Mathematical Snapshots by Gian-Carlo Rota Killian Faculty Achievement Award Lecture, March 5, 1997

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0. Introduction.

When I was in high school, my English teacher gave me to read an essay by James Thurber, called "The secret life of Walter Mitty". After rereading this essay every few years, I decided that everyone has a Walter Mitty complex. One way to understand a person might be to discover that person's Walter Mitty fantasies .

My own Walter Mitty complex has few variations. I imagine myself trying new lines of attack on problems I will never solve, or lecturing on subjects in which I am incompetent. It never occurred to me that I might one day receive the Killian Faculty Achievement Award, and thus, no drafts of this lecture were attempted in my Walter Mitty daydreaming. When David Benney phoned me last May and ordered me to fly back immediately from Strasbourg to receive the award, I was caught without a premeditated text, and with barely a year to decide what not to say in the Killian lecture.

The philosopher José Ortega y Gasset wrote that most of the tasks we undertake in the course of our lives are impossible. Nevertheless, he added, we must try to carry them out .

The Killian lecture is an example of the impossibility Ortega y Gasset probably had in mind, especially when the lecture deals with mathematics. The language and the results of mathematics seem nowadays to lie farther from the mainstream of science than they have ever been.

The only argument against this opinion would be to make the results of mathematics available in a language from which scientists will benefit. But mathematicians' inability to make themselves understood is not a recent phenomenon. It is the thoroughly documented plight of mathematics throughout history, since Pythagoras.

One reason for the mathematicians' difficulty in communicating is the mathematicians' concept of nature. The mathematicians' concept of nature is at variance with the concept of nature that is shared by other scientists.

To a scientist, nature is a primeval forest to be explored, rich in surprising and unpredictable fauna, endowed with mysterious laws that scientists bravely wrest from the jungle. Once discovered, the laws of nature are written up by scientists for the benefit of posterity, in a language that sometimes - but not always - happens to be the language of mathematics. A scientist need not be fluent in that peculiar language that is called mathematics, just as he or she need not be fluent in Urdu or Gaelic.

Mathematicians do not agree with this view.

Galileo wrote the famous sentence: "the great book of nature is written in the language of mathematics" . Galileo was a great scientist, one of the greatest perhaps. But Galileo was also a practical joker. His practical jokes got him into trouble from time to time. Could it be the case that Galileo's sentence was written tongue in cheek? This insinuation will be indignantly rejected by every mathematician. To mathematicians, Galileo's sentence is sculpted in marble. Every discovery of a new scientific fact is a challenge to uncover the uderlying mathematical structure . This structure is not "abstracted" from nature, as psychologists would have us believe. It is the basic makeup of nature, it was always there, waiting to be told and staring at us all the time.

The natural laws discovered by scientists will be refined like a metal, polished like a jewel and finally stored as theorems in the archives of mathematics.

Mathematicians triumphantly point to mechanics as the example of a theory that began as an empirical science, and that eventually made its way into mathematics as a generalized geometry, geometry with time added. Mathematicians believe that every science will sooner or later meet the fate that befell mechanics.

This is the mathematicians' faith. It is also a reason why some scientists find mathematicians difficult persons to deal with.

1. The dark side of mathematics.

Unfortunately, this is not the only reason why mathematicians are regarded with suspicion. Another reason is the teaching of mathematics.

Some students who learn higher mathematics are turned off the subject . Why? They feel that some of the mathematics they are taught belabors the obvious and pursues the preposterous.

As a matter of fact, much mathematical research done in the first half of this century was concerned with finding preposterous examples of innocent-looking definitions. Regions of the plane without an area, nowhere differentiable functions, continuous curves that fill a whole square were taken with the utmost seriousness in that bygone age.

These curios are now stored in the attic. However, the pursuit of the preposterous had a beneficial consequence. It trained mathematicians to look for unexpected instances of intuitive definitions. Some of these unexpected instances are now turning out to be downright useful.

I should like to review the discovery of such an unexpected instance. It is obtained by analyzing the everyday notion of volume, or, in abstract terms, measure. Contrary to the rules that speakers are expected to follow, I will give away the punch line. We will see that volume is characterized by four axioms, and we will find a new measure that fits these axioms, after a slight twist.

