (ii) the timelike components of vectors are measured in 3D imaginary space, while the spacelike components are attributed to 3D real space; (iii) timelike and spacelike componenst of all Q-vectors, and primarily of the basic BQ-vector (27), should be mutually orthogonal, which provides the conventional Lorents norm squared given by Eq. (10). These three features ensure that the squares of the Q-model vectors have a form appropriate to the conventional relativity. But about 70 years ago Fueter [10] noticed that the BQ-vector of electromagnetic (EM) field strength

$$\mathbf{F} = (B_k + iE_k)\,\mathbf{q}_k \tag{32}$$

has an "unconventional" square form:

$$\mathbf{F}^2 = (B_k + iE_k)\,\mathbf{q}_k(B_n + iE_n)\,\mathbf{q}_n$$
$$= E^2 - B^2 - 2iE_kB_k. \tag{33}$$

Electromagnetic invariants are recognized in Eq. (33), which gives a real number if only $E_kB_k = 0$, e.g., as in a plane wave, but the second invariant by no means always vanishes in electromagnetic field theory. Thus the orthogonality condition of the type (11) sometimes does not hold, the BQ-vector (32) has, in general, no definite norm, but this does not impede deduction of a correct Q-version of the Maxwell equations from Eq. (32). A question arises, if removing of the orthogonality condition is admissible for the BQ-interval expression and how it may affect the description of space-time geometry? To analyze the situation, consider a BQ-interval built of 6D coordinates similarly to Eq. (32):

$$\mathbf{z} = (x_k + it_k)\,\mathbf{q}_k, \qquad x_kt_k \neq 0; \tag{34}$$

its square is a complex number:

$$\mathbf{z}\mathbf{z} = (x_k + it_k)\,\mathbf{q}_k(x_n + it_n)\,\mathbf{q}_n$$
$$= t^2 - x^2 - 2it_kx_k = z^2 \tag{35}$$

with a special relativity-type interval as the real part. It is initially implied that the observer with his frame $\{\mathbf{q}_k\}$ is located in ordinary 3D-space R_3 (configuration space), and it is confirmed by the form of the BQ-vector (34), where the R_3 coordinates x_k are real; hence all real-valued magnitudes are to be attributed to the R_3 world. The time coordinates of the observer, according to Eq. (34), are measured in the "parallel" (dual) 3D-space T_3 which is reciprocally imaginary with R_3:

$$T_3 = i\,R_3;$$

similarly, all imaginary-valued magnitudes are attributed to the T_3 world. This means that the real and imaginary parts of Eq. (35) are the R_3 Lorentz-invariant SR-interval and its dual T_3 replica, respectively, measured by the same observer. It is worth noting that the T_3 interval is "seen" from R_3 as if settled on the light cone bordering the two worlds. Indeed, the transformations to null coordinates

$$u_k = (t_k + x_k)/\sqrt{2}, \qquad y_k = (t_k - x_k)/\sqrt{2},$$
$$t_k = (u_k + y_k)/\sqrt{2}, \qquad x_k = (u_k - y_k)/\sqrt{2}$$

[10] Force per unit mass of the body.

deliver the observer onto the light cone, and from this viewpoint the intervals (35) are measured as

$$z^2 = 2u_k y_k - i\,(u^2 - y^2),$$

i.e., the former SR interval remains real but of light-cone-type (in the conventional version of relativity, it vanishes due to condition $u_k y_k = 0$), while the T_3 interval is "seen" from this position in the standard diagonal form. Another characteristic transformation, "imaginary reflections" of the coordinates,

$$\xi_k = -it_k, \qquad \tau_k = ix_k,$$
$$t_k = i\xi_k, \qquad x_k = -i\tau_k \tag{36}$$

forces the observer to "change the world" $R_3 \rightarrow T_3$: real magnitudes become imaginary and vice versa. This transformation leaves intact the form of the interval (35):

$$z^2 = \tau^2 - \xi^2 - 2i\tau_k\xi_k \tag{37}$$

now measured by a T_3-observer. Eq. (36) means that the former R_3-distance becomes T_3-time and the former R_3 time turns to T_3-distance, while similarity of Eqs. (35) and (37) makes one conclude that the R_3 observer and the T_3 observer see absolutely identical space-times and are unable to distinguish one 3D world from another.

We finally note that the suggested unconventional interval is not fit to application of the standard 4D Lorentz group; it is easily shown that the BQ-vector (34) is form-invariant under the full group SO(3,C) of rotating frames without restrictions of the type (23), (24), while scalar squares given by Eqs. (35) and (36) are invariant under "imaginary reflections" (36) and also under SO(3,C) transformations of the involved coordinates. Thus, using a BQ-vector of the type (34) for construction of relativity theory in a way analogous to that suggested in Sec. 3, would evidently lead to an unconventional model, different from Einstein's relativity. But the 3D world duality conjecture may be helpful in constructing a procedure of 6D extension of electromagnetic theory using methods suggested by Fueter [10].

4. Discussion

The scheme presented here, initiating the Q-model of relativity as a 6D BQ-vector square root from standard space-time line element, above all helps a clearer understanding of its origin and existence. But the developed method of building BQ-vectors, whose squares give quadratic forms with an indefinite metric, may be applied to analyzing the nature and probably revise the theories where expressions of the type "difference of squares" are essential. These are not only intervals of special and general relativity (in tangent space) or the electromagnetic field invariant. Operators in differential equations of hyperbolic type, including the d'Alembert and Klein-Gordon operators, obviously have a similar mathematical form, and so they also should admit "extraction" of BQ-square roots representing first-order vector differential operators. This may lead to another development of the theory and result in its extension or in alternative models. Natural candidates for square-root analysis are electromagnetic and spinor field equations. Such a study will be described elsewhere.

References

[1] A.P. Yefremov, *in:* "Space-Time Structure, Algebra and Geometry", Russian Hypercomplex Society, Lilia Print, Moscow, 2007, pp. 403-421 (in Russian).

[2] F. Smarandache and V. Christianto, *Progr. in Phys.* **1**, 42–45 (2007).

[3] V. Christianto and F. Smarandache, *Progr. in Phys.* **3**, 24–26 (2007).

[4] A.P. Yefremov, *Progr. in Phys.* **2**, 93–96 (2007).

[5] A.P. Yefremov, "Quaternionic relativity. I. Inertial motion", *Grav. & Cosmol.* **2**, 77–83 (1996).

[6] A.P. Yefremov, "Quaternion Spaces, Frames and Fields", RUDN Publ., Moscow, 2005 (in Russian).

[7] E.F. Taylor and J.A. Wheeler, "Space-Time Physics", W.H. Freeman & Co., San Francisco–London, 1966.

[8] Ch.W. Misner, K.S. Thorne and J.A. Wheeler, "Gravitation", W.H. Freeman & Co., San Francisco–London, 1973.

[9] A.P. Yefremov, *in:* "Quantization in Astrophysics, Brownian Motion, and Supersymmetry", Chennai Ed., 2007, pp. 440-457.

[10] R. Fueter, *Comm. Math. Hel.* **B 7**, 307–330 (1934–1935); ibid., **B 8**, 371–378 (1936–1937); ibid., **B 9**, 320–334 (1936–1937); ibid., **B 10**, 306–315 (1937–1938).