CMPS 102
Introduction to Analysis of Algorithms
Fall 2003

Asymptotic Growth of Functions

We introduce several types of asymptotic notation which are used to compare the performance and efficiency of algorithms. As we'll see, the asymptotic run time of an algorithm gives a simple, and machine independent, characterization of it's complexity.

Definition Let $g(n)$ be a function. The set $O(g(n))$ is defined as

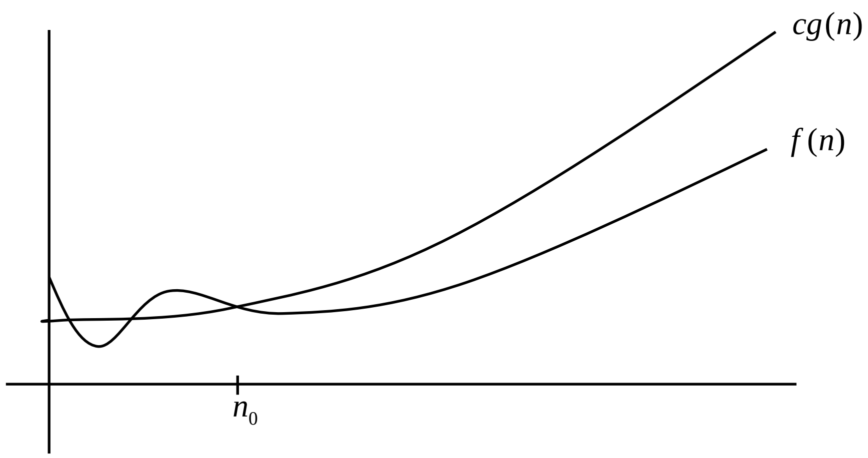
$$O(g(n)) = \{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \} .$$

In other words, $f(n) \in O(g(n))$ if and only if there exist positive constants *c*, and n_0, such that for all $n \geq n_0$, the inequality $0 \leq f(n) \leq cg(n)$ is satisfied. We say that $f(n)$ is *Big O* of $g(n)$, or that $g(n)$ is an *asymptotic upper bound* for $f(n)$.

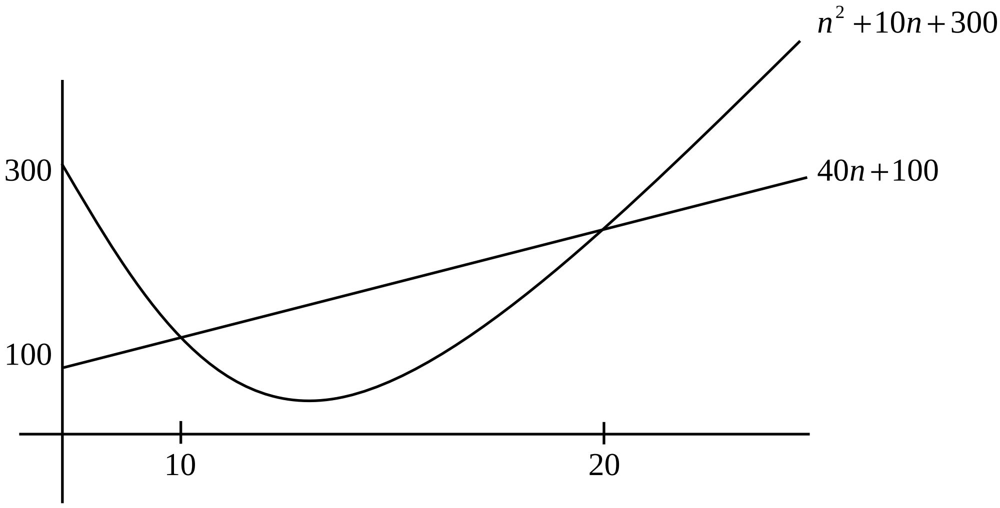
We often abuse notation slightly by writing $f(n) = O(g(n))$ to mean $f(n) \in O(g(n))$. Actually $f(n) \in O(g(n))$ is also an abuse of notation. We should really write $f \in O(g)$ since what we have defined is a set of functions, not a set of numbers. The notational convention $O(g(n))$ is useful since it allows us to refer to the set $O(n^3)$ say, without having to introduce a function symbol for the polynomial n^3. Observe that if $f(n) = O(g(n))$ then $f(n)$ is *asymptotically non-negative*, i.e. $f(n)$ is non-negative for all sufficiently large *n*, and likewise for $g(n)$. We make the blanket assumption from now on that all functions under discussion are asymptotically non-negative.