Measure is defined by two axioms. A measure v on a family of subsets, for example, subsets of ordinary space, is a real number which is assigned to subsets $A, B, \ldots$ in the family, and which satisfies :

Axiom 1.

$$v(\emptyset) = 0,$$

where $\emptyset$ is the empty set.

This axiom looks like a triviality, but it has unexpected payoffs.

Axiom 2. If A and B are two sets, then

$$v(A \cup B) = v(A) + v(B) - v(A \cap B).$$

The picture shows that this axiom states that measure is additive. In particular, if sets A and B are disjoint, then

$$v(A \cup B) = v(A) + v(B).$$

This property extends to any finite family F of sets. Let us record it:

Axiom 2'.

$$v(\cup_{A \in F} A) = \sum_{A \in F} v(A),$$

provided any two sets in the family F are disjoint.

The best example of measure is the volume $v(A)$ of a solid A in space.

Axioms 1 and 2 do not single out volume among all measures. To this end, two axioms must be added:

Axiom 3.

The volume of a set A is independent of the position of A. In other words, if a set A in three-dimensional Euclidean space can be rigidly moved onto a set B, then A and B have the same volume.

In technical language, volume is invariant under the group of Euclidean motions.

Axiom 4.

If P is a parallelotope with orthogonal sides of lengths x_1, x_2, x_3, then

$$v(P) = x_1 x_2 x_3.$$

These four axioms, together with suitable continuity conditions, determine volume. By an approximation process such as one finds in an advanced calculus textbook, one shows that that these axioms imply that the volume of a sphere S_r of radius r is given by the known formula

$$\frac{v(S_r) = 4\pi r^3}{3.}$$

A similar characterization of volume holds in n-dimensional Euclidean space for any finite dimension n. The fourth axiom is changed to

Axiom 4n.

$$v(P) = x_1 x_2 ... x_n$$

whenever P is a parallelotope with orthogonal sides equal to $x_1, x_2, ..., x_n$.

What happens if we keep the first three axioms, but tamper with the fourth axiom? To answer this question, we appeal to the basic tools of combinatorial mathematics.

The basic tools of combinatorial mathematics are the elementary symmetric functions, that is, the following polynomials in n variables:

$$e_1(x_1, x_2, ...x_n) = x_1 + x_2 + ... + x_n.,$$

$$e_2(x_1, x_2, ...x_n) = x_1x_2 + x_1x_3 + ... + x_{n-1}x_n,$$

...

$$e_{n-1}(x_1, x_2, ..., x_n) = x_2x_3...x_n + x_1x_3x_4...x_n + ... + x_1x_2...x_{n-1},$$

$$e_n(x_1, x_2, ..., x_n) = x_1x_2...x_n.$$

For $n = 3$, there are three elementary symmetric functions:

$$e_1(x_1, x_2, x_3) = x_1 + x_2 + x_3,$$

$$e_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3,$$

$$e_3(x_1, x_2, x_3) = x_1x_2x_3.$$

Observe an interesting coincidence. The last of these three symmetric functions is also the formula for the volume of a parallelotope. Axiom 4 can be rewritten as

Axiom 4.

$$v(P) = e_3(x_1, x_2, x_3).$$

On the right side of this formula, let us replace the symmetric function of degree three by the symmetric function of degree two. In other words, let us replace Axiom 4 by

Axiom 4':

$$a(P) = a_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$$

The new measure a satisfies axioms 1,2, and 3, with v changed to a, but instead of satisfying axiom 4 it satisfies axiom 4'.

Does the measure a make any sense? Of course it does. The right hand side equals, except for a factor of two, the formula for the surface area of the parallelotope P. Again, the surface areas of solids can be computed starting from axioms 1, 2, 3, and 4', by continuity considerations. The surface area of a sphere S_r of radius r is given - except for a factor of two - by the known formula

$$a(S_r) = 2\pi r^2.$$

Now you are thinking: there he goes. He is indulging in that favored activity of mathematicians, belaboring the obvious. We have axiomatized two well known mathematical notions. Big deal. But let us take the next step.

