Actions of tori and finite fans

Sonia L. Rueda
Departamento de Matemáticas. E.T.S. Arquitectura
Universidad Politécnica de Madrid
Madrid, Spain
E-mail:srueda@aq.upm.es

February 17, 2004

Abstract

Let k be an algebraically closed field of characteristic 0, $X = k^r \times (k^\times)^s$ and let G be an algebraic torus acting diagonally on X. We construct a fan Δ such that the quotient $Y//G$ is isomorphic to the toric variety determined by Δ and $\mathcal{D}(X) = \mathcal{D}(Y)$, for a distinguished G-invariant open subset Y of k^n. The main goal of this construction is to give necessary and sufficient conditions on Δ for $\mathcal{D}(X)^G$ to have enough simple finite dimensional modules.

Let G be a reductive group acting on the smooth affine variety $X = k^r \times (k^\times)^s$, with k an algebraically closed field k of characteristic 0. We denote the ring of regular functions on X by $\mathcal{O}(X)$ and the ring of differential operators by $\mathcal{D}(X)$. Let $\mathcal{D}(X)^G$ be the subring of $\mathcal{D}(X)$ of invariants under the action of G. There are several papers where actions of tori and finite fans are related, [4], Chaper VI, [5], [11] and more recently [2], [3]. We would like to associate a finite fan of cones to the action of G on X, in such a way that the study of the fan will allow us to get conclusions about the finite dimensional $\mathcal{D}(X)^G$-modules.

Suppose that $n = r + s$. Given a fan (N, Δ), we consider the following open subset of k^n, $Y = \{x \in k^n | x_i \neq 0 \text{ for } i \notin \{1, \ldots, r\}\}$, where r is the number of one dimensional cones of Δ. We say that a finite fan Δ is *associated to the action* of G on X, if the following conditions hold,

1. (A1) the quotient variety $Y//G$ is isomorphic to the toric variety determined by the fan Δ, $X(\Delta)$,
2. (A2) codim $X \backslash Y \geq 2$.

In this paper, G will be a finite dimensional algebraic torus acting diagonally on X.

Proposition A *There exists a fan Δ associated to the action of the algebraic torus G on X.*

The proof of Proposition A is constructive, we give a method to obtain a fan associated to the action of G on X. This result was motivated by the work of I.M. Musson in [11]. Given a finite fan Δ he gives an action of G on Y such that the variety of closed orbits $Y//G$ is isomorphic to the toric variety determined by Δ . A similar result was proved by D.A. Cox in [5]. We consider Proposition A a converse of this results, since our point of departure is the action of G on X. The variety Y is relevant to us because it serves as a bridge between the action of G on X and the fan Δ, condition (A1) explains this connection. In fact, we define Y as in [11]. Furthermore, Y is a toric subvariety of X, which is a toric variety for a dense torus T and G is a subtorus of T. The variety Y admits a good quotient by the action of G. The existence of good quotients of a toric variety by a subtorus action was studied in a recent paper by A. A'Campo-Neuen and J. Hausen, [2]. Also, the open subsets of a normal variety which admit a good quotient by a torus action have been described in [9].

We call $V = k^r, W = (k^\times)^s$ then $X = V \times W \subseteq k^n$, where $n = r + s$. Let us suppose that G acts transitively on X. The connected component H^o of the identity in H is a torus but we may have $H \neq H^o$ and then H/H^o is a finite group. Let H be the stabilizer of w in W. The following result reflects the connection existing between the action of G on X and the action of H on V.

Proposition B *Δ is a fan associated to the action of G on X if and only if Δ is a fan associated to the action of H on V.*

The main goal of this paper is to give necessary and sufficient conditions on Δ for $\mathcal{D}(X)^G$ to have a nonzero finite dimensional module. In a recent work with Musson [13], we show that if $\mathcal{D}(X)^G$ has a nonzero finite dimensional module then $\mathcal{D}(X)^G$ has enough simple finite dimensional modules. We say that a k-algebra R has enough simple finite dimensional modules if $\cap ann_R M = 0$, where the intersection is taken over all simple finite dimensional R-modules, [13].

Condition (A2) implies that $\mathcal{D}(X) = \mathcal{D}(Y)$, so we can transfer our attention to the study of $\mathcal{D}(Y)^G$-modules. We say that a finite fan is contained in a half-space if the intersection of its dual cones is not zero.

Proposition C *The $\mathcal{D}(Y)^G$-module $\mathcal{O}(Y)^G$ is finite dimensional if and only if the fan Δ is not contained in a half-space.*

This will allow us to prove the following theorem.

Theorem D *The following conditions are equivalent.*

1. *$\mathcal{D}(X)^G$ has a nonzero finite dimensional module.*

2. *There exists a fan Δ not contained in a half-space and associated to the action of G on X.*

When $V^{H^o} = 0$, we can modify a fan associated to the action of G on X to get a fan which is not contained in a half-space and it is associated to an action that is different from the original one but gives the same invariant differential operators. This fact allowed us to realize that $V^{H^o} = 0$ is a necessary and sufficient condition for $\mathcal{D}(X)^G$ to have a nonzero finite dimensional module, as proved in [13] without the use of fans.

The paper is organized as follows. In § 1, we introduce some notation about actions of tori, finite fans and rings of differential operators. Section 2 contains a method to construct fans that will be proved to be associated to the action of G on X. We prove Proposition B in § 3. In § 4, we prove Proposition C and Theorem D. The last section, contains a description of the members of the family of finite dimensional simple $\mathcal{D}(X)^G$-modules $\{\mathcal{O}(Y)_\chi\}_{\chi \in \mathbb{Z}^m}$, in terms of the fan. This family was proved to have enough members in [13]. We show that the dimension of $\mathcal{O}(Y)_\chi$ is the number of lattice points inside a certain polytope (i.e. a bounded polyhedron). This computation can be done with LattE.