In practice we will be concerned with integer valued functions of a (positive) integer *n* ($g : \mathbf{Z}^+ \to \mathbf{Z}^+$). However, in what follows, it is useful to consider *n* to be a continuous real variable taking positive values and *g* to be real valued function ($g : \mathbf{R}^+ \to \mathbf{R}^+$).

Geometrically $f(n) = O(g(n))$ says:



Example $40n+100 = O(n^2+10n+300)$. Observe that $0 \le 40n+100 \le n^2+10n+300$ for all $n \ge 20$, as can be easily verified. Thus we may take $n_0 = 20$ and $c = 1$ in the definition.



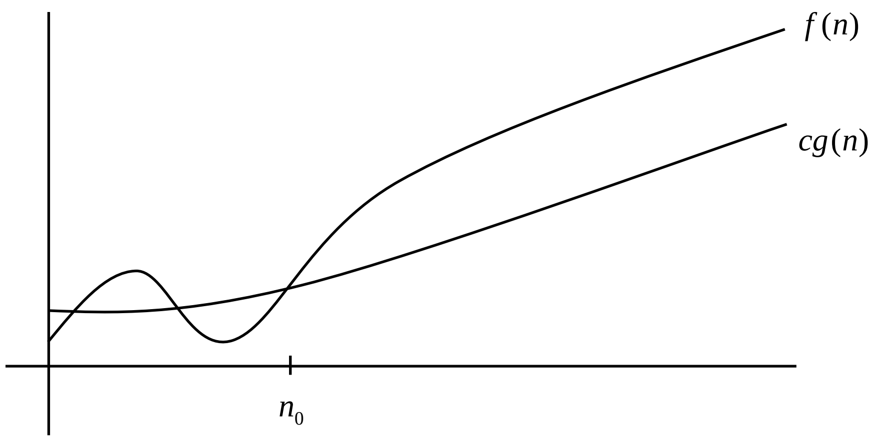
Note that in this example, any value of n_0 greater than 20 will also work, and likewise any value of c greater than 1 works. In general if there exist positive constants n_0 and c such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0$, then infinitely many such constants also exist. In order to prove that $f(n) = O(g(n))$ it is not necessary to find the smallest possible n_0 and c making the $0 \le f(n) \le cg(n)$ true. It is only necessary to show that at least one pair of such constants exist.

Generalizing the last example, we will show that $an+b = O(cn^2+dn+e)$ for any constants *a*-*e*, and in fact $p(n) = O(q(n))$ whenever $p(n)$ and $q(n)$ are polynomials with $\deg(p) \le \deg(q)$.

Definition Let $g(n)$ be a function and define the set $\Omega(g(n))$ to be

$$\Omega(g(n)) = \{\, f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \ge n_0 :\ 0 \le cg(n) \le f(n) \,\}.$$

We say $f(n)$ is *big Omega* of $g(n)$, and that $g(n)$ is an *asymptotic lower bound* for $f(n)$. As before we write $f(n) = \Omega(g(n))$ to mean $f(n) \in \Omega(g(n))$. The geometric interpretation is:



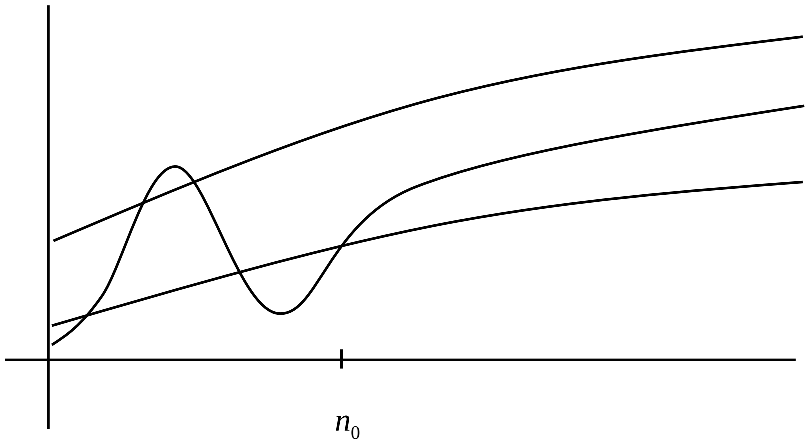
Lemma $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.

