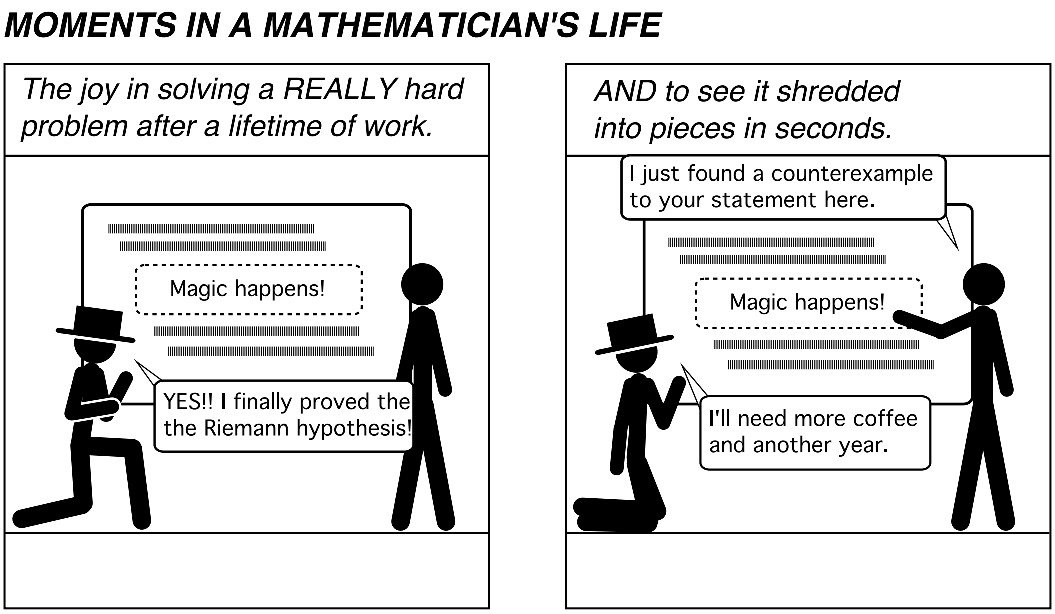
Issue 1, 2010

Paradox

THE MAGAZINE OF THE MELBOURNE UNIVERSITY MATHEMATICS AND STATISTICS SOCIETY



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Paradox

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In this edition of Paradox

Regulars

Features

Articles

On the Cover:
An original Paradox comic, created by PhD student Tharatorn Supasiti. See page 12 for more.

Words from the Editor

Welcome to the first edition of Paradox for 2010. This edition is packed with useful information and interesting maths, to keep you entertained and up-to-date on what's happening in MUMS this semester.

For those interested in taking part in the infamous MUMS Puzzle Hunt, this edition features a useful introductory guide. The Hunt can be slightly intractable for those who are uninitiated, so we hope that this will give you a good start on how to attack the puzzles. And since the guide is written by the chief organiser of this year's Hunt, it would be wise to pay close attention to his hints!

This edition also features a return of original comics to the Paradox line-up. For those not old enough to remember, Paradox has twice had resident cartoonist on its books, first with the adventures of Paradox Kid (1999 – 2001) and later with the adventures of Knot Man (2002 – 2004). These can be found on the Paradox archive on the MUMS website, and are both well worth checking out. Now another artist has risen to the challenge of these illustrious forebears, with an as-yet-unnamed series of maths-related comics. We wish him luck, and hope he keeps the comics coming.

Finally, Paradox would like to re-iterate that it is a magazine run for maths students, by maths students. We encourage contributions from all our readers, no matter how large or small. If you have recently heard a good maths joke (they all are), seen an interesting article about maths in the news, or have anything else to tell us, please drop all contributions into the Paradox drop-box, just inside the door to the MUMS room.

So whether you're a first year, a final year, a post-grad, or even a professor checking up on what the *young ones* are up to these days, and whether you are interested in how maths can save your life from *monsters*, or want to find out how to get (square) roots, Paradox hopes you enjoy this edition.

— Stephen Muirhead

Words from the President

Once again I've been ordered to write some President's Words: 'just write something random like you always do'. Something random eh? Using my statistical package of choice *R*, I generated a random number between 0 and 100 and got 56. Now, using my 'encyclopedia' of choice, we discover many important facts about this number. Rather amazingly I managed to get a number that is the sum of six consecutive primes: (3 + 5 + 7 + 11 + 13 + 17) = 56. But it doesn't stop there! Adding up the divisors of 1 through to 8 equals 56 and the maximal determinant of an 8 x 8 matrix of zeros and ones is also 56. Something that wasn't listed in this 'encyclopedia' that I discovered purely by myself is that 56 is also equal to 7x8. Pretty amazing eh?

I suppose I should put in a word here about our annual Puzzle Hunt, but I'm sure many of you are already gearing up for a week of no sleep. For those who don't know what it is, it's a week long event where we release puzzles daily, and at the end of the week the solutions of these puzzles will point towards somewhere on campus where a 'treasure' can be found. Did I mention first prize is $200? Besides the all-mighty Puzzle Hunt, we'll also be running our weekly seminars, so please come along to those too. While it might sound nerdy to tell your friends you're spending Friday afternoon listening to a dude talk about maths, trust me, some of the talks coming up will be crackers. And you can always just lie and say you're going to go learn how to skateboard. . . while eating ice cream in a shark tank. Though you may need to be a little more subtle than me.

Maths in the News:
'The fiancée formula: academics work out best time to propose' (*The Daily Mail*, 27/2/2010). Based on the theory of optimum stopping times, a professor of probability from UNSW in Sydney has worked out that $e+0.368\,(l-e)$ is the approximate optimum age to propose, where e is the earliest age you would consider getting married, and l is the latest. This balances, on the one hand, the need to gather experience, and on the other, the less opportunities you have if you wait too long. Professor Dooley recognises that 'probability isn't the most romantic basis for a marriage' but insists the formula 'does seem to fit a lot of couples, whether through accident or design.'

The MUMS Logo

While the exact origin of the MUMS logo is shrouded in mystery, the inspiration it drew on is obvious, merging as it does two of the most well-known mathematical concepts: the symbol for infinity and the Möbius band. For the unaware, the Möbius band is a one-sided surface which can be formed by taking a strip of paper, putting a twist in it, and gluing the ends together.

It's unsurprising, then, that approximations of the MUMS logo turn up in a variety of places:

from jewellery to furniture,

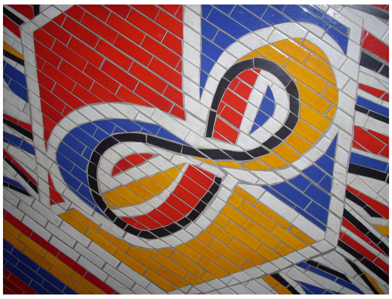
and from art

to recycling

to footwear.

But the MUMS logo also turns up in some very *unlikely* places. Here are just a few places the logo has been spotted by Paradox:

Decoration on the Paris Métro,



a neck tattoo in Serbia,

and a take-away shop in Balwyn.

Paradox would like to hear about other unlikely places you've spotted the MUMS logo; e-mail locations or photos to the Editor. The best ones will get published in the next edition.

Paradox **Wallpaper Challenge**

Following up on the challenge set in the last edition, Paradox is pleased to announce Quynh-Chi Nguyen as the winner of the Wallpaper Challenge and its $20 prize. Quynh-Chi found examples of nine of the 17 possible wallpaper groups on the Melbourne Uni campus, as follows:

Group	Location on campus
pmm	Brickwork on the exterior of the Chemistry building.
cmm	Classic alternating style of brickwork, multiple locations.
p4g	Thatched style of brickwork, multiple locations.
p2	Brickwork on the north wall of the Biology North Extension, as well as wire meshing in multiple locations.
p4m	Tiled floor in the Richard Berry building.
p1	Tiling in the ground floor café in The Spot (the new Commerce building across University Square).
pgg	Wooden parquetry in the Richard Berry Building.
p6m	Covering of an air-vent in the Basement of Union House.
pm	Department pigeon holes next to the Front Office in Richard Berry building.

Jokes and Trivia

Q: What is the physicist's definition of a vector space?
A: Any set V satisfying the axiom that for any x in V, x has a little arrow drawn over it.

It was mentioned on CNN recently that they had just established a new record for the largest known prime number. They said it was four times bigger than the previous record.

Q: Why didn't the Romans find algebra very challenging?
A: Because X was always 10.

Combinatorists do it as many ways as they can.
Algebraists do it with multiple roots.
Topologists do it openly.
Statisticians do all the standard deviations.
Mathematical Analysts do it in π-somes: larger than threesomes and they go on forever.

Q: What did the logician choose when offered a choice between a sausage roll and eternal bliss in the afterlife?
A: The sausage roll! He reasoned that nothing is better than eternal bliss in the afterlife, and a sausage roll is better than nothing.

Q: What does a mathematician call his dog?
A: Cauchy – because it leaves a residue at every pole.

A mathematician organizes a raffle in which the prize is an infinite amount of money paid over an infinite amount of time. Of course, with the promise of such a prize, his tickets sell like hot-cakes. When the winning ticket is drawn, and the jubilant winner comes to claim his prize, the mathematician explains the mode of payment: one dollar now, half a dollar next week, a third of a dollar the week after that…

An infinite crowd of mathematicians enters a bar. The first one orders a pint, the second one a half pint, the third one a quarter pint…'I understand,' says the bartender, who proceeds to pour two pints.

∞

Three professors, a physicist, a chemist, and a statistician, are sitting in the staff-tea room when the microwave suddenly catches fire.

The physicist says, 'I know what to do! We must cool down the materials until their temperature is lower than the ignition temperature and then the fire will go out.'

The chemist says, 'No! No! We must cut off the supply of oxygen so that the fire will go out due to the lack of one of the reactants.'

While the physicist and chemist debate which course to take, they are both alarmed to see the statistician running around the room starting other fires. They both scream, 'What are you doing?' To which the statistician replies, 'I'm trying to get an adequate sample size.'

∞

Physicist Max Born is Olivier Newton-John's grandfather.

Euler and Copernicus both feature on the Lutheran Calendar of Saints, and both on 24 May.

At age three Erdös could already multiply three digit numbers together in his head.

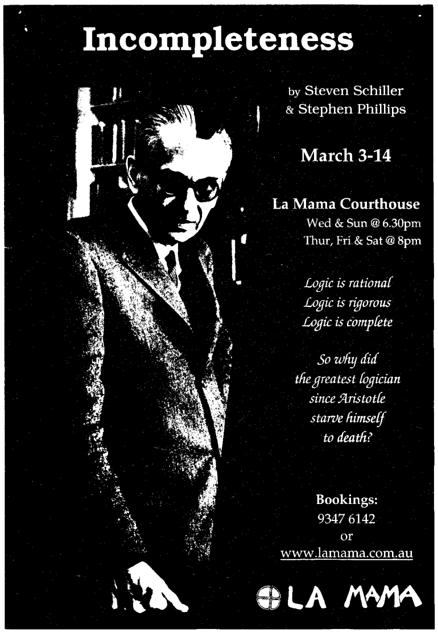
Mathematician André Weil was the brother of philosopher Simone Weil.

Reclusive mathematician Alexander Grothendieck has not been seen in public since 1991.

Henri Poincaré's cousin, Raymond Poincaré, served as both Prime Minister and President of France.

Review – *Incompleteness*

I met a woman once. It was at one of those interminable parties. She remarked that all men are liars. I told her that I agreed. . . she didn't seem to understand the joke.



There are very few people whose work challenges the very nature of human existence. Kurt Gödel (1906-1978), often considered the greatest logician since Aristotle, is one such person. In demonstrating the incompleteness of axiomatic systems – and hence the existence of statements that can neither be proved nor disproved – his work places an upper limit on mankind's capacity for knowledge, forcing us to confront our shortcomings head-on. Yet interestingly Gödel, too, was in a sense 'incomplete': he suffered from paranoia and hypochondria, spent the later part of his career in isolation, and ultimately starved himself to death.

Incompleteness (La Mama Courthouse, Carlton, 3rd – 14th March 2010) seeks to explore the parallels between Gödel's work and his life, at once immersing the spectator in the realm of logic and revealing its limitations.

The striking achievement of *Incompleteness* is its ability to create a tangible atmosphere of 'logic', which permeates every aspect of the performance. We open with Gödel (Stephen Phillips), alone on the stage, speaking entirely in formal logical propositions. Departing from an initial vocabulary of {it, is, that, exists, assume, therefore, prove} he progresses, through assumption and definition, to statements of increasing complexity. His movements are equally methodical; an alternating array of boxes covers the stage, permitting only deliberate steps from one empty space to the next. Here we have, in visual form, the co-existence of two separate worlds: a surface reality, and a heavily regulated underworld of abstract logic, intermeshed but kept entirely separate.

These devices rapidly become claustrophobic, immediately suggestive of the restrictiveness of purely logical thought. This plays out through our glimpses of Gödel's life, becoming increasingly isolated, shunned by his peers and seeking refuge in his work. At one point he shares his private torments: 'I cannot stop. I must either go back to the start, or continue onwards.' Gödel's social inadequacies culminate at a party, where he fumbles an explanation of the 'Liar's paradox' (see quote above), humiliating both himself and the logic in which he places so much faith.

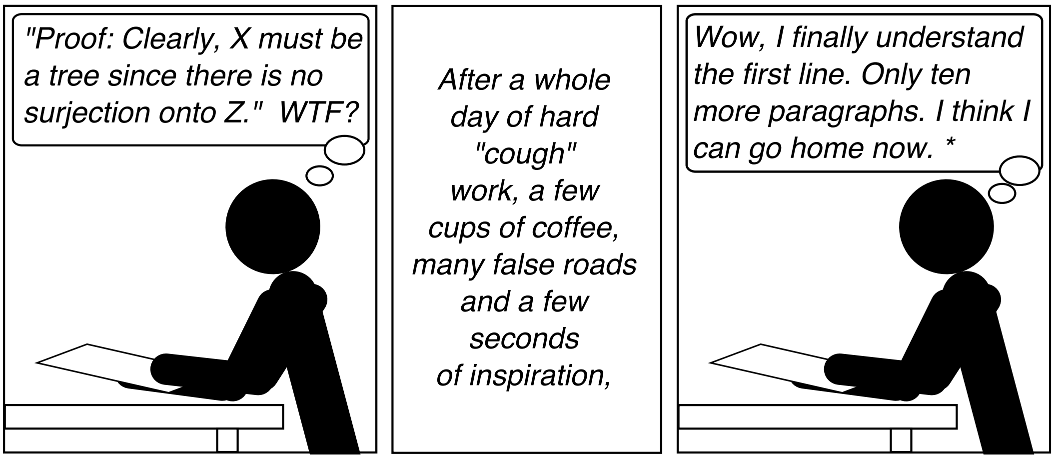
The play concludes with Gödel's realisation that 'absence', both in proof and in life, 'weighs heavier than presence'. With this Gödel emerges from the logical underworld, stepping for the first time into the 'surface' reality. From this vantage he realises that there are indeed truths that cannot be derived by logic alone. Faced with the proposition that humanity has cried a 'river of tears' Gödel proceeds to calculate: roughly 100 billion humans, living for an average of 50 years, with the volume of a teardrop in the order of 10^{-7} cubic metres. Yet his logic fails at the crucial step: it cannot explain how often, and by implication why, the uniquely human act of shedding tears occurs.

Ultimately *Incompleteness* is suggesting that humanity transcends reason, and that the complexity of human interaction is ample evidence of the 'incompleteness' of logical thought. Yet *Incompleteness* does not seek to undermine the importance and ongoing relevance of logic. As Gödel makes clear, 'though some of us can hold our breath for longer, we can all go beneath the surface'. The caveat lies in the logical implication: that from time to time we must all come up for air.

— Stephen Muirhead

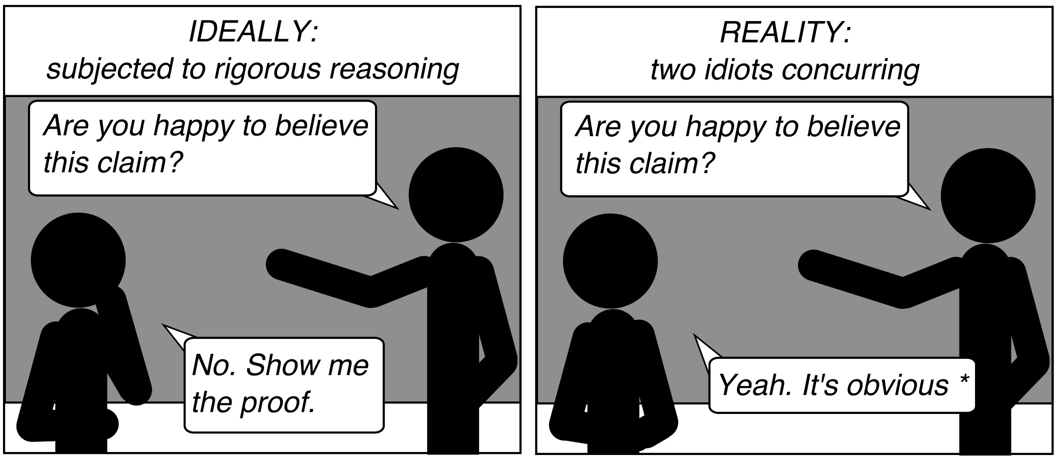
Paradox **Comics**

DECIPHERING A MATHS PAPER



** At this rate, he will finish this paper in the year 2049.*

COLLABORATIVE RESEARCH



** A simple check and he'd find a mistake.*

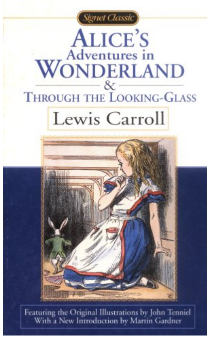
— Tharatorn Supasiti

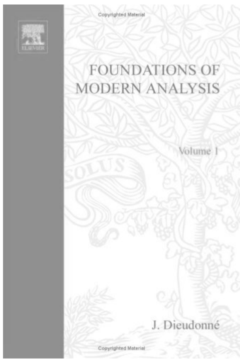
Books Every Maths Student Should Read

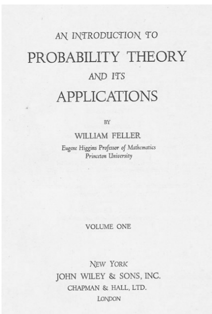
For this edition Paradox surveyed our learned professors and lecturers about their favourite books on mathematics and statistics. The resulting list is eclectic, comprising topics ranging from the history of mathematics to brainteasers, mathematicians' biographies to seminal textbooks, all of which will provide hours of amusement. Most of these books are available for borrowing at either the Mathematical Sciences Library, the ERC (Eastern Resource Centre) or the Baillieu Library.

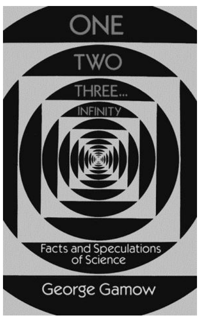
Special thanks go to Barry Hughes, Owen Jones, Jerry Koliha, Guoqi Qian, Arun Ram and Andrew Robinson for their thoughtful contributions, and happy reading!

Eric T Bell, *Mathematics: Queen and servant of science* (1951)	
	Regarded by some as essential reading for all students of mathematics, this book is for those interested in a thorough yet engaging story of how the field of mathematics came to be. From early beginnings with Euclid to more recent applications of mathematics, this classic is filled with inspiring accounts of how mathematics has buttressed the scientific and technological development of modern civilisation.

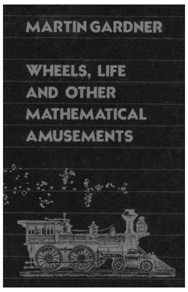
Lewis Carroll, *Alice's Adventures in Wonderland / Through the Looking Glass*	
	Accompanied by Sir John Tenniel's iconic illustrations, *Alice in Wonderland* is a classic novel that has enamoured mathematicians and Common Folk alike for more than a century. Carroll – a mathematician himself – takes the reader through a vivid surrealist fantasy brimming with strange characters in a nonsensical world. And while there are scores of interpretations of the story, a 2009 article by *New Scientist* explores the mathematical allusions in the book (see Melanie Bayley, 'Alice's Adventures in Algebra: Wonderland solved', available online). Note: Tim Burton's recent film adaptation is no substitute for reading the book.

<table>
<tr><td colspan="2">Jean Dieudonné, Foundations of Modern Analysis (1969)</td></tr>
<tr><td>
</td><td>‘It is a book without which I could not imagine my mathematical education. The author is the principal founder of the Bourbaki group, yet the book is un-Bourbaki-like.’ – Jerry Koliha</td></tr>
</table>

<table>
<tr><td colspan="2">William Feller, An Introduction to Probability Theory and its Applications (1968)</td></tr>
<tr><td>
</td><td>‘Now in its third edition, reprinted in 1971 with minor corrections, this book which first came out in 1950 was probably the first decent text on probability written in English. The balance of probabilistic reasoning and analytical techniques is marvellous and though specialists will regard it as dated, and it probably is, I retain enormous affection for it. There is a second volume, which came somewhat later (1st edition 1966; 2nd edition 1971), which contains more mathematical technicalities and is also excellent.’ – Barry Hughes</td></tr>
</table>

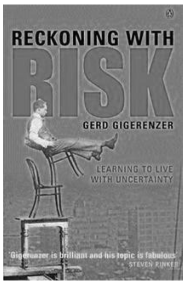
<table>
<tr><td colspan="2">George Gamow, One Two Three… Infinity: Facts and speculations of science (1988)</td></tr>
<tr><td>
</td><td>Written in layman’s English, Gamow’s book has drawn the curiosities of countless budding mathematicians and physicists with its exploration of famous unsolved problems (some of which have been solved since his death) and accessible explanations of big ideas such as ‘the size of infinity’.</td></tr>
</table>

Martin Gardner, *Wheels, Life and Other Mathematical Amusements* (1983)



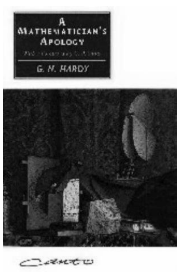
Filled with mathematical brainteasers, and spanning a wide range of topics including chess and electricity, Martin Gardner's book is prime entertainment for those who love engaging in thorny mental gymnastics. Perfect practice for the MUMS Puzzle Hunt in April!

Gerd Gigerenzer, *Reckoning with Risk: Learning to live with uncertainty* (2003)



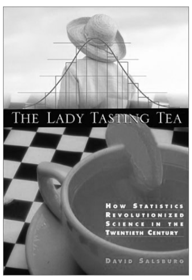
'Gigerenzer is at the forefront of statistical reasoning and cognition. This book is about using statistical tools for thinking in everyday life.' - Andrew Robinson

Godfrey H Hardy, *A Mathematician's Apology* (1948)



Acclaimed as 'one of the most eloquent descriptions in our language of the pleasure and power of mathematical invention' by the New Yorker, Hardy's memoir is a poignant depiction of his fading mathematical abilities at the end of his career, and his profound passion for the field. An inspiring book for undergraduates and practicing mathematicians alike.

David Salsburg, *The Lady Tasting Tea: How statistics revolutionized science in the twentieth century* (2001)



'[A] pleasant and very readable stroll through statistical inventions and controversies of the last 100 years.' – Andrew Robinson

A (Rough) Guide to Undergraduate Maths

Well, what sort of mathematician are you?

At Melbourne University, mathematics is broadly split into four fields: pure mathematics, applied mathematics, statistics and operations research. The question is, what's the difference between these fields? If you're a first year, fresh out of high school (to whom this article is mostly targeted), there's a good chance that the only two fields of mathematics that you're familiar with are *Specialist Mathematics* and *Mathematical Methods*. Which to be quite frank are the most useless titles for subjects ever. You might as well have called them *Maths* and *Harder Maths*. There's nothing particularly *special* or *methodical* about either. High school has hardly prepared you for the diversity and broad utility of university mathematics. So here I'll list just a few basic pointers that may help you with deciding just what sort of mathematician you are.

Pure Mathematics

Are you the sort of person who finds the elegance of mathematics attractive? Do you enjoy proving seemingly useless but nevertheless interesting results? If you answered yes to these questions, then pure maths may be the way forward for you.

In the pure maths specialisation, more than any other, you'll discover why maths is regarded as an art as well as a science (though, quite possibly, only mathematicians put it like this). Pure mathematics is about studying the underlying concepts that make all maths work. And besides that, it's just really cool stuff.

The variety of material in the pure maths specialisation makes it particularly interesting. You'll find out that solving polynomials isn't just as simple as using the quadratic formula. In fact, you'll even see why there is no quintic formula. You'll discover that a punctured torus (a donut surface with a hole in it) is essentially the same as two circles joined at a point. Just don't tell any bakers that one, it may blow their minds and result in some strange looking donuts later on. These are just tiny fragments of what you'll learn studying pure maths, but just as a small warning, this specialisation is not for the faint of heart.

Applied Mathematics

Do you really like formulae? Would you like to see how maths can be used in the real world? And most importantly, do you really *really* like calculus? If the answers to these questions are yes, then you should be looking into applied maths.

Applied mathematics is about trying to model complicated systems and then poking around with the inputs to see how things change under certain conditions. Applied mathematics has applications in just about every field you can think of. In the applied maths specialisation, you learn techniques and skills that will enable you to solve certain types of equations which commonly crop up in the real world, such as modeling river flows or how human cells reproduce. Just be prepared for a lot of calculus.

Probability, Statistics and Stochastic Processes

You see statistics all the time. Figures, percentages and ratios are thrown up all the time in the modern world. But do you ever wonder how meaningful these numbers are? When you play a card game, do you ever wonder, "well that was unlikely, but exactly *how* unlikely was it?" We all know that smoking is bad for you, but how exactly do you *prove* this? If these are things that you've wondered about, then you should be looking into probability, statistics and stochastic processes.

In probability you'll learn how to calculate the probability of certain events happening, and study various distributions occurring naturally in the real world. An important use of probability is its application to statistics and stochastic processes. In statistics you'll learn how to properly analyse a data set. By creating statistical models you'll be able to test the effects of certain variables on others. Stochastic processes is about modeling random processes that occur in the world. For example, you can model the number of people who walk into a shopping centre. You can even attempt to make money by modeling financial markets, though personally I wouldn't recommend this off just your undergraduate subjects.

Discrete Mathematics and Operations Research

So we all spend plenty of time bagging our the government for being slow, inefficient, wasteful, or more often than not all of the above. But how would you make it better? Do you spend time thinking about how you could make processes faster, more efficient and just better in general? If these questions appeal to you, then you should be looking into discrete mathematics and operations research.

This specialisation is all about decision making. And decision making is hard. Just think about a can of baked beans, and the path it travels from the farm where the original beans are grown, to your dinner plate. There's at least a dozen different processes that have to happen before it reaches you. Now the question is, what's the best way to do this? You'd want to reduce time, but also costs, and then on top of that increase quality. All of a sudden your choices aren't all so clear cut. Operations research deals with these sorts of issues in a scientific manner to help with decision making. And with society becoming more complex, and processes becoming more numerous, there's no doubt that this field is important.

Now what?

So maybe I've given you some idea about what the different specialisation are. The question that remains is what subjects to do. The best advice I can give is to pick up a copy of the course advice booklet produced by the maths department. It's bright orange and you can pick it up from the front office in the Richard Berry building. If you're a first year, the choices are actually remarkably simple.

First year subjects

In first year, maths students, regardless of your specialisation, will take a first year mathematics and statistics package. The first year package is a good little set of subjects that will give you an introductory glimpse into all the fields of maths. You are then able to narrow your focus on any of the specialisations. What you pick depends on how you did in high school. Most packages are two subjects, one in first semester, one in second, and there's the option of also taking the breadth subject 800-101 Critical Thinking with Data.

For students who did Specialist Maths, it depends on what your raw study score was:

- ≥ 38 – take 620-157 Accelerated Mathematics 1 in semester 1 and 620-158 Accelerated Mathematics 2 in semester 2.
- ≥ 27 – take 620-155 Calculus 2 in semester 1 and 620-156 Linear Algebra in semester 2.
- < 27 or not Specialist Maths – take 620-154 Calculus 1 before taking 620-155 Calculus 2 and 620-156 Linear Algebra.

After first year

Once you get into second or third year you'll need to specialise into one of the four fields I've talked about above. Because I'm lazy, I'm not going to go into all the subjects but instead refer you to the department's course advice booklet.

Disclaimers, advice and conclusion

As I'm sure my law student friends will tell me, I need to put in a disclaimer thingie here. The material above is all purely my opinion, I can not stress enough that if you are looking for course advice there are people, very friendly and nice people even, in the Maths and Stats Learning Centre (MSLC) who are far more qualified than yours truly to help you out with subject selection. Understandably, some people find it easier to talk to peers, so while we will always recommend that you speak with the MSLC, also feel free to drop into the MUMS room in the maths building and have a chat with some of us. We're also very friendly and nice people!

The field of mathematics and statistics is an enormous field with an incredible variety of content. Hopefully I've given some people an insight into how awesome maths is at uni, and offered some helpful advice. I've certainly had no regrets doing maths at uni, and I encourage you all to do as much maths as possible, though admittedly, I may be slightly biased.

— Han Liang Gan

How To Get Roots and Other Things

In this digressive article we shall examine how various entities, sorted in ascending order of complexity, compute the humble square root function. We start with an algorithm done by hand, suitable for a mere human being. Then, we look at how our future masters – computers and their less-able calculator cousins – outdo us with fancy technology. Finally, we move on to the *pure mathematician*, who prefers a convergent, albeit fairly useless, method to achieve his noble goal.

Humans

Issue 1 of the 2005 edition of Paradox already describes how to find square roots by hand (you can download it from the Paradox website, which you have no doubt bookmarked). But why does the algorithm work? To quickly recap, here's an example of how one might find the square root of 2:

	1.	4	1	4	2	…
1	2.	00	00	00	00	…
	1					
24	1	00				
		96				
281		4	00			
		2	81			
2824		1	19	00		
		1	12	96		
28282			6	04	00	
			5	65	64	
…				38	36	…

We divide the decimal digits of 2 into pairs, and at each step perform something akin to a long division. The catch is, the numbers down the left column are constructed by taking the 'partial root' a obtained on the top row, and the largest integer d such that $(20a + d) \times d \leq$ the last remainder. d is then the new digit in the partial root and $20a + d$ is the new entry on the left column.

For example, 28 (in bold) is obtained by doubling the partial root, 14, and d is required such that $(280 + d) \times d \leq 400$. The maximal d is thus 1.

The algorithm works due to the identity $(10a + d)^2 = 100a^2 + (20a + d)d$. Here a is a partial root and d is the next digit, so together they produce a sequence of digits that approximate $\sqrt{2}$. So in the above, $a = 14$, $(10a + d)^2 = 140^2 + (280 + d) \times d$. Now (inductively) 140^2 has been calculated by the last partial root (14), and $20000 - 140^2$ is precisely the remainder (400). Therefore we need $(280 + d) \times d$ close to the remainder to keep $(140 + d)^2$ close to 20000. So by induction it works.

How can this algorithm help you in life, you might ask? Only some decades ago, in many countries it was a part of the high school curriculum to compute square roots by hand (or on the abacus). In some places the computation of cube roots by hand was mandatory. This task is not as fearsome as it sounds, for we can adapt $(10a + b)^3$ along the above lines to derive an algorithm.

More interestingly, it might be handy for 'problem solving' or 'lateral thinking' questions in job interviews by quirky employers. I'm sure you've heard of some of those questions. I was certainly at an interview once and was suddenly asked to give a value for the square root of 1000. I was later told that many applicants responded with '100'.

However, its most important application is to combat the many evils of a constantly diminishing high school syllabus. Once I had an evil maths teacher who, among other things, encouraged the usage of calculators in year 7 maths classes. When I told her that *nothing* taught in year 7 required a calculator, she blurted out some challenge tantamount to finding the square root of 265. While she gallantly fumbled for a calculator and struggled to press the right buttons, I was able to produce a few correct digits. So there's a reason to teach your kids how to find square roots.

Computers and calculators

Of course I hold no hard feelings against calculators, apart from the fact that many of them are redundant. Many programmable models capable of symbolic computations are too bulky to be easily portable, their interfaces too convoluted for entering commands quickly, and their processors too weak (compared to laptops) to solve many serious problems.

Nevertheless, some degree of ingenuity is needed to make a hand held calculator perform a square root computation. One reason different algorithms are used is because each may suffer from loss of precision in pathological cases. Many models use the algorithm in the previous section with a few speedups (for instance, using binary, which makes guessing d easier). Others use the fact that $\sqrt{x} = \exp(1/2 \log x)$, where both $\exp$ and $\log$ have fast converging series (see the next section). This is, of course, similar to what you would do if you used log tables or slide rules, if you are old enough to remember those things but not quite old enough to forget them.

Some calculators utilise the CORDIC (coordinate rotation digital computer) algorithms. Apart from having a horrible acronym, the algorithms are quite nice, but to see this we first take a detour and see how a calculator may compute sin and cos.

Suppose you want to find $\cos(t)$, that is, you want to find the x-coordinate of a point on the unit circle that makes t^c with the x-axis. We can start from the point $(1, 0)$, rotate it by a suitably chosen angle, then by a smaller angle in the right direction, etc, until we get very close to t. Rotation by s can be done easily by matrix multiplication, the matrix used being

$$\begin{pmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{pmatrix} = \frac{1}{\sqrt{1+\tan^2 s}} \begin{pmatrix} 1 & -\tan(s) \\ \tan(s) & 1 \end{pmatrix}.$$

We judiciously pick s to be the sequence $\arctan(2^{-k}), k = 0, 1, 2, \ldots$, so the entries in the matrix become very easy to compute (especially in binary). Also, the factor at the front can be calculated in advance and stored in a table. As the set of angles we choose decrease monotonically to 0, we are bound to get close to t.

If we water down the CORDIC algorithm and only take the 'rotational' part, then we have an algorithm for calculating square roots in binary. Say we want to know the square root of $n = 100001001_2$. Our first guess is 100000_2, which we square and compare with n. It's too big, so we step down ('rotate') and try 10000_2, whose square is less than n. We then step up a little, testing 11000_2, whose square is too big. We then try 10100_2, etc, stepping up and down appropriately, until we get $10000.01\ldots$, which says the square root of 265 is about 16.25. At each stage we get an extra digit, and only need squaring, and this is fast; it can also be used to quickly test if a number is a perfect square.

But wait, are multiplications (e.g. squaring) really fast? Well, if you are only multiplying 2 digit numbers, which algorithm you use makes no practical difference; but under the standard algorithm, it takes the order of n^2 1-digit multiplications to multiply 2 n-digit numbers. Karatsuba, a Russian mathematician with a Japanese sounding name, had a better idea:

First note that addition and subtraction are easy. Now observe that, for instance, $(10a+b)\times(10c+d) = 100ac+10(ad+bc)+bd = 100ac+10((a+b)(c+d)-ac-bd)+bd$.

All the hard work lies in calculating ac, bd, and $(a+b)(c+d)$ and remembering each of them. That is, at the cost of extra memory, we can now multiply them using 3, not 4, 1-digit multiplications. This procedure can be implemented recursively to multiply two n-digit numbers in the order of $n^{\log_2 3}$ operations.

Now that you are convinced the multiplication algorithm humans regularly use is not optimal (in fact, we still don't know what the optimal algorithm is), you might also believe that our current best method is the fiendishly clever fast Fourier transform multiplication (FFT). Its full scope is too complicated to be stated here, but the basic idea is that when two polynomials are multiplied, the process is very similar to a convolution. Now the Fourier transform of a convolution is the product of the transforms. But to perform each (discrete) Fourier transform, we can divide the input into two parts (even and odd indices), effectively making it 2 transforms of half the input size, with an extra multiplication. Such divide-and-conquer approaches generally have a time of $O(n \log n)$ which also holds here. To apply this to multiplying numbers, just write the number as a polynomial with the variable set to the base.

We need one more piece of information before we explain how a computer can take square roots using arbitary precision arithmetic. Newton's method (or the Newton-Raphson method) is a good all-purpose root finding algorithm. To numerically solve $f(x) = 0$, we make a guess x_0, and iterate using $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. To see why this works, note that x_{n+1} is the the x-intercept of the tangent to $f(x)$ at x_n. So if x_n lies near a solution, unless the $f(x)$ is pathological, then graphically we see that x_{n+1} lies closer to the solution. In fact, it is routine to see that the convergence is quadratic (so the each error is approximately the square of the last one).

So if we try $f(x) = x^2 - a$, we quickly see that $x_{n+1} = \frac{1}{2}(x_n - a/x_n)$. This converges very quickly to $\sqrt{a}$ and was known to the Babylonians. However, it is not conducive to FFT because one division per step is required.[1] To avoid this, we find $1/\sqrt{a}$, using $f(x) = 1/x^2 - a$. Now we have $x_{n+1} = x_n + \frac{1}{2}x_n(1 - ax_n^2)$, involving only multiplications, and only one division is needed at the end to get the answer.

The final trick with Newton's method is this: Suppose we want d digits of precision in the final answer, should we run each iteration with d digits? Surprisingly, no. The algorithm approximately doubles the number of correct digits each time, so we can start with very low precision (say 1 digit), and double the precision each time. It doesn't matter if we lose some digits in this process, as the next step of the algorithm sets it right anyway. This reduces the total precision used from the order of $\log d$ to a constant, and makes earlier steps tremendously fast.

Pure mathematicians

But speed is of no concern to the pure mathematician. From the binomial theorem, we know that

$$\sqrt{1+x} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots ,$$

which can be used to compute $\sqrt{a}$ for $0 < a < 2$. For larger a, we can use the largest integer n such that $n^2 \leq a$ and look at $\sqrt{a} = n\sqrt{a/n^2}$ where a/n^2 is certainly less than 2. This is surely the pure mathematician's algorithm of choice, for it is elegant, expressible in closed form, convergent, and has a known error (the error is less than the first unused term in the series). Indeed, haven't you come across a first year exercise along the lines of 'estimate the square root of 26'? We proceed thus: $\sqrt{26} = 5\sqrt{1 + 1/25} \approx 5(1 + 1/25/2) = 5.1$.

To approximate how fast the error decreases, we just look at the ratio of the $(k+1)$th term and the kth term, which is $|\frac{1/2-k}{k+1}|x$. For large k this is approximately x (linear), which is not as good as Newton's method. Of course, the pure mathematician is in tune with infinity and does not care about such trifles.

[1] Divisions can be done using Newton's method: to find $1/b$, let $f(x) = 1/x - b$, then $x_{n+1} = x_n + x_n(1 - bx_n)$.

Series methods can, however, have occasional uses. There is a story of the physics Nobel laureate Richard Feynman soundly defeating the abacus. At a restaurant, a man quite proficient on the abacus came in and challenged the waiters and customers in a contest of speed addition. Not surprising, the abacus won, even against Feynman. When the man raised the bar to long division, he only managed to tie with Feynman. Finally, he made the mistake of moving up to cube roots, and the number he picked was 1729.03. Now, Feynman did not miss the convenient series $\sqrt[3]{1729.03} = \sqrt[3]{12^3 + 1.03} \approx 12 + \frac{12 \cdot 1.03}{3 \cdot 12^3} - \cdots$. So the great physicist simply sat there and slowly wrote down the answer correct to 5 decimal places, while the sweating, distaught abacus operator barely got 12.0.

We shall wrap up with a related story of how Feynman fooled a group of pure mathematicians from Princeton University. We know that $\exp(x) = 1 + x + x^2/2! + x^3/3! + \cdots$, and the radius of convergence is ∞. Feynman mumbled to the mathematical luminaries that it is easy to calculate powers of e using that series. Not convinced, one of them asked for the value of $e^{3.3}$. Quickly came an answer, 27.11. While they looked it up in a table, Feynman produced more correct digits. Not admitting defeat, they then asked for e^3, and Feynman gave 20.085. Still not believing that he could sum a series that fast, they asked for $e^{1.4}$. Reluctantly but quickly, the approximation 4.05 was given and Feynman walked out.

Of course, not even one as smart as Feynman could sum series instantly in his head (around 15 terms are required to get 4 digits of accuracy for $e^{3.3}$). He misled them into thinking he was summing the series, while in fact he memorised a few useful constants: $\log 10 \approx 2.3026, \log 2 \approx 0.69315$, and $e \approx 2.71828$.[2] Therefore, $e^{3.3} = e^{2.3}e \approx 27.18, e^3 = e^{2.3}e^{0.7} \approx 20$, and $e^{1.4} = e^{0.7}e^{0.7} \approx 4$, and some quick mental gymnastics made the appropriate corrections possible.

— James Wan

[2]Not every pure mathematician knows the numerical value of π, e, or γ.

A Guide to the MUMS Puzzle Hunt

Do you enjoy puzzles? Do you get a thrill when solving them? Do you like winning prizes? If you answered yes to any of the above, then you might want to participate in the annual MUMS Puzzle Hunt. This year the Puzzle Hunt will be held in the week beginning 19th April 2010. It's going to be a week involving nothing but puzzles!

But let's not get ahead of ourselves. First, here's a complete breakdown on what the Puzzle Hunt is. And to do that we need to first figure out what a puzzle is.

Wikipedia defines a puzzle as a problem or enigma that tests the ingenuity of the solver. Indeed, if you think about the different kinds of puzzles that exist – brainteasers, sudokus, jigsaws, crosswords, anagrams, sokoban, rubik's cube, tetris, etc – the list goes on and on!

Unless you've been living inside a black hole for the last twenty years, chances are you would have heard of or tried at least some of these puzzles. In fact, there's a good chance you might even have enjoyed them. Some even go so far as to grab the daily newspaper just for the puzzles section, choosing to ignore the news.

Enter the MUMS Puzzle Hunt. The Hunt ties in all kinds of puzzles you can think of, and then even more, into a week of action-packed puzzle-solving. 25 puzzles will be released throughout the week (five a day), and the objective is to solve as many of these as possible with minimal hints. You can have up to ten people on your team to help you with this task, or if you prefer you can enter by yourself. And the Hunt, except for the final challenge, takes place entirely online, so you can solve the puzzles in your own time, from wherever you happen to be.

Sounds simple, you might say, especially if you're an avid puzzler. But here's the twist – these puzzles come with no instructions. It is entirely up to your lateral thinking processes to decide what steps to take, and how to interpret the puzzle as intended by the puzzle writer.

Sounds random, you might say. And yet there is method in this madness. What looks like a bunch of symbols at first could very well turn into something elegant when viewed in the right perspective. Things will just start to click and fall into place, and that's where the thrill comes in.

Without further ado, let's jump right into a puzzle! One of my personal favourites is called *Pairs Repaired* (2009 Hunt, #5.2). It basically consists of eight boxes arranged in two rows, and looks something like this:

 T R Y S T A P P L Y L O O K S	 D I N E D B R I A R G A F F E	 S O U P S U D D E R T A R O T	 M I M E S H O O F S T I G H T
C L O C K A N N U L B U S K S 	F I F E R T R U S T M E M O S 	C R O O N A P N E A C E L L S 	Y E L L S H O O T S R A T E R

And that's the whole puzzle. Yes, it has absolutely no instructions, and your task as the puzzle solver is to find an answer within it. The answer is always a word, or bunch of words. But how do you know what the right answer is? And how would you even begin to solve it?

So stare at the puzzle for a while. Look for things that are unusual, unique, familiar, or simply out of place. Remember, you are always looking for a pattern that the puzzle writer intended. And then when you find it, fine-tune it to see how you might get an answer string out of it.

You might start to notice things, like the fact each box has three 5-letter words or that they all have a blank row. Now what? Is the blank row meant to be filled with another 5-letter word? And how? Maybe you stare some more and notice that all the words have a repeated letter!

At this point you might be tempted to just extract every repeated letter and see if you can make something out of it:

TPO	DRF	SDT	MOT
CNS	FTM	OAL	LOR

Nope, it doesn't seem to spell anything useful. Maybe we might want to pair up the letters again? (Remember, the title is usually a clue, and Pairs Repaired seems to imply pairing more than once.) Unfortunately, there's only one P, one C, and in general it just doesn't seem to work.

And then it hits you. You want to pair the letters not only within a word, but probably between different words in the same box. A quick check reveals that TRYST and APPLY shares a Y, TRYST and LOOKS shares an S, and APPLY and LOOKS share an L. Furthermore, this works for all eight boxes!

In fact, you can eliminate all the pairs in this way to leave just three letters in each box. And then maybe, just maybe, you have to find another five-letter word that uses these three letters, plus another repeated letter not already used in the same box. Let's see how that works:

RAK	NBG	PEA	EFG
OAB	IUO	RPS	YHA

And now, we just find the missing repeated letter in each set, to give us the following words:

CRACK	BONGO	LAPEL	FUGUE
MAMBO	UNION	USURP	HAPPY

Which you can get by adding the letters C, O, L, U, M, N, U, P respectively. Wait a second, COLUMN UP. Could that be the final answer? It does look a bit unusual. In fact, you might spend a guess on this answer. Nothing wrong with that. But you decide to decode it instead.

COLUMN UP. Hmmm. . . could that be telling you to read up a column? Reading up the columns give nonsense six-letter phrases like BACLAT, UNLOPR, SNOOPY. Hold on, SNOOPY is a real word. And so are MUFFIN, CACTUS and RHYTHM! We got four new words out of this, but how do we get an answer out of all this?

Indeed, the idea is to do the exact same thing all over again, except with six-letter words this time. Removing all duplicate letters gives P, I, A, R. Another quick dictionary search reveals PILLAR as the only six-letter word giving us what we need. (Pillar also means a column standing up, in fact.)

And so you proceed to solve the puzzle with PILLAR as the answer, and the answer checker warmly congratulates you on solving it correctly. Congrats! You have just solved a Puzzle Hunt puzzle! As you can see, none of the steps are impossible or even that difficult when considered individually, it's just a lot of pattern matching.

Now, if you were really going through my entire demonstration of how the puzzle works, you might have wondered one thing. In fact, you might have wondered many more things, but there really is one finer point which was just hand-waved completely. Just like in mathematics.

I'm referring of course, to the part where you find the five-letter words, i.e. from `RAK` to `CRACK`. Any good puzzler always needs a good bag of tools, and in fact there are quite a number of useful tools that will suit almost every puzzler: Google, Wikipedia, Babelfish, Wordsmith, Crossword Solver etc.

In this case, you would probably need to use a crossword solver or an anagram finder fairly frequently, just to test whether different sets of letters might produce valid 5-letter words. Or if you were more technical, you might even write a script that automatically did all the dictionary checking for you.

In any case, the Puzzle Hunt is an annual event we organise, and is one of the largest events of the calendar, with hundreds of people participating from all over the world! For more information regarding the Puzzle Hunt, please visit: `http://www.ms.unimelb.edu.au/~mums/puzzlehunt/`. The website also contains all the puzzles from previous Hunts, as well as hints and full solutions, so if you want to really prepare then this is a great place to start. Good luck!

— Muhammad Adib Surani

Maths in the News(2):
'Rom-coms, period dramas are rubbish: mathematical proof' (*The Registry*, 24/2/2010). According to James Cuttin of Cornