An ordinal indexed hierarchy of separation properties

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Abstract

We refine and stratify the standard separation properties to produce a descending hierarchy between T_3 and T_1. The interpolated properties are related to the patch properties and the Vietoris modifications of the parent space.

Key words: separation property, stacking property

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Preamble

We take another look at the standard separation properties

$$T_3 \Longrightarrow T_2 \Longrightarrow T_1 \Longrightarrow T_0$$

and show how the step from T_3 to T_2 can be continued below T_2. The new separation properties are related to the patch properties of the parent space, and to the nature of its Vietoris hyperspaces. We attach to each ordinal α three separation properties, α-neat, α-regular, and α-trim, such that

$$\alpha\text{-neat} \Longrightarrow \alpha\text{-regular} \Longrightarrow \alpha\text{-trim} \Longrightarrow (\alpha+1)\text{-neat}$$

for all α. In general, none of these implications is an equivalence. Here we concentrate on the neat and regular properties, but the trim properties are worth noting. For a T_0 space S we have

$$\begin{array}{lcl} S \text{ is 0-neat} & \Longleftrightarrow & S \text{ is empty} \\ S \text{ is 0-regular} & \Longleftrightarrow & S \text{ is } T_3 \\ S \text{ is 1-neat} & \Longleftrightarrow & S \text{ is } T_2 \\ S \text{ is 1-regular} & \Longleftrightarrow & S \text{ is ??} \\ S \text{ is 2-neat} & \Longleftrightarrow & S \text{ is ??} \\ & \vdots & \end{array}$$

where the later properties seem not to have been described before, but they certainly become progressively weaker. For instance, the maximal compact topology, Example 99 in [10], is 1-regular but not T_2. In Section 8 we give a whole family of examples which illustrate the differences between the properties we develop. As the survey [7] shows, many separation properties have been invented. However, in that work none of these seem to be arranged in an ordinal indexed family. More recently two ordinal hierarchies have been described in [8]. One of these is related to, but not the same as, our neat hierarchy. When we were doing the work for this paper we were unaware of [8]. We say more about the relationship between the two hierarchies at the end of Section 4.

We attach to each compact saturated subset Q of a space S a certain operation, a derivative ∂_Q, on the family $\mathcal{C}S$ of closed sets of S. (At least that is what we do when S is sober. For a non-sober space we use a more general approach.) This operation can be iterated through the ordinals, and eventually stabilizes. The length of the iteration gives an ordinal rank α, and it is this that is used in the hierarchies.

We find that neatness, being α-neat for some α, decomposes into two properties.

$$\text{neat} = \text{packed} + \text{stacked}$$

The first, being packed, is well known but unnamed, and is concerned with the behaviour of the patch space of the parent space. The second, being stacked, is concerned with the behaviour of certain Vietoris hyperspaces of the parent space. It is the stacking properties that are stratified by the ordinals.

These results are taken from [9], mainly Chapters 8 and 11. That account is written from a point-free perspective. However, to make this account accessible to a wider readership we develop the material in a point-sensitive way. (We do make the occasional remark about the point-free approach, but these can be ignored if you wish.)

Contents

1 Introduction

We outline the aims of this paper, and describe where the results come from. As mentioned in the preamble, originally we used point-free methods (and, in part, obtained more general results). Here we use only point-sensitive methods. However, in this introduction it will be necessary to allude to the point-free approach. If you are not familiar with these methods, it will not detract from your understanding of the results presented later.

Let S be a topological space. This has families $\mathcal{O}S$ of open subsets and $\mathcal{C}S$ of closed subsets. Typically we let

$$U, V, W \ldots \text{range over } \mathcal{O}S \qquad X, Y, Z \ldots \text{range over } \mathcal{C}S$$

and we write E° for the interior and E^- for the closure of an arbitrary subset $E \subseteq S$.

We use the standard separation properties T_3, T_2, T_1, T_0 where $T_3 = T_0$+regular. As explained in the preamble, our aim is to show that T_3 and T_2 are the initial steps in a hierarchy which descends to somewhere above T_1.

The specialization order on a space S is the comparison of points given by

$$p \leq q \Longleftrightarrow p \in q^-$$

for $p, q \in S$. This is always a pre-order, is a partial order precisely when the space is T_0, and is equality precisely when the space is T_1.

A saturated set is an upper section of this comparison. Every open set is saturated but there may be many non-open saturated sets. We write $E^{\uparrow}$ for the saturation (upwards closure) of a subset $E \subseteq S$.

Let $\mathcal{Q}S$ be the family of compact saturated sets. We use Q as a typical member of $\mathcal{Q}S$. Observe that the saturation $K^{\uparrow}$ of a each compact set K is in $\mathcal{Q}S$. Much of what we do can be seen as an analysis of the way $\mathcal{Q}S$ influences the more general properties of S.

It is a standard exercise that for a T_2 space S we have $\mathcal{Q}S \subseteq \mathcal{C}S$. For later we state this in the form of a separation property and sketch the proof.

1.1 LEMMA. *Let S be a T_2 space and consider $p \in S$ and $Q \in \mathcal{Q}S$ with $p \notin Q$. Then*

$$p \in U \qquad Q \subseteq V \qquad U \cap V = \emptyset$$

for some $U, V \in \mathcal{O}S$.

Proof. Fix $p \notin Q \in \mathcal{Q}S$. For each $q \in Q$ the T_2 separation gives

$$p \in U_q \qquad q \in V_q \qquad U_q \cap V_q = \emptyset$$

for some $U_q, V_q \in \mathcal{O}S$. Letting q vary through Q produces an open cover of Q which, by the compactness, refines to a finite cover, and so produces the required $U, V \in \mathcal{O}S$. ■

This suggests that in a T_2 space S the sets $Q \in \mathcal{Q}S$ are trying to behave like points. We remember this when we discuss the V-modifications (Vietoris hyperspaces) of a space.

The property $\mathcal{Q}S \subseteq \mathcal{C}S$ of a space doesn't seem to have a name, so we give it one.

1.2 DEFINITION. A space S is packed if each compact saturated set is closed.

A space S is tightly packed if each compact saturated set is closed and finite. ■

Another property of T_2 spaces does have a name.

1.3 DEFINITION. For a space S a closed subset Z is irreducible if it is non-empty and

$$(Z \text{ meets } U) \text{ and } (Z \text{ meets } V) \Longrightarrow Z \text{ meets } U \cap V$$

for each $U, V \in \mathcal{O}S$. For each $p \in S$ the closure $Z = p^- = \{p\}^-$ is is irreducible, and we say p is a generic point of Z. A space S is sober if it is T_0 and each of its closed irreducible sets has a generic point (which, by the T_0 property, is unique). ■

Almost trivially both the implications

$$(1) \quad T_2 \Longrightarrow T_1\text{+sober+packed} \qquad\qquad T_0\text{+packed} \Longrightarrow T_1 \quad (2)$$

hold. Not so trivially both are strict. An appropriate counter-example for (1) can be found in [10]. (This book doesn't deal explicitly with sobriety and packedness. The dissertation [6] fills in some of the missing details.) The two conditions T_1 and sobriety are incomparable, a space can have one without the other. Notice that a space is T_1+sober precisely when each closed irreducible set is a singleton. Such a space need not be T_2.

Both sobriety and packedness can be viewed as desirable properties of a space. Thus each T_2 space is acceptable, but there are defective spaces. As an attempt to correct one or other of the defects we can attach to a space S an appropriate space

$$\sigma : S \longrightarrow {}^{s}S \qquad \pi : {}^{p}S \longrightarrow S$$

using a continuous map σ or π. The left hand space ${}^{s}S$ is the sober reflection of S. For a T_0 space its points are the closed irreducible subsets of S with 'essentially the same' topology. The space ${}^{s}S$ is always sober and, in particular, S is sober precisely when σ is a homeomorphism. The right hand space ${}^{p}S$ is the patch space of S. This has the same points as S but with more open sets. (We simply declare that each $Q \in \mathcal{Q}S$ is now closed.) In particular, S is packed precisely when π is a homeomorphism. Unfortunately, in general, the space ${}^{p}S$ need not be packed, and the construction has to be repeated.

Both these constructions use point-sensitive methods (that is the standard methods of point set topology). There are also point-free methods available. We need to say a few words about these without going into any details.

Let ***Top*** be the category of topological spaces and continuous maps. This is connected with another category ***Frm*** of a more algebraic nature. A frame is a certain kind of complete lattice. These are the objects of ***Frm*** and the arrows are the appropriate morphisms. There is a pair of contravariant functors connecting ***Top*** and ***Frm***. For a space S the topology $\mathcal{O}S$ is a frame. Each frame A has a point space $\mathsf{pt}(A)$ in ***Top*** (obtained by a kind of spectral construction) together with a surjective morphism

$$A \longrightarrow \mathcal{O}\mathsf{pt}(A)$$

which need not be an isomorphism.

Many constructions in ***Top*** can be mimicked in ***Frm***. Sometimes this gives essentially the same results, sometimes it gives better results, and sometimes it just misses the point. For instance, by converting a space S into its topology $\mathcal{O}S$ and then taking the point space $\mathsf{pt}(\mathcal{O}S)$ we obtain the sober reflection of S. Thus $\sigma : S \longrightarrow {}^{s}S$ is one of the units of the contravariant adjunction between ***Top*** and ***Frm***.

There is also a point-free version of the patch construction. For a sober space S there is a bijective correspondence between $\mathcal{Q}S$ and the Scott-open filters on $\mathcal{O}S$. This correspondence is discussed in Section 3. For any frame A there is a process of modifying A by 'formally adjoining' its Scott-open filters. This produces a larger frame

$$A \longrightarrow PA$$

and an embedding. The point set content of the construction is explained in Section 2.

We apply this construction to the topology of a space S to obtain a pair of morphisms

$$f : \mathcal{O}S \longrightarrow P\mathcal{O}S \qquad g : P\mathcal{O}S \longrightarrow \mathcal{O}{}^{p}S$$

in ***Frm***. The composite is essentially the map ${}^{p}S \longrightarrow S$ viewed as a frame morphism. In particular, when S is packed, that is ${}^{p}S = S$, we obtain a pair of morphisms

$$f : \mathcal{O}S \longrightarrow P\mathcal{O}S \qquad g : P\mathcal{O}S \longrightarrow \mathcal{O}S$$

where the composite $g \circ f$ is the identity on $\mathcal{O}S$. The other composite $f \circ g$ need not be the identity on $P\mathcal{O}S$. An analysis of this led to the result presented here.

Roughly speaking we say a space is neat if the embedding f is an isomorphism. The topological content of this notion is discussed in Section 4. We find that

$$(3) \quad T_0\text{+neat} \Longrightarrow T_1\text{+sober+packed}$$

but this is not an equivalence.

What topological properties ensure neatness? From III(1.2)(iii) of [5] we have

$$T_3 \Longrightarrow T_0\text{+neat}$$

(and, in fact, a more general point-free result). We improved this by weakening the hypothesis to T_2, and this with (3) gives (1) above. An analysis of neatness led to the ordinal indexed hierarchy of separation properties, as described in Section 4 and 5.

The implication (3) is not an equivalence. What more is needed on the right hand side? The missing property, that of being stacked, is discussed in Section 6. It turns out that this is concerned with the nature of the V-modifications of the parent space.

For a space S the point-sensitive Vietoris hyperspaces use certain collections of subsets as points. The most common collection is $\mathcal{Q}S$, but there are other larger collections. There is also a point-free version of this construction, originally described in [3, 4]. This produces an even larger set $\mathcal{V}S$ of points which, in general, are not just certain subsets of S. The stacking property is concerned with the differences between these various V-modifications. The background to these results is described in more detail in Section 7.

Section 8 contains a collection of examples which illustrate the notions developed here.

2 Derivatives on a space

At first sight you may think the following gadgets look a little odd.