1 Notation

1.1 Actions of Tori

Set $\mathbb{X}(G) = Hom(G, k^\times)$, $\mathbb{Y}(G) = Hom(k^\times, G)$, the groups of characters and one-parameter subgroups of G, respectively.

A *diagonal action* of a torus G on X is an action that extends to a diagonal action on k^n. Such an action is given by an embedding of G into the group T of diagonal matrices in $GL(n)$. Details about this action are given in [13], §1.1 and the following concepts are described. There exist $\eta_1, \ldots, \eta_n \in \mathbb{X}(G)$ such that *G acts on X with weights $\eta_1, \ldots, \eta_n$* . Identify G with $(k^\times)^m$ and $\mathbb{X}(G)$ with $\mathbb{Z}^m$. We think of $\mathbb{X}(G)$ as a space of column vectors with integer entries. We call L the $n \times m$ matrix whose i-th column vector is η_i, $i = 1, \ldots, n$. We say that *G acts on X by the matrix L.*

Let $\psi : \mathbb{X}(T) \longrightarrow \mathbb{X}(G)$ be the restriction map. This map is given by multiplication by L. There is a natural bilinear pairing

$$(\ , \ ) : \mathbb{X}(T) \times \mathbb{Y}(T) \longrightarrow \mathbb{Z}. \tag{1}$$

defined by the requirement that

$$(a \circ b)(\lambda) = \lambda^{(a,b)} \tag{2}$$

for all $a \in \mathbb{X}(T)$, $b \in \mathbb{Y}(T)$ and $\lambda \in k^\times$.

We will assume that G acts faithfully on X. Therefore L has rank m. Let $l = n - m$.

Lemma 1.1. *Assume that $\{\eta_{r+1}, \ldots, \eta_n\}$ are linearly independent. There exist matrices $\Gamma \in GL_m(\mathbb{Z})$, $\Delta \in GL_n(\mathbb{Z})$ such that*

$$\Gamma L \Delta = \begin{bmatrix} b_{11} & \ldots & b_{1l} & d & 0 & \ldots & 0 \\ b_{21} & \ldots & b_{2l} & 0 & d & \ldots & 0 \\ \vdots & \ddots & \vdots & & & \ddots & \\ b_{m1} & \ldots & b_{ml} & 0 & 0 & \ldots & d \end{bmatrix}. \tag{3}$$

where d is a nonzero integer.

Proof. Let $m' = m - s$. Since $\{\eta_1, \ldots, \eta_n\}$ contains m linearly independent vectors, there exist $\eta_{i_1}, \ldots, \eta_{i_{m'}} \in \{\eta_1, \ldots, \eta_n\}$ such that $\eta_{i_1}, \ldots, \eta_{i_{m'}}, \eta_{r+1}, \ldots, \eta_n$ are linearly independent. There exists $\Delta \in GL_n(\mathbb{Z})$ such that the last m' columns of $L\Delta$ are $\eta_{i_1}, \ldots, \eta_{i_{m'}}$. Let Γ' be the $m \times m$ matrix whose i-th column vector is the $(l+i)$-ith column of $L\Delta$, $i = 1, \ldots, m$. Then $d := |det\Gamma'| \neq 0$. Let $\Gamma = d\Gamma'^{-1}$, then the $m \times n$ matrix with integer coefficients $\Gamma L\Delta$ will look like (3). $\square$

If $\{\eta_{r+1}, \ldots, \eta_n\}$ are linearly independent, by Lemma 1.1 and [13], equations (15) and (16), we assume that the matrix L has the special form (3).

1.2 Finite fans

As far as possible we follow the notation of [8] , Chapter 1. Let $N \simeq \mathbb{Z}^l$ be the l-dimensional lattice. Let $(N, \mathit{\Delta})$ be a fan in N. Recall that each $\sigma \in \mathit{\Delta}$ is a strongly convex rational polyhedral cone in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $M = Hom_{\mathbb{Z}}(N, \mathbb{Z})$ and $< \_ , \_ >: M \times N \to \mathbb{Z}$ the natural bilinear pairing. For each $\sigma \in \mathit{\Delta}$, let

$$\Lambda_\sigma = M \cap \sigma^\vee = \{u \in M | < u, v > \geq 0 \text{ for all } v \in \sigma\} \tag{4}$$

and $U_\sigma = Speck[\Lambda_\sigma]$ is a semigroup algebra. By [8], Theorem 1.4 we can glue U_σ to obtain a toric variety $X(\mathit{\Delta})$.

Denote by $\mathit{\Delta}(1)$ the set of cones of $(N, \mathit{\Delta})$ with dimension one. Given $v \in N$ let $\tau_v = \mathbb{R}_+ v$ be the ray generated by $v \in N$. Let $v, v' \in N$, if $v = cv'$ with $c > 0$ then $\tau_v = \tau_{v'}$. Suppose that $\mathit{\Delta}(1) = \{v_1, \ldots, v_r\}$. Given $\sigma \in \mathit{\Delta}$ we define $[\sigma] = \{i \in \{1, \ldots, r\} | \tau_{v_i} \text{ is a face of } \sigma\}$. Then $\sigma = \sum_{i \in [\sigma]} \tau_{v_i}$.

If $u \in M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, a subset of the form

$$H_u = \{v \in N_{\mathbb{R}} | < u, v > \geq 0\} \tag{5}$$

with $u \neq 0$ is called a *half-space* in $N_{\mathbb{R}}$, see [12], §1. We will say that the fan $(N, \mathit{\Delta})$ is *contained in a half-space* if we can find $0 \neq u \in M_{\mathbb{R}}$ such that $\sigma \subseteq H_u$ for all $\sigma \in \mathit{\Delta}$. Equivalently, if the intersection of its dual cones is not zero.

