Khovanov Homology of Links Embedded in I-Bundles

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Introduction

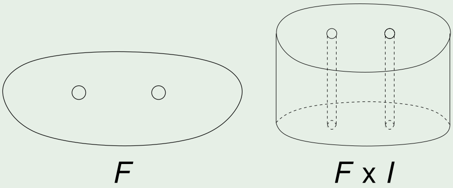
M. Asaeda, J. Przytycki and A. Sikora introduced a theory for links embedded in I-bundles. We will replicate their theory using surfaces to generate the chain groups instead of diagrams.

I-bundle over an orientable surface

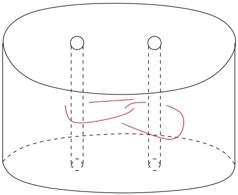
Definition

An **I-bundle** over an orientable surface is a 3-manifold F x I where F is an orientable surface and I is the unit interval.

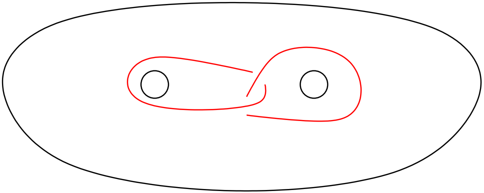
Example



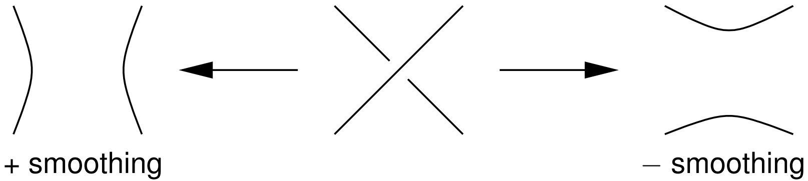
We can place a link in the 3-manifold F x I



and this can be represented by a diagram in F

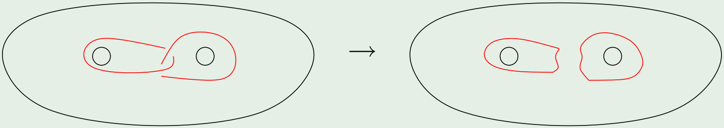


Given a crossing in a diagram we can smooth it in two ways:



Thus we can smooth the diagram in F.

Example



We have chosen the + smoothing at each crossing.

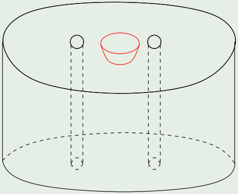
Definition

A choice of smoothing at each crossing is called a **state**.

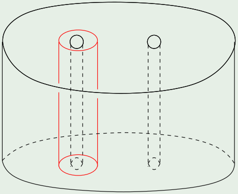
Definition

Let S be a surface properly embedded in a 3-manifold M. Boundary curves of S are called **inessential** if they bound a disk in M and they are called **essential** otherwise.

Example



Inessential Boundary Curve

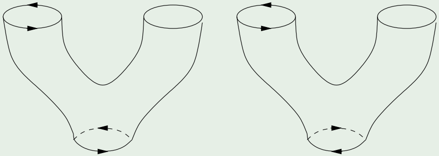


Essential Boundary Curves

Definition

Given an oriented surface S and an oriented boundary curve c, S and c are **compatible** if the orientation on c agrees with the orientation on c coming from the boundary orientation on S. Two oriented boundary curves of a connected component are said to be **compatible** if they are compatible with the same orientation on the connected component.

Example



Not Compatible Compatible

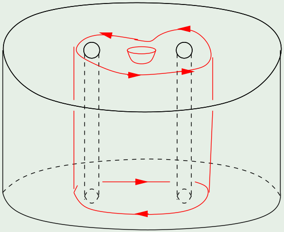
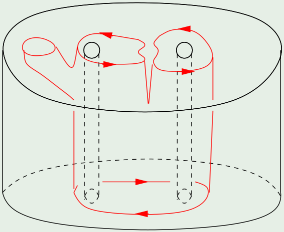
Definition

A properly embedded orientable surface S is said to be **F-oriented** if S has the following properties:

- Whenever any essential boundary curve of S is oriented then all other essential boundary curves on the same component are oriented.
- The oriented boundary curves on each component of S are compatible with one another.
- Inessential boundary curves of S are not oriented.