Proof: If $f(n) = O(g(n))$ then there exist positive numbers c_1, n_1 such that $0 \le f(n) \le c_1 g(n)$ for all $n \ge n_1$. Let $c_2 = 1/c_1$ and $n_2 = n_1$. Then $0 \le c_2 f(n) \le g(n)$ for all $n \ge n_2$, proving $g(n) = \Omega(f(n))$. The converse is similar and we leave it to the reader. ///

Definition Let $g(n)$ be a function and define the set $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$. Equivalently

$$\Theta(g(n)) = \{\, f(n) \mid \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \ge n_0 :\ 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \,\}.$$

We write $f(n) = \Theta(g(n))$ and say the $g(n)$ is an *asymptotically tight bound* for $f(n)$, or that $f(n)$ and $g(n)$ are *asymptotically equivalent*. We interpret this geometrically as:



Exercise Prove that if c is a positive constant, then $cf(n) = \Theta(f(n))$.

Exercise Prove that $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.

Example Prove that $\sqrt{n+10} = \Theta\left(\sqrt{n}\right)$.

Proof: According to the definition, we must find positive numbers c_1, c_2, n_0, such that the inequality $0 \le c_1\sqrt{n} \le \sqrt{n+10} \le c_2\sqrt{n}$ holds for all $n \ge n_0$. Pick $c_1 = 1$, $c_2 = \sqrt{2}$, and $n_0 = 10$. Then if $n \ge n_0$ we have:

$$
\begin{array}{lccc}
 & -10 \le 0 & \text{and} & 10 \le n \\
\therefore & -10 \le (1-1)n & \text{and} & 10 \le (2-1)n \\
\therefore & -10 \le (1-c_1^2)n & \text{and} & 10 \le (c_2^2-1)n \\
\therefore & c_1^2 n \le n+10 & \text{and} & n+10 \le c_2^2 n, \\
\therefore & & c_1^2 n \le n+10 \le c_2^2 n, & \\
\therefore & & c_1\sqrt{n} \le \sqrt{n+10} \le c_2\sqrt{n}, &
\end{array}
$$

as required. ///

The reader may find our choice of values for the constants c_1, c_2, n_0 in this example somewhat mysterious. Adequate values for these constants can usually be obtained by working backwards algebraically from the inequality to be proved. Notice that in this example there are many valid choices. For instance one checks easily that $c_1 = \sqrt{1/2}$, $c_2 = \sqrt{3/2}$, and $n_0 = 20$ work equally well.

Exercise Let *a*, *b* be real numbers with $b > 0$. Prove directly from the definition (as above) that $(n+a)^b = \Theta(n^b)$. (In what follows we learn a much easier way to prove this.)

Lemma If $f(n) \le h(n)$ for all sufficiently large *n*, and if $h(n) = O(g(n))$, then $f(n) = O(g(n))$.
Proof: The above hypotheses say that there exist positive numbers c and n_1 such that $h(n) \le cg(n)$ for all $n \ge n_1$. Also there exists n_2 such that $0 \le f(n) \le h(n)$ for all $n \ge n_2$. (Recall $f(n)$ is assumed to be asymptotically non-negative.) Define $n_0 = \max(n_1, n_2)$, so that if $n \ge n_0$ we have both $n \ge n_1$ and $n \ge n_2$. Thus $n \ge n_0$ implies $0 \le f(n) \le cg(n)$, and therefore $f(n) = O(g(n))$. ///

Exercise Prove that if $h_1(n) \le f(n) \le h_2(n)$ for all sufficiently large *n*, where $h_1(n) = \Omega(g(n))$ and $h_2(n) = O(g(n))$, then $f(n) = \Theta(g(n))$.

Example Let $k \ge 1$ be a fixed integer. Prove that $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$.

Proof: Observe that $\sum_{i=1}^{n} i^k \le \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1} = O(n^{k+1})$, and

$$\sum_{i=1}^{n} i^k \ge \sum_{i=\lceil n/2 \rceil}^{n} i^k \ge \sum_{i=\lceil n/2 \rceil}^{n} (n/2)^k \ge \lceil n/2 \rceil \cdot (n/2)^k \ge (n/2)(n/2)^k = (1/2)^{k+1} n^{k+1} = \Omega(n^{k+1}).$$

By the result of the preceding exercise, we conclude $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$. ///

When asymptotic notation appears in a formula such as $T(n) = 2T(n/2) + \Theta(n)$ we interpret $\Theta(n)$ to stand for some anonymous function in the class $\Theta(n)$. For example $3n^3 + 4n^2 - 2n + 1 = 3n^3 + \Theta(n^2)$. Here $\Theta(n^2)$ stands for $4n^2 - 2n + 1$, which belongs to the class $\Theta(n^2)$.