1.3 Coordinate rings and rings of differential operators.

In this section, we gather some definitions and results from [13], §2. Note that X is a toric variety with a dense torus $T = (k^\times)^n \subseteq X$. Write Q_i for the character e_i considered as a regular function on T. Then

$$\mathcal{O}(X) = k[Q_1, \ldots, Q_r, Q_{r+1}^{\pm 1}, \ldots, Q_n^{\pm 1}]. \tag{6}$$

We consider the action of G on $\mathcal{O}(T)$ (or $\mathcal{O}(X)$) given by right translation. This convention implies that Q_i has weight η_i. Let $P_i = \partial / \partial Q_i$,

$$\mathcal{D}(X) = k[Q_1, \ldots, Q_r, Q_{r+1}^{\pm 1}, \ldots, Q_n^{\pm 1}, P_1, \ldots, P_n]. \tag{7}$$

If $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^r \times \mathbb{Z}^s, \mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$, set $Q^\lambda = Q_1^{\lambda_1} \dots Q_n^{\lambda_n}$, and $P^\mu = P_1^{\mu_1} \dots P_n^{\mu_n}$. The elements $Q^\lambda P^\mu \in \mathcal{D}(X)$, with $L\lambda = L\mu$, form a basis of $\mathcal{D}(X)^G$.

Let Y be an open subset of X. Define the codimension of $X \backslash Y$ in k^n, to be

$$\text{codim } X \backslash Y = \dim k^n - \dim X \backslash Y.$$

Proposition 1.2. *If codim $X \backslash Y \geq 2$, then $\mathcal{O}(X) = \mathcal{O}(Y)$ and $\mathcal{D}(X) = \mathcal{D}(Y)$.*

Proof. The result follows from [10], Proposition II.2.2. □

2 Fans associated to the action of G.

Let us describe Y in detail.

2.1 The set Y.

Let (N, Δ) be a finite fan. For every $\sigma \in \Delta$ we define $x^{\hat{\sigma}} = \prod_{i \notin [\sigma]} x_i$ and we consider the T-invariant open sets

$$V_\sigma = k^n - Z(x^{\hat{\sigma}}) \tag{8}$$

where $Z(x^{\hat{\sigma}}) = \{x \in k^s | x^{\hat{\sigma}} = 0\}$. Let

$$Z = \cap_{\sigma \in \Delta} Z(x^{\hat{\sigma}}). \tag{9}$$

Hence Z is closed and T-invariant. We have an open subset

$$Y = k^n - Z = \cup_{\sigma \in \Delta} V_\sigma \tag{10}$$

of an affine space k^n. Note that Y might no longer be affine. These sets were introduced in [11], §1.3. See also [5], Theorem 2.1.

We determine the irreducible components of Z. For $I \subseteq \{1, \dots, n\}$ set $Z_I = \{x \in k^n | x_i = 0 \text{ if } i \in I\}$.

Lemma 2.1. *Any T-invariant irreducible closed set in k^n is some Z_I.*

Proof. See [8], §3.1. □

By Lemma 2.1, Z is a union of irreducible closed subsets Z_I. Observe that when $I \subseteq J$ then $Z_J \subseteq Z_I$ for $I, J \subseteq \{1, \dots, n\}$. Therefore, the irreducible components that occur in Z are the ones in the family $\mathcal{I}$ of subsets of $\{1, \dots, n\}$ verifying the following statements.

1. $Z_I \subseteq Z$ and;

2. I is minimal verifying the previous condition, i.e. there is no $J \subseteq \{1, \dots, n\}$, $J \subsetneq I$ such that $Z_J \subseteq Z$.

Thus, $Z = \cup_{I \in \mathcal{I}} Z_I$.

2.2 Construction of the fan associated to the action.

We will use the following lemma to develop our construction.

Lemma 2.2. *There exists an $n \times l$ matrix E that satisfies the following statements.*

1. *The rows of E generate N as a group.*
2. *The columns of E are a $\mathbb{Z}$-basis of ker ψ.*

Proof. By [1], Theorem 12.4.3, there exist matrices $Q \in GL_m(\mathbb{Z})$ and $P \in GL_n(\mathbb{Z})$ such that

$$L' = QLP = \begin{bmatrix} d_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & d_2 & & 0 & & & \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & & d_m & 0 & \dots & 0 \end{bmatrix} \tag{11}$$

with $d_i \neq 0$ for all $i = 1, \dots m$. Let I_l be the identity $l \times l$ matrix and E' the $n \times l$ matrix with I_l in the last l rows and zeroes in the first m rows. Then $L'E' = 0$. We define $E := PE'$. Let us prove that E satisfies statements 1 and 2.

1. Let $\bar{P}$ be the matrix obtained by deleting the first m rows of P^{-1}. From the definition of E we get easily that $I_l = \bar{P}E$. This proves that the rows of E generate N as a group.
2. The columns of E are elements of ker ψ because $LE = 0$. Given any $\lambda \in$ ker ψ then $L'P^{-1}\lambda = 0$. The columns of E' are a $\mathbb{Z}$-basis of the kernel of L'. Then there exist $z_1, \dots, z_l \in \mathbb{Z}$ such that

$$P^{-1}\lambda = E' \begin{bmatrix} z_1 \\ \vdots \\ z_l \end{bmatrix}, \text{ therefore } \lambda = E \begin{bmatrix} z_1 \\ \vdots \\ z_l \end{bmatrix}.$$

   This proves that the columns of E generate ker ψ as a group and since ker ψ has rank l the result follows.

□

Let E be an $n \times l$ matrix satisfying the statements of Lemma 2.2. We can identify $B = \mathbb{Y}(T)$ with $\mathbb{Z}^n$ and think of it as a space of row vectors with integer entries. Define

$$\varphi : B \longrightarrow N \tag{12}$$

by $\varphi(e) = eE$ for all $e \in B$. By Lemma 2.2(1) φ is onto. Let $K =$ ker φ. Then K is a free abelian group of rank m.