2.1 DEFINITION. A derivative on a space S is an operator ∂ on $\mathcal{C}S$ which is deflationary, monotone, and respects joins, that is

$$(d) \quad \partial(X) \subseteq X \qquad (m) \quad Y \subseteq X \Longrightarrow \partial(Y) \subseteq \partial(X) \qquad (j) \quad \partial(Y \cup X) \subseteq \partial(Y) \cup \partial(X)$$

for each $X, Y \in \mathcal{C}S$. Because of (m) the comparison of (j) is, in fact, an equality. ■

There is one very well known example of a derivative that is worth remembering.

2.2 EXAMPLE. Let S be a T_0 space. Recall that a point $p \in X \in \mathcal{C}S$ is isolated in the closed set if $X \cap U = \{p\}$ for some $U \in \mathcal{O}S$. Let $\boldsymbol{lim}(X)$ be the set of non-isolated points of X, the limit points of X. This $\boldsymbol{lim}$ is the CB-derivative on $\mathcal{C}S$. ■

We compare derivatives in a pointwise fashion, that is

$$\partial_1 \leq \partial_2 \Longleftrightarrow (\forall X \in \mathcal{C}S)[\partial_1(X) \subseteq \partial_2(X)]$$

for derivatives ∂_1, ∂_2 (on the same space). Notice also that the composite $\partial_1 \circ \partial_2$ of two derivatives is itself a derivative, and is smaller than its two components.

The CB-derivative is used to extract the perfect part of a closed set. This is done by iteration, and a similar process is available with any derivative.

2.3 DEFINITION. Let ∂ be any derivative on a space S. The family

$$(\partial^{\alpha} \mid \alpha \in \mathrm{Ord})$$

of ordinal iterates of ∂ is generated by

$$\partial^{0}(X) = X \qquad \partial^{\alpha+1}(X) = \partial(\partial^{\alpha}(X)) \qquad \partial^{\lambda}(X) = \bigcap\{\partial^{\alpha}(X) \mid \alpha < \lambda\}$$

for each $X \in \mathcal{C}S$, each ordinal α, and each limit ordinal λ. ■

Notice that each ∂^{α} is a derivative. Thus we obtain a descending chain of derivatives

$$\partial^{0} \geq \partial^{1} = \partial \geq \partial^{2} \geq \cdots \geq \partial^{\alpha} \geq \cdots$$

which, on cardinality grounds, must eventually stabilize. In other words, there is a sufficiently large ordinal $\theta = |\partial|$ such that $\partial \circ \partial^{\theta} = \partial^{\theta}$. This is the rank or closure ordinal of ∂, and in due course, we will look at the size of this for some particular derivatives. Until then it is convenient to write '∞' for this closure ordinal. Of course, the value of ∞ depends on the parent derivative ∂, but when this matters we will be more precise.

For instance $\boldsymbol{lim}^{\infty}$ is the perfect part extractor on the parent space S. The size of this ∞ is an indicator of some of the pathological properties of S. In a similar way we will use the rank of other derivatives to determine other properties of S.

2.4 EXAMPLES. Let S be an arbitrary space. For an arbitrary subset K of S, setting

$$\langle K\rangle(X) = (K \cap X)^{-}$$

for each $X \in \mathcal{C}S$ produces an idempotent derivative $\langle K\rangle$ on $\mathcal{C}S$. In particular, when K is closed we have $\langle K\rangle(X) = K \cap X$ for each $X \in \mathcal{C}S$. ■

For a derivative ∂ the set $K = \partial^{\infty}(S)$ is closed and we find that $\partial^{\infty} \leq \langle K\rangle$. We look at certain derivatives ∂ such that $\langle L\rangle \leq \partial^{\infty}$ for some associated saturated set L.

The topology $\mathcal{O}S$ of a space S is a lattice, and so the notion of a filter ∇ on $\mathcal{O}S$ makes sense. Such a filter ∇ is a family of open sets of S, is closed under binary intersections, and is upwards closed. (Do not confuse this notion of filter with the more usual notion of filter used in point set topology. The two notions are related but are not the same.)

2.5 DEFINITION. For a space S and a filter ∇ on $\mathcal{O}S$, we set

$$\partial_{\nabla}(X) = \bigcap\{(W \cap X)^{-} \mid W \in \nabla\}$$

for each $X \in \mathcal{C}S$ to obtain the associated derivative ∂_{∇} of ∇. ■

The fact that ∂_{∇} is deflationary and monotone is immediate, but the preservation of joins needs to be checked. By way of contradiction suppose

$$\partial_{\nabla}(X \cup Y) \nsubseteq \partial_{\nabla}(X) \cup \partial_{\nabla}(Y)$$

for some $X, Y \in \mathcal{C}S$. Thus

$$p \in \partial_{\nabla}(X \cup Y) \quad p \in (U \cap X)^{-\prime} \quad p \in (V \cap Y)^{-\prime}$$

for some point $p \in S$ and $U, V \in \nabla$. Let $W = U \cap V$ so that $W \in \nabla$ and

$$p \in (W \cap (X \cup Y))^{-} \quad p \in (W \cap X)^{-\prime} \quad p \in (W \cap Y)^{-\prime}$$

which leads to a contradiction.

Each such derivative ∂_∇ has a closure ordinal $\infty = |\nabla|$ which enables us to attach an ordinal rank to the parent filter. For later use we refine this process.

2.6 DEFINITION. Let ∇ be a filter on the open sets $\mathcal{O}S$ of the space S. We set

$$\nabla(0) = S \quad \nabla(\alpha+1) = \partial_\nabla(\nabla(\alpha)) \quad \nabla(\lambda) = \bigcap\{\nabla(\alpha) \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ. In other words we set $\nabla(\alpha) = \partial_\nabla^\alpha(S)$ for each ordinal α. This produces a descending chain of closed sets

$$S = \nabla(0) \supseteq \nabla(1) \supseteq \cdots \supseteq \nabla(\alpha) \supseteq \cdots$$

which stabilizes at (or before) $\nabla(\infty) = \partial_\nabla^\infty(S)$ using the closure ordinal of ∂_∇. ■

For each filter ∇ of $\mathcal{O}S$ the set $\nabla(\infty)$ is closed and $\partial^\infty \leq \langle \nabla(\infty) \rangle$. The intersection $\bigcap \nabla$ is saturated since each member of ∇ is open and hence saturated. A simple calculation shows that

$$\bigcap \nabla \subseteq X \Longrightarrow \bigcap \nabla \subseteq \partial_\nabla(X)$$

for each closed set X, and hence $\bigcap \nabla \subseteq \nabla(\infty) = \partial_\nabla^\infty(S)$. We can refine this comparison.

2.7 LEMMA. *For each space S and filter ∇ on $\mathcal{O}S$ we have*

$$\langle \bigcap \nabla \rangle \leq \partial_\nabla^\infty \leq \langle \nabla(\infty) \rangle$$

the sandwich of the associated derivative ∂_∇^∞.

Proof. For convenience let $\partial = \partial_\nabla$. By the remarks above we have $\partial^\infty \leq \langle \nabla(\infty) \rangle$. To obtain the lower bound of ∂^∞ let $L = \bigcap \nabla$. For each $W \in \nabla$ and $X \in \mathcal{C}S$ we have $L \cap X \subseteq W \cap X$ and hence $\langle L \rangle(X) \subseteq \partial(X)$ holds. A simple induction shows $\langle L \rangle(X) \subseteq \partial^\alpha(X)$ for each ordinal α, and hence $\langle L \rangle \leq \partial^\infty$ as required. ■

In general ∂_∇^∞ will lie strictly between these two extremes.

2.8 EXAMPLE. Let S be a T_1 space and let ∇ be the filter of cofinite sets (each of which is open). Then $\partial_\nabla = \boldsymbol{lim}$ and hence ∂_∇^∞ is the perfect part extractor. In particular, $\nabla(\infty)$ is the perfect part of S, which could be S. At the other extreme we have $\bigcap \nabla = \emptyset$. In other words, these two extremes tell us almost nothing about the parent space. ■

We look at a certain filters for which $\bigcap \nabla = \nabla(\infty)$, and so the sandwich collapses.

3 Scott-open filters

We describe a slight refinement of the Hofmann-Mislove result which relates Scott-open filters with compact saturated sets.

Recall that a family $\mathcal{U}$ of open sets is (upwards) directed if it is non-empty and for each $U, V \in \mathcal{U}$ there is some $W \in \mathcal{U}$ with $U \cup V \subseteq W$.

3.1 DEFINITION. For a space S, a filter ∇ on the topology $\mathcal{O}S$ is Scott-open if

$$\bigcup \mathcal{U} \in \nabla \Longrightarrow \mathcal{U} \cap \nabla \neq \emptyset$$

for each directed family $\mathcal{U} \subseteq \mathcal{O}S$. In other words, ∇ is Scott-open when its complement $\mathcal{O}S - \nabla$ is closed under directed unions. ■

Being Scott-open is a compactness property, as the standard example illustrates.

3.2 EXAMPLES. Let S be a T_0 space.

(a) Let $Q \in \mathcal{Q}S$. The open neighbourhood filter $\nabla(Q)$ of Q given by

$$U \in \nabla(Q) \Longleftrightarrow Q \subseteq U$$

(for $U \in \mathcal{O}S$) is Scott-open. Since Q is saturated we find that $Q = \bigcap \nabla(Q)$.

(b) Suppose Z is closed irreducible in S. Then

$$U \in \nabla \Longleftrightarrow Z \text{ meets } U$$

(for $U \in \mathcal{O}S$) defines a Scott-open filter. In fact, this ∇ satisfies $\mathcal{U} \cap \nabla \neq \emptyset$ for every (not just directed) family $\mathcal{U}$ of open sets with $\bigcup \mathcal{U} \in \nabla$. The space is sober precisely when each such Z is a point closure q^{-}, in which case $\nabla = \nabla(q^{\uparrow})$. ■

When the space S is sober the Hofmann-Mislove result shows that each Scott-open filter ∇ on $\mathcal{O}S$ has the form $\nabla(Q)$ for a (unique) $Q \in \mathcal{Q}S$, namely $Q = \bigcap \nabla$. For later use we need to extract a bit more information out of a proof of this result. To do that we set up a bit of notation which we carry through to the end of Theorem 3.5.

Let S be a T_0 space (which, as yet, need not be sober), and let ∇ be a Scott-open filter on $\mathcal{O}S$. Let $\mathcal{M}$ be the set of maximal members of $(\mathcal{O}S - \nabla)$. Since this difference is closed under directed unions an application of Zorn's Lemma gives the following.

3.3 LEMMA. *For each $U \in (\mathcal{O}S - \nabla)$ there is some $P \in \mathcal{M}$ with $U \subseteq P$.*

It often happens that a use of Zorn's Lemma is combined with a proof that the extracted maximal elements are 'prime' in some sense. The same thing happens here.

3.4 LEMMA. *Each member of $\mathcal{M}$ is the complement of a closed irreducible set.*

We now assume that S is sober. Thus each $P \in \mathcal{M}$ has the form $m^{-\prime}$ for some (unique) $m \in S$. Let M be the set of these points. Thus

$$\mathcal{M} = \{m^{-\prime} \mid m \in M\}$$

is the set of maximal members of $\mathcal{O}S - \nabla$.

3.5 THEOREM. *Suppose the space S is sober and let ∇ be a Scott-open filter on $\mathcal{O}S$. Then there is some $Q \in \mathcal{Q}S$ with $\nabla = \nabla(Q)$. Furthermore, $Q = \bigcap \nabla$.*

Proof. Consider the set $M \subseteq S$ extracted above. We first show

$$U \in \nabla \Longleftrightarrow M \subseteq U$$

for $U \in \mathcal{O}S$. In fact, we show the contrapositive.

Suppose $U \in (\mathcal{O}S - \nabla)$. By Lemma 3.3 and the construction of M we have $U \subseteq m^{-\prime}$ for some $m \in M$. In particular $M \nsubseteq U$. Conversely, suppose $M \nsubseteq U$, so that $m \in U'$ for some $m \in M$, to give $U \subseteq m^{-\prime} \notin \nabla$, and hence $U \notin \nabla$.

