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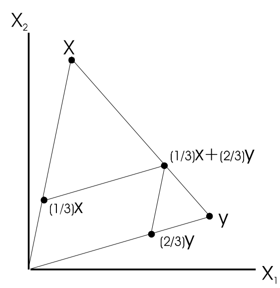
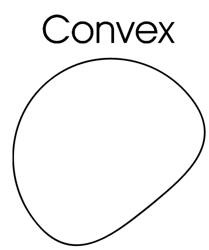
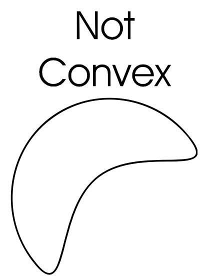
Concave and Quasi-Concave Functions

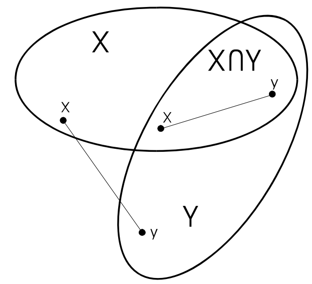
A set $X \subset \mathbb{R}^n$ is *convex* if $x, y \in X$ implies $\lambda x + (1-\lambda)\, y \in X$ for all $\lambda \in [0,1]$.

Geometrically, if $x, y \in \mathbb{R}^n$, then $\{z \in \mathbb{R}^n : z = \lambda x + (1-\lambda)\, y \text{ for } \lambda \in [0,1]\}$ constitutes the straight line connecting x and y. So a convex set is any set that contains the entire line segment between any two vectors in the set.









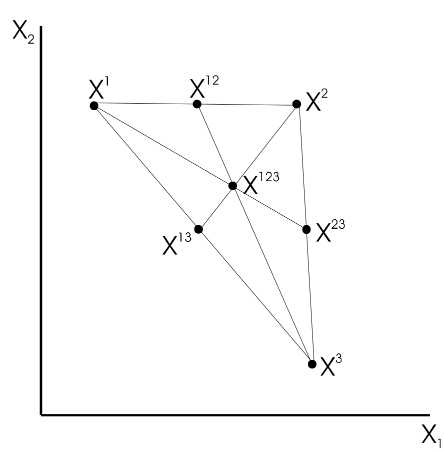
- The intersection of two convex sets is convex. Can you prove this?
- The union of two convex sets is not necessarily convex. Why not?

A vector $z \in \mathbb{R}^n$ is a *convex combination* of $x^1, ..., x^m \in \mathbb{R}^n$ if

$$z = \sum_{j=1}^{m} \lambda_j x^j \quad \text{for some } \lambda_1, ..., \lambda_m \geq 0 \text{ with } \sum_{j=1}^{m} \lambda_j = 1.$$

In the figure below:

$$\begin{aligned} x^{12} &= \tfrac{1}{2}x^1 + \tfrac{1}{2}x^2, \quad x^{13} = \tfrac{1}{2}x^1 + \tfrac{1}{2}x^3, \quad x^{23} = \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 \\ x^{123} &= \tfrac{2}{3}x^{12} + \tfrac{1}{3}x^3 = \tfrac{2}{3}x^{13} + \tfrac{1}{3}x^2 = \tfrac{2}{3}x^{23} + \tfrac{1}{3}x^1 = \tfrac{1}{3}x^1 + \tfrac{1}{3}x^2 + \tfrac{1}{3}x^3 \end{aligned}$$



Theorem 1: A set $X \subset \mathbb{R}^n$ is convex if and only if it contains any convex combination of any vectors $x^1, ..., x^m \in X$.

Proof. (if) If X contains any convex combination of its vectors, then as a special case, $\lambda x + (1-\lambda)\, y \in X$ for all $x, y \in X$ and $\lambda \in [0,1]$ ..

(only if) The proof is by mathematical induction on m. For $m = 1$, the only convex combination of vector x is x itself. So the basis statement for $m = 1$ is true. The induction step is to suppose that the proposition is true for $m - 1 > 0$ vectors, and then to show that this implies the proposition is true for m vectors. So consider any convex combination $\sum_{j=1}^{m} \lambda_j x^j$ of m vectors contained in X. Since $m \geq 2$ and each $\lambda_j \geq 0$ with $\sum_{j=1}^{m} \lambda_j = 1$, we may suppose WLOG that $\lambda_m < 1$. Then since

$$\sum_{j=1}^{m-1} \frac{\lambda_j}{1-\lambda_m} = \frac{\sum_{j=1}^{m-1} \lambda_j}{1-\lambda_m} = \frac{1-\lambda_m}{1-\lambda_m} = 1,$$

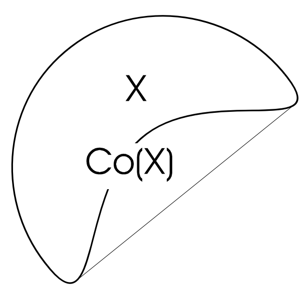
the induction hypothesis implies that $y \equiv \sum_{j=1}^{m-1} \left(\frac{\lambda_j}{1-\lambda_m}\right) x^j \in X$. Then the definition of a convex set implies

$$\sum_{j=1}^{m} \lambda_j x^j = \sum_{j=1}^{m-1} \lambda_j x^j + \lambda_m x^m = (1-\lambda_m) \sum_{j=1}^{m-1} \left(\frac{\lambda_j}{1-\lambda_m}\right) x^j + \lambda_m x^m = (1-\lambda_m)\, y + \lambda_m x^m \in X.$$

The proposition then follows from mathematical induction. ■

Given any set $X \subset \mathbb{R}^n$, the *convex hull* $Co(X)$ is the intersection of all convex sets that contain X.

- Since the intersection of any two convex sets is convex, it follows that the convex hull is the smallest convex set that contains X.



- If X is convex, then $Co(X) = X$. Why?

Theorem 2: Suppose $X \subset \mathbb{R}^n$. Then $Co(X)$ is the set of all convex combinations of vectors in X.

Proof. See Appendix. ■

Concave Functions

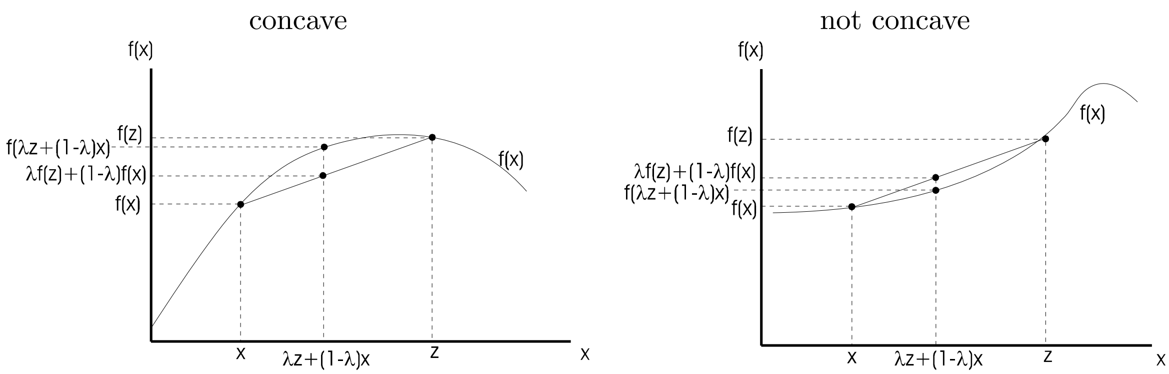
For the remainder of these notes, we suppose that $X \subset \mathbb{R}^n$ is a convex set.

$f : X \to \mathbb{R}$ is *concave* if for any $x, z \in X$, we have, for all $\lambda \in (0, 1)$,

$$f(\lambda z + (1 - \lambda)\, x) \geq \lambda f\,(z) + (1 - \lambda)\, f(x).$$

$f : X \to \mathbb{R}$ is *strictly concave* if for any $x, z \in X$ with $x \neq z$, we have, for all $\lambda \in (0, 1)$,

$$f(\lambda z + (1 - \lambda)\, x) > \lambda f\,(z) + (1 - \lambda)\, f(x).$$



- A constant function is concave. Why?
- A linear function is concave. Why?
- $f : X \to \mathbb{R}$ is concave if and only if $f(\lambda(z - x) + x) \geq \lambda\,(f(z) - f(x)) + f(x)$ for all $x, z \in X$ and $\lambda \in (0, 1)$. Why?
- $f : X \to \mathbb{R}$ is concave if and only if $f(\lambda \Delta x + x) \geq \lambda\,(f(x + \Delta x) - f(x)) + f(x)$ for all $x, (x + \Delta x) \in X$ and $\lambda \in (0, 1)$. Why?

Theorem 3: (Jensen's Inequality) Suppose that $f : X \to \mathbb{R}$ is concave. If $x^1, \ldots, x^m \in X$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}_+$ with $\sum_{i=1}^m \alpha_i = 1$, then $f(\sum_{i=1}^m \alpha_i x^i) \geq \sum_{i=1}^m \alpha_i f(x^i)$.

Proof. Left as an exercise [Hint: use the definition of a concave function and mathematical induction.] ■

Linear Combinations of Concave Functions

Consider a list of functions $f_i : X \to \mathbb{R}$ for $i = 1, ..., n$, and a list of numbers $\alpha_1, ..., \alpha_n$. The function $f \equiv \sum_{i=1}^n \alpha_i f_i$ is called a *linear combination* of $f_1, ..., f_n$. If each of the weights $\alpha_i \geq 0$, then f is a *nonnegative linear combination* of $f_1, ..., f_n$.

The next proposition establishes that any nonnegative linear combination of concave functions is also a concave function.

Theorem 4: Suppose $f_1, ..., f_n$ are concave functions. Then for any $\alpha_1, ..., \alpha_n$, for which each $\alpha_i \geq 0$, $f \equiv \sum_{i=1}^n \alpha_i f_i$ is also a concave function. If, in addition, at least one f_j is strictly concave and $\alpha_j > 0$, then f is strictly concave.

Proof. Consider any $x, y \in X$ and $\lambda \in (0,1)$. If each f_i is concave, we have

$$f_i(\lambda x + (1-\lambda)y) \geq \lambda f_i(x) + (1-\lambda) f_i(y)$$

Therefore,

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\equiv \sum_{i=1}^n \alpha_i f_i(\lambda x + (1-\lambda)y) \geq \sum_{i=1}^n \alpha_i (\lambda f_i(x) + (1-\lambda) f_i(y)) \\ &= \lambda \sum_{i=1}^n \alpha_i f_i(x) + (1-\lambda) \sum_{i=1}^n \alpha_i f_i(y) \equiv \lambda f(x) + (1-\lambda) f(y). \end{aligned}$$

This establishes that f is concave. If some f_i is strictly concave and $\alpha_i > 0$, then the inequality is strict. ■

Since a constant function is concave, Theorem 4 implies:

- If f is concave, then any affine transformation $\alpha f + \beta$ with $\alpha \geq 0$ is also concave.
- If f is strictly concave, then any affine transformation $\alpha f + \beta$ with $\alpha > 0$ is also strictly concave.

Theorem 5: Let $I \subset \mathbb{R}$ be an interval and let $X = I^n$. Suppose $f : X \to \mathbb{R}$ is defined by $f(x) = \sum_{i=1}^n \phi_i(x_i)$, where each $\phi_i : I \to \mathbb{R}$.
(a) If each ϕ_i is concave, then f is concave.
(b) If each ϕ_i is strictly concave, then f is strictly concave.

Proof. Left as an exercise. ■

A function $f : X \to \mathbb{R}^n$ of the form $f(x) = \sum_{i=1}^n \phi_i(x)$ is sometimes called a *linearly separable* function.

- Why does Theorem 5 require that each ϕ_i be strictly concave to ensure that f is strictly concave, while Theorem 4 requires only one f_i be strictly concave to ensure that f is is strictly concave?

Concave Functions of an Affine Function

Theorem 6: Let $X \subset \mathbb{R}^n$ be convex and $f : X \to \mathbb{R}$ is a concave function.
(a) Let $g : \mathbb{R}^m \to \mathbb{R}^n$ be defined by $g(y) = Ay + b$, where A is an $n \times m$ matrix and suppose $g[Y] \subset X$. Then $h = (f \circ g) : Y \to \mathbb{R}$ is a concave function.
(b) If f is strictly concave and g is 1-1, then h is strictly concave.

Proof. Left an exercise. ■

Quasi-Concave Functions

$f : X \to \mathbb{R}$ is *quasi-concave* if for any $x, z \in X$, we have $f(\lambda z + (1-\lambda)x) \geq \min\{f(x), f(z)\}$ for all $\lambda \in (0,1)$.

f is *strictly quasi-concave* if for any $x, z \in X$ and $x \neq z$, we have $f(\lambda z + (1-\lambda)x) > \min\{f(x), f(z)\}$ for all $\lambda \in (0,1)$.

Theorem 7: A (strictly) concave function is (strictly) quasi-concave.

Proof. The theorem follows immediately from the observation that if f is concave, then for all $x, z \in X$, we have

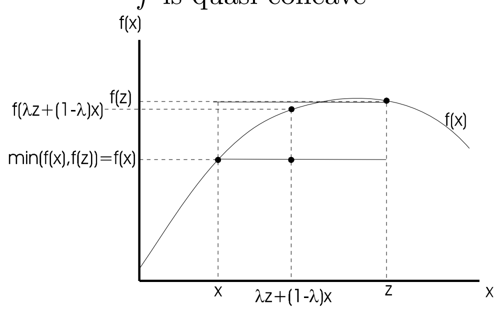
$$f(\lambda z + (1-\lambda)x) \geq \lambda f(z) + (1-\lambda) f(x) \geq \min\{f(x), f(z)\} \text{ for all } \lambda \in (0,1).$$

If f is strictly concave, we have for all $x, z \in X$ and $x \neq z$,

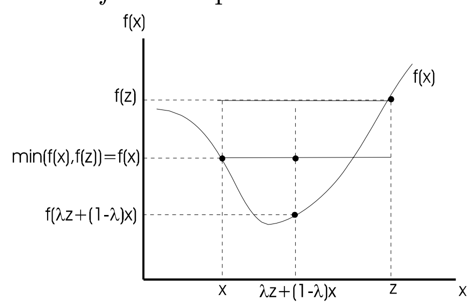
$$f(\lambda z + (1-\lambda)x) > \lambda f(z) + (1-\lambda) f(x) \geq \min\{f(x), f(z)\} \text{ for all } \lambda \in (0,1).$$

■

f is quasi-concave



f is not quasi-concave



- If $X \subset \mathbb{R}$, then $f : X \to \mathbb{R}$ is quasi-concave if and only if it is either monotonic or first nondecreasing and then nonincreasing. Why?

Our next theorem states that any monotone nondecreasing transformation of a quasi-concave function is quasi-concave.

Theorem 8: Suppose $f : X \to \mathbb{R}$ is quasi-concave and $\phi : f(X) \to R$ is nondecreasing. Then $\phi \circ f : X \to \mathbb{R}$ is quasi-concave. If f is strictly quasi-concave and ϕ is strictly increasing, then $\phi \circ f$ is strictly quasi-concave.

Proof. Consider any $x, y \in X$. If f is quasi-concave, then $f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}$. Therefore, ϕ nondecreasing implies

$$\phi(f(\lambda x + (1-\lambda)y)) \geq \phi(\min\{f(x), f(y)\}) = \min\{\phi(f(x)), \phi(f(y))\}.$$

If f is strictly quasi-concave, then for $x \neq y$, we have $f(\lambda x + (1-\lambda)y) > \min\{f(x), f(y)\}$. Therefore, if ϕ is strictly increasing, we have

$$\phi(f(\lambda x + (1-\lambda)y)) > \phi(\min\{f(x), f(y)\}) = \min\{\phi(f(x)), \phi(f(y))\}.$$

■

Recall that for any $x \in X$, $P(x) \equiv \{z \in X : f(z) \geq f(x)\}$ is called the *better set* of x.

Theorem 9: $f : X \to \mathbb{R}$ is quasi-concave if and only if $P(x)$ is a convex set for each $x \in X$.

Proof. (only if) Suppose f is quasi-concave. Choose an arbitrary $x^0 \in X$. To show that $P(x^0)$ is convex, consider any $x, y \in P(x^0)$. Then, $f(x), f(y) \geq f(x^0)$ and the quasi-concavity of f imply that

$$f(\lambda x + (1-\lambda)\, y) \geq \min\{f(x), f(y)\} \geq f(x^0)$$

which implies that $\lambda x + (1-\lambda)\, y \in P(x^0)$ for any $\lambda \in (0,1)$.

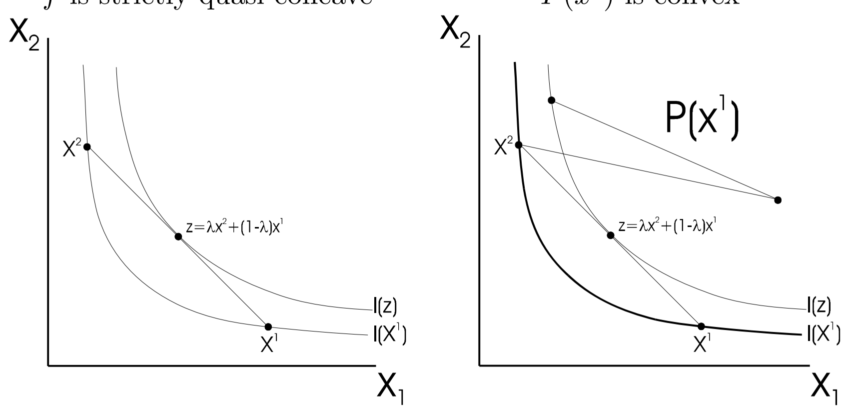
(if) Suppose $P(x^0)$ is convex for each $x^0 \in X$. Now consider any $x, y \in X$. WLOG, suppose that $f(x) \leq f(y)$. Then letting $x = x^0$, we have that $x, y \in P(x^0)$ and therefore $\lambda y + (1-\lambda)\, x \in P(x^0)$ for any $\lambda \in (0,1)$. It then follows from the definition of $P(x^0)$ that

$$f(\lambda y + (1-\lambda)\, x) \geq f(x) = \min\{f(x), f(y)\}.$$

■

Theorem 9 is illustrated below for an increasing function $f : \mathbb{R}^2_+ \to \mathbb{R}$. Notice that all convex combinations of vectors in $P(x^1)$ are also elements of $P(x^1)$. Also notice that if f is strictly quasi-concave, then the level set can contain no straight line segments.

f is strictly quasi-concave $\qquad$ $P(x^1)$ is convex



Corollary 1: Suppose $f : X \to \mathbb{R}$ attains a maximum on X. (a) If f is quasi-concave, then the set of maximizers is convex. (b) If f is strictly quasi-concave, then the maximizer of f is unique.

Proof. (a) Let x^0 be a maximizer of f. Then $f(x) \leq f(x^0)$ for all $x \in X$ implies that $P(x^0)$ is the set of maximizers of f. If f is quasi-concave, then Theorem 7 implies that $P(x^0)$ is convex.

(b) If f is strictly quasi-concave, suppose x, y are both maximizers of f. Then $x \neq y$ implies $f(\frac{1}{2}x + \frac{1}{2}y) > \min\{f(x), f(y)\} = f(x)$ which implies that x is not a maximizer of f.

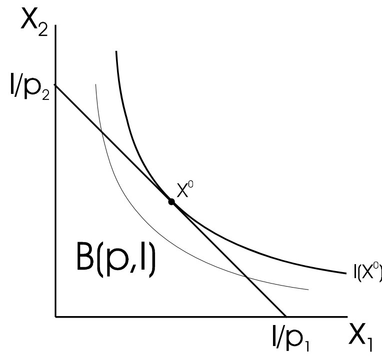
■

Example: Let $I > 0$ be the income of some household and let $p = (p_1, ..., p_n) \in \mathbb{R}^n_+$ denote the vector of prices of the n goods. Then

$$B(p, I) \equiv \left\{x \in \mathbb{R}^n_+ : px \leq I\right\}$$

defines its budget set – the set of all possible nonnegative bundles of goods it may purchase within its budget. You can verify that $y, z \in B(p, I)$ implies $\lambda y + (1 - \lambda) z \in B(p, I)$ for all $\lambda \in [0, 1]$. Therefore, $B(p, I)$ is a convex set.

Now suppose that the household preferences are given by some utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$. Then Corollary 1 implies that if u is strictly quasi-concave, there is a unique bundle $x \in B(p, I)$ that maximizes $u : B(p, I) \to \mathbb{R}$.



Convex and Quasi-Convex Functions

If we reverse the inequality sign in the definitions of concave and quasi-concave functions we obtain convex and quasi-convex functions.

$f : X \to \mathbb{R}$ is a *convex* function if for any $x, y \in X$, we have, for all $\lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

$f : X \to \mathbb{R}$ is a *strictly convex* function if for any $x, y \in X$ where $x \neq y$, we have, for all $\lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda) y) < \lambda f(x) + (1 - \lambda) f(y)$$

NOTE: A convex set and a convex function are two distinct concepts.

$f : X \to \mathbb{R}$ is *quasi-convex* if for any $x, y \in X$, we have $f(\lambda x + (1 - \lambda) y) \leq \max\{f(x), f(y)\}$ for all $\lambda \in (0, 1)$.

f is *strictly quasi-convex* if for any $x, y \in X$ and $x \neq y$, we have $f(\lambda x + (1 - \lambda) y) < \max\{f(x), f(y)\}$ for all $\lambda \in (0, 1)$.

The following proposition is an immediate consequence of the definitions.

Theorem 10: (a) f is a (strictly) convex function if and only if $-f$ is a (strictly) concave function.

(b) f is a (strictly) quasi-convex function if and only if $-f$ is a (strictly) quasi-concave function.

Theorem 6 allows us to easily translate all of our propositions for concave and quasi-concave functions to the analogues for convex and quasi-convex functions, which are provided here for easy reference.

- Suppose $f_1, ..., f_n$ are convex functions. Then for any $\alpha_1, ..., \alpha_n$, for which each $\alpha_i \geq 0$, then $f \equiv \sum_{i=1}^{n} \alpha_i f_i$ is also a convex function. If, in addition, at least one f_j is strictly convex and $\alpha_j > 0$, then f is strictly convex.
- A linear function is both concave and convex.
- A (strictly) convex function is (strictly) quasi-convex.
- Suppose $f : X \to \mathbb{R}$ is quasi-convex and $\phi : f(X) \to \mathbb{R}$ is nondecreasing. Then $\phi \circ f : X \to \mathbb{R}$ is quasi-convex. If f is strictly quasi-convex and ϕ is strictly increasing, then $\phi \circ f$ is strictly quasi-convex.
- A function $f : X \to \mathbb{R}$ is quasi-convex if and only if for each $x \in X$, $W(x)$ is convex.
- Suppose $f : X \to \mathbb{R}$ attains a minimum on X. (a) If f is quasi-convex, then the set of minimizers is convex. (b) If f is strictly quasi-convex, then the minimizer of f is unique.

Appendix

Theorem 2: Suppose $X \subset \mathbb{R}^n$. Then $Co(X)$ is the set of all convex combinations of vectors in X.

Proof. Theorem 1 implies that any convex combination of elements $x^1, ..., x^m \in X$ must be contained in $Co(X)$. To show that $Co(X)$ contains only vectors that are convex combinations of some $x^1, ..., x^m \in X$, we need to show that $Y \equiv \left\{x \in \mathbb{R}^n : x \text{ is a convex combination of some } x^1, ..., x^m \in X\right\}$ is a convex set. So consider any $y, z \in Y$. Then, by definition, there is a set of vectors $y^1, ..., y^m \in X$ and list of nonnegative numbers $\alpha_1, ..., \alpha_m$ with $\sum_{i=1}^{m} \alpha_i = 1$ such that $y = \sum_{i=1}^{m} \alpha_i y^i$. Similarly, there is a set of vectors $z^1, ..., z^r \in X$ and list of nonnegative numbers $\beta_1, ..., \beta_r$ with $\sum_{i=1}^{r} \beta_i = 1$ such that $z = \sum_{i=1}^{r} \beta_i z^i$. Then for any $\lambda \in [0, 1]$, we have

$$\begin{aligned} \lambda y + (1 - \lambda) z &= \lambda \sum_{i=1}^{m} \alpha_i y^i + (1 - \lambda) \sum_{i=1}^{r} \beta_i z^i \\ &= \sum_{i=1}^{m} \lambda \alpha_i y^i + \sum_{i=1}^{r} (1 - \lambda) \beta_i z^i \end{aligned}$$

which, since $\lambda \sum_{i=1}^{m} \alpha_i + (1 - \lambda) \sum_{i=1}^{r} \beta_i = \lambda + (1 - \lambda) = 1$, implies that $\lambda y + (1 - \lambda) z$ is a convex combination of $y^1, ..., y^n, z^1, ..., z^r \in X$. ■