## Mathematical Analysis of the Quadratic Koch Island and the Koch Snowflake

## Introduction

The previous chapter shows that one of the ways to improve a heat exchange is to increase the heat transfer area. One of the properties of fractals is an increase in the circumference and hence increased heat transfer area. The area and circumference of the fractal heat exchanger is dependent on the fractal being applied and the number of times it is applied. Thus, the correlation between the area, circumference and the amount of times the fractal is applied needs to be found. This chapter will derive equations for the fractal heat exchanger geometry based on the Koch snowflake and Koch island.

## Quadratic Koch Island

Following will be some figures showing the relevant dimensions needed for the analysis. The first figure is the original square with no fractals applied yet. The next figure shows the modified square with one fractal application. The definition of the length of the fractal is shown after that and finally the resultant quadratic island is shown.

Figure 1 shows the dimension $x_{0}$, which is the original length of the square.


Figure 1: The original square
The fractal with section lengths of $l$ is applied to each side of the square and the result is shown in Figure 2.


Figure 2: Quadratic Koch curve after one iteration

The length of the fractal, $l_{f}$ is shown in Figure 3 and is the total length of all the fractal segments.


Figure 3: Length of a fractal
The overall length, $L$ is shown in Figure 4 in the completed fractal image.


Figure 4: The resultant quadratic Koch curve
The cross sectional area, circumference, fractal section length and the overall length will now be derived for the quadratic Koch island.

## Cross Sectional Area

The overall area of the block (side length of $x_{0}$ ) does not change, since when a fractal area is added the same area is subtracted elsewhere.
Thus

$$
\begin{equation*}
A(n)=A_{0}=x_{0}^{2} \tag{1}
\end{equation*}
$$

where $n$ is the amount of time the fractal was applied.

## Fractal Section Length

The length of each fractal section, $l(n)$, decreases with each application of the fractal. Thus the fractal section length is described by the following equation.

$$
\begin{equation*}
l(n)=(1 / 4)^{n} x_{0} \tag{2}
\end{equation*}
$$

## PROOF

The length of the initial fractal section is a $1 / 4$ of the block side length $\left(x_{0}\right)$. With each application of the fractal the newly created line becomes a $1 / 4$ smaller. Thus the new fractal is the size of

$$
\begin{equation*}
l_{2}=\frac{1}{4} \cdot \frac{1}{4} x_{0}=\frac{1}{16} x_{0} \tag{3}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
l_{1}=\frac{1}{4} x_{0} \tag{4}
\end{equation*}
$$

$$
\begin{equation*}
l_{2}=\frac{1}{4^{2}} x_{0} \tag{5}
\end{equation*}
$$

So that

$$
\begin{equation*}
l_{n}=\frac{1}{4^{n}} x_{0} \tag{6}
\end{equation*}
$$

The final equation shows that the fractal section length decreases after every iteration. When the amount of iterations goes to infinity, the section length decreases to zero.

## Circumference

The circumference of the block increases every time the fractal is applied and is equal to

$$
\begin{equation*}
C=4\left(2^{n} x_{0}\right) \tag{7}
\end{equation*}
$$

## PROOF

The total length of the fractal, $l_{f}$ is 8 times the section length, which was just determined in the previous section. The amount of fractal sections increases by $8^{n}$.

Applied once: $\quad l_{f}=\frac{8}{4} x_{0}=2 x_{0}$
Applied twice: $\quad l_{f}=\frac{64}{16} x_{0}=2^{2} x_{0}$
Applied $n$ times: $\quad l_{f}=\frac{8^{n}}{4^{n}} x_{0}=2^{n} x_{0}$

## Overall Length

The overall length, $L$ (shown in Figure 4) of the fractal is the original length, $x_{0}$ plus twice all the segment lengths.

$$
\begin{align*}
& L=x_{0}+2\left(\frac{1}{4} x_{0}+\frac{1}{16} x_{0}+\ldots\right)  \tag{11}\\
& L=x_{0}+2 x_{0} \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^{k}=x_{0}+2 x_{0}\left(\sum_{k=1}^{\infty} \frac{1}{4}\left(\frac{1}{4}\right)^{k-1}\right) \tag{12}
\end{align*}
$$