Example

These surfaces are F-oriented.

Foams

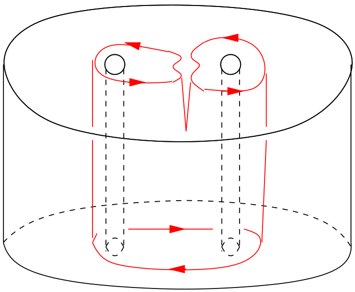
Let D be a diagram of a link in a compact orientable surface F such that the crossings of D are enumerated.

Definition

A **foam** is an F-oriented orientable compact surface properly embedded in F x I.

- Foams have a state of the diagram D as the boundary on the top (F x $\{0\}$) and essential disjoint oriented circles as the boundary on the bottom (F x $\{1\}$).
- Foams may be marked with dots.
- Two foams are equivalent if they are isotopic to one another relative to the boundary. Thus the dots on foams may move within components, but dots may not switch components.

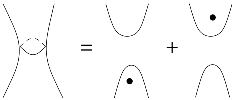
Examples of Foams



Relations

The foams are subject to the following relations

- Neck-Cutting



Relations

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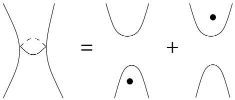
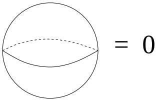
- Neck-Cutting 
- Sphere bounding ball = 0 

Relations

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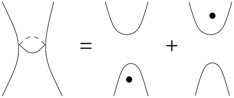
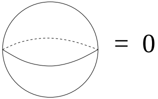
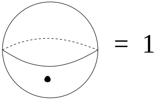
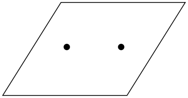
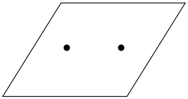
- Neck-Cutting 
- Sphere bounding ball = 0 
- Sphere bounding ball with dot = 1 

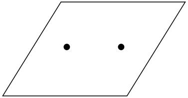
- A component with two dots = 0



$= 0$

- A component with two dots = 0  $= 0$

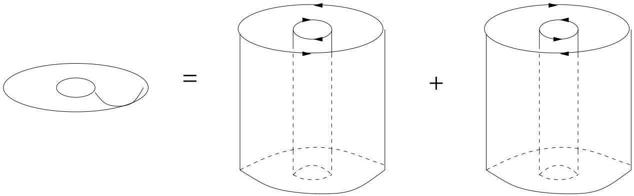
- A foam with an incompressible component that isn't a disk or a sphere, but has a dot equals zero.

- A component with two dots = 0  $= 0$

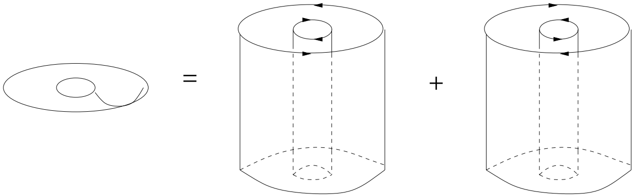
- A foam with an incompressible component that isn't a disk or a sphere, but has a dot equals zero.

- A foam with an incompressible component with negative euler characteristic equals zero.

- If the annulus below on the left is incompressible then



- If the annulus below on the left is incompressible then



- An annulus with its boundary completely in the bottom and without a dot equals one.

Summary of Relations

That's a lot of relations to keep track of, so here is a quick summary:

1. By the neck-cutting relation we can reduce all foams down to incompressible surfaces. If anything besides disks and annuli remain, that foam is trivial.
2. Dots may only be on disks after everything is incompressible or the foam is trivial.
3. If there is an unoriented annulus with both boundary components in the top, then replace that foam with the sum of two foams each with appropriate vertical oriented annuli.

In this manner we can consider all foams to consist entirely of oriented annuli without dots and disks with or without dots.

Defining the Degree

Now we introduce the tri-grading that is used.

Definition

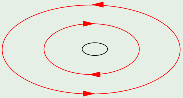
Let S be a Foam.
$I(S) = \{$# of positive smoothings in the state corresponding to the top boundary of $S\} -$
$\{$# of negative smoothings in the state corresponding to the top boundary of $S\}$
$J(S) = I(S) + 2(2d - \chi(S))$ where d = the number of dots S has.