Next we check that M is compact. To this end suppose $M \subseteq \bigcup \mathcal{U}$ where $\mathcal{U}$ is a directed family of open sets. If $\bigcup \mathcal{U} \notin \nabla$ then, again by Lemma 3.3, we have

$$M \subseteq \bigcup \mathcal{U} \subseteq m^{-\prime}$$

for some $m \in M$. This is impossible so we have $\bigcup \mathcal{U} \in \nabla$, and hence, since ∇ is Scott-open, there is some $U \in \mathcal{U}$ with $U \in \mathcal{U}$. The equivalence above ensures $M \subseteq U$.

Each open set is saturated, so $Q = M^{\uparrow}$ is a compact saturated set with

$$U \in \nabla \Longleftrightarrow Q \subseteq U$$

for $U \in \mathcal{O}S$. Trivially we have $Q \subseteq \bigcap \nabla$, so it remains to verify the converse inclusion.

Consider any $p \in \bigcap \nabla$. Then $p^{-\prime} \notin \nabla$ (otherwise $p \in p^{-\prime}$), so that $Q \nsubseteq p^{-\prime}$, and hence there is some $q \in Q \cap p^{-}$. We have $q \leq p$, so that $p \in Q$, since Q is saturated. ■

If we start from a compact saturated set, then we obtain the following result.

3.6 THEOREM. *Suppose the space S is sober, and consider $Q \in \mathcal{Q}S$. Let M be the set of minimal members of Q (relative to the specialization order). Then for each $q \in Q$ there is some $m \in M$ with $m \leq q$, the set M is compact, and $M^{\uparrow} = Q$.*

We call the M the minimal generating set of Q. It will be important in Section 7.

4 Neat spaces

By Lemma 2.7 we have a sandwich

$$\langle \bigcap \nabla \rangle \leq \partial_{\nabla}^{\infty} \leq \langle \nabla(\infty) \rangle$$

for each filter ∇ on the topology $\mathcal{O}S$ of a space S. By Example 2.8 these three components can be distinct. We investigate certain filters where the components can be the same.

The following result is essentially a point-sensitive version of Lemma 2.4(ii) of [3] or Lemma 3.4(ii) of [4]. We give a proof since the technique used here is important.

4.1 LEMMA. *Let ∇ be a Scott-open filter on the topology $\mathcal{O}S$ of the space S. Then*

$$\nabla(\infty) \subseteq U \Longrightarrow U \in \nabla$$

for each $U \in \mathcal{O}S$.

Proof. We approach the result in four phases. In each we use the derivative $\partial = \partial_\nabla$. For the first phase we show

$$\partial(X)' \in \nabla \Longrightarrow X' \in \nabla$$

for $X \in \mathcal{C}S$. Thus suppose $\partial(X)' \in \nabla$. Then

$$\bigcup\{(W \cap X)^{-\prime} \mid W \in \nabla\} = \partial(X)' \in \nabla$$

and the left hand union is directed. Thus, since ∇ is Scott-open we have

$$(W' \cup X')^\circ = (W \cap X)^{-\prime} \in \nabla$$

for some $W \in \nabla$. This gives

$$(W' \cup X')^\circ \cap W \in \nabla$$

which, since

$$(W' \cup X')^\circ \cap W = ((W' \cup X') \cap W)^\circ = X' \cap W$$

gives $X' \cap W \in \nabla$, and hence $X' \in \nabla$.

For the second phase we show

$$\partial^\alpha(X)' \in \nabla \Longrightarrow X' \in \nabla$$

for $X \in \mathcal{C}S$ and each ordinal α. We proceed by induction on α. The base case, $\alpha = 0$, is trivial. For the induction step, $\alpha \mapsto \alpha + 1$, since $\partial^{\alpha+1} = \partial \circ \partial^\alpha$ we have

$$\partial^{\alpha+1}(X)' \in \nabla \Longrightarrow \partial^\alpha(X)' \in \nabla \Longrightarrow X' \in \nabla$$

using the observation of the first phase and the induction hypothesis. The induction leap to a limit ordinal follows by another use of the Scott-openness.

For the third phase we observe that

$$\partial^\infty(U') = \emptyset \Longleftrightarrow U \in \nabla$$

for $U \in \mathcal{O}S$. The implication $\Rightarrow$ is a particular instance of the second phase, and the implication $\Leftarrow$ is immediate (since $U \in \nabla$ gives $\partial(U') \subseteq (U \cap U')^-$).

Finally, we can obtain the required result. Thus suppose $\nabla(\infty) \subseteq U$ for some $U \in \mathcal{O}S$. Then we have both

$$\partial^\infty(U') \subseteq \nabla(\infty) \subseteq U \qquad \partial^\infty(U') \subseteq U'$$

to give $\partial^\infty(U') \subseteq U \cap U' = \emptyset$ and hence $U \in \nabla$ by the third phase. ■

With this observation we can characterize when the sandwich collapses.

4.2 THEOREM. *Let ∇ be a Scott-open filter on the topology $\mathcal{O}S$ of a space S, and let $\partial = \partial_\nabla$. The following four conditions on ∇ are equivalent.*

(i) $\partial^\infty = \langle K \rangle$ *for some* $K \in \mathcal{C}S$.

(ii) $(\forall U \in \mathcal{O}S)[U \in \nabla \Longleftrightarrow \nabla(\infty) \subseteq U]$

(iii) $\bigcap \nabla = \nabla(\infty)$

(iv) $\langle \bigcap \nabla \rangle = \partial^\infty = \langle \nabla(\infty) \rangle$

Proof. (i)⇒(ii). Suppose $\partial^\infty = \langle K \rangle$ for some $K \in \mathcal{C}S$. Then

$$K = \langle K \rangle(S) = \partial^\infty(S) = \nabla(\infty)$$

to show what K must be. Consider any $U \in \nabla$. Then

$$\partial(U') \subseteq (U \cap U')^- = \emptyset$$

so that

$$\nabla(\infty) \cap U' = \langle K \rangle(U') = \partial^\infty(U') = \emptyset$$

to give $\nabla(\infty) \subseteq U$. The converse implication holds by Lemma 4.1.

(ii)⇒(iii). Assuming (ii) we have $\nabla(\infty) \subseteq \bigcap \nabla$, and the converse always holds.

(iii)⇒(iv). This follows from the sandwich of ∂^∞.

(iv)⇒(i). This is trivial. ■

For a Scott-open filter ∇ the sandwich collapses precisely when $\nabla(\infty) \subseteq \bigcap \nabla$. Here ∞ is some ordinal which depends on ∇. We take note of this value.

4.3 DEFINITION. Let S be a space and let α be an ordinal.

(a) A Scott-open filter ∇ on $\mathcal{O}S$ is α-**neat** if $\nabla(\alpha) \subseteq \bigcap \nabla$, and hence $\nabla(\alpha) = \bigcap \nabla$.

(b) The space is α-**neat** if each of its Scott-open filters is α-neat.

(c) A Scott-open filter or a space is **neat** if it is α-neat for some α. ■

For ordinals $\alpha \leq \beta$ we have $\nabla(\alpha) \subseteq \nabla(\beta)$ so that

$$\alpha\text{-neat} \Longrightarrow \beta\text{-neat}$$

for both Scott-open filters and spaces.

4.4 THEOREM. *(0) A space is 0-neat precisely when it is empty.*

(1) A T_0 space is 1-neat precisely when it is T_2.

Proof. (0) Suppose the space S is 0-neat, and consider the improper filter $\nabla = \mathcal{O}S$. This is Scott-open (trivially) and hence the 0-neatness gives $S = \nabla(0) \subseteq \bigcap \nabla = \emptyset$. The converse is immediate.

(1) Suppose the space S is T_0 and 1-neat, and consider distinct $p, q \in S$. Since S is T_0 we may suppose $q \notin p^-$, so that $p \notin q^\uparrow \in \mathcal{Q}S$. Using the Scott-open filter $\nabla = \nabla(q^\uparrow)$ the 1-neatness gives

$$\bigcap \{W^- \mid q \in W \in \mathcal{O}S\} = \nabla(1) \subseteq \bigcap \nabla = q^\uparrow$$

so there is an open set W with $p \notin W^-$ and $q \in W$. This is the required T_2-separation.

Conversely, suppose the space is T_2, and consider any Scott-open filter ∇. Since S is T_2, it is sober, and hence Theorem 3.5 gives us some $Q \in \mathcal{Q}S$ with $\nabla = \nabla(Q)$ and $\bigcap \nabla = Q$. We have $\bigcap \nabla = Q \subseteq \nabla(1)$, and we must improve this to an equality.

By way of contradiction, suppose there is a point $p \in (\nabla(1) - Q)$. By Lemma 1.1 there are $U, V \in \mathcal{O}S$ with

$$p \in U \qquad Q \subseteq V \qquad U \cap V = \emptyset$$

so that $p \in \nabla(1) \subseteq V^- \subseteq U'$ which is the contradiction. ■

As far as we can ascertain 2-neatness (or any $(2+\alpha)$-neatness) does not seem to be a standard separation property.

What kind of spaces are neat? The following gives a necessary condition.

4.5 THEOREM. *Each T_0+neat space is T_1+sober+packed.*

Proof. Let S be a space that is T_0 and neat.

We show first that S is packed. Consider $Q \in \mathcal{Q}S$, and let $\nabla = \nabla(Q)$. Then

$$Q \subseteq \nabla(\infty) \subseteq \bigcap \nabla = Q$$

where the first inclusion is a general property of $\mathcal{Q}S$, the second is the assumed neatness, and the equality is another general property. This gives $Q = \nabla(\infty) \in \mathcal{C}S$ as required.

Since S is T_0+packed, it is T_1. Thus it remains to show that S is sober.

To this end consider any closed irreducible subset Z of S, and this time let ∇ be the Scott-open filter given by

$$U \in \nabla \iff Z \text{ meets } U$$

(for $U \in \mathcal{O}S$). We show that Z is a singleton.

By way of contradiction suppose p, q are distinct members of Z. Since S is T_1 we have $p \in U$ and $q \notin U$ for some $U \in \mathcal{O}S$. The point p ensures that Z meets U, so that $U \in \nabla$, and hence $\nabla(\infty) \subseteq U$ by the assumed neatness of S. Since

$$q \in U' \subseteq \nabla(\infty)' \in \mathcal{O}S$$

we see that Z meets $\nabla(\infty)$, and hence $\nabla(\infty) \subseteq \nabla(\infty)'$ by a second use of neatness. Thus $\nabla(\infty) = \emptyset$, so that $\emptyset \in \nabla$ by Lemma 4.1, and hence Z meets $\emptyset$. ■

We will improve this result in Section 6.

Before we continue with our analysis let's briefly consider a possible weakening of the notion of α-neatness.

Suppose we attach to each space S a selected family $\mathcal{F}S$ of Scott-open filters. Then we may say a space S is α-$\mathcal{F}$-neat if each $\nabla \in \mathcal{F}S$ is α-neat. We use a particular case of this idea.

For each point $q \in S$ the filter $\nabla(q^\uparrow)$ given by

$$U \in \nabla(q^\uparrow) \iff q^\uparrow \subseteq U \iff q \in U$$

(for $U \in \mathcal{O}S$) is Scott-open with $\bigcap \nabla(q^\uparrow) = q^\uparrow$. This filter gives us a derivative

$$\partial_q = \partial_{\nabla(q^\uparrow)}$$

on $\mathcal{C}S$, and we may say S is α-**point-neat** if $\partial_q^\alpha(S) = q^\uparrow$ for each point $q \in S$. This notion has appeared elsewhere.

In [8] Sequeira attached to each point q of a space S an ordinal indexed family $\Delta_\alpha(q)$ of subsets of S. The space S is α-**step-Hausdorff** if $\Delta_\alpha(q) = \{q\}$ for each $q \in S$. In fact

$$\Delta_\alpha(q) = \partial_q^\alpha(S)$$

and it doesn't take too long to prove the following.

