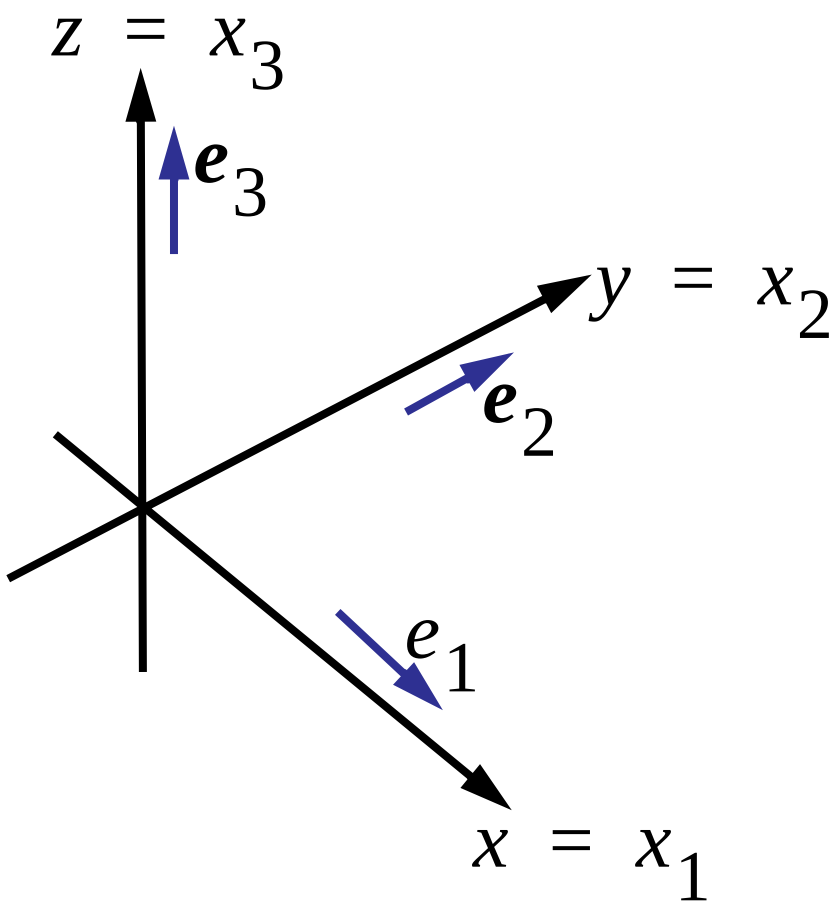
Cartesian Tensors

Reference: H. Jeffreys *Cartesian Tensors*


$z = x_3$
e_3
$y = x_2$
e_2
e_1
$x = x_1$

Coordinates

$$x_i \qquad i = 1, 2, 3$$

$$x_1 = x \qquad x_2 = y \qquad x_3 = z \tag{1}$$

Unit vectors

$$\boldsymbol{e}_i \qquad i = 1, 2, 3$$

$$\boldsymbol{e}_1 = \boldsymbol{e}_x = \boldsymbol{i} \qquad \boldsymbol{e}_2 = \boldsymbol{e}_y = \boldsymbol{j} \qquad \boldsymbol{e}_3 = \boldsymbol{e}_z = \boldsymbol{k} \tag{2}$$

General vector (formal definition to follow) denoted by components e.g. $\boldsymbol{u} = u_i$

Notation

The boldface notation for vectors is referred to as ***dyadic*** notation

The subscript notation is ***tensor*** notation.