The expression $\sum_{i=1}^{n} \Theta(i)$ can be puzzling. On the surface it stands for $\Theta(1) + \Theta(2) + \Theta(3) + \cdots + \Theta(n)$, which is meaningless since $\Theta(\text{constant})$ consists of all functions which are bounded above by some constant. We interpret $\Theta(i)$ in this expression to stand for a *single* function $f(i)$ in the class $\Theta(i)$, evaluated at $i = 1, 2, 3, \ldots, n$.

Exercise Prove that $\sum_{i=1}^{n} \Theta(i) = \Theta(n^2)$. The left hand side stands for a single function $f(i)$ summed for $i = 1, 2, 3, \ldots, n$. By the previous exercise it is sufficient to show that $h_1(n) \le \sum_{i=1}^{n} f(i) \le h_2(n)$ for all sufficiently large *n*, where $h_1(n) = \Omega(n^2)$ and $h_2(n) = O(n^2)$.

Definition $o(g(n)) = \{\, f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \ge n_0 : 0 \le f(n) < cg(n) \,\}$. We say that $g(n)$ is a *strict Asymptotic upper bound* for $f(n)$ and write $f(n) = o(g(n))$ as before.

Lemma $f(n) = o(g(n))$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

Proof: Observe that $f(n) = o(g(n))$ if and only if $\forall c > 0, \exists n_0 > 0, \forall n \ge n_0 : 0 \le \frac{f(n)}{g(n)} < c$, which is the very definition of the limit statement $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. ///

Example $\lg(n) = o(n)$ since $\lim_{n\to\infty} \frac{\lg(n)}{n} = 0$. (Apply l'Hopitals rule.)

Example $n^k = o(b^n)$ for any $k > 0$ and $b > 1$ since $\lim_{n\to\infty} \frac{n^k}{b^n} = 0$. (Apply l'Hopitals rule $\lceil k \rceil$ times.) In other words, any exponential grows strictly faster than any polynomial.

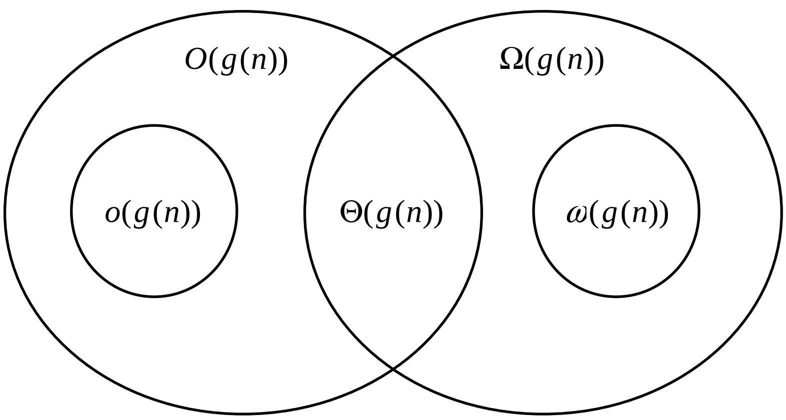
By comparing definitions of $o(g(n))$ and $O(g(n))$ one sees immediately that $o(g(n)) \subseteq O(g(n))$. Also no function can belong to both $o(g(n))$ and $\Omega(g(n))$, as is easily verified (exercise). Thus $o(g(n)) \cap \Omega(g(n)) = \varnothing$, and therefore $o(g(n)) \subseteq O(g(n)) - \Theta(g(n))$.

Definition $\omega(g(n)) = \{\, f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \ge n_0 : 0 \le cg(n) < f(n) \,\}$. Here we say that $g(n)$ is a *strict asymptotic lower bound* for $f(n)$ and write $f(n) = \omega(g(n))$.

Exercise Prove that $f(n) = \omega(g(n))$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.

Exercise Prove $\omega(g(n)) \cap O(g(n)) = \varnothing$, whence $\omega(g(n)) \subseteq \Omega(g(n)) - \Theta(g(n))$.

The following picture emerges:



Lemma If $\lim_{n\to\infty}\frac{f(n)}{g(n)} = L$, where $0 \le L < \infty$, then $f(n) = O(g(n))$.