Emboldened by our success with two symmetric functions, we now replace axiom 4 by yet another axiom, that uses the one symmetric function that we have so far left out. We try to define a new measure that satisfies axioms 1, 2, and 3, together with

Axiom 4".

$$w(P) = e_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

with v replaced by w. In particular, if P' is the face of the parallelotope with sides equal to x_1, x_2, then we have

$$w(P') = x_1 + x_2.$$

Does this make any sense?

When a definition is proposed, the fundamental condition to be verified is its consistency. The French mathematician Henri Poincaré put it elegantly when he wrote: "in mathematics, to be is to be consistent".

Look at the two parallelotopes P_1 and P_2. The first parallelotope has sides equal to x_1, x_2, x_3 , and the second parallelotope has sides equal to y, x_2, x_3. The two parallelotopes have the common face P' . If the measure $w(P)$ is to be consistent, then by axiom 2 we must have

$$w(P_1 \cup P_2) = w(P_1) + w(P_2) - w(P_1 \cap P_2) = w(P_1) + w(P_2) - w(P')$$

(*)

Let us check this equality.

The left side is computed by observing that the parallelotope $P_1 \cup P_2$ has sides equal to x_1 ,x_2 and $x_3 + y$. Therefore , Axiom 4" tells us that

$$w(P_1 \cup P_2) = x_1 + x_2 + x_3 + y.$$

Now let us compute the right side . We have

$$w(P_1) = x_1 + x_2 + x_3$$

$$w(P_2) = x_1 + x_2 + y$$

$$w(P') = x_1 + x_2,$$

Adding the right sides of the above three formulas, we see that the right side of (*) equals

$$w(P_1)+w(P_2)-w(P_1 \cap P_2) = x_1+x_2+x_3+x_1+x_2++y-(x_1+x_2) = x_1+x_2+x_3+y.$$

The two sides of equation (*) agree , thereby convincing us that the definition of w is consistent.

Again, continuity considerations enable us to compute the measure $w(A)$ when A is any reasonable solid in ordinary space. The limiting process required to carry out such computation is only a little more complex than the limiting processes we teach in 18.02.

What is the meaning of the new measure w ?

The definition of $w(P)$ for a parallelotope P has a geometric interpretation. When multtiplied by 4, it equals the perimeter of the parallelotope P, that is, the sum of the lengths of all the edges of the parallelotope P.

But, one may object, $w(P)$ makes sense for a parallelotope P, because a parallelotope has a well defined perimeter. What about $w(A)$ when A is a solid that does not have a well defined perimeter, a sphere for example? Defining the perimeter of a sphere seems to fly in the face of common sense.

Einstein wrote : "Common sense is the residue of the prejudices that were instilled into us before the age of seventeen". Since the new measure w is well defined, common sense will have to adjust to reality.

The measure w is called the mean width, a misnomer that has been kept for historical reasons. The formula for the mean width of a sphere of radius r is

$$w(S_r) = 4r.$$

For a tetrahedron T of side x, we have

$$v(T) = \frac{\sqrt{2}x^3}{12},$$

$$a(T) = \frac{x^2\sqrt{3}}{2},$$

$$w(T) = \frac{3\alpha x}{\pi},$$

where $cos\alpha = -1/3$.

The mean width is a new measure on three dimensional solids that enjoys equal rights with volume and surface area.

In n dimensions, each of the n elementary symmetric functions leads similarly to a generalization of volume. We thereby obtain n invariant measures in n-dimensional space. These measures are called the intrinsic volumes.

The intrinsic volumes are independent of each other, except for certain inequalities they satisfy which remain to be discovered. We know little about the intrinsic volumes , because they have not been around for long.

I know of no person who has an intuitive feeling for the mean width, similar to the intuitive feeling we have for volume and area. The closest we can come to date to an intuitive interpretation of the mean width is the following probabilistic interpretation in the very special case of convex sets.

Take two convex sets A and B in three dimensional Euclidean space, and suppose that A is contained in B. Again, let us begin by belaboring the obvious. Suppose that we take a point at random belonging to the larger set B. What is the probability that the point shall belong to the smaller set A? The answer is obvious: such a probability equals the volume of A divided by the volume of B.

Now let us take a leap of reason. Instead of choosing a point at random, let us choose a straight line at random in space. Assuming that such a straight line meets the larger set B, what is the probability that such a straight line will also meet the smaller set A?