4.6 THEOREM. *A space is α-step-Hausdorff precisely when it is T_0 and α-point-neat. Furthermore, each such space is T_1.*

A more detailed analysis of the connection between neatness and the properties discussed in [8] will appear elsewhere.

5 Three interlacing hierarchies

We know that each α-neat space is $(\alpha+1)$-neat, but not conversely. What does an α-neat space have that an $(\alpha+1)$-neat doesn't? It has a certain amount of regularity.

5.1 DEFINITION. Let α be an ordinal.

(r) A space S is α-regular if for each $U, V \in \mathcal{O}S$ with $U \not\subseteq V$, there is some $W \in \mathcal{O}S$ such that

$$(*) \quad W \subseteq U \qquad W \not\subseteq V \qquad (W \cap \nabla(\alpha))^{-} \subseteq U$$

for each Scott-open filter ∇ with $U \in \nabla$.

(t) A space S is α-trim if for each $U, V \in \mathcal{O}S$ with $U \not\subseteq V$, and for each Scott-open filter ∇ with $U \in \nabla$, there is some $W \in \mathcal{O}S$ such that $(*)$ holds. ■

The idea is the same for both notions. Given $U, V \in \mathcal{O}S$ with $U \not\subseteq V$, there is a $W \in \mathcal{O}S$ which separates U and V in a certain way. The difference between α-regularity and α-trimness is the dependencies amongst the quantifiers. In particular, we have

$$\alpha\text{-regular} \Longrightarrow \alpha\text{-trim}$$

for these trivial reasons. It seems that in general this implication is not an equivalence. However, the case $\alpha = 0$ is different. The trick is that $\nabla(0) = S$ for every filter ∇, and this makes the dependency on ∇ illusory.

The following is the analogue of Theorem 4.4.

5.2 THEOREM. *For a space S the three properties*

$$(i) \quad S \text{ is regular} \qquad (ii) \quad S \text{ is 0-regular} \qquad (iii) \quad S \text{ is 0-trim}$$

are equivalent.

Proof. $(i) \Rightarrow (ii)$. Suppose S is regular, consider any $U, V \in \mathcal{O}S$ with $U \not\subseteq V$, and consider any witnessing point $p \in U, p \in V'$. By the regularity of S we have

$$p \in W \subseteq W^{-} \subseteq U$$

for some $W \in \mathcal{O}S$. For each Scott-open filter ∇ we have $\nabla(0) = S$, so that

$$(W \cap \nabla(0))^{-} = W^{-} \subseteq U$$

as required.

$(ii) \Rightarrow (iii)$. This is immediate.

$(iii) \Rightarrow (i)$. Suppose S is 0-trim and consider any $p \in U \in \mathcal{O}S$. Let $V = p^{-\prime}$, so that $U \nsubseteq V$. With the Scott-open filter $\nabla = \mathcal{O}S$, since $\nabla(0) = S$ the 0-trimness gives

$$W \subseteq U \qquad W \nsubseteq p^{-\prime} \qquad W^{-} \subseteq U$$

for some $W \in \mathcal{O}S$. In particular $p \in W$, which is enough to show that S is regular. ■

The notion of α-regularity has an inbuilt uniformity in the selection of the separating set W. Because of this the weaker notion of α-trimness may seem more natural. However, the stronger notion of α-regularity is related to a notion of α-well-inside in the same way that standard regularity is related to the standard well-inside comparison. We won't develop that relationship here, but we will give just a hint of the difference.

Suppose S is α-trim, let $U \in \mathcal{O}S$, and let ∇ be a Scott-open filter with $U \in \nabla$. Let

$$V = \bigcup\{W \in \mathcal{O}S \mid W \subseteq U, (W \cap \nabla(\alpha))^{-} \subseteq U\}$$

so that $V \subseteq U$. The α-trimness ensures that $V = U$. This decomposition of U depends on the filter ∇. A similar use of α-regularity will produce a filter independent decomposition.

5.3 THEOREM. *The three implications*

$$\alpha\text{-}neat \Longrightarrow \alpha\text{-}regular \Longrightarrow \alpha\text{-}trim \Longrightarrow (\alpha+1)\text{-}neat$$

hold for each ordinal α.

Proof. For the left hand implication suppose the space S is α-neat, and consider $U, V \in \mathcal{O}S$ with $U \nsubseteq V$. Let $W = U$, so that

$$W \subseteq U \qquad W \nsubseteq U$$

(trivially). Consider any Scott-open filter ∇ with $U \in \nabla$. The α-neatness gives $\nabla(\alpha) \subseteq U$, so that

$$(W \cap \nabla(\alpha))^{-} \subseteq \nabla(\alpha)^{-} \subseteq \nabla(\alpha) \subseteq U$$

to give the required result.

The central implication follows by a trivial manipulation of quantifiers.

For the right hand implication suppose the space S is α-trim, consider a Scott-open filter ∇, and consider any open set $U \in \nabla$. We require $\nabla(\alpha+1) \subseteq U$. By the decomposition observation above we have

$$U = \bigcup\{W \in \mathcal{O}S \mid W \subseteq U, (W \cap \nabla(\alpha))^{-} \subseteq U\}$$

and, by a simple calculation, this union is directed. Since ∇ is Scott-open and $U \in \nabla$, this gives some $W \in \nabla$ with

$$\nabla(\alpha+1) \subseteq (W \cap \nabla(\alpha))^{-} \subseteq U$$

as required. ■

For $\alpha = 0$ this result and Theorem 5.2 show that each regular space is 1-neat. By Theorem 4.4, for T_0 spaces this gives $T_3 \Rightarrow T_2$ (which is not exactly unknown). Theorem 5.3 is a kind of generalization of this implication.

6 Stacking properties

By Theorem 4.5 each T_0+neat space is T_1+sober+packed, but not conversely. Neat spaces have extra properties. Consider $Q \in \mathcal{Q}S$. This generates a Scott-open filter $\nabla(Q)$ with $Q = \bigcap \nabla(Q)$, and so gives a derivative $\partial_Q = \partial_{\nabla(Q)}$. To simplify the notation we set

$$Q(\alpha) = \partial^{\alpha}_{\nabla(Q)}(S) = \nabla(Q)(\alpha)$$

for each ordinal α. The filter $\nabla(Q)$ is neat precisely when $Q = Q(\infty)$. The packedness of the space is concerned with an equality $Q^{-} = Q$.

Each $Q \in \mathcal{Q}S$ gives us three sets

$$Q \subseteq Q^{-} \subseteq Q(\infty)$$

and comparisons between four derivatives

$$\begin{array}{cccc} & \langle Q^{-} \rangle & \cdots & \langle Q(\infty) \rangle & \\ \text{Packed} & \vdots & & \vdots & \text{Neat} \\ & \langle Q \rangle & \cdots & \partial^{\infty}_{Q} & \end{array}$$

where each dotted line indicates a comparison, either left to right or from bottom to top. Packedness ensures an equality on the left and neatness is concerned with an equality on the right. What about equalities at the top or bottom?

6.1 DEFINITION. Let S be a space.

(a) A set $Q \in \mathcal{Q}S$ is stacked if $Q^{-} = Q(\infty)$, and is strongly stacked if $\langle Q \rangle = \partial^{\infty}_{Q}$.

(b) The space S is stacked or strongly stacked if each of its compact saturated sets is stacked or strongly stacked, respectively. ■

These names may look odd. They are explained by the second part of the following.

6.2 THEOREM. *(a) All three of the implications*

$$\textit{neat} \Longrightarrow \textit{strongly stacked} \Longrightarrow \textit{stacked} \qquad T_1\textit{+stacked} \Longrightarrow \textit{sober}$$

hold for spaces.

(b) A T_0 space is neat precisely when it is packed and stacked.

Proof. (a) The two left hand implications are immediate.

For the right hand implication consider any space S which is T_0+stacked (not yet T_1). Consider any closed irreducible subset Z of S. We obtain some information about Z.

Consider $q \in Z$, let $Q = q^{\uparrow}$ (so that $Q \in \mathcal{Q}S$), and let $\partial = \partial_Q$. We show first that $Z \subseteq \partial(Z)$ and hence $\partial(Z) = Z$. To this end consider $W \in \mathcal{O}S$ with $q \in W$. We show $Z \subseteq (W \cap Z)^{-}$, and then take the intersection over all such W. Consider any $p \in Z$ and, by way of contradiction, suppose

$$p \in V = (W \cap Z)^{-\prime} = (W' \cup Z')^{\circ}$$

holds. Since Z meets both W and V (at q and p, respectively) we have $Z \cap W \cap V \neq \emptyset$ which, by a simple calculation, is nonsense.

Since $Z = \partial(Z)$ we have

$$q \in Z = \partial^\infty(Z) \subseteq \partial^\infty(S) = Q(\infty) = Q = q^\uparrow$$

where $Q(\infty) = Q$ holds since S is stacked. Finally, if we assume that S is T_1 then $q \in Z \subseteq q^\uparrow = \{q\}$ so that Z is a singleton, which is more than enough to show S is sober.

(b) The implication

$$T_0\text{+neat} \Longrightarrow \text{packed+stacked}$$

follows by Theorem 4.5 and part (a).

Conversely, suppose the space S is T_0+packed+stacked. The first two properties ensure that S is T_1, and hence S is sober by part (a). Consider any Scott-open filter ∇ on $\mathcal{O}S$. We have $\nabla = \nabla(Q)$ for some $Q \in \mathcal{Q}S$. Since S is packed and stacked we have

$$\nabla(\infty) = Q(\infty) = Q^- = Q = \bigcap \nabla$$

and hence S is neat. ■

The Alexandroff topology on a poset gives a T_0+strongly stacked space which is neat precisely when the poset is discrete.

7 V-points

Let S be a sober space. We wish to construct a hyperspace of S, a new space where the points are certain subsets of S. The construction we use here is as follows.

7.1 DEFINITION. Let S be any topological space and let $\mathcal{K}S$ be any set of compact subsets K. For $U \in \mathcal{O}S$ let $\Diamond(U)$ and $\Box(U)$ be the subsets of $\mathcal{K}S$ given by

$$K \in \Diamond(U) \Longleftrightarrow K \text{ meets } U \qquad K \in \Box(U) \Longleftrightarrow K \subseteq U$$

(for $K \in \mathcal{K}S$). We use these to topologize $\mathcal{K}S$. We take the smallest topology in which each of these sets is open. This is the V-modification of S on $\mathcal{K}S$. ■

We use the given topology $\mathcal{O}S$ to doubly index a subbase of the topology constructed on $\mathcal{K}S$. (We needn't worry about whether or not $\emptyset$ should be in $\mathcal{K}S$.)

By varying the choice of $\mathcal{K}S$ we produce different V-modifications. For instance, when $\mathcal{K}S$ is the set of singletons we merely reproduce the parent space. In the original example (as described by Vietoris) the space S is compact T_2 and $\mathcal{K}S = \mathcal{C}S$. An obvious extension of this is to take $\mathcal{K}S = \mathcal{Q}S$. A more recent extension uses the compact lenses of S.

A lens of S is a subset of the form $L = R \cap Y$ where R is saturated and Y is closed. Thus L is a certain kind of convex subset in the specialization order. Let $\mathcal{L}S$ be the set of the compact lenses, the lenses of S which are also compact subsets.

For any $Q \in \mathcal{Q}S$ and $X \in \mathcal{C}S$ the lens $L = Q \cap X$ is compact. In particular, Q is a compact lens (since $Q = Q \cap Q^-$). Thus we have an inclusion $\mathcal{Q}S \subseteq \mathcal{L}S$ which, with the V-topology on each of $\mathcal{Q}S$ and $\mathcal{L}S$, is a topological embedding.