The Third Index

To define the third index it is necessary to specify that one of the orientations possible on each homotopy class of essential simple closed curve in the bottom is the positive one and the opposite orientation is the negative one.

Example

If we say the inner curve in the figure has the positive orientation then the outer curve has the negative orientation.



Thus for each homotopy class of simple closed curve we have chosen a positive orientation and a negative orientation.

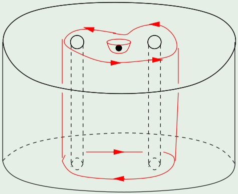
Definition

Let $\gamma_1, \ldots, \gamma_n$ be a family of disjoint simple closed curves in the bottom of a foam S. If γ_i and γ_j are parallel then $\gamma_i = \gamma_j$. Then we have
$K(S) = \sum_1^n k_i \gamma_i$, where

$$k_i = \begin{cases} 1, & \text{if } \gamma_i \text{ is oriented in the positive direction} \\ -1, & \text{if } \gamma_i \text{ is oriented in the negative direction} \end{cases}$$

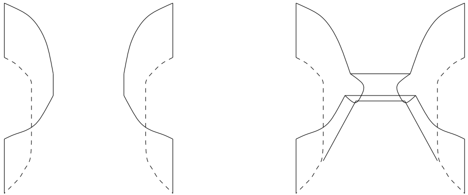
A foam S is a generator of the Chain Group $C_{i,j,k}(D)$ if S has a representation that consists only of vertical oriented annuli and disks, and $I(S) = i$, $J(S) = j$ and $K(S) = k$.

Example



This foam has $I = -2$, $J = -2 + 2(2 * 1 - \chi) = -2 + 2 = 0$, $K = \alpha$ if α is the curve in the bottom oriented in the positive direction. Thus this foam is a generator of $C_{-2,0,\alpha}(D)$.

Locally we define $\bar{d}_p(S)$ at the p-th crossing by

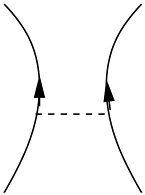


S $\bar{d}_p(S)$

Informally this operation will be called placing a bridge at the p-th crossing.

There are two situations when $\bar{d}_p$ cannot be applied to a foam.

1. Placing a bridge could make it impossible to preserve existing boundary orientations. This occurs if the p-th crossing is as in the following figure.

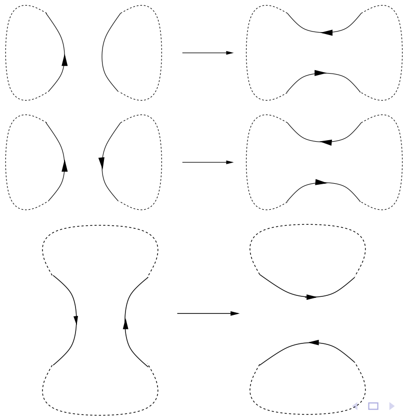


   Notationally this will be referred to as (EO).

2. Placing a bridge could create non-orientable surfaces. This will be referred to as (NOS).

When $\bar{d}_p$ is applied to a foam the orientation on boundary curves changes as below with the additional condition that inessential boundary curves of $\bar{d}_p(S)$ are unoriented.

Boundary of S Boundary of $\bar{d}_p(S)$



We define

$$d_p(S) = \begin{cases} 0, & \text{if } p\text{-th crossing is smoothed negatively} \\ 0, & \text{if (EO) occurs} \\ 0, & \text{if (NOS) occurs} \\ \bar{d}_p(S), & \text{else} \end{cases}$$

Thus we have the boundary operator $d : C_{i,j,s}(D) \to C_{i-2,j,s}(D)$, defined by

$$d(S) = \sum_{p \text{ a crossing}} (-1)^{t(S,p)} d_p(S),$$

where $t(S,p) = |\{$ j a crossing of D : $j > p$ and j is smoothed negatively in the state corresponding to the top boundary of S$\}|$

$d^2 = 0$

Theorem

$d^2 = 0$, so this theory produces homology.

Proof. (Sketch)

By how the negative signs are distributed in the definition of the boundary operator it is sufficient to show that $d_i(d_j(S)) = d_j(d_i(S))$ for all foams S and all crossings i and j.

This is immediate if we disregard orientation, (EO) occurring and (NOS) occurring. Thus we only need to address when these situations occur and show that in these cases the partial boundary operators commute.