Let e_i be the ith standard basis vector for B and call $v_i = \varphi(e_i)$ the ith row vector of E , $1 \leq i \leq n$. The matrix E has rank l; hence the subset $\{v_1, \dots, v_n\}$ of $N = \mathbb{Z}^l$ contains l linearly independent vectors. Observe that $\{v_1, \dots, v_n\}$ could contain elements that are equal and also the zero element.

Let Δ be any fan in N with $\Delta(1) = \{\tau_{v_i} | i = 1, \dots, r\}$. We will prove that such a fan is associated to the action of G on X.

Example 2.3. *Let $r = 4$, $s = 0$, $m = 2$; then $l = 2$. Let*

$$L = \begin{bmatrix} 3 & 3 & 2 & 0 \\ 4 & 4 & 0 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} -1 & -2 \\ 1 & 0 \\ 0 & 3 \\ 0 & 4 \end{bmatrix},$$

hence $v_1 = (-1, -2)$, $v_2 = (1, 0)$, $v_3 = (0, 3)$ and $v_4 = (0, 4)$. Then Δ could be the fan with maximal cones $\sigma_1, \sigma_2, \sigma_3$, where $[\sigma_1] = \{1\}$, $[\sigma_2] = \{2\}$ and $[\sigma_3] = \{3, 4\}$.

2.3 Proof of Proposition A.

1. Let $K^{\perp} = \{\lambda \in \mathbb{X}(T) | (\lambda, K) = 0\}$. Then $K^{\perp} = \ker \psi$. There is an isomorphism $w : M \to K^{\perp}$ given by
$$< x, \varphi(b) >= (w(x), b) \tag{13}$$
for all $x \in M$, $b \in B$. By equation (13), it can be proved in the same way as [11], Theorem 1 that the variety of closed orbits $Y//G$ is isomorphic to $X(\Delta)$.

2. Consider the family $\mathcal{I}' = \{I \in \mathcal{I} | |I| = 1\}$ and define
$$\hat{Z} := \cup_{I \in \mathcal{I}'} Z_I. \tag{14}$$
Since $\cup_{\sigma \in \Delta} [\sigma] = \{1, \ldots, r\}$, then $X = k^n - \hat{Z}$ and $X \backslash Y = Z \backslash \hat{Z}$. By (14), $X \backslash Y = \cup_{I \in \mathcal{I}''} Z_I$ with $\mathcal{I}'' = \mathcal{I} \backslash \mathcal{I}' = \{I \in \mathcal{I} | |I| \geq 2\}$. We also have codim $\cup_{I \in \mathcal{I}''} Z_I = \inf_{I \in \mathcal{I}''}$codim Z_I and codim $Z_I = |I| \geq 2$ for all $I \in \mathcal{I}''$. Therefore codim $X \backslash Y \geq 2$.

Remark 2.4. *There is a canonical morphism $p : Y \longrightarrow X(\Delta)$ such that $X(\Delta)$ is isomorphic to the geometric quotient $Y//G$. We have a covering U_σ of $X(\Delta)$ with U_σ isomorphic to $V_\sigma//G$, for each $\sigma \in \Delta$. Also, $p_{|V_\sigma} : V_\sigma \longrightarrow U_\sigma$ is the categorical quotient of G restricted to V_σ. Therefore, the morphism p is a good quotient as defined in [2], §3.*

3 Fans associated to the action of H on V.

Given a finite fan Δ, for each $\sigma \in \Delta$, define $V'_\sigma = \{x \in k^r | x_i \neq 0 \text{ if } i \notin [\sigma]\}$. Then $V_\sigma = V'_\sigma \times W$, recall that $n = r + s$.

Suppose G is a torus acting faithfully on V_σ with weights $\eta_1, \ldots, \eta_n$. We assume that G acts transitively on W, then by [13], Lemma 3.1, $\eta_{r+1}, \ldots, \eta_n$ are linearly independent. Let $w = (w_{r+1}, \ldots, w_n)$ be an element of W. Then $H = G_w = \cap_{i=r+1}^{n} \ker \eta_i$. It can be proved in the same way as [13], Lemma 3.2, that the slice representation at w, [7], [15], is isomorphic to (H, V_σ).

Consider the H-invariant open subset of V, $Y' = \cup_\sigma V'_\sigma$. This is the variety defined in (10) for the case $n = r$.

Theorem 3.1. *The varieties $Y//G$ and $Y'//H$ are isomorphic.*

Proof. Given $\sigma \in \mathit{\Delta}$. Part of the Luna slice theorem states that there is a closed H-stable subvariety S_σ containing w and a G-equivariant étale map $G \times^H S_\sigma \longrightarrow V_\sigma$. Taking $S_\sigma = V'_\sigma + w$ we get a G-equivariant isomorphism $\delta_\sigma : G \times^H S_\sigma \longrightarrow V_\sigma$ and this map induces an isomorphism between $V_\sigma//G$ and $V'_\sigma//H$, this can be proved as [13], Theorem 6.2.

If τ is a face of σ, then $V_\tau \subseteq V_\sigma$, $V'_\tau \subseteq V'_\sigma$ and the isomorphism $V_\sigma//G \cong V'_\sigma//H$ restricts to the isomorphism $V_\tau//G \cong V'_\tau//H$. Thus, we may identify $Y//G = \cup_\sigma V_\sigma//G$ with $Y'//H = \cup_\sigma V'_\sigma//H$. □

3.1 Proof of Proposition B

By Theorem 3.1, $Y//G$ is isomorphic to $X(\mathit{\Delta})$ if and only if $Y'//H$ is. Let as prove that $\operatorname{codim} X\backslash Y \geq 2$ if and only if $\operatorname{codim} V\backslash Y' \geq 2$.