This is the point-sensitive version of the V-modification (or as much of it as we need to know here). There is also a point-free version. This was introduced in [3, 4], and a simplified account for compact T_2 spaces is given in [5]. We give an outline of the general constuction, but for what we do here we can take the results of Theorem 7.2 and Lemma 7.3 as the definition of the associated space $\mathcal{V}S$ of a sober space S.

Starting from the parent space we view the topology $\mathcal{O}S$ as a lattice of a certain kind. We generate a larger lattice (which must satisfy certain equalities), and hit this with a hull-kernel (spectral) construction to produce a space. This is the new V-modification. We write $\mathcal{V}S$ for the points of the new space. As you can probably guess these points have something to do with $\mathcal{Q}S$. The points of $\mathcal{V}S$ are described in [3, 4], but not in the most transparent manner.

Consider any $Q \in \mathcal{Q}S$. As in Section 3 this has a minimal generating subset M. Thus we have three closed sets $M^- \subseteq Q^- \subseteq Q(\infty)$ and a derivative ∂_Q associated with Q.

7.2 THEOREM. *For a sober space S the points in $\mathcal{V}S$ are the pairs (Q, X) where*

$$Q \in \mathcal{Q}S \qquad X \in \mathcal{C}S \qquad M^- \subseteq X \subseteq Q(\infty) \qquad \partial_Q(X) = X$$

hold. Furthermore, for $Q \in \mathcal{Q}S$, each of (Q, M^-), (Q, Q^-), $(Q, Q(\infty))$ is in $\mathcal{V}S$.

This shows that each $Q \in \mathcal{Q}S$ always produces at least one point in $\mathcal{V}S$ (namely (Q, Q^-)). There can be many points in $\mathcal{V}S$ of the form (Q, X), and in Section 8 we illustrate that the family of all such Q-points can be extremely complicated.

The topology on $\mathcal{V}S$ is produced automatically (since it is a hull-kernel construction). We find that it does look very like the more traditional V-topologies.

7.3 LEMMA. *For each sober space S the doubly indexed families given by*

$$(Q, X) \in \blacklozenge(U) \iff X \text{ meets } U \qquad (Q, X) \in \blacksquare(U) \iff Q \subseteq U$$

(for V-points (Q, X) and $U \in \mathcal{O}S$) form a subbase of the topology on $\mathcal{V}S$.

It is easy to check that the assignment

$$Q \longmapsto (Q, Q^-) : \mathcal{Q}S \longrightarrow \mathcal{V}S$$

is a topological embedding. This show that $\mathcal{V}S$ has a rather curious structure.

There is a copy of $\mathcal{Q}S$ running through $\mathcal{V}S$, and $\mathcal{V}S$ is bundled over $\mathcal{Q}S$. However, some points such as (Q, M^-) lie on one side of the sliver, and other points such as $(Q, Q(\infty))$ lie on the other side. Not much seems to be known about this geometric structure (for it is surely more than just a space), but we can begin to see how the stacking properties of the parent space simplify the structure.

7.4 THEOREM. *A space S is T_1+stacked precisely when it is sober and its V-modifications $\mathcal{Q}S$ and $\mathcal{V}S$ are canonically homeomorphic.*

Proof. Suppose that S is T_1+stacked. By Theorem 6.2 the space is sober, so we may use the material outlined above. Consider any $Q \in \mathcal{Q}S$ and point $(Q, X) \in \mathcal{V}S$. But S is T_1 so that $M = Q$, and S is stacked so that $Q^- = Q(\infty)$. Thus, by Theorem 7.2, we have $X = Q^-$, to give the required result.

Conversely, suppose S is sober and $\mathcal{Q}S = \mathcal{V}S$, that is

$$(Q, M^-) = (Q, Q^-) = (Q, Q(\infty))$$

for each $Q \in \mathcal{Q}S$. In particular, we have $Q^- = Q(\infty)$, so that S is stacked. Consider any $m \in S$ and let $Q = m^\uparrow$. We have $m^- = Q^-$ which, by a simple argument, shows that m^- is a singleton. Thus S is T_1. ■

The relationship between $\mathcal{L}S$ and $\mathcal{V}S$ is more interesting.

7.5 THEOREM. *Let S be a sober space. For each $L \in \mathcal{L}S$ we have $(L^\uparrow, L^-) \in \mathcal{V}S$.*

Proof. We have $L = R \cap Y$ for some saturated R and closed Y, and we know that L is compact. We first massage this information into a more canonical form.

Let $Q = L^\uparrow$ and $X = L^-$, so that $Q \in \mathcal{Q}S$ (since the saturation of any compact set is compact) and $X \in \mathcal{C}S$. We have $L \subseteq R$ and $L \subseteq Y$ so that

$$Q = L^\uparrow \subseteq R^\uparrow = R \qquad X = L^- \subseteq Y^- = Y$$

and hence $L \subseteq Q \cap X \subseteq R \cap Y = L$ to give

$$L = Q \cap X = L^\uparrow \cap L^-$$

which is the canonical representation of L.

We have $L^\uparrow = Q \in \mathcal{Q}S$ and $L^- = X \in \mathcal{C}S$, so it remains to verify the other two conditions of Theorem 7.2.

Let M be the minimal generating set of Q. We show $M \subseteq L$, so that $M^- \subseteq L^- = X$. Consider any $m \in M$ and, aiming for a contradiction, suppose $L \subseteq m^{-\prime}$. Then, since $m^{-\prime}$ is saturated, we have $m \in M \subseteq L^\uparrow \subseteq m^{-\prime}$ which is false. This shows that L meets m^-, to give some $l \in L \cap m^-$. But now $l \leq m \in M$ with $l \in L$, so that $l = m$ by the minimality of m. Thus $m = l \in L$, and hence $M \subseteq L$.

Finally, consider any $W \in \mathcal{O}S$ and $Q \subseteq W$. Then $L \subseteq Q \cap X \subseteq W \cap X$ so that $L^- \subseteq (W \cap X)^-$ to give $X = L^- \subseteq \partial_Q(X)$ to complete the proof. ■

This result shows how each compact lens gives a V-point. By looking at the two constructions it is easy to check that the assignment

$$L \longmapsto (L^\uparrow, L^-) : \mathcal{L}S \longrightarrow \mathcal{V}S$$

is a topological embedding. Naturally, we call the pair $(L^\uparrow, L^-)$ a focal point of S.

7.6 LEMMA. *Let S be a sober space. A V-point (Q, X) of S is a focal point if and only if $X = (Q \cap X)^-$.*

Proof. Suppose first that (Q, X) is a focal point of S. Thus there is some $L \in \mathcal{L}S$ with $Q = L^\uparrow$ and $X = L^-$. But now, since L is a lens, as in the proof of Theorem 7.5, we have $L = (L^- \cap L^\uparrow)$, so that

$$X = L^- = (L^- \cap L^\uparrow)^- = (Q \cap X)^-$$

as required.

Conversely, suppose (Q, X) is a V-point with $X = (Q \cap X)^-$. We show that $L = Q \cap X$ is a lens with $L^\uparrow = Q$ and $L^- = X$. The equality $L^- = X$ holds by hypothesis, and the inclusion $L \subseteq Q$ gives $L^\uparrow \subseteq Q$. We require the converse inclusion.

Let M be the minimal generating set of Q. We have $M \subseteq X$ (since (Q, X) is a V-point) so that $M \subseteq L$. In particular, we have

$$L \subseteq U \Longrightarrow M \subseteq U \Longrightarrow Q \subseteq U$$

for each $U \in \mathcal{O}S$. Now consider an arbitrary $q \in Q$. We have $Q \nsubseteq q^{-\prime}$, so that $L \nsubseteq q^{-\prime}$, which gives some $p \in L$ with $p \leq q$, and hence $q \in L^{\uparrow}$. Thus $Q \subseteq L^{\uparrow}$ and hence $L^{\uparrow} = Q$.

Finally $L \subseteq L^{-} \cap L^{\uparrow} = X \cap Q = L$ to give $L = L^{-} \cap L^{\uparrow}$, and hence L is a lens. ■

With this characterization we quickly obtain the focal analogue of Theorem 7.2.

7.7 THEOREM. *For a sober space S the focal points of S are the pairs (Q, X) where*

$$Q \in \mathcal{Q}S \qquad X \in \mathcal{C}S \qquad M^{-} \subseteq X \subseteq Q^{-} \qquad (Q \cap X)^{-} = X$$

hold. Furthermore, for $Q \in \mathcal{Q}S$, both (Q, M^{-}) and (Q, Q^{-}) is a focal point.

This shows when S is T_1 its focal points are essentially the members of $\mathcal{Q}S$. However, lenses are useful only when the specialization order is important (and not just equality).

The stacking properties of a space has an impact on its focal properties.

7.8 THEOREM. *Both the implications*

$$\textit{strongly stacked} \Longrightarrow (\mathcal{V}S = \mathcal{L}S) \Longrightarrow \textit{stacked}$$

hold for each sober space S.

Proof. For the left hand implications suppose the sober space S is strongly stacked and consider any V-point (Q, X). Since $\partial_Q = \langle Q \rangle$ a use of Theorem 7.2 gives

$$X = \partial_Q(X) = (Q \cap X)^{-}$$

and hence (Q, X) is a focal point by Theorem 7.7.

For the right hand implications suppose for the sober space S each V-point is focal. In particular, for $Q \in \mathcal{Q}S$ the V-point $(Q, Q(\infty))$ is focal, and hence $Q(\infty) \subseteq Q^{-}$ by Theorem 7.7, to show that Q is stacked. ■

The precise relationship between these three properties is not known to us.

8 Boss spaces

We describe a family of examples which illustrate the variety of stacking properties a space may have. For each space we start with a tree $\mathbb{S}$ furnished with some extra gadgetry. We refer to a member of $\mathbb{S}$ as a node. We attach to $\mathbb{S}$ a new point $\star$, the boss point, to obtain

$$S = \{\star\} \cup \mathbb{S}$$

the set of points of the space. The furnishings of $\mathbb{S}$ induce the topology $\mathcal{O}S$ on S. This gives us a boss space with its boss topology. The boss point controls many of the topological properties of S, and yet doesn't seem to do anything useful. Each such space is T_1+sober+tightly packed but the stacking properties can differ widely.

- For each ordinal α there is a boss space which is $(\alpha+1)$+neat but not α-neat.
- There is a boss space S with $\partial_\star(S) = S$ and such that for many subtrees $\mathbb{T}$ of $\mathbb{S}$ the pair $(\{\star\},\ \{\star\} \cup \mathbb{T})$ is a V-point of S.

All the examples we used are based on the most common type of tree (well-founded with height no more that ω), but the methods work for a much larger arboretum.

8.1 DEFINITION. Let $\mathbb{S}$ be a poset with comparison $\leq$ and, as usual, with $<$ as the associated strict comparison. We say $\mathbb{S}$ is a tree if for each node $x \in \mathbb{S}$ the set

$$P(x) = \{z \in \mathbb{S} \,|\, z \leq x\}$$

of predecessors of x is linearly ordered (by the comparison).

For each node $x \in \mathbb{S}$ let

$$I(x)$$

be the set of immediate successors of x, the set of all those $y \in \mathbb{S}$ such that $x < y$ and there is no $z \in \mathbb{S}$ with $x < z < y$. ■

You may be more familiar with the smaller class of trees where for each node x the set $P(x)$ is well-ordered (not just linearly ordered). We don't need that restriction here.

Notice that

$$\left.\begin{matrix} y \leq x \\ z \leq x \end{matrix}\right\} \Longrightarrow y \leq z \text{ or } z \leq y \qquad I(x) \cap I(y) \neq \emptyset \Longrightarrow x = y$$

for all nodes x, y, z of a tree.