Proof: The definition of the above limit is $\forall\varepsilon > 0, \exists n_0 > 0, \forall n \ge n_0 : \left|\frac{f(n)}{g(n)} - L\right| < \varepsilon$. Thus if we let $\varepsilon = 1$, there exists a positive number n_0 such that for all $n \ge n_0$:

$$\left|\frac{f(n)}{g(n)} - L\right| < 1$$

$$\therefore \quad -1 < \frac{f(n)}{g(n)} - L < 1$$

$$\therefore \quad \frac{f(n)}{g(n)} < L + 1$$

$$\therefore \quad f(n) < (L+1)\cdot g(n).$$

Now take $c = L + 1$ in the definition of O, so that $f(n) = O(g(n))$ as claimed. ///

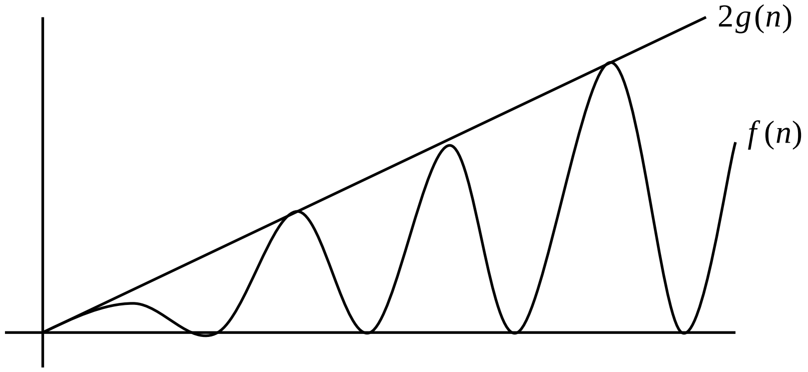
Lemma If $\lim_{n\to\infty}\frac{f(n)}{g(n)} = L$, where $0 < L \le \infty$, then $f(n) = \Omega(g(n))$.

Proof: The limit statement implies $\lim_{n\to\infty}\frac{g(n)}{f(n)} = L'$, where $L' = 1/L$ and hence $0 \le L' < \infty$. By the previous lemma $g(n) = O(f(n))$, and therefore $f(n) = \Omega(g(n))$. ///

Exercise Prove that if $\lim_{n\to\infty}\frac{f(n)}{g(n)} = L$, where $0 < L < \infty$, then $f(n) = \Theta(g(n))$.

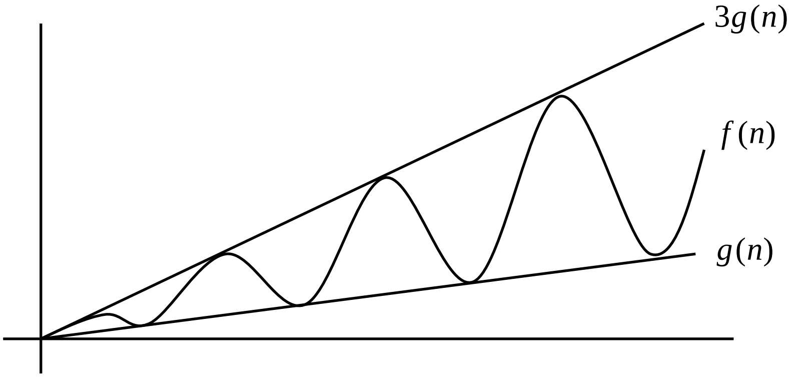
Although $o(g(n))$, $\omega(g(n))$, and a certain subset of $\Theta(g(n))$ are characterized by limits, the full sets $O(g(n))$, $\Omega(g(n))$, and $\Theta(g(n))$ have no such characterization as the following examples show.

Example A Let $g(n) = n$ and $f(n) = (1 + \sin(n))\cdot n$.



Clearly $f(n) = O(g(n))$, but $\frac{f(n)}{g(n)} = 1 + \sin(n)$, whose limit does not exist. This example shows that the containment $o(g(n)) \subseteq O(g(n)) - \Theta(g(n))$ is in general strict since $f(n) \ne \Omega(g(n))$ (exercise). Therefore $f(n) \ne \Theta(g(n))$, so that $f(n) \in O(g(n)) - \Theta(g(n))$. But $f(n) \ne o(g(n))$ since the limit does not exist.