Since $\sum_{k=1}^{n} a r^{k-1}=a\left(\frac{1-r^{n}}{1-r}\right)$

$$
\begin{equation*}
L(n)=x_{0}\left(1+\frac{2}{3}\left(1-\frac{1}{4^{n}}\right)\right) \tag{13}
\end{equation*}
$$

For the case $n \rightarrow \infty$

$$
\begin{equation*}
L=\frac{5}{3} x_{0} \tag{15}
\end{equation*}
$$

## Summary

The results are summarised in Table 1.

| Iteration | Segment <br> length | Circumference | Cross sectional <br> area | Overall length <br> $\boldsymbol{L}$ (Figure 4) |
| :---: | :---: | :---: | :---: | :---: |
| 0 | $x_{0}$ | $4 x_{0}$ | $x_{0}{ }^{2}$ | $x_{0}$ |
| 1 | $\frac{1}{4} x_{0}$ | $8 x_{0}$ | $x_{0}{ }^{2}$ | $\frac{3}{2} x_{0}$ |
| 2 | $\frac{1}{16} x_{0}$ | $16 x_{0}$ | $x_{0}{ }^{2}$ | $\frac{13}{8} x_{0}$ |
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $n$ | $\frac{1}{4^{n}} x_{0}$ | $4\left(2^{n} x_{0}\right)$ | $x_{0}{ }^{2}$ | $x_{0}\left(1+\frac{2}{3}\left(1-\frac{1}{4^{n}}\right)\right)$ |
| $\infty$ | 0 | $\infty$ | $x_{0}{ }^{2}$ | $\frac{5}{3} x_{0}$ |

Table 1: Geometry of the quadratic island
When the amount of iterations approaches infinity, the segment length goes to zero, while the circumference tends to infinity. The cross sectional area stays the same, while the overall length approaches a constant value.

## Koch Snowflake

The above process is now repeated for the Koch snowflake. The different figures are presented below, showing the original length and the fractal segment length.

The original triangle has side lengths of $y_{0}$ as shown below in Figure 5.


Figure 5: The original triangle
The fractal segment length is $l$ and is shown in Figure 6. The correlations for the segment length, circumference and area will now be derived.


Figure 6: Triangle after first fractal application

## Segment Length

After each fractal iteration, the length of the sides is a third of the previous one.

$$
\begin{align*}
& l(0)=y_{0}  \tag{16}\\
& l(1)=1 / 3 \quad y_{0}  \tag{17}\\
& l(2)=1 / 3(1 / 3) l=1 / 9 l=(1 / 3)^{2} y_{0}
\end{align*}
$$

Thus

$$
\begin{equation*}
l(n)=(1 / 3)^{n} y_{0} \tag{19}
\end{equation*}
$$

## Circumference

The circumference of the triangle is 3 times the length of one side. The length of one side can be established as follows:
After the first iteration there are 4 segments of the same length ( $1 / 3 y_{0}$ ). Thus the side has a length of $4\left(1 / 3 y_{0}\right)$. After the second iteration there are $4^{2}$ segments of length $(1 / 3)^{2} y_{0}$. After the $n$ 'th iteration there are $4^{n}$ segments of length $(1 / 3)^{n} y_{0}$.
Thus the circumference is

$$
\begin{equation*}
C=3 \cdot 4^{n}\left(\frac{1}{3}\right)^{n} y_{0}=4^{n}\left(\frac{1}{3}\right)^{n-1} y_{0} \tag{20}
\end{equation*}
$$

## Area

The area of a triangle is
Area $=1 / 2 \times$ base $\times$ perpendicular height
Thus the area of the original triangle with side length $y_{0}$ is

$$
\begin{equation*}
\operatorname{Area}(0)=\frac{1}{2} \cdot y_{0} \cdot \frac{\sqrt{3}}{2} y_{0}=\frac{\sqrt{3}}{4} y_{0}^{2} \tag{21}
\end{equation*}
$$

For the first iteration the area increases with 3 triangles with side lengths of $1 / 3 y_{0}$ :

$$
\begin{equation*}
\operatorname{Area}(1)=\operatorname{Area}(0)+3 \cdot\left(\frac{1}{2} \cdot \frac{1}{3} y_{0} \cdot \frac{\sqrt{3}}{6} y_{0}\right)=\frac{\sqrt{3}}{4} y_{0}^{2}+\frac{\sqrt{3}}{12} y_{0}^{2}=\frac{\sqrt{3}}{3} y^{2} \tag{22}
\end{equation*}
$$