Summation convention

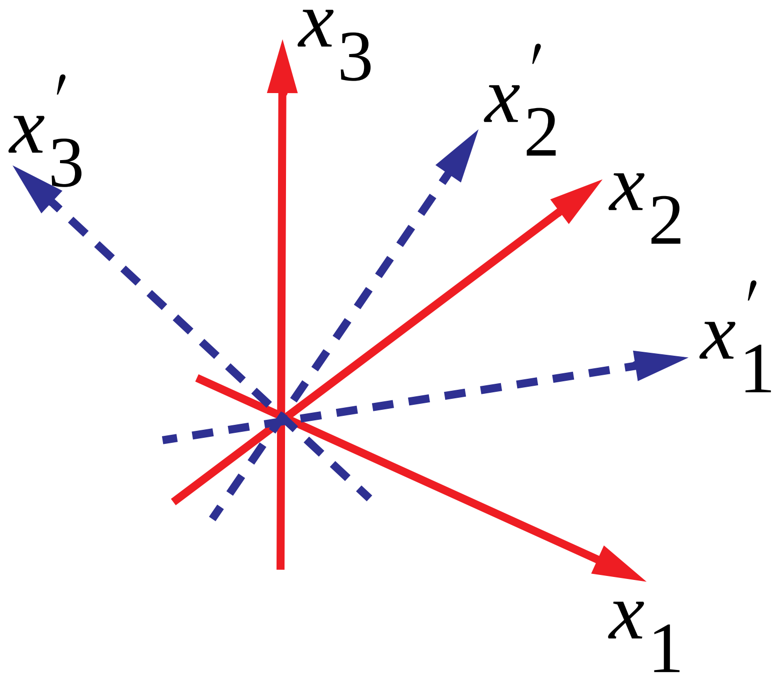
Einstein: repeated index means summation:

$$u_i v_i = \sum_{i=1}^{3} u_i v_i$$
$$u_{ii} = \sum_{i=1}^{3} u_{ii} \qquad (3)$$

2 Orthogonal Transformations of Coordinates

The behaviour of quantities under orthogonal transformations of the coordinate system is the basis of Cartesian tensors.

We want to formulate equations in such a way that they are independent of the specific coordinate system.



General linear transformation

$$x_i' = a_{ij} x_j$$

$$a_{ij} = \text{Transformation Matrix}$$

Position vector

Consider the position vector expressed in terms of coordinates and unit vectors in two related coordinate systems

$$r = x_i \boldsymbol{e}_i = x_i' \boldsymbol{e}_i' \tag{4}$$

In view of the transformation from the unprimed to the primed system:

$$\boldsymbol{r} = a_{ij} x_j \boldsymbol{e}_i' = x_j (a_{ij} \boldsymbol{e}_i') \tag{5}$$

Therefore we can write:

$$\boldsymbol{e}_j = a_{ij} \boldsymbol{e}_i' \tag{6}$$

so that we have the two companion transformations:

$$x_i' = a_{ij} x_j \qquad \boldsymbol{e}_i = a_{ji} \boldsymbol{e}_j' \tag{7}$$

Kronecker Delta

$$\delta_{ij} = 1 \text{ if } i = j$$
$$= 0 \text{ otherwise}$$

In matrix form

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Substitution property

$$\delta_{ij} T_{jk\ldots} = T_{ik\ldots} \tag{9}$$

In the summation over j the only term of the sum that makes any contribution is that for which $j = i$.

2.1 Orthogonal transformation

So far, what we have described is valid for any ***linear transformation.***

Now impose the condition that both the original and the primed reference frames are ***orthonormal***

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \quad \text{and} \quad e_i' \cdot e_j' = \delta_{ij} \tag{10}$$

Use transformation of the unit vectors:

$$\begin{aligned} \boldsymbol{e}_i \cdot \boldsymbol{e}_j &= a_{ki}\boldsymbol{e}'_k \cdot a_{lj}\boldsymbol{e}'_l \\ &= a_{ki}a_{lj}\boldsymbol{e}'_k \cdot \boldsymbol{e}'_l \\ &= a_{ki}a_{lj}\delta_{kl} \\ &= a_{ki}a_{kj} \end{aligned} \quad (11)$$

NB the last operation is an example of the substitution property of the Kronecker Delta.

Since $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$, then the orthonormal condition on a_{ij} is

$$a_{ki}a_{kj} = \delta_{ij} \quad (12)$$

In matrix notation:

$$\boldsymbol{a}^T\boldsymbol{a} = \boldsymbol{I} \tag{13}$$

We also have as a consequence of the properties of matrices, that

$$\boldsymbol{a}\boldsymbol{a}^T = \boldsymbol{I} \tag{14}$$

In tensor notation:

$$(\boldsymbol{a}\boldsymbol{a}^T)_{ij} = a_{ik}a_{jk} = \delta_{ij} \tag{15}$$

Any of equations (12), (13), (14) or (15) defines an orthogonal transformation.

2.2 Reverse transformations

$$x_i' = a_{ij}x_j \Rightarrow a_{ik}x_i' = a_{ik}a_{ij}x_j = \delta_{kj}x_j = x_k$$

$$\therefore x_k = a_{ik}x_i' \Rightarrow x_i = a_{ji}x_j'$$

i.e. the reverse transformation is simply determined by the transpose.

Similarly, following from

$$\boldsymbol{e}_j = a_{ij}\boldsymbol{e}_i' \tag{16}$$

we have

$$\boldsymbol{e}_i' = a_{ij}\boldsymbol{e}_j \tag{17}$$

Transformations for both coordinates and basis vectors

The complementary set of transformations is then

$$x_i' = a_{ij} x_j \qquad \boldsymbol{e}_i' = a_{ij} \boldsymbol{e}_j \tag{18}$$

2.3 Interpretation of the matrix a_{ij}

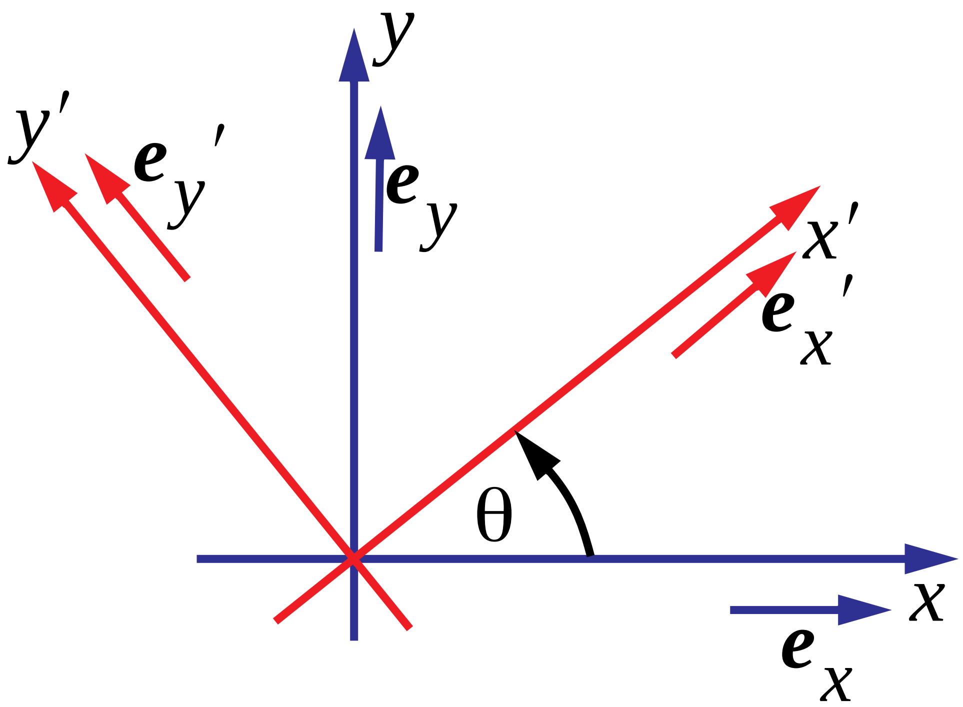
Since

$$\boldsymbol{e}_i' = a_{ij} \boldsymbol{e}_j$$

then the a_{ij} are the components of $\boldsymbol{e}_i'$ wrt the unit vectors in the original system.e.g.

$$\boldsymbol{e}_1' = a_{11}\boldsymbol{e}_1 + a_{12}\boldsymbol{e}_2 + a_{13}\boldsymbol{e}_3 \tag{19}$$

2.4 Example: 2D rotation



It is easiest here to determine the relationship between the unit basis vectors:

$$\begin{aligned} \boldsymbol{e}_x{}' &= \cos\theta\,\boldsymbol{e}_x + \sin\theta\,\boldsymbol{e}_y \\ \boldsymbol{e}_y{}' &= -\sin\theta\,\boldsymbol{e}_x + \cos\theta\,\boldsymbol{e}_y \end{aligned} \tag{20}$$

In matrix form:

$$\begin{bmatrix} e_x{}' \\ \boldsymbol{e}_y{}' \\ \boldsymbol{e}_z{}' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} \tag{21}$$

Then the transformation equation for the coordinates is:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{22}$$

3 Scalars, Vectors & Tensors

We define these objects by the way in which they transform with respect to orthogonal coordinate transformations.

3.1 Scalar (f):

$$f(x'_i) = f(x_i) \tag{23}$$

Example of a scalar is $f = r^2 = x_i x_i$. Examples from fluid dynamics are the density and temperature.

3.2 Vector (u):

Prototype vector: x_i

General transformation law:

$$x_i' = a_{ij} x_j \Rightarrow u_i' = a_{ij} u_j \tag{24}$$

as the transformation law for a generic vector.

3.3 Gradient operator

Suppose that f is a scalar. The gradient of f is defined by

$$(\text{grad } f)_i = (\nabla f)_i = \frac{\partial f}{\partial x_i} \tag{25}$$

Need to show this is a vector by its transformation properties.

$$\frac{\partial f}{\partial x_i'} = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial x_i'} \tag{26}$$

Since,

$$x_j = a_{kj} x_k' \tag{27}$$

then

$$\frac{\partial x_j}{\partial x_i'} = a_{kj}\delta_{ki} = a_{ij}$$

$$\text{and} \quad \frac{\partial f}{\partial x_i'} = a_{ij}\frac{\partial f}{\partial x_j} \tag{28}$$

Hence the gradient operator satisfies our definition of a vector.

Scalar Product

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{29}$$

is the scalar product of the vectors u_i and v_i.

Exercise:

Show that $\boldsymbol{u} \cdot \boldsymbol{v}$ is a scalar.

3.4 Tensor

Prototype second rank tensor $x_i x_j$

General definition by transformation of components:

$$T'_{ij} = a_{ik} a_{jl} T_{kl} \tag{30}$$

Exercise:

Show that $u_i v_j$ is a second rank tensor if u_i and v_j are vectors.

Exercise:

$$u_{i,j} = \frac{\partial u_i}{\partial x_j}$$

is a second rank tensor. (Introduces the comma notation for partial derivatives.) In dyadic form this is written as grad $\boldsymbol{u}$ or $\nabla\boldsymbol{u}$.

3.5 Divergence

Exercise:

Show that the quantity

$$\nabla \cdot v = \text{div } v = \frac{\partial v_i}{\partial x_i} \qquad (31)$$

is a scalar.

4 Products and Contractions of Tensors

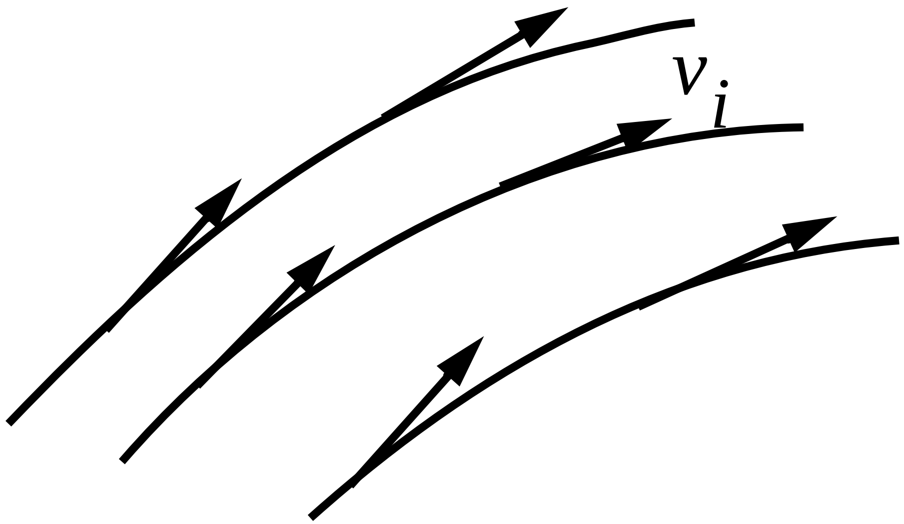
It is easy to form higher order tensors by multiplication of lower rank tensors, e.g. $T_{ijk} = T_{ij}u_k$ is a third rank tensor if T_{ij} is a second rank tensor and u_k is a vector (first rank tensor). It is straightforward to show that T_{ijk} has the relevant transformation properties.

Similarly, if T_{ijk} is a third rank tensor, then T_{ijj} is a vector. Again the relevant transformation properties are easy to prove.

5 Differentiation following the motion

This involves a common operator occurring in fluid dynamics. Suppose the coordinates of an element of fluid are given as a function of time by

$$x_i = x_i(t) \tag{32}$$



The velocities of elements of fluid at all spatial locations within a given region constitute a vector field, i.e. $v_i = v_i(x_j, t)$

If we follow the trajectory of an element of fluid, then on a particular trajectory $x_i = x_i(t)$. The acceleration of an element is then given by:

$$f_i = \frac{dv_i}{dt} = \frac{d}{dt} v_i(x_j(t), t) = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j}\frac{dx_j}{dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \quad (33)$$

Exercise: Show that f_i is a vector.

6 The permutation tensor ε_{ijk}

$$
\begin{aligned}
\varepsilon_{ijk} &= 0 \quad \text{if any of } i, j, k \text{ are equal} \\
&= 1 \quad \text{if } i, j, k \text{ unequal and in cyclic order} \\
&= -1 \quad \text{if } i, j, k \text{ unequal and not in cyclic order}
\end{aligned}
\tag{34}
$$

e.g.

$$
\varepsilon_{112} = 0 \qquad \varepsilon_{123} = 1 \qquad \varepsilon_{321} = -1 \tag{35}
$$

Is ε_{ijk} a tensor?

In order to show this we have to demonstrate that ε_{ijk}, when defined the same way in each coordinate system has the correct transformation properties.

Define

$$
\begin{aligned}
\varepsilon'_{ijk} &= \varepsilon_{lmn} a_{il} a_{jm} a_{kn} \\
&= \varepsilon_{123} a_{i1} a_{j2} a_{k3} + \varepsilon_{312} a_{i3} a_{j1} a_{k2} + \varepsilon_{231} a_{i2} a_{j3} a_{k1} \\
&\quad + \varepsilon_{213} a_{i2} a_{j1} a_{k3} + \varepsilon_{321} a_{i3} a_{j2} a_{k1} + \varepsilon_{132} a_{i1} a_{j3} a_{k2} \\
&= a_{i1}(a_{j2} a_{k3} - a_{j3} a_{k2}) - a_{i2}(a_{j1} a_{k3} - a_{j3} a_{k1}) \\
&\qquad + a_{i3}(a_{j1} a_{k2} - a_{j2} a_{k1}) \\
&= \begin{vmatrix} a_{i1} & a_{i2} & a_{i3} \\ a_{j1} & a_{j2} & a_{j3} \\ a_{k1} & a_{k2} & a_{k3} \end{vmatrix}
\end{aligned}
$$

In view of the interpretation of the a_{ij}, the rows of this determinant represent the components of the primed unit vectors in the unprimed system. Hence:

$$\varepsilon'_{ijk} = \boldsymbol{e}'_i \cdot (\boldsymbol{e}'_j \times \boldsymbol{e}'_k)$$

This is zero if any 2 of i, j, k are equal, is +1 for a cyclic permutation of unequal indices and -1 for a non-cyclic permutation of unequal indices. This is just the definition of ε'_{ijk}. Thus ε_{ijk} transforms as a tensor.

6.1 Uses of the permutation tensor

Cross product

Define

$$c_i = \varepsilon_{ijk} a_j b_k \tag{36}$$

then

$$\begin{aligned} c_1 &= \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2 \\ c_2 &= \varepsilon_{231} a_3 b_1 + \varepsilon_{213} a_1 b_3 = a_3 b_1 - a_1 b_3 \\ c_3 &= \varepsilon_{312} a_1 b_2 + \varepsilon_{321} a_2 b_1 = a_1 b_2 - a_2 b_1 \end{aligned} \tag{37}$$

These are the components of $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$.

6.2 Triple Product

In dyadic notation the triple product of three vectors is:

$$t = \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) \tag{38}$$

In tensor notation this is

$$t = u_i \varepsilon_{ijk} v_j w_k = \varepsilon_{ijk} u_i v_j w_k \tag{39}$$

6.3 Curl

$$(\mathrm{curl}\ \boldsymbol{u})_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \tag{40}$$

e.g.

$$(\text{curl } \boldsymbol{u})_1 = \varepsilon_{123}\frac{\partial u_3}{\partial x_2} + \varepsilon_{132}\frac{\partial u_2}{\partial x_3} = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \quad (41)$$

etc.

6.4 The tensor $\varepsilon_{iks}\varepsilon_{mps}$

Define

$$T_{ikmp} = \varepsilon_{iks}\varepsilon_{mps} \quad (42)$$

Properties:

- If $i = k$ or $m = p$ then $T_{ikmp} = 0$.

- If $i = m$ we only get a contribution from the terms $s \neq i$ and $k \neq i, s$. Consequently $k = p$. Thus $\varepsilon_{iks} = \pm 1$ and $\varepsilon_{mps} = \varepsilon_{iks} = \pm 1$ and the product $\varepsilon_{iks}\varepsilon_{iks} = (\pm 1)^2 = 1$.
- If $i = p$, similar argument tells us that we must have $s \neq i$ and $k = m \neq i$. Hence, $\varepsilon_{iks} = \pm 1$, $\varepsilon_{mps} = \mp 1 \Rightarrow \varepsilon_{iks}\varepsilon_{mps} = -1$.

So,

$$i = m, k = p \Rightarrow 1 \quad \text{unless} \quad i = k \Rightarrow 0$$

$$i = p, k = m \Rightarrow -1 \quad \text{unless} \quad i = k \Rightarrow 0$$

These are the components of the tensor $\delta_{im}\delta_{kp} - \delta_{ip}\delta_{km}$.

$$\therefore \varepsilon_{iks}\varepsilon_{mps} = \delta_{im}\delta_{kp} - \delta_{ip}\delta_{km} \qquad (43)$$

6.5 Application of $\varepsilon_{iks}\varepsilon_{mps}$

$$
\begin{aligned}
[\mathrm{curl}\ (\boldsymbol{u}\times\boldsymbol{v})]_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\varepsilon_{klm}u_l v_m) \\
&= \varepsilon_{ijk}\varepsilon_{klm}\frac{\partial}{\partial x_j}(u_l v_m) \\
&= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\left(\frac{\partial u_l}{\partial x_j}v_m + u_l\frac{\partial v_m}{\partial x_j}\right)
\end{aligned}
\tag{44}
$$

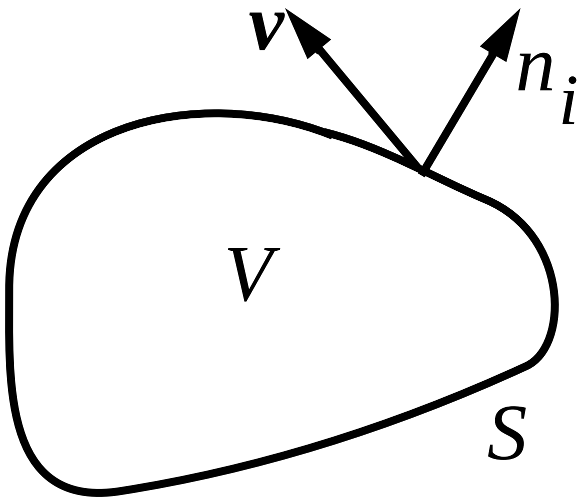
We then use the substitution property of δ_{ij} to show that:

$$
\begin{aligned}
[\mathrm{curl}\ (\boldsymbol{u} \times \boldsymbol{v})]_i &= \frac{\partial u_i}{\partial x_m} v_m - v_i \frac{\partial u_j}{\partial x_j} + u_i \frac{\partial v_m}{\partial x_m} - u_j \frac{\partial v_i}{\partial x_j} \\
&= v_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial v_i}{\partial x_j} + u_i \frac{\partial v_j}{\partial x_j} - v_i \frac{\partial u_j}{\partial x_j} \qquad (45) \\
&= (\boldsymbol{v} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{v} + \boldsymbol{u} \nabla \cdot \boldsymbol{v} - \boldsymbol{v} \nabla \cdot \boldsymbol{u})_i
\end{aligned}
$$

The Laplacean

$$
\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = \frac{\partial^2 \phi}{\partial x_i \partial x_i} \qquad (46)
$$

7.1 Green's Theorem



In dyadic form:

$$\int_V (\nabla \cdot \boldsymbol{v}) dV = \int_S (\boldsymbol{v} \cdot \boldsymbol{n}) dS \qquad (47)$$

In tensor form:

$$\int_V \frac{\partial v_i}{\partial x_i} dV = \int_S v_i n_i dS = S \qquad (48)$$

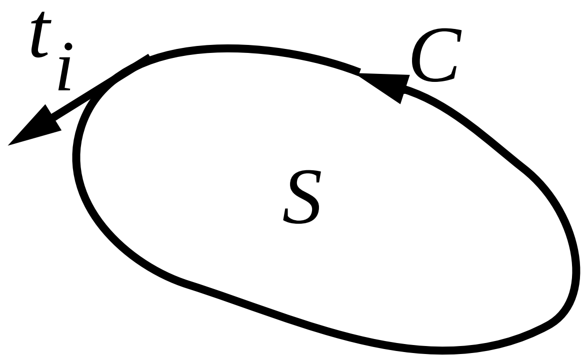
$= \text{Flux of } \boldsymbol{v} \text{ through S}$

Extend this to tensors:

$$\int_V \frac{\partial T_{ij}}{\partial x_j} dV = \int_S T_{ij} n_j dS \tag{49}$$

$$= \text{Flux of } T_{ij} \text{ through } S$$

7.2 Stoke's Theorem



In dyadic form:

$$\int_S (\text{curl } \boldsymbol{u}) \cdot \boldsymbol{n} dS = \int_C \boldsymbol{u} \cdot \boldsymbol{t} ds \qquad (50)$$

In tensor form:

$$\int_S \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} n_i dS = \int_C u_i t_i ds \qquad (51)$$