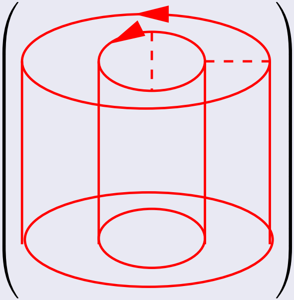
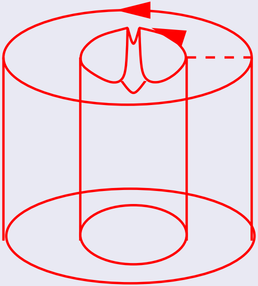
Proof. (cont'd)

Assume $d_i(d_j(S)) = 0$ because (EO) occurs.

The idea behind this part of the proof is that if a bridge cannot be placed at j because of (EO), then placing a bridge at i first either does not change the fact that a bridge cannot be placed at j or the bridge connects an annulus to a disk with a dot. This results in a foam that has an incompressible annulus with a dot which is trivial in the quotient.

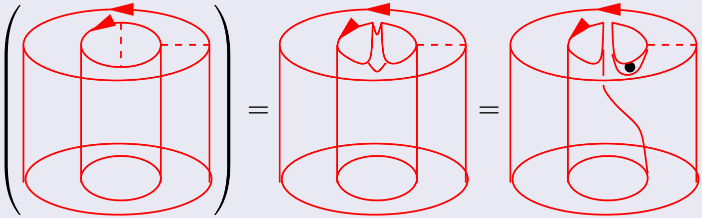
Proof. (cont'd)

In the figure j is the outer crossing and i is the inner crossing. Thus either

d_i  $=$ 

where $d_j(d_i(S)) = 0$ by (EO)

Proof. (cont'd)

d_i 

or $d_j(d_i(S))$ is represented by a foam with an annulus with a dot.

In either case the partials commute.

To proceed we need the following Lemma:

Lemma

The number of boundary curve stays constant when a bridge is placed if and only if (NOS) occurs. Otherwise the number of boundary curves changes by one.

Now we continue with the proof of the theorem:

Proof. (cont'd)

Assume $d_i(d_j(S)) = 0$ because (NOS) occurs.

By the lemma, if (NOS) doesn't occur

{# of boundary curves of S} $\equiv$ {# of boundary curves of $d_j(d_i(S))$} mod 2

Proof. (cont'd)

If (NOS) occurs exactly once when placing the bridges we have

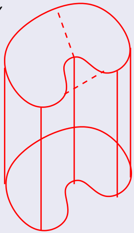
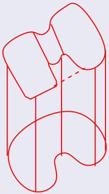
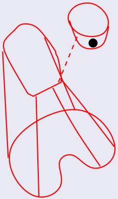
{# of boundary curves of S} $\neq$ {# of boundary curves of $d_j(d_i(S))$} mod 2

Since {# of boundary curves of $d_j(d_i(S))$} = {# of boundary curves of $d_i(d_j(S))$} then if (NOS) occurs exactly once through one of the orderings of the crossings, it must occur through the other ordering of crossings as well.

Proof. (cont'd)

Thus we must address the case where (NOS) occurs at each crossing through one ordering. This can be reduced to the case where we start with an annulus and locally $d_i(S)$ has two boundary curves and $d_j(d_i(S))$ has one boundary curve again.

Proof. (cont'd)

d_i () $=$  $=$ trivial in quotient or $=$ 

If placing the first bridge creates an incompressible pair of pants, then it is trivial in the quotient. If not, then one of the top boundary curves is inessential. Therefore there is a compressing disk and then the second bridge connects an annulus with a disk with a dot, so after bridging this is trivial in the quotient.

Proof. (cont'd)

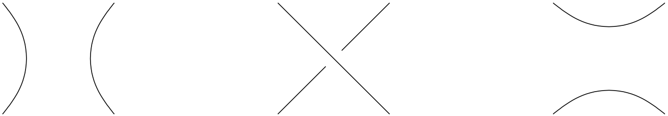
Claim: The orientation on boundary curves of $d_j(d_i(S))$ must agree with the orientation on the boundary curves of $d_i(d_j(S))$.

If we start with oriented annuli and place bridges without compressing, the oriented boundary curves are present in the bottom and we are dealing with one connected component. Thus only one possible orientation of boundary curves is possible for $d_j(d_i(S))$ and $d_i(d_j(S))$.

$\square$

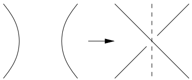
We now show the homology theory that we have produced is the same as the theory from [APS]. To do this we use Viro's short exact sequence and the five lemma.

Let p be a crossing of the diagram D. Consider the skein triple in F:

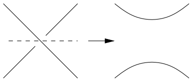


D_∞ D_p D_0

Now define:
$\alpha_0 : C_{i,j,s}(D_\infty) \to C_{i-1,j-1,s}(D_p)$ is the natural embedding as depicted in the figure below.



$\beta : C_{i,j,s}(D_p) \to C_{i-1,j-1,s}(D_0)$ is the natural projection where foams with a negative smoothing at p are sent to 0 and foams with a positive smoothing are affected as in the figure below.



Let $\alpha : C_{i,j,s}(D_\infty) \to C_{i-1,j-1,s}(D_p)$ be defined by $\alpha(S) = (-1)^{t'(S)}\alpha_0(S)$ where $t'(S) =$ the number of negative crossings in the ordering before p.

Viro's Short Exact Sequence

Theorem

α and β are chain maps and the sequence

$$0 \to C_{i+1,j+1,s}(D_\infty) \xrightarrow{\alpha} C_{i,j,s}(D_p) \xrightarrow{\beta} C_{i-1,j-1,s}(D_0) \to 0$$

is exact.

Proof.

α and β are chain maps by some careful algebra.

Since α is an embedding it is 1-1. The image of α is all foams in $C_{i,j,s}(D_p)$ that have a state as the top boundary smoothed negatively at p. The kernel of β is precisely these foams. Since β is a projection, it is onto $C_{i-1,j-1,s}(D_0)$. Thus the sequence is exact. $\square$

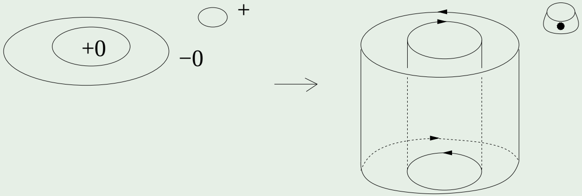
Definition

Let $\bar{H}_{i,j,k}(D)$ be the homology defined by APS and let $H_{i,j,k}$ be the homology of the chain complex we have constructed.

Define $\Phi : \bar{C}_{i,j,s} \to C_{i,j,s}$ by taking an enhanced state in $\bar{C}_{i,j,s}$ and changing each circle as follows to get a foam with the same state as its top boundary in $C_{i,j,s}$:

Trivial circle marked with a $+ \to$ disk with a dot
Trivial circle marked with a $- \to$ disk without a dot
Nontrivial circle marked with a $+0 \to$ vertical annulus with the positive orientation
Nontrivial circle marked with a $-0 \to$ vertical annulus with the negative orientation

Example



Lemma

$\Phi : \bar{C}_{i,j,s} \rightarrow C_{i,j,s}$ is a chain map $\forall i, j, s$.

Proof. (Sketch)

The proof is done by examining how the boundary operator for $\bar{H}$ affects enhanced states for each possible case and ensuring this is preserved by Φ. $\square$

Theorem

$\bar{H}_{i,j,k}(D) \cong H_{i,j,k}(D)\ \forall\ i, j, k$, that is to say the homology theory created with embedded surfaces coincides with the theory developed by Asaeda, Przytycki, and Sikora for I-bundles over orientable surfaces.

Proof.

The proof is by induction on the number of crossings.

Assume D has 0 crossings. Therefore the boundary maps are all the zero map. Thus the chain groups are also the homology groups. Note Φ is an isomorphism on the chain groups since it takes generators to generators, so it is also an isomorphism on homology in this case.

Now assume Φ_* is an isomorphism for all diagrams having less than n crossings and D_p is a diagram with n crossings.