The answer is satisfying. Such a probability equals the surface area of the set A, divided by the surface area of the set B.

You can tell what is coming next. We take a random plane in space. Assuming that the plane meets the larger set B, what is the probability that it will also meet the smaller set A? Such a probability equals the mean width of A, divided by the mean width of B.

It is likely that when scientists become aware of the existence of the mean width, they will find interpretations and applications of this measure.

2. The hidden side of mathematics.

At this point the mathematician will ask: are the intrinsic volumes defined by the elementary symmetric functions all the invariant measures in Euclidean n-dimensional space?

The answer to this question is negative, but do not be alarmed. We are missing only one measure.

In discovering the missing measure, we face a critical stage in mathematical research. We are forced to retrace our steps and to focus on something that we have missed not because it was hidden, but because it was too evident.

Be prepared for a letdown. We have missed the symmetric function of degree zero, which equals the constant one, in symbols

$$e_0 = 1.$$

Can we define a new measure that equals one on all subsets of space? No, because axiom 1 would be violated. Let us do the next best thing: we try to define a measure χ by setting

$$\chi(P) = 1$$

if the parallelotope P is non empty, and $\chi(\emptyset) = 0$. Actually, we might as well set

Axiom 5.

$$\chi(A) = 1$$

whenever A is an arbitrary non empty closed convex set.

It takes some work to prove that χ is consistently defined by axioms 1,2,3 and 5, with χ in place of v. The difficulty is that a polyhedron R may be assembled out of convex pieces in infinitely many ways, and one has to show that no matter which family of pieces we use to assemble R , the computation of $\chi(R)$ carried out by axioms 2 and 5 always yields the same number.

For example, take R to be a square in the plane from which a smaller square has been removed.

R equals the union of four convex sets A, B, C, D . Additivity requires that the measure of the union of these four sets be computed as follows:

$$\chi(R) = \chi(A \cup B \cup C \cup D) = \chi(A) + \chi(B) + \chi(C) + \chi(D) - \chi(A \cap B) - \chi(B \cap C) - \chi(C \cap D) - \chi$$

On the right hand side four terms equal to one, and four terms equal to minus one. Therefore, $\chi(R) = 0$.

Consistency demands that we obtain zero no matter how we decompose R into convex polyhedra.

Consistency can be established, and thus χ is a measure, a new intrinsic volume. It is called the Euler characteristic.

In conclusion: in Euclidean n-dimensional space there are exactly $n + 1$ invariant measures, namely, the Euler characteristic and the intrinsic volumes corresponding to the n elementary symmetric functions. The discovery of this fact is an achievement of mathematics in the latter half of the twentieth century.

The measure χ leads to an immediate understanding of one of the best known formulas of mathematics.

When we remove all the faces on the boundary of a convex polyhedron Q, we are left with the interior $int(Q)$ of the convex polyhedron Q. A delicate computation reminiscent of the evaluation of an integral shows that the Euler characteristic of the interior $int(Q)$ is given by the formula

$$\chi(int(Q)) = (-1)^k,$$

(**)

where k is the dimension of the convex polyhedron Q.

For example, if Q is a straight line segment, then $int(Q)$ consists of the same straight line segment with the two endpoints removed, and we have $\chi(intQ) = -1$ by formula (**). If Q is a square, then $int(Q)$ consists of the same square with the four boundary edges removed, and the formula (**) tells us that $\chi(int(Q)) = +1$.

Suppose now that a polyhedron R , not necessarily convex, is partitioned into faces in an arbitrary way. Every face must be a convex polyhedron, and the interiors of any two faces must be disjoint. The union of all interiors of all the faces equals the polyhedron R, in symbols:

$$R = \cup_{Q \in F} int(Q).$$

Here, we denote by F the family of all faces in our partition of the polyhedron R.

Since the interiors of any two faces are disjoint, we may apply axiom 2', with χ replacing v, thereby obtaining

$$\chi(R) = \sum_{Q \in F} \chi(int(Q)).$$

If the family F of faces contains f_0 faces of dimension 0, tf_1 faces of dimension 1, and so forth, then by collecting the summands on the right that correspond to faces of the same dimension and by using formula (**) for the Euler characteristic of the interior $int(Q)$, the right hand side simplifies to

$$\chi(R) = f_0 - f_1 + f_2 - f_3 + ... - ...$$

We have proved that the alternating sum of the number of faces of each dimension gives the Euler characteristic of the polyhedron R, no matter how we subdivide the polyhedron into faces. This formula is the classical Euler-Poincaré formula.