8.2 EXAMPLE. Let I be an arbitrary set, thought of as an alphabet. Let $\mathbb{S}$ be the set of all words on I, including the empty word $\perp$. Thus each $x \in \mathbb{S}$ is a list

$$x = \perp i_1 \ldots i_l$$

of letters $i_1, \ldots, i_l \in I$. Here l is the length of x and $l = 0$ is allowed. These words are partially ordered by extension, thus $x \leq y$ (for words x, y) precisely when

$$y = x i_1 \ldots i_l$$

for some sequence $i_1, \ldots, i_l$ of letters (and again $l = 0$ is allowed). This makes $\mathbb{S}$ a well-founded tree of height ω (or 0 if I is empty). It is the full I-splitting tree.

For each word x

$$I(x) = \{xi \,|\, i \in I\}$$

is the set of immediate successors of x. Each set $I(x)$ is essentially the same as I. ■

When I is empty, we have $\mathbb{S} = \{\perp\}$. When I is a singleton, S is essentially the natural numbers. When I is a pair, S is the Cantor tree. When I is countably infinite, S is the Baire tree. Eventually we will look at the case where I is uncountable.

8.3 DEFINITION. For a set I a notion of smallness on I is an ideal $\mathcal{S}$ of $\mathcal{P}I$. ■

Thus $\mathcal{S}$ is a collection of subsets of I with $\emptyset \in \mathcal{S}$ and such that

$$F \subseteq E \in \mathcal{S} \Longrightarrow F \in \mathcal{S} \qquad E, F \in \mathcal{S} \Longrightarrow E \cup F \in \mathcal{S}$$

for $E, F \subseteq I$. We think of the members of $\mathcal{S}$ as the small subsets of I. There are some rather uninteresting notions of smallness on I. We could take either of the two extremes $\mathcal{S} = \{\emptyset\}$ or $\mathcal{S} = \mathcal{P}I$, or any principal ideal. There are many more sophisticated notions of 'small', especially when I carries some extra 'calibrating' structure.

8.4 DEFINITION. Let $\mathcal{S}$ be a notion of smallness on a set I. We say $\mathcal{S}$ is fresh if $E \in \mathcal{S}$ for each finite $E \in \mathcal{P}I$. We say $\mathcal{S}$ is stringent if $E \in \mathcal{S}$ for each countable $E \in \mathcal{P}I$. ■

If I is countable then $\mathcal{P}I$ is the only stringent notion of smallness. However, when I is uncountable there can be many stringent ideals.

8.5 DEFINITION. Let $\mathbb{S}$ be a tree. We say $\mathbb{S}$ is dressed if for each node $x \in \mathbb{S}$ the set $I(x)$ carries a notion of smallness $\mathcal{S}(x)$. We say $\mathbb{S}$ is freshly dressed or stringently dressed if each $\mathcal{S}(x)$ is fresh or stringent, respectively. ■

For a dressed tree $\mathbb{S}$, as x varies through $\mathbb{S}$ the notions of smallness $\mathcal{S}(x)$ may be quite unrelated. However, the full I-splitting tree of Example 8.2 can be dressed in a uniform way. We select a notion of smallness $\mathcal{S}$ for the alphabet I.

8.6 DEFINITION. Let $\mathbb{S}$ be a dressed tree and set

$$S = \{\star\} \cup \mathbb{S}$$

(where $\star \notin \mathbb{S}$). Let $\mathcal{O}S$ be the family of all those subsets U of S such that both

$$\star \in U \Longrightarrow (\forall x \in \mathbb{S})[I(x) - U \in \mathcal{S}(x)] \qquad (\forall x \in U)[I(x) - U \in \mathcal{S}(x)]$$

hold. ■

It is easy to check that $\mathcal{O}S$ is a topology. A set $X \subseteq S$ is closed precisely when both

$$(\exists x \in \mathbb{S})[I(x) \cap X \notin \mathcal{S}(x)] \Longrightarrow \star \in X \qquad (\forall x \in \mathbb{S})[I(x) \cap X \notin \mathcal{S}(x) \Longrightarrow x \in X]$$

hold. We will return to the first implications when we look at the CB properties of S.

To give some examples of open and closed sets we first sort out some notation.

As earlier for an arbitrary space we write $(\cdot)^{\uparrow}$ and $(\cdot)^{\downarrow}$ for the upwards and the downwards closures relative to the specialization order. As we will see, each interesting boss space S is T_1, so these operations are not needed. However, S is based on a tree $\mathbb{S}$ which carries a different comparison. For $E \subseteq \mathbb{S}$ we write

$$\uparrow E \qquad \downarrow E$$

for the upwards and the downwards closures of E in $\mathbb{S}$ relative to the carried comparison.

8.7 EXAMPLES. (a) For each $y \in \mathbb{S}$ the set

$$U = \uparrow y = \{z \in \mathbb{S} \mid y \le z\}$$

is open. By construction $\star \notin U$, so it suffices to consider those $x \in U$. For such an x we have $I(x) \subseteq U$ so that $I(x) - U = \emptyset \in \mathcal{S}(x)$.

(b) Suppose $\mathbb{S}$ is freshly dressed. For $V \in \mathcal{O}S$ with $\star \notin V$ and $y \in \mathbb{S}$, the set

$$U = V - \uparrow y$$

is open. Since $\star \notin U$ it suffices to consider those $x \in U$. For such an x we have

$$I(x) - U = (I(x) - V) \cup (I(x) \cap \uparrow y)$$

and the first component is in $\mathcal{S}(x)$ (since $x \in V \in OS$). Consider any z in the second component. Thus $x < z$ with nothing between these. But also $y \le z$ and hence $x < y \le z$ (since $\mathbb{S}$ is a tree and $y \not\le x$). Thus $z = y$, and the second component is no more than a singleton.

(c) Suppose $\mathbb{S}$ is freshly dressed. Then each finite $X \subset \mathbb{S}$ is closed (since $I(x) \cap X \in \mathcal{S}(x)$ for every node x). Since $\{\star\}$ is closed (almost vacuously), this shows that S is T_1.

If $\mathbb{S}$ is stringently dressed then each countable $X \subset \mathbb{S}$ is closed. This second remark will be crucial in the proof of Theorem 8.8.

(d) Suppose $\mathbb{S}$ is stringently dressed, consider any countable $Z \subseteq \mathbb{S}$, and any $Y \subseteq \downarrow Z$. We show that

$$(\forall x \in \mathbb{S})[I(x) \cap Y \in \mathcal{S}(x)]$$

and hence Y is closed.

Fix $x \in \mathbb{S}$. If $I(x) \cap Y$ is empty, then it is certainly in $\mathcal{S}(x)$. Thus we may suppose this intersection is non-empty. For each $y \in I(x) \cap Y$ we have $x < y \le z$ for some $z \in Z$. By choice this sets up a selection function $f : I(x) \cap Y \longrightarrow Z$ with $x < y \le f(y) \in Z$ for each $y \in I(x) \cap Y$. We show that f is injective, so that $I(x) \cap Y$ is countable, and hence is in $\mathcal{S}(x)$.

Consider $y_1, y_2 \in I(x) \cap Y$ with $f(y_1) = f(y_2) = z \in Z$. Since $\mathbb{S}$ is a tree we have $x < y_1 \le y_2 \le z$ (say). But $y_1 \in I(x)$ so there is nothing between x and y_2, to give $y_1 = y_2$, as required. ■

The construction of Definition 8.6 works for any dressed tree. However, for our purposes we need a stringent dressing. This is used in the proof of the following crucial result. We will indicate precisely where.

8.8 THEOREM. *Let $\mathbb{S}$ be a stringently dressed tree. Then the associated boss space S is T_1+sober+tightly packed.*

Proof. We look at the three properties in turn.

Example 8.7(c) shows that S is T_1. (This requires only that $\mathbb{S}$ is freshly dressed.)

To show that S is sober consider any closed irreducible subset Z. We aim to show that Z is a singleton, but first we show that $Z \cap \mathbb{S}$ is no more than a singleton.

By way of contradiction suppose there are distinct $x, y \in Z \cap \mathbb{S}$. Since

$$Z \text{ meets } U = \uparrow x \text{ at } x \qquad Z \text{ meets } V = \uparrow y \text{ at } y$$

and U, V are open, by Example 8.7(a), the irreducibility ensures that Z meets $U \cap V$ to give some $z \in Z$ with $x, y \leq z$. Since $\mathbb{S}$ is a tree we may suppose $x < y \leq z$ (by symmetry). The set $W = \uparrow x - \uparrow y$ is open, by Example 8.7(a,b), and

$$Z \text{ meets } W = \uparrow x \text{ at } x \qquad Z \text{ meets } V = \uparrow y \text{ at } y$$

so that Z meets $W \cap V$ (by a second use of the irreducibility). Since

$$W \cap V \subseteq V' \cap V = \emptyset$$

this is the contradiction.

This shows that either Z is a singleton or has the form $\{\star, x\}$ for some $x \in \mathbb{S}$. We must exclude this second case.

Suppose $Z = \{\star, x\}$. Then Z meets $\uparrow x$ at x and Z meets $\{x\}'$ at $\star$, so that Z meets $\uparrow x \cap \{x\}'$ (by a third use of the irreducibility). This give some $z \in Z \cap \mathbb{S}$ with $x < z$, which can not be since $Z \cap \mathbb{S}$ is no more than a singleton.

It remains to show that S is tightly packed. This is where the stringent dressing of $\mathbb{S}$ is crucial. We invoke Example 8.7(c).

Let $Q \in \mathcal{Q}S$. To show that Q is finite we first show something weaker.

Let L be any antichain of $\mathbb{S}$. We show that $Q \cap L$ is finite.

By way of contradiction suppose $Q \cap L$ is infinite. Thus there is a countably infinite subset $X \subseteq Q \cap L$. Since the dressing is stringent, this set X is closed by Example 8.7(c). Thus, using Example 8.7(a) we see that

$$X' \cup \{\uparrow x \,|\, x \in X\}$$

is an open cover of S (since $\star \in X'$). The compactness of Q gives

$$X \subseteq Q \subseteq X' \cup \uparrow x_1 \cup \cdots \cup \uparrow x_m$$

for some $x_1, \ldots, x_m \in X$, and hence

$$X \subseteq L \cap (\uparrow x_1 \cup \cdots \cup \uparrow x_m)$$

holds. Consider any $y \in X$. Then $y \in L$ and $x_i \leq y$ for some $1 \leq i \leq m$. But $x_i \in X \subseteq L$ and L is an antichain, so that $y = x_i$. This gives $X \subseteq \{x_1, \ldots, x_m\}$ which is the contradiction (since X is supposed to be infinite).

We use this observation for sets $L \subseteq I(x)$ for $x \in \mathbb{S}$.

Consider any subset $H \subseteq S$ and any $x \in \mathbb{S}$ and let $L = H \cap I(x)$. By the observation

$$Q \cap L = Q \cap H \cap I(x)$$

is finite and hence in $\mathcal{S}(x)$. This shows that $Q \cap H$ is closed. In particular, for each $q \in Q$ the set $X_q = Q \cap \{q\}'$ is closed, and hence $U_q = X_q' = Q' \cup \{q\}$ is open. But

$$\{U_q \,|\, q \in Q\}$$

is an open cover of S, so that the compactness of Q gives

$$Q \subseteq Q' \cup \{q_1, \ldots, q_m\}$$

for some $q_1, \ldots, q_m \in Q$. Thus $Q \subseteq \{q_1, \ldots, q_m\}$ to give the required result. ■

Let us fix some boss space S with a stringent dressing. Eventually we want to take some $Q \in \mathcal{Q}S$ and see what ∂_Q does. In particular, we want to determine $Q(\infty)$ and to estimate the closure ordinal. It turns out that the boss has his finger in everything.

8.9 DEFINITION. For each closed set $X \in \mathcal{C}S$ let $\ell(X)$ be the subset of S given by

$$\star \in \ell(X) \Longleftrightarrow \star \in X \qquad x \in \ell(X) \Longleftrightarrow I(x) \cap X \notin \mathcal{S}(x)$$

for each $x \in \mathbb{S}$. ■

The characterization of closed sets given just before Examples 8.7 shows that

$$\ell(X) \neq \emptyset \Longrightarrow \star \in X \qquad \ell(X) \subseteq X$$

for each $X \in \mathcal{C}S$. It is easy to check that ℓ is a derivative on S but we don't need to do that here. However, we do observe that ℓ is related to a derivative on S.