We can relate the short exact sequences coming from the two theories by the Φ map in the following commutative diagram where both rows are exact.

$$\begin{tikzcd}
0 \arrow[r] & \bar{C}_{i+1,j+1,k}(D_\infty) \arrow[r, "\bar{\alpha}"] \arrow[d, "\Phi"] \arrow[dr, phantom, "\circlearrowright"] & \bar{C}_{i,j,k}(D_p) \arrow[r, "\bar{\beta}"] \arrow[d, "\Phi"] \arrow[dr, phantom, "\circlearrowright"] & \bar{C}_{i-1,j-1,k}(D_0) \arrow[r] \arrow[d, "\Phi"] & 0 \\
0 \arrow[r] & C_{i+1,j+1,k}(D_\infty) \arrow[r, "\alpha"] & C_{i,j,k}(D_p) \arrow[r, "\beta"] & C_{i-1,j-1,k}(D_0) \arrow[r] & 0
\end{tikzcd}$$

The commutative diagram of short exact sequences induces the following commutative diagram of long exact sequences.

$$\begin{tikzcd}
\ldots \bar{H}_{i+1,j-1,k}(D_0) \arrow[r, "\partial"] \arrow[d, "\Phi_*"'] \arrow[dr, phantom, "\circlearrowleft"] & \bar{H}_{i+1,j+1,k}(D_\infty) \arrow[r, "\bar{\alpha}_*"] \arrow[d, "\Phi_*"] \arrow[dr, phantom, "\circlearrowleft"] & \bar{H}_{i,j,k}(D_p) \arrow[r, "\bar{\beta}_*"] \arrow[d, "\Phi_*"] \arrow[dr, phantom, "\circlearrowleft"] & \bar{H}_{i-1,j-1,k}(D_0) \arrow[r, "\partial"] \arrow[d, "\Phi_*"] \arrow[dr, phantom, "\circlearrowleft"] & \bar{H}_{i-1,j+1,k}(D_\infty) \arrow[r] \arrow[d, "\Phi_*"] & \ldots \\
\ldots H_{i+1,j-1,k}(D_0) \arrow[r, "\partial"] & H_{i+1,j+1,k}(D_\infty) \arrow[r, "\alpha_*"] & H_{i,j,k}(D_p) \arrow[r, "\beta_*"] & H_{i-1,j-1,k}(D_0) \arrow[r, "\partial"] & H_{i-1,j+1,k}(D_\infty) \arrow[r] & \ldots
\end{tikzcd}$$

Note all Φ_* above are between diagrams with fewer than n crossings except for the middle one. Thus by the inductive assumption these are all isomorphisms. By that fact and by the commutativity of the diagram the five lemma can be applied. Therefore the middle Φ_* is also an isomorphism.

Thus by induction Φ_* is an isomorphism on diagrams with any number of crossings and the theorem holds.

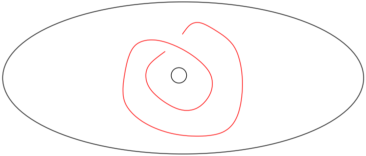
Since Asaeda, Przytycki and Sikora proved invariance for the $\bar{H}(D)$ homology and $\bar{H}(D) \cong H(D)$ by the previous theorem we obtain,

Corollary

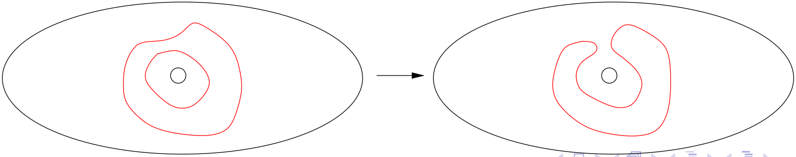
$H(D)$ is an invariant under Reidemeister moves 2 and 3 and a Reidemeister 1 move shifts the indices in a predictable way.

Example

Let's compute the homology for a simple example.
Here is the diagram in the surface.



This is how the boundary state changes under the boundary operator.



Finding the generators

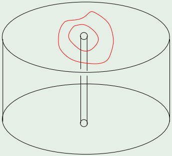
Here is the process to find all of the generators associated to a state of the diagram.

1. Place the state in the top of F x I.
2. For every essential curve in the state put a copy of that curve in the bottom and connect them with a vertical annulus.
3. Cap every inessential curve off with a disk that lies in F x I.
4. Choose an orientation for every annulus and decide which disks to place dots on.

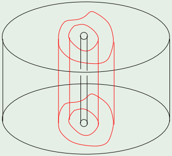
Example

Here we will construct one generating foam from a state.

- First we place a state in the top of F x I

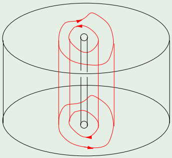


- These are essential curves, so they are the tops of vertical annuli



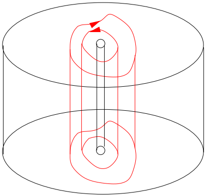
Example (cont'd)

They must be oriented to be generators of the chain groups, thus we choose an orientation for each one.

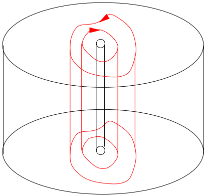


Applying this technique we can see that $C_{1,1,*}$ is generated by:

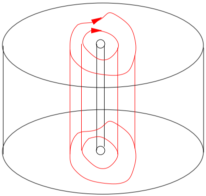
$A =$



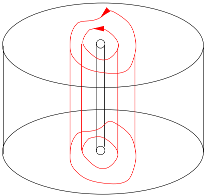
$B =$



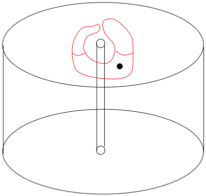
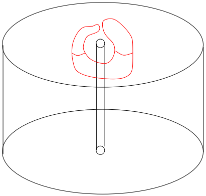
$C =$



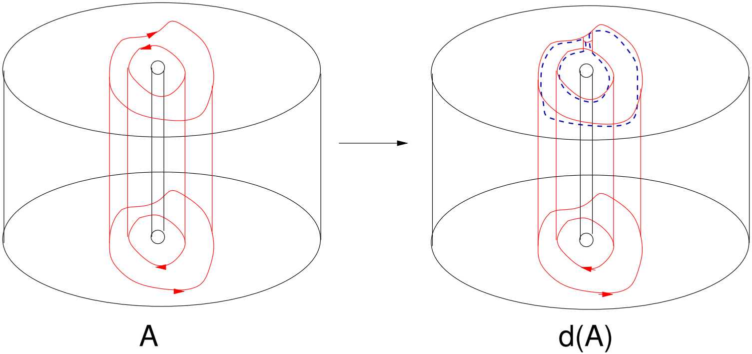
$D =$



and $C_{-1,*,*}$ is generated by:

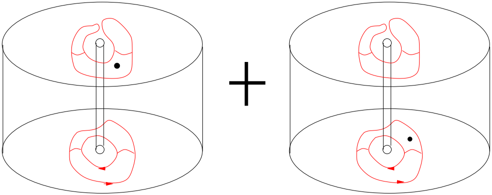
$E =$  $F =$ 

Apply the boundary operator

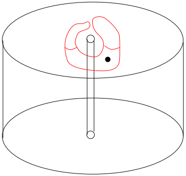


Note $d(A)$ has a compressing disk outlined by the dotted blue line.

Compressing upon this disk using the neck-cutting relation yields



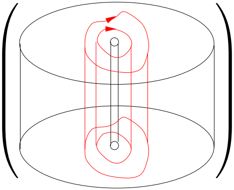
which reduces to

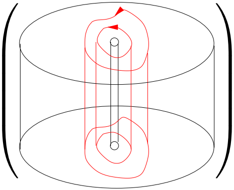


$$= E$$

In the same manner as $d(A)$, $d(B) = E$.

Also,

$d(C) = d$  $= 0$ and

$d(D) = d$  $= 0$ both by (EO).

Results

Under the boundary operator we have:
$A \to E$, $B \to E$, $C \to 0$ and $D \to 0$.

Note, E is in the image of d, but F is not.

Thus we have:

$H_{1,1,0} = ker(d_{1,1,0})/im(d_{-1,1,0}) =< A - B > /\{0\} \cong \mathbb{Z}$

$H_{1,1,2\alpha} = ker(d_{1,1,2\alpha})/im(d_{-1,1,2\alpha}) =< C > /\{0\} \cong \mathbb{Z}$

$H_{1,1,-2\alpha} = ker(d_{1,1,-2\alpha})/im(d_{-1,1,-2\alpha}) =< D > /\{0\} \cong \mathbb{Z}$

$H_{-1,-3,0} = ker(d_{-1,-3,0})/im(d_{-3,-3,0}) =< F > /\{0\} \cong \mathbb{Z}$

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