In the above example , we count 12 vertices, 16 edges and 4 faces, giving 12-16+4 = 0, as expected.

Similar combinatorial formulas can be derived for the other intrinsic volumes. However, they involve angles as well as faces.

3. The bright side of mathematics.

As you know, Cassandra was one of the forty children of the king of Troy. She was condemned to foresee the future and never to be believed. Had she not lived in the doomed city of Troy, she might have made some optimistic predictions, though she would still not have been believed.

Mathematics has played the role of Cassandra. Future revolutions in our picture of the universe could have been foretold, if we had not lent a deaf ear to the mathematics of the time.

Freeman Dyson has brilliantly documented this phenomenon in his essay "Missed opportunities".

Are there any revolutions in the making? What is the mathematics of today trying to tell us? Will there soon be a change of paradigm, as our late colleague Tom Kuhn might have put it?

I would not be asking this question, if the answer were negative.

If any of you were to name one fundamental idea of science in this century, you would be likely to say: symmetry. The physical world is ruled by symmetry, by groups. Again, let me give away the punch line: symmetry is about to undergo a drastic overhaul.

What is symmetry?

It has been known for a long time that every function of two variables can be written as the sum of a symmetric function and a skew symmetric function. In symbols,

$$f(x_1, x_2) = f_s(x_1, x_2) + f_a(x_1, x_2),$$

where

$$f_s(x_1, x_2) = \frac{f(x_1, x_2) + f(x_2, x_1)}{2}$$

$$f_a(x_1, x_2) = \frac{f(x_1, x_2) - f(x_2, x_1)}{2}.$$

The function $f_s(x_1, x_2)$ is a symmetric function:

$$f_s(x_1, x_2) = f(x_2, x_1).$$

The function $f_a(x_1, x_2)$ is a skew symmetric function:

$$f_a(x_1, x_2) = -f_a(x_2, x_1).$$

Around the turn of the century, mathematicians realized that the analog of this decomposition of a function as the sum of a symmetric and a skew symmetric function no longer holds for functions of three or more variables. The correct decomposition for functions of any number of variables was discovered almost simultaneously by Alfred Young in Cambridge - the other Cambridge - and Issai Schur in Berlin.

I cannot help digressing on a piece of historical gossip. Alfred Young was a very shy mathematician. Several years ago, some of us decided to reprint a photograph of Young as the frontispiece of a book we were editing in his honor. To our surprise, we were not able to find any such photograph; as a backup, we had to use a photograph of his grave.

Everybody at MIT is familiar with the eigenvectors of a matrix. Physicists working in quantum mechanics and psychologists working in factor analysis have equally benefited from the reduction of symmetric matrices to diagonal form. The eigenvectors of a matrix provide the decomposition of a vector space into subspaces, each of which is invariant under the matrix. Forgive me for bringing up these elementary facts.

The core of the theory of symmetry is a generalization of the idea of eigenvector. Instead of a matrix, we have the group of all permutations of a set $\{1, 2, ..., n\}$ of n elements. The vector space is the space of functions $f(x_1, x_2, ..., x_n)$ of n variables. A permutation σ maps the function $f(x_1, x_2, ..., x_n)$ to $f(x_{\sigma 1}, x_{\sigma 2}, ..., x_{\sigma}$

The fundamental result is the decomposition of the space of functions of n variables into subspaces, called symmetry classes. The symmetry classes play the role of eigenvectors. Each symmetry class is invariant under all permutations. Every function of n variables can be written as the sum of p_n functions, each one belonging to a different symmetry class. The integer p_n equals the number of partitions of the integer n.

The theory of symmetry classes is an achievement of mathematics in this century. It was completed in the latter half, by both physicists and mathematicians. The idea is understood from a couple of examples. What happens for $n = 3$?