We have $Y = \{x \in k^n | x_i \neq 0 \text{ for } i \notin \cup[\sigma]\}$ and $Y' = \{x \in k^r | x_i \neq 0 \text{ for } i \notin \cup[\sigma]\}$. If $\operatorname{codim} X\backslash Y \geq 2$ then $\mathcal{O}(X) = \mathcal{O}(Y)$, therefore $\cup[\sigma] = \{1, \ldots, r\}$. By the proof of Proposition A (2) for the case $n = r$ then $\operatorname{codim} V\backslash Y' \geq 2$. Conversely if $\operatorname{codim} V\backslash Y' \geq 2$, then $\mathcal{O}(V) = \mathcal{O}(Y')$ so $\cup[\sigma] = \{1, \ldots, r\}$ and the by proof of Proposition A (2) the result follows.

3.2 $\mathcal{D}(X(\mathit{\Delta}))$-modules.

Set $\mathfrak{h} = Lie(H) \subseteq \mathfrak{g} = Lie(G)$. For $\lambda \in \mathfrak{g}^*$, $\mu \in \mathfrak{h}^*$ we set

$$\mathcal{B}_\lambda(X) = \mathcal{D}(X)^G/(\mathfrak{g} - \lambda(\mathfrak{g}))\,, \qquad \mathcal{B}_\mu(V) = \mathcal{D}(V)^H/(\mathfrak{h} - \mu(\mathfrak{h})). \tag{15}$$

Here $(\mathfrak{g} - \lambda(\mathfrak{g}))$ is the ideal generated by all elements of the form $x - \lambda(x)$, with $x \in \mathfrak{g}$, and $(\mathfrak{h} - \mu(\mathfrak{h}))$ is defined similarly. Let $i^* : \mathfrak{g}^* \longrightarrow \mathfrak{h}^*$ be the map obtained from the inclusion $i : \mathfrak{h} \longrightarrow \mathfrak{g}$.

By [13], Proposition C, there is an injective algebra homomorphism $\mathcal{D}(V)^H \longrightarrow \mathcal{D}(X)^G$. If $\lambda \in \mathfrak{g}^*$ and $\mu = i^*(\lambda)$, the previous map induces an isomorphism $\mathcal{B}_\mu(V) \cong \mathcal{B}_\lambda(X)$ and by [11], Theorem 5 they are isomorphic to $\mathcal{D}(X(\mathit{\Delta}))$. Note that any simple $\mathcal{D}(X)^G$-module is a $\mathcal{B}_\lambda(X)$-module for some $\lambda \in \mathfrak{g}^*$. So we can reduce the study of finite dimensional simple $\mathcal{D}(X)^G$-modules to that of finite dimensional simple $\mathcal{D}(V)^H$-modules and also to the study of $\mathcal{D}(X(\mathit{\Delta}))$-modules.

In [14] it is shown that the category of $\mathcal{D}(X(\mathit{\Delta}))$-modules is equivalent to a category of graded $\mathcal{D}(V)$-modules modulo $\mathfrak{b}$-torsion, with $\mathfrak{b} = Z$ defined by equation (9) for $s = 0$.

4 Fans not contained in a half-space.

In this section we include some lemmas that will be used to prove Proposition C and Theorem D.

Suppose $I \subseteq \{1, \ldots, r\}$. For $1 \leq i \leq n$, set

$$\varsigma_i = \begin{cases} -\eta_i \text{ if } i \in I \\ \eta_i \text{ if } i \notin I \end{cases}. \tag{16}$$

Let L_I be the matrix with columns $\varsigma_1, \ldots, \varsigma_n$. Then G_I denotes the m-dimensional torus acting on X by the matrix L_I. By [13], Lemma 5.2, the map $\sigma_I : \mathcal{D}(X) \to \mathcal{D}(X)$ defined by

$$\sigma_I(Q_i) = \begin{cases} -P_i \text{ if } i \in I \\ Q_i \text{ if } i \notin I \end{cases} \qquad \sigma_I(P_i) = \begin{cases} Q_i \text{ if } i \in I \\ P_i \text{ if } i \notin I \end{cases} \tag{17}$$

$i = 1, \ldots, n$ is an isomorphism between $\mathcal{D}(X)^G$ and $\mathcal{D}(X)^{G_I}$. Therefore G_I and G have the same invariant differential operators.

Lemma 4.1. *When the matrix L is of the special kind (3), then $v_1, \ldots, v_l$ are linearly independent.*

Proof. By Lemma 2.2, $LE = 0$ and the rows $v_1, \ldots, v_n$ of E generate N as a group. The equation $LE = 0$ means that for $i = 1, \ldots, m$

$$dv_{l+i} = -\sum_{j=1}^{l} b_{ij} v_j. \tag{18}$$

Thus $v_{l+1}, \ldots, v_n$ belong to the $\mathbb{R}$-span of $v_1, \ldots, v_l$. The result follows from this. $\square$

Let us suppose that L is of the special kind (3) and let Δ be a fan as in § 2.2. By Lemma 4.1, $\mathcal{B} = \{v_1, \ldots, v_l\}$ is a basis of $N_{\mathbb{R}}$. With respect to $\mathcal{B}$ the vectors $v_{l+1}, \ldots, v_n$ have coordinates

$$v_j = (-\frac{1}{d} b_{j-l,1}, \ldots, -\frac{1}{d} b_{j-l,l}), \qquad j = l+1, \ldots, n. \tag{19}$$

Let $m' = r - l$. For $i = 1, \ldots, l$, let ρ_i be the vector in $\mathbb{Z}^{m'}$ obtained deleting the last $m - m'$ entries of η_i.

Lemma 4.2. *If $\rho_i = 0$ for some $i \in \{1, \ldots, l\}$, then Δ is contained in a half-space.*

Proof. Consider the basis $\mathcal{B}$ in N. Let $u \in M_{\mathbb{R}}$ such that $< u, v_j >= 0$ if $j \neq i$, $j \in \{1, \ldots l\}$ and $< u, v_i >= 1$. Then $< u, v_j >= 0$, for all $j = l+1, \ldots, n$. Therefore Δ is contained in the half-space H_u. $\square$

Lemma 4.3. *If $\mathcal{O}(Y)^G = k$, then $\eta_{r+1}, \ldots, \eta_n$ are linearly independent.*

Proof. It follows from Proposition 1.2 and [13], Lemma 4.1. $\square$

4.1 Proof of Proposition C.

Let

$$\phi_\sigma := \{\lambda \in K^\perp | (\lambda, e_i) \geq 0 \text{ for all } i \in [\sigma]\}. \tag{20}$$

Then $\mathcal{O}(V_\sigma)^G = k[\phi_\sigma]$. Hence $\mathcal{O}(Y)^G = k$ if and only if $\cap_{\sigma \in \Delta} \phi_\sigma = 0$. Furthermore, $w(\Lambda_\sigma) = \phi_\sigma$. Hence $0 \neq u \in \cap_{\sigma \in \Delta} \sigma^\vee$ if and only if Δ is contained in the half-space H_u. This proves the result.

Remark 4.4. *Let us call G' the m-dimensional torus acting on X by a matrix L'. Let Δ' be a fan associated to the action of G'. Suppose that $\mathcal{O}(X)^G = \mathcal{O}(X)^{G'}$. By Proposition C, Δ is contained in a half-space if and only if Δ' is.*

4.2 Proof of Theorem D.

(1)⇒(2) By [13], Theorem B and Lemma 5.1, there is a subset I of $\{1,\ldots,r\}$ such that $\mathcal{O}(X)^{G_I}=k$. By Proposition A, there exists a fan Δ associated to the action of G_I on X. By Proposition C, Δ is not contained in a half-space.

(2)⇒(1) By Proposition C, $\mathcal{O}(Y)^G=k$. By Lemma 4.3, Remark 4.4, and Lemma 4.2, $\rho_i\neq 0$ for all $i=1,\ldots,r$. By [13], Lemma 3.3 and Theorem B the result follows.

4.3 Construction of an associated fan not included in a half-space.

By [13], Theorem B, if $V^{H^o}=0$ then $\mathcal{D}(X)^G$ has a nonzero finite dimensional module and by Theorem D there exists a fan Δ associated to the action of G on X and not contained in a half-space. By [13], Lemma 3.3., $V^{H^o}=0$ if and only if $\rho_i\neq 0$ for all $i=1,\ldots r$.

Suppose that $\rho_i\neq 0$ for all $i=1,\ldots,l$, then L is of the special kind (3). We give a construction of a fan associated to the action of G and not contained in a half-space.

Let $v_1^*,\ldots,v_l^*$ be the dual basis of $\mathcal{B}$. Given $j\in\{l+1,\ldots,r\}$, let

$$I_j^0=\{i\in\{1,\ldots,l\}|<v_i^*,v_j>=0\},\tag{21}$$

$$I_j^+=\{i\in\{1,\ldots,l\}|<v_i^*,v_j>>0\},\tag{22}$$

$$I_j^-=\{i\in\{1,\ldots,l\}|<v_i^*,v_j><0\},\tag{23}$$

and

$$I_j=I_j^+\cup I_j^-.\tag{24}$$

Then there exists $J\subseteq\{l+1,\ldots,r\}$ such that

$$\cup_{j\in J}I_j=\{1,\ldots,l\}\tag{25}$$

because $\rho_i\neq 0$,

$$\rho_i=\begin{bmatrix} b_{1i}\\ \vdots\\ b_{m'i}\end{bmatrix}\ \text{and}\ \frac{-1}{d}b_{j-l,i}=<v_i^*,v_j>,\ i=1,\ldots,l,\ j=l+1,\ldots,r.$$

Take J to be minimal verifying (25), and let $J=\{j_1,\ldots,j_c\}$ with $c\leq m'$ and

$$|I_{j_h}|\leq|I_{j_{h+1}}|\qquad h=1,\ldots,c-1.\tag{26}$$

These two assumptions will make the next computation shorter. We take a subset I of $\{1,\ldots,l\}$ in the following way:

$$I:=I_{j_1}^+\cup_{h=2}^{c}[(\cap_{t=1}^{h-1}I_{j_t}^0)\cap I_{j_h}^+]=\tag{27}$$

$$=I_{j_1}^+\cup(I_{j_1}^0\cap I_{j_2}^+)\cup(I_{j_1}^0\cap I_{j_2}^0\cup I_{j_3}^+)\cup\ldots\cup(I_{j_1}^0\cap\ldots\cap I_{j_{c-1}}^0\cap I_{j_c}^+).\tag{28}$$

Define

$$v_i^I = \begin{cases} -v_i \text{ if } i \in I \\ v_i \text{ if } i \notin I \end{cases}, i = 1, \ldots, r. \tag{29}$$

Let Δ_I be a fan in N with $\Delta_I(1) = \{\tau_{v_i^I} | i = 1, \ldots, r\}$. This fan is associated to the action of G_I on X.

Proposition 4.5. *Δ_I is not contained in a half-space.*

Proof. Suppose Δ_I is contained in the half-space H_u for some $u \in M_{\mathbb{R}}$, $u \neq 0$. Then $v_i^I \in H_u$ for all $i = 1, \ldots, r$. Let $u = u_1 v_1^* + \ldots + u_l v_l^*$. Then $u_i \geq 0$ for $i \notin I$ and $u_i \leq 0$ for $i \in I$.

Suppose $1 \leq i \leq r$ and consider three cases:

If $i \in I_{j_1}^0$, then $< v_i^*, v_{j_1} >= 0$.
If $i \in I_{j_1}^+$, then $< v_i^*, v_{j_1} > > 0$ and $u_i \leq 0$.
If $i \in I_{j_1}^-$, then $< v_i^*, v_{j_1} > < 0$ and $u_i \geq 0$.