8.10 LEMMA. *For each $X \in \mathcal{C}S$ we have*

$$\ell(X) \cap \mathbb{S} = \boldsymbol{lim}(X) \cap \mathbb{S}$$

where $\boldsymbol{lim}$ *is the CB-derivative on S.*

Proof. Fix $X \in \mathcal{C}S$.

Firstly, consider any $x \in \ell(X) \cap \mathbb{S}$ and, by way of contradiction, suppose x is isolated in X. Thus $X \cap U = \{x\}$ for some $U \in \mathcal{O}S$. In particular, $X \subseteq U' \cup \{x\}$ so that

$$I(x) \cap X \subseteq (I(x) - U) \cup (I(x) \cap \{x\}) = I(x) - U$$

(since $x \notin I(x)$). But $x \in U$, so that $I(x) - U \in \mathcal{S}(x)$, which is the contradiction (since $x \in \ell(X)$).

Secondly, consider any $x \in \mathbb{S}$ with $x \in X - \ell(X)$. We have $x \in X$ and $I(x) \cap X \in \mathcal{S}(x)$, and we must show that x is isolated in X. To this end consider

$$U = (\mathbb{S} \cap X') \cup \{x\}$$

so that $X \cap U = \{x\}$ and hence it suffices to show that U is open.

By construction $\star \notin U$. Consider any $y \in U$. We require $I(y) - U \in \mathcal{S}(y)$. But $I(y) - U \subseteq I(y) \cap X$ so that $I(y) \cap X \in \mathcal{S}(y)$ will suffice. Since $y \in U$ we have either $y \in X'$ or $y = x$. Since X' is open this first alternative gives $I(y) \cap X \in \mathcal{S}(y)$. For the second alternative we have $y = x \notin \ell(X)$, and hence $I(y) \cap X \in \mathcal{S}(y)$, as required. ■

This result can not be strengthened. If $\star \in X$ then $\star \in \ell(X)$, but $\star$ may be isolated in X (for instance if $X = \{\star\}$).

Consider any $Q \in \mathcal{Q}S$, and let $\partial = \partial_Q$. We have $Q \cap X \subseteq \partial(X)$ for each $X \in \mathcal{C}S$. The operator ℓ gives us an upper bound for ∂.

8.11 LEMMA. *Let $Q \in \mathcal{Q}S$ and let $\partial = \partial_Q$. Then for each $X \in \mathcal{C}S$ we have*

$$\partial(X) \subseteq \ell(X) \cup (Q \cap X)$$

with equality if $\star \in Q$.

Proof. We first show the inclusion and then deal with the equality.

If $\star \in \partial(X)$ then $\star \in X$, and hence $\star \in \ell(X)$. Also $\partial(X) \subseteq X$, so it suffices to show

$$x \in \partial(X) \Longrightarrow x \in \ell(X) \text{ or } x \in Q$$

(for $x \in \mathbb{S}$). In fact, we show the contrapositive.

Suppose

$$(i) \quad x \notin \ell(X) \qquad (ii) \quad x \notin Q$$

for $x \in \mathbb{S}$. Let $U = \{x\}' \in \mathcal{O}S$, so that $Q \subseteq U$ (by (ii)), and hence $\partial(X) \subseteq (U \cap X)^-$. We show that $U \cap X$ is closed, so that $\partial(X) \subseteq U \cap X \subseteq U$ and hence $x \notin \partial(U)$.

Suppose

$$I(y) \cap U \cap X \notin \mathcal{S}(y)$$

for some $y \in \mathbb{S}$. We require $\{\star, y\} \subseteq U \cap X$. By enlargement we have $I(y) \cap X \notin \mathcal{S}(y)$ so that $\{\star, y\} \subseteq X$. Also $\star \in U$ by construction. From (i) we have $I(x) \cap X \in \mathcal{S}(x)$ so that $y \neq x$, and hence $y \in U$, as required.

For the second part of the proof we suppose $\star \in Q$. For $X \in \mathcal{C}S$ we have $Q \cap X \subseteq \partial(X)$, so it suffices to show $\ell(X) \subseteq \partial X$. If $\ell(X) = \emptyset$, then we are done. Thus we may suppose $\ell(X) \neq \emptyset$, and hence $\star \in X$ (since X is closed).

Consider any open set U with $Q \subseteq U$. We require $\ell(X) \subseteq (U \cap X)^-$. To this end let $Y = (U \cap X)^-$, so that $X \cap U \cap Y' = \emptyset$ and hence $X \subseteq U' \cup Y$ to give

$$I(x) \cap X \subseteq (I(x) - U) \cup (I(x) \cap Y)$$

for each $x \in \mathbb{S}$. Since $\star \in Q \subseteq U$ we have $I(x) - U \in \mathcal{S}(x)$ for each such x. Thus, since $Y \in \mathcal{C}S$, we have

$$x \in \ell(X) \Longrightarrow I(x) \cap X \notin \mathcal{S}(x) \Longrightarrow I(x) \cap Y \notin \mathcal{S}(x) \Longrightarrow x \in Y$$

to give $\ell(X) \subseteq Y$, as required. ■

If $\star \notin X$ then $\ell(X) = \emptyset$, and we have the following.

8.12 COROLLARY. *Let $Q \in \mathcal{Q}S$ and $\partial = \partial_Q$. Then*

$$\partial(X) = Q \cap X$$

for each $X \in \mathcal{C}S$ with $\star \notin X$.

We said earlier that the operator ℓ is a derivative but didn't prove it. Here is why. The singleton $\{\star\}$ is in $\mathcal{Q}S$ and so gives a derivative $\partial_\star = \partial_{\{\star\}}$. By Lemma 8.11 we have

$$\partial_\star(X) = \ell(X) \cup (\{\star\} \cap X)$$

for each $X \in \mathcal{C}S$. But either $\star \in X$ in which case $\star \in \ell(X)$, or $\star \notin X$, and hence

$$\partial_\star(X) = \ell(X) \cup \{\star\} = \ell(X) \qquad \partial_\star(X) = \ell(X) = \emptyset$$

in these two cases. Thus we have the following.

8.13 COROLLARY. *We have $\partial_\star = \ell$.*

This with Lemma 8.11 begins to show how the boss point controls the stacking properties of S. To obtain more precise information we extend the lemma.

8.14 LEMMA. *Let $Q \in \mathcal{Q}S$ and $\partial = \partial_Q$. For each $X \in \mathcal{C}S$ and ordinal α we have*

$$Q \cap X \subseteq \partial^\alpha(X) \subseteq \ell^\alpha(X) \cup (Q \cap X)$$

where the right hand inclusion is an equality if $\star \in Q$.

Proof. The left hand inclusion is immediate. For the right hand inclusion we proceed by induction on α. The base case and the induction leap to a limit ordinal are straight forward. For the induction step, $\alpha \mapsto \alpha + 1$, by an immediate use of the induction hypothesis we have

$$\begin{aligned}
\partial^{\alpha+1}(X) &\subseteq \partial\big(\ell^\alpha(X) \cup (Q \cap X)\big) \\
&= \partial\big(\ell^\alpha(X)\big) \cup \partial(Q \cap X) \\
&\subseteq \big(\ell^{\alpha+1}(X) \cup (Q \cap \ell^\alpha(X)\big) \cup \big(\ell(Q \cap X) \cup (Q \cap X)\big) \\
&= \ell^{\alpha+1}(X) \cup (Q \cap X)
\end{aligned}$$

to give the required result. Here the third step (the second inclusion) follows by Lemma 8.11, and the final equality holds since ℓ is deflationary. ■

As a particular case of the this result we may take $X = S$ to get

$$Q \subseteq Q(\alpha) \subseteq \ell^\alpha(S) \cup Q$$

for each ordinal α. Since $\ell = \partial_\star$, for S to be neat we must at least have $\ell^\theta(S) = \{\star\}$ for some ordinal θ. But then

$$Q \subseteq Q(\theta) \subseteq \{\star\} \cup Q$$

for each $Q \in \mathcal{Q}S$, which almost shows neatness. Certainly if $\star \in Q$ then $Q(\theta) = Q$. When Q does not contain $\star$ the two extremes differ by just one point. In particular, if $\star \notin Q(\theta)$ then $Q = Q(\theta)$ and the space is neat. We show that this must be the case.

8.15 LEMMA. *Let $Q \in \mathcal{Q}S$ with $\star \notin Q$. Then $Q = Q(2)$.*

Proof. We show first that $Q(1) \subseteq \{\star\} \cup \downarrow Q$. To this end consider $x \in \mathbb{S} - \downarrow Q$. By Examples 8.7(a,b) both $U = \uparrow x$ and $V = \mathbb{S} - \uparrow x$ are open, and these sets are disjoint. For each $q \in Q$ we have $x \not\leq q$, so that $q \in V$. Thus $Q \subseteq V$, to give $Q(1) \subseteq V^- \subseteq U'$ and hence $x \notin Q(1)$.

We have $Q(1) \subseteq \{\star\} \cup X$ where $X = \downarrow Q$ and, by Example 8.7(d), the set X is closed. In particular $\partial(X) = Q \cap X = Q$ by Corollary 8.12. Also $Q \subseteq \mathbb{S} \in \mathcal{O}S$ (since $\star \notin Q$) so that $\partial(\{\star\}) \subseteq (\mathbb{S} \cap \{\star\})^- = \emptyset$ and hence

$$Q \subseteq Q(2) = \partial(Q(1)) \subseteq \partial(\{\star\}) \cup \partial(X) = Q$$

to give the required result. ■

With this result we can complete the calculation we started earlier.

8.16 THEOREM. *Let $\mathbb{S}$ be a stringently dressed tree with boss space S, and suppose $\ell^{\theta}(S) = \{\star\}$ for some ordinal θ. Then $Q(\theta) = Q$ for each $Q \in \mathcal{Q}S$, and S is neat.*

Proof. Consider $Q \in \mathcal{Q}S$.

If $\star \in Q$ then $Q(\theta) = Q$ by the calculation above.

If $\star \notin Q$ then $Q(2) = Q$ by Lemma 8.15. Thus only if $\theta \leq 1$ can a discrepancy arise. However, in this case we have

$$Q \subseteq Q(1) = \partial(S) \subseteq \ell(S) \cup Q = \{\star\} \cup Q$$

so that either $Q = Q(1)$ or $Q(1) = \{\star\} \cup Q$. Since $\star \notin Q$ we must have $Q = Q(1)$. (The case $\theta \leq 1$ can arise only if S is discrete.) ■

This result does not show that every stringent boss space is neat, for we also need $\ell^{\infty}(S) = \{\star\}$ to achieve that. Later we will see an example where $\ell^{\infty}(S) = S$ with S very large. However, the result does suggest a way of ensuring neatness.

8.17 THEOREM. *Let $\mathbb{S}$ be a stringently dressed tree in which each strictly ascending chain is finite. Then the associated boss space is neat.*

Proof. Let $X = \ell^{\infty}(S)$, so that X is closed with $\ell(X) = X$. By Theorem 8.16 it suffices to show $X = \{\star\}$. By way of contradiction suppose there is some $x \in X \cap \mathbb{S}$. Then $x \in \ell(X)$ so that $I(x) \cap X \notin \mathcal{S}(x)$ and, in particular, $I(x) \cap X \neq \emptyset$. This gives us some $x < y \in X \cap \mathbb{S}$. By iterating this construction we generate a strictly ascending chain of elements of $X \cap \mathbb{S}$. This is the contradiction. ■

This result gives us a simple way of producing neat spaces. For convenience, let us say a tree is standard if is is well-founded and each of its branches is finite. Then each stringently dressed standard tree has a neat boss space. Of course, what the result doesn't do is to estimate the degree of neatness (the stacking length). This can be small even when the tree is large. For instance, if the dressing is such that $S(x) = \mathcal{P}I(x)$ for each node x then the the boss space is discrete (and hence T_2). In the remainder of this section we show that even with a standard tree the stacking length can be arbitrarily large. To achieve this the tree must have a lot of splitting.