Every function of three variables can be written as the sum of a symmetric function, a skew symmetric function, and a cyclic symmetric function, in symbols

$$f(x_1, x_2, x_3) = f_s(x_1, x_2, x_3) + f_a(x_1, x_2, x_3) + f_c(x_1, x_2, x_3).$$

A symmetric function satisfies the equations

$$f_s(x_{\sigma 1}, x_{\sigma 2}, x_{\sigma 3}) = f_s(x_1, x_2, x_3)$$

for every permutation σ of the set $\{1, 2, 3\}$.

Similarly, a skew symmetric function satisfies the equations

$$f_s(x_{\sigma 1}, x_{\sigma 2}, x_{\sigma 3}) = (sign\sigma) f_s(x_1, x_2, x_3)$$

where the sign is $+1$ or -1 according as the permutation is even or odd.

There is a third symmetry class of functions, cyclic symmetric functions, defined by the equation

$$f_c(x_1, x_2, x_3) + f_c(x_3, x_1, x_2) + f_c(x_2, x_3, x_1) = 0.$$

It is true but not obvious that the family of cyclic symmetric functions is invariant under permutations.

For functions $f(x_1, x_2, x_3, x_4)$ of four variables, the plot thickens. There are five symmetry classes of functions. The first three are the symmetric functions, skew symmetric functions and the analogs in four variables of cyclic symmetric functions., which satisfy the equations

$$f(x_1, x_2, x_3, x_4) + f(x_1, x_4, x_2, x_3) + f(x_1, x_3, x_4, x_3) = 0,$$

$$f(x_1, x_2, x_3, x_4) + f(x_4, x_2, x_1, x_3) + f(x_3, x_2, x_4, x_1) = 0,$$

$$f(x_1, x_2, x_3, x_4) + f(x_4, x_1, x_3, x_2) + f(x_2, x_4, x_3, x_1) = 0,$$

$$f(x_1, x_2, x_3, x_4) + f(x_3, x_1, x_2, x_4) + f(x_2, x_3, x_1, x_4) = 0.$$

The last two symmetry classes are defined by equations that are far from intuitive, and will not be written down. Only crystallographers and particle physicists have an intuitive feeling for the last two symmetry classes.

Every function of four variables is the sum of five functions, each one belonging to one of these symmetry classes, and so on for every n.

The decomposition of a function into a sum of functions belonging to symmetry classes has found a great many applications, and we are tempted to let it go to our heads and philosophize about it, as Hermann Weyl did.

However, the latest bulletins are negative. The message is that something more basic lies hidden underneath this theory.

Once more, we are forced to retrace our steps and make ourselves aware of phenomena that we have been taking for granted. Something is wrong with the common sense visualization of a permutation.

The simplest permutation, or transposition of the set $\{1, 2\}$, sends 1 to 2 and 2 to 1. If we iterate this transposition we obtain the trivial permutation that leaves both integers fixed.

We can visualize this fact by another picture.

But wait a minute: are we telling the truth? If the two arrows are what they are supposed to be, then we must specify which of the two arrows goes underneath the other. After we decide which arrow goes underneath, then we see that the iteration of a transposition is no longer the identity permutation. One of the strings keeps winding around the other as we iterate.

We are not entitled to asssume that the iteration of a trasnposition gives the identity permutation. This is the unwarranted assumption that we made for a hundred years. What happens when we take the bold step of dropping this assumption? Why, permutations are replaced by new objects , called braids, as in the figure.

Products of braids are taken in the same way as products of permutations, by placing the diagram of a braid underneath the other. The inverse of a braid is obtained by flipping the diagram of the braid, and the product of a braid with its inverse is the identity braid.

The theory of symmetry is now being revamped after the advent of braids. This is the cutting edge of mathematics. New theories are sprouting up: quantum groups, the Yang-Baxter equations, monoidal categories, and what not. The end is nowhere in sight. It is not known at present what will replace the old symmetry classes. In technical terms, no one has yet determined the irreducible representations of the braid group, and even the concepts of group and representation may get overhauled.

It is likely that these new theories will have a domino effect on our picture of the physical world. Other ingrained prejudices about space will be dealt a fatal blow. A new world is in the making.

When we meet again in the new world ten years from now, we will marvel at how we could have ever entertained such prejudices, while the truth was always there, waiting to be told and staring at us all the time.

Thank you.