For $n$ iterations the area can be calculated as follows:
$\operatorname{Area}(n)=\operatorname{Area}(0)+\operatorname{Area}(1)+\cdots+\operatorname{Area}(n)$

$$
\begin{align*}
& \operatorname{Area}(n)=\frac{\sqrt{3}}{4} y_{0}^{2}+3\left[\left(\frac{1}{3}\right)^{2} \frac{\sqrt{3}}{4} y_{0}^{2}\right]+12\left[\left(\frac{1}{3}\right)^{4} \frac{\sqrt{3}}{4} y_{0}^{2}\right]+\cdots+3 \cdot 4^{n-1}\left[\left(\frac{1}{3}\right)^{2 n} \frac{\sqrt{3}}{4} y_{0}^{2}\right] \\
& \operatorname{Area}(n)=\frac{\sqrt{3}}{4} y_{0}^{2}\left[1+\frac{3}{4} \cdot 4\left(\frac{1}{9}\right)^{1}+\frac{3}{4} \cdot 4^{2}\left(\frac{1}{9}\right)^{2}+\cdots+\frac{3}{4} \cdot 4^{n}\left(\frac{1}{9}\right)^{n}\right]  \tag{23}\\
& \operatorname{Area}(n)=\frac{\sqrt{3}}{4} y_{0}^{2}\left[1+\frac{3}{4} \sum_{k=1}^{n}\left(\frac{4}{9}\right)^{k}\right]=\frac{\sqrt{3}}{4} y_{0}^{2}\left[1+\frac{3}{5}\left(1-\frac{4}{9}\right)^{n}\right]
\end{align*}
$$

PROOF

$$
\begin{equation*}
\sum_{k=1}^{n}\left(\frac{4}{9}\right)^{k}=\sum_{k=1}^{n}\left(\frac{4}{9}\right)\left(\frac{4}{9}\right)^{k-1}=\frac{4}{9} \cdot \frac{1-\left(\frac{4}{9}\right)^{n}}{1-\frac{4}{9}}=\frac{4}{5}\left(1-\left(\frac{4}{9}\right)^{n}\right) \tag{26}
\end{equation*}
$$

## Summary

The results are summarised in Table 2.

| Iteration | Segment length | Circumference | Cross sectional area |
| :---: | :---: | :---: | :---: |
| 0 | $y_{0}$ | $3 y_{0}$ | $\frac{\sqrt{3}}{4} y_{0}^{2}$ |
| 1 | $\frac{1}{3} y_{0}$ | $4 y_{0}$ | $\frac{\sqrt{3}}{3} y_{0}^{2}$ |
| 2 | $\frac{1}{9} y_{0}$ | $\frac{16}{3} y_{0}$ | $\frac{10 \sqrt{3}}{27} y_{0}^{2}$ |
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $n$ | $\frac{1}{3^{n}} y_{0}$ | $\frac{4^{n}}{3^{n-1}} y_{0}$ | $\frac{\sqrt{3}}{4} y_{0}^{2}\left[1+\frac{3}{5}\left(1-\left(\frac{4}{9}\right)^{n}\right)\right]$ |
| $\infty$ | 0 | $\infty$ | $\frac{2 \sqrt{3}}{5} y_{0}^{2}$ <br> (Peitgen et al., 1993) |

Table 2: Geometry of the Koch snowflake
The above table shows that when the amount of iterations approaches infinity, the segment length tends to zero, while the circumference goes to infinity. This is the same result as for the quadratic Koch fractal. The cross sectional area is however not constant, the area converges to a constant value.

## References

Peitgen, H., Jürgens, H. and Saupe, D., 1993, "Fractals for the classroom, Part one," First Edition, Second printing, Springer-Verlag, New York.

## Nomenclature

A Area $\left[\mathrm{m}^{2}\right]$
$a \quad$ Constant
C Circumference [m]
$k \quad$ Amount of iterations
$l \quad$ Segment length [m]
$l_{f} \quad$ Fractal length [m]
$L \quad$ Overall length [m]
$n \quad$ Amount of iterations
$r$ Constant
$x$ Side length [m]
$x_{0} \quad$ Original length of one of the square's sides [m]
$y_{0} \quad$ Original length of one of the triangle's sides [m]
Subscripts
$o$ - Original