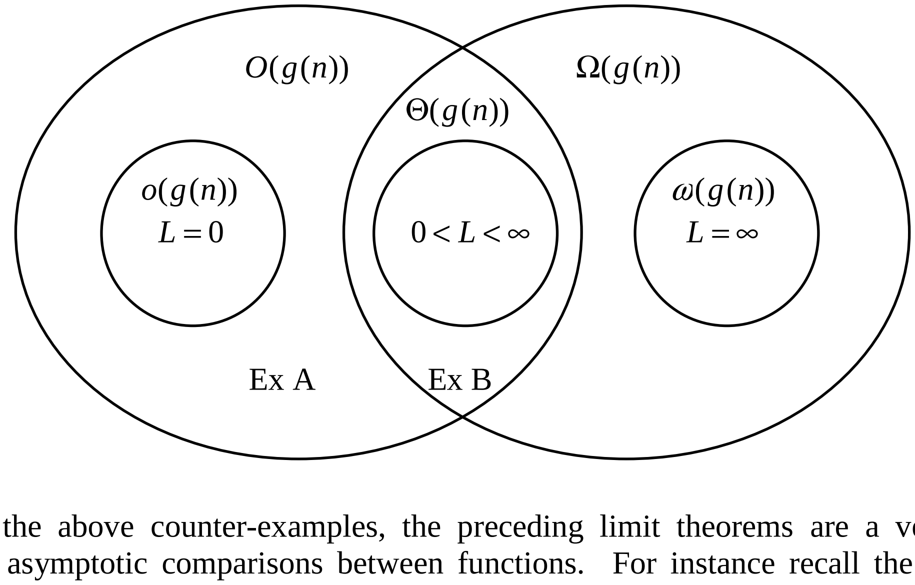
Example B Let $g(n) = n$ and $f(n) = (2 + \sin(n)) \cdot n$.



Since $n \le (2 + \sin(n)) \cdot n \le 3n$ for all $n \ge 0$, we have $f(n) = \Theta(g(n))$, but $\dfrac{f(n)}{g(n)} = 2 + \sin(n)$ whose limit does not exist.

Exercise Find functions $f(n)$ and $g(n)$ such that $f(n) \in \Omega(g(n)) - \Theta(g(n))$, but $\lim_{n\to\infty} \dfrac{f(n)}{g(n)}$ does not exist (even in the sense of being infinite), so that $f(n) \ne \omega(g(n))$.

The preceding limit theorems and counter-examples can be summarized in the following diagram. Here *L* denotes the limit $L = \lim_{n\to\infty} \dfrac{f(n)}{g(n)}$, if it exists.



In spite of the above counter-examples, the preceding limit theorems are a very useful tool for establishing asymptotic comparisons between functions. For instance recall the earlier exercise to show $(n + a)^b = \Theta(n^b)$ for real numbers *a*, and *b* with $b > 0$. The result follows immediately from

$$\lim_{n\to\infty} \frac{(n + a)^b}{n^b} = \lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^b = 1^b = 1,$$

since $0 < 1 < \infty$.

Exercise Use limits to prove the following:

a. $n\lg(n) = o(n^2)$ (here $\lg(n)$ denotes the base 2 logarithm of *n*.)

b. $n^5 2^n = \omega(n^{10})$.

c. If $P(n)$ is a polynomial of degree $k \geq 0$, then $P(n) = \Theta(n^k)$.

d. $f(n) + o(f(n)) = \Theta(f(n))$. (One can always disregard lower order terms)

e. $(\log n)^k = o(n^\varepsilon)$ for any $k > 0$ and $\varepsilon > 0$. (Polynomials grow faster than logs.)

f. $n^\varepsilon = o(b^n)$ for any $\varepsilon > 0$ and $b > 1$. (Exponentials grow faster than polynomials.)

There is an analogy between the asymptotic comparison of functions $f(n)$ and $g(n)$, and the comparison of real numbers *x* and *y*.

$$
\begin{array}{lcl}
f(n) = O(g(n)) & \sim & x \leq y \\
f(n) = \Theta(g(n)) & \sim & x = y \\
f(n) = \Omega(g(n)) & \sim & x \geq y \\
f(n) = o(g(n)) & \sim & x < y \\
f(n) = \omega(g(n)) & \sim & x > y
\end{array}
$$

Note however that this analogy is not exact since there exist pairs of functions which are not comparable, while any two real numbers are comparable. (See problem 3-2c, p.58.)