To begin this last part we produce an ascending chain

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_r \subseteq \cdots \subseteq S_\omega \quad (r < \omega)$$

of boss spaces where, for $r < \omega$, the space S_{r+1} is $(r+1)$-neat but not r-neat. The properties of

$$S_\omega = \bigcup\{S_r \mid r < \omega\}$$

are in contrast to this. Later we will indicate how to obtain $(\alpha + 1)$-neatness for $\alpha \geq \omega$.

Let I be some uncountable alphabet and let $\mathbb{S}_\omega$ be the full I-splitting tree of Example 8.2. Let $\mathcal{S}$ be the ideal of countable subsets of I. We use this as a uniform notion of smallness on $\mathbb{S}_\omega$. Of course, this is a stringent dressing.

The tree $\mathbb{S}_\omega$ splits naturally into layers. Thus let

$$\mathbb{L}_0 = \{\perp\} \qquad \mathbb{L}_{r+1} = \{xi \mid x \in \mathbb{L}_r, i \in I\}$$

for each $r < \omega$. With these let

$$\mathbb{S}_0 = \emptyset \qquad \mathbb{S}_{r+1} = \mathbb{S}_r \cup \mathbb{L}_r$$

(for $r < \omega$) to produce the ascending chain

$$\mathbb{S}_0 \subseteq \mathbb{S}_1 \subseteq \cdots \subseteq \mathbb{S}_r \subseteq \cdots \subseteq \mathbb{S}_\omega \quad (r < \omega)$$

of layered lower sections of $\mathbb{S}_\omega$. For each $r < \omega$ we have

$$\mathbb{S}_{r+1} = \mathbb{L}_0 \cup \cdots \cup \mathbb{L}_r$$

and $\mathbb{L}_r$ is the top layer of $\mathbb{S}_{r+1}$.

As well as $\mathbb{S}_\omega$, each $\mathbb{S}_r$ is a tree, so we may set

$$S_r = \{\star\} \cup \mathbb{S}_r \qquad S_\omega = \{\star\} \cup \mathbb{S}_\omega$$

to produce the ascending chain of boss spaces.

Each tree $\mathbb{S}_r$ is standard (but $\mathbb{S}_\omega$ is not), so each space S_r is neat. We have

$$S_0 = \{\star\} \qquad S_1 = \{\star, \bot\}$$

and both are discrete. In particular, S_1 is 1-neat (that is T_2) but not 0-neat (non-empty). More generally, we will show that

$$\ell^r(S_{r+1}) = \{\star, \bot\} \qquad \ell^{r+1}(S_{r+1}) = \{\star\}$$

so that S_{r+1} is $(r+1)$-neat but not r-neat.

We use the layering of $\mathbb{S}_\omega$

8.18 LEMMA. *For each $r < \omega$ the three conditions*

$$(i) \quad x \in \mathbb{S}_r \qquad (ii) \quad I(x) \subseteq \mathbb{S}_{r+1} \qquad (iii) \quad I(x) \cap \mathbb{S}_{r+1} \neq \emptyset$$

are equivalent for each $x \in \mathbb{S}_\omega$.

Proof. $(i) \Rightarrow (ii)$ Suppose $x \in \mathbb{S}_r$, so that $x \in \mathbb{L}_s$ for some $s < r$, and hence

$$I(x) \subseteq \mathbb{L}_{s+1} \subseteq \mathbb{S}_{r+1}$$

as required.

$(ii) \Rightarrow (iii)$ This is trivial (since $I(x) \neq \emptyset$).

$(iii) \Rightarrow (i)$ Suppose $y \in I(x) \cap \mathbb{S}_{r+1}$, so that $x < y \in \mathbb{S}_{r+1}$. Then $y \in \mathbb{L}_s$ for some $s \leq r$, and hence $x \in \mathbb{L}_t$ for some $t < r$, to give $x \in \mathbb{S}_r$. ■

The calculations of this last proof are done in the top space S_ω. However, in order to determine its properties some calculations have to be done in a lower space such as S_{r+1}. We must do this with a little care. For instance, for a node $x \in S_{r+1} \cap \mathbb{S}_\omega = \mathbb{S}_{r+1}$, what is the set of immediate successors in $\mathbb{S}_{r+1}$? It is either $I(x)$ if $x \in \mathbb{S}_r$ or is empty if $x \in \mathbb{L}_r$, the top layer of $\mathbb{S}_{r+1}$. In other words the set is $I(x) \cap \mathbb{S}_{r+1}$. We could develop a notation to reflect this, such as $I_{r+1}(x)$, but this won't be necessary provided we do take some care. Here is an example of this.

8.19 COROLLARY. *For each $r < \omega$ the set S_r is closed in all the higher spaces $S_r, \dots, S_\omega$.*

Proof. This is trivial for the case $r = 0$. Thus it suffices to show that each set S_{r+1} is closed in all its higher spaces.

We have $\star \in S_{r+1}$. Consider any node x such that

$$I(x) \cap \mathbb{S}_{r+1} = I(x) \cap S_{r+1} \notin \mathcal{S}(x)$$

holds in some higher space. Then $I(x) \cap \mathbb{S}_{r+1} \neq \emptyset$ (otherwise this intersection is in $\mathcal{S}(x)$), so that Lemma 8.18 gives $x \in \mathbb{S}_{r+1} \subseteq S_{r+1}$, as required. ■

Each of the spaces $S_0, S_1, \dots, S_\omega$ carries its own special derivative as given by Definition 8.9. For the time being let us write ℓ_{s+1} for that carried by S_{s+1}, and reserve ℓ for that carried by S_ω. Thus, for each $X \in \mathcal{C}S_{s+1}$ we have

$$\star \in \ell_{s+1}(X) \iff \star \in X \qquad x \in \ell_{s+1}(X) \iff I(x) \cap \mathbb{S}_{s+1} \cap X \notin \mathcal{S}(x)$$

for each $x \in \mathbb{S}_{s+1}$. Remember that $I(x) \cap \mathbb{S}_{s+1}$ is the set of immediate successors of x in $\mathbb{S}_{s+1}$. In particular, with $X = S_{r+1}$ for some $r \leq s$ we have

$$x \in \ell_{s+1}(S_{r+1}) \iff I(x) \cap \mathbb{S}_{r+1} \notin \mathcal{S}(x)$$

for each $x \in \mathbb{S}_{s+1}$. For such x we have $x \in \mathbb{L}_t$ for some $t \leq s$. But then $I(x) \subseteq \mathbb{L}_{t+1}$ to give

$$I(x) \cap \mathbb{S}_{r+1} = \begin{cases} I(x) & \text{if } t < r \\ \emptyset & \text{if } r \leq t \end{cases}$$

which shows that this intersection is either uncountable and not in $\mathcal{S}(x)$ (if $t < r$) or is empty and in $\mathcal{S}(x)$ (if $r \leq t$). Thus we have

$$x \in \ell_{s+1}(S_{r+1}) \iff (\exists t < r)[x \in \mathbb{L}_t] \iff x \in \mathbb{S}_r$$

for each $x \in \mathbb{S}_{s+1}$.

In other words

$$\ell_{s+1}(S_{r+1}) \cap \mathbb{S}_{s+1} = \mathbb{S}_r$$

which, since $\star \in S_r \subseteq S_{r+1}$, proves the following. In the statement of the result we have dropped the subscript of ℓ_{s+1}, but have included a warning.

8.20 LEMMA. *For each $r < \omega$ we have $\ell(S_{r+1}) = S_r$ where this calculation may take place in any higher space $S_{r+1}, \dots, S_\omega$.*

The explanation before this result shows that when calculating $\ell(S_{r+1})$ we don't need to worry about what the parent space is, the result is the same for each of $S_{r+1}, \dots, S_\omega$. In the same way we have the following.

8.21 LEMMA. *For each $r < \omega$ we have $\ell^r(S_{r+1}) = S_1 = \{\star, \bot\}$.*

Proof. We proceed by induction on r. The base case, $r = 0$, is trivial. For the induction step, $r \mapsto r + 1$, we have

$$\ell^{r+1}(S_{r+2}) = \ell^r(\ell(S_{r+2})) = \ell^r(S_{r+1}) = S_1$$

using first Lemma 8.20 and then the induction hypothesis. ■

This, with Theorem 8.17, more or less proves the following.

8.22 THEOREM. *For each $r < \omega$ the boss space S_{r+1} is $(r + 1)$-neat but not r-neat.*

Proof. By Theorem 8.17 the space S_{r+1} is certainly neat, so it suffices to determine its stacking rank. By Lemma 8.21 we have $\ell^r(S_{r+1}) = S_1 \neq \{\star\}$ so that space is not r-neat. But this also gives $\ell^{r+1}(S_{r+1}) = \ell(S_1) = \{\star\}$ so that space is $(r + 1)$-neat. ■

This result shows that the neatness hierarchy does not collapse before ω. Later we will indicate why the hierarchy never collapses. Before that let's look at the space S_ω.

It is not too hard to show that $\ell(S_\omega) = S_\omega$, and hence this space is far from neat. We generalize this observation. A subtree $\mathbb{T}$ of $\mathbb{S}_\omega$ is just a lower section. Many of these can be used to produce closed sets.

8.23 DEFINITION. A $\mathbb{T}$ of $\mathbb{S}_\omega$ is rampant if $I(x) \cap \mathbb{T} \notin \mathcal{S}(x)$ for each $x \in \mathbb{T}$. ■

In other words a rampant subtree is 'almost all' of $\mathbb{S}_\omega$. Such subtrees are easy to generate. For each uncountable $J \subseteq I$ we may view the full J-splitting tree as a subtree of $\mathbb{S}_\omega$. Such a subtree is rampant. There are many other rampant subtrees. For instance, start from the root $\perp$, and select an uncountable subset of $I(\perp)$. For each x in this subset select an uncountable subset of $I(x)$. Then for each y in each of these sets select an uncountable subset of $I(y)$. Repeat this process through all levels.

The following result is a trivial consequence of being rampant.

8.24 THEOREM. *Let $\mathbb{T}$ be a rampant subtree of $\mathbb{S}_\omega$. Then $X = \{\star\} \cup \mathbb{T}$ is closed with $\ell(X) = X$, and the pair $(\{\star\}, \{\star\} \cup \mathbb{T})$ is a V-point of S_ω.*

To conclude we indicate how the higher levels of neatness can be achieved.

Suppose for some ordinal α we have an example of a tree $\mathbb{S}$ which produces a $(\alpha+1)$-neat space which is not α-neat. We form a new tree $\mathbb{S}^+$ by sticking uncountable many copies of $\mathbb{S}$ above a new root. (This is the process which takes us from $\mathbb{S}_{r+1}$ to $\mathbb{S}_{r+2}$.) A few calculations shows that $\mathbb{S}^+$ produces $(\alpha + 2)$-neat space which is not $(\alpha + 1)$-neat.

Suppose λ is a limit ordinal and suppose for $\alpha < \lambda$ we have a tree $\mathbb{S}_\alpha$ which produces a a $(\alpha+1)$-neat space which is not α-neat. (We certainly have such examples for $\lambda = \omega$.) Form a new tree $\mathbb{S}_\lambda$ by taking uncountably many copies of each $\mathbb{S}_\alpha$ and stick them above a new root. Again it can be checked that $\mathbb{S}_\lambda$ produces a space which is $\lambda+1$-neat but not λ-neat. The details of these calculations can be found in [9].

You may have spotted that we haven't exhibited a space that is ω-neat but not r-neat for each $r < \omega$. Such an example can be found in [9]. It is interesting to note that this is the boss space of a non-well-founded tree.

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