In all cases we have $u_i < v_i^*, v_{j_1} >\leq 0$. Therefore $< u, v_{j_1} >\leq 0$. But $v_{j_1} \in H_u$ so $< u, v_{j_1} >= 0$. Thus $u_i = 0$ for all $i \in I_{j_1}$.

Analogously we can prove that $< u, v_{j_2} >= 0$ and therefore $u_i = 0$ for all $i \in I_{j_2} \backslash I_{j_1}$. Hence $u_i = 0$ for all $i \in I_{j_2} \cup I_{j_1}$. In this way we get that $u_i = 0$ for all $\cup_{j \in J} I_j = \{1, \ldots, l\}$. □

Example 4.6. *Let $n = r = 6$ and $m = 2$. The action of G on $kQ_1 + \ldots + kQ_6$ is given by the matrix*

$$L = \begin{bmatrix} 0 & -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}. \tag{30}$$

Then $v_1 = (1,0,0,0)$, $v_2 = (0,1,0,0)$, $v_3 = (0,0,1,0)$, $v_4 = (0,0,0,1)$, $v_5 = (0,1,-2,0)$, $v_6 = (-1,0,1,1)$. Also $J = \{5,6\}$ and $I = I_5^+ \cup (I_5^0 \cap I_6^+)$, with $I_5^+ = \{2\}$, $I_5^0 = \{1,4\}$ and $I_6^+ = \{3,4\}$. Therefore $I = \{2,4\}$.

5 Finite Polytopes.

Let us suppose that $\mathcal{D}(X)^G$ has a nonzero finite dimensional module. We can assume that L is of the special kind (3). Let Δ be a fan associated to the action of G on X and not contained in a half-space. Let Y be as in § 2.1. Define $\Lambda \subseteq \mathbb{Z}^m$ by $\Lambda = \{L\alpha | \alpha \in \mathbb{N}^r \times \mathbb{Z}^s\}$. For $\chi \in \Lambda$ define

$$\mathcal{O}(Y)_\chi = span\{Q^\lambda \in \mathcal{O}(Y) | L\lambda = \chi\}. \tag{31}$$

It is easy to see that

$$\mathcal{O}(Y) = \oplus_{\chi \in \Lambda} \mathcal{O}(Y)_\chi. \tag{32}$$

For each $\chi = (\chi_1, \ldots, \chi_m) \in \Lambda$, $\mathcal{O}(Y)_\chi$ is a simple $\mathcal{D}(Y)^G$-module by [13], Lemma 4.3 and Lemma 1.2. By [13], Lemma 4.1., $\mathcal{O}(Y)_\chi$ is finite dimensional. Let

$\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{N}^r \times \mathbb{Z}^s$ such that $L\varphi = \chi$. Given $\sigma \in \Delta$ and the $\mathcal{D}(Y)^G$-module $\mathcal{O}(V_\sigma)$, we can easily see that $\mathcal{O}(V_\sigma) = \oplus_{\chi \in \Lambda} \mathcal{O}(V_\sigma)_\chi$. Then

$$\mathcal{O}(V_\sigma)_\chi = span\{Q^\lambda \in \mathcal{O}(V_\sigma) | \lambda \in \varphi + K^\perp\}. \tag{33}$$

Let

$$\phi_{\sigma,\chi} := \{\lambda \in \varphi + K^\perp | (\lambda, e_i) \geq 0 \text{ for all } i \in [\sigma]\}. \tag{34}$$

We can write

$$\phi_{\sigma,\chi} = \{\varphi + \mu \in \varphi + K^\perp | (\mu, e_i) \geq -\varphi_i \text{ for all } i \in [\sigma]\}. \tag{35}$$

Observe that $\mathcal{O}(V_\sigma)_\chi = k[\phi_{\sigma,\chi}]$, by (8) $V_\sigma = \{x \in k^n | x_i \neq 0 \text{ for all } i \notin [\sigma]\}$; see also (33) and (34). Therefore

$$\mathcal{O}(Y)_\chi = \cap_{\sigma \in \Delta} k[\phi_{\sigma,\chi}] \tag{36}$$

since $Y = \cup_{\sigma \in \Delta} V_\sigma$.

Let us consider the following $r \times l$ matrix,

$$P = \begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \\ b_{11} & \ldots & b_{1l} \\ \vdots & & \vdots \\ b_{m'1} & \ldots & b_{m'l} \end{bmatrix}.$$

We denote by P_i the i-th row vector of P. Let $b = (b_1, \ldots, b_n) \in \mathbb{N}^r \times \mathbb{Z}^s$ such that

$$b_i = \begin{cases} \varphi_i \text{ if } i \in \{1, \ldots, l\} \\ d\varphi_i \text{ if } i \in \{l+1, \ldots, n\} \end{cases}. \tag{37}$$

Theorem 5.1. *The dimension of $\mathcal{O}(Y)_\chi$ is the number of lattice points inside the polytope*

$$\{x \in M_\mathbb{R} | < x, P_i > \leq b_i, i = 1, \ldots, r\}. \tag{38}$$

Proof. Define the sets

$$\psi_{\sigma,\chi} := \{\lambda \in K^\perp | (\lambda, e_i) \geq -\varphi_{i,\chi}, \text{ for all } i \in [\sigma]\}. \tag{39}$$

Then $\phi_{\sigma,\chi} = \varphi + \psi_{\sigma,\chi}$ where $\phi_{\sigma,m}$ is the set given in (34). Also $k[\phi_{\sigma,\chi}] = Q^\varphi k[\psi_{\sigma,\chi}]$. Therefore $\mathcal{O}(Y)_\chi = Q^\varphi(\cap_\sigma k[\psi_{\sigma,\chi}])$, by (36). Let

$$\Lambda_{\sigma,\chi} := \{x \in M | < x, v_i > \geq -\varphi_{i,\chi} \text{ for all } i \in [\sigma]\}. \tag{40}$$

Then $\psi_{\sigma,\chi} = w(\Lambda_{\sigma,\chi})$, with w as in (13), and $k[\Lambda_{\sigma,\chi}] \cong k[\psi_{\sigma,\chi}]$. Therefore, the dimension of $\mathcal{O}(Y)_\chi$ is the number of lattice points in the set $\cap_{\sigma \in \Delta} \Lambda_{\sigma,\chi}$. Henceforth the dimension of $\mathcal{O}(Y)_\chi$ is the number of lattice points in the polytope

$$\{x \in M_\mathbb{R} | < x, v_i > \geq -\varphi_i \text{ for all } i = 1, \ldots, r\}.$$

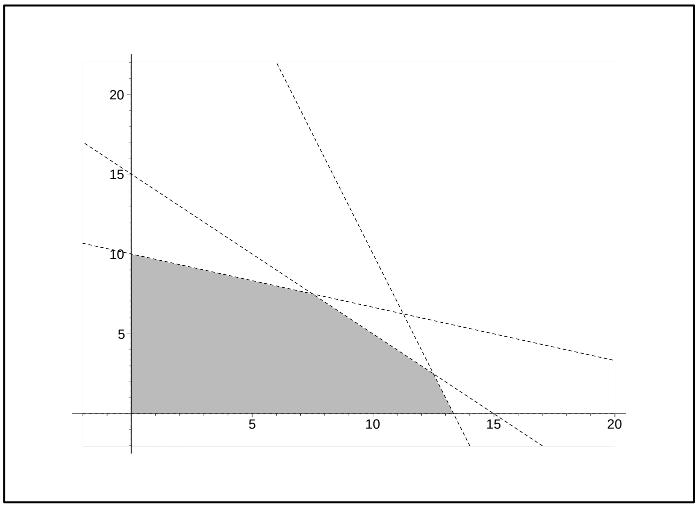
It can be easily seen that this polytope coincides with (38) setting $\mathcal{B}$ as $N_\mathbb{R}$ basis. □

5.1 Example

Assume that $\dim G = 3$ and $X = k^5$. Then $\mathcal{D}(X) = A_5$ is the 5-th Weyl algebra. Let the action of G on X be given by the matrix

$$L = \begin{bmatrix} 2 & 2 & 1 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 \end{bmatrix}. \tag{41}$$

We consider the A_5^G-module $\mathcal{O}(Y)_\chi$ with $\chi = (30, 30, 40)$. Then $\dim \mathcal{O}(Y)_\chi = 108$, the number of lattice points inside the polytope $\{(x_1, x_2) \in \mathbb{Z}^2 | x_1 \geq 0, x_2 \geq 0, 2x_1 + 2x_2 \leq 30, x_1 + 3x_2 \leq 30, 3x_1 + x_2 \leq 40\}$. The number of points inside the polytope was obtained with LattE, which is a recent computer package for lattice point enumeration [6]. The following picture show this polytope.



Acknowledgements

This is part of the author's PhD thesis written at the Mathematics Department of the University of Wisconsin-Milwaukee under the supervision of Professor Ian M. Musson. I would like to thank him for helpful comments on earlier drafts of this paper.

References

[1] M. Artin, *Algebra* (Prentice Hall, 1991).

[2] A. A'Campo-Neuen and J. Hausen, Quotients of toric varieties by the action of a subtorus, *Tohoku Math. Journal* **51** (1999).

[3] A. A'Campo-Neuen and J. Hausen, Toric prevarieties and subtorus actions, *Geom. Dedicata* **87** (2001), 35-64.

[4] M. Audin, *The Topology of Torus Actions on Symplectic Manifolds*, Progress in Mathematics, Vol. 93 (Birkhäuser, Basel, 1991).

[5] D.A. Cox, The homogeneous coordinate ring of a toric variety, *J. Algebraic Geom.* **4** (1995), no. 1, 17-50.

[6] J.A. De Loera, R. Hemmecke, J. Tauzer and R. Yoshida, Effective Lattice Point Counting in Rational Convex Polytopes, available via http://www.math.ucdavis.edu/ latte/theory.html.

[7] D. Luna, Slices étales. Sur les groupes algébriques, *Bull. Soc. Math. France* **33** (Soc. Math. France, Paris, 1973), 81-105.

[8] W. Fulton, *Introduction to toric varieties* (Princeton University Press, 1993).

[9] J. Hausen, Geometric invariant theory based on Weil divisors. Preprint available at (arXiv:amth.AG/0301204v2) 2003.

[10] T. Levasseur, Anneaux d'opérateurs différentiels, in: P. Dubreil et M.-P. Malliavin, eds., Séminaire d'Algébre, *Lecture Notes in Mathematics* **867** (Springer, 1981) 157-173.

[11] I.M. Musson, Differential operators on toric varieties, *J. Pure and Applied Algebra* **95** (1994), 303-315.

[12] I.M. Musson, Rings of differential operators on invariant rings of tori, *Trans. Amer. Math. Soc.* **303** (1987), 805-827.

[13] I.M. Musson and S.L. Rueda, Finite dimensional representations of invariant differential operators, *Trans Amer. Math. Soc.*, (accepted for publication). Preprint available at (arXiv:amth.RT/0305279v1) 2003.

[14] M. Mustaţă, G.G. Smith, H. Tsai and U. Walther, $\mathcal{D}$-modules on smooth toric varieties, *J. of Algebra* **240** (2001), 744-770.

[15] P. Slodowy, Der Scheibensatz für algebraische Transformationsgruppen (pp. 89–113); Algebraische Transformationsgruppen und Invariantentheorie. Edited by H. Kraft, P. Slodowy and T. A. Springer. DMV Seminar, **13**. Birkhuser Verlag, Basel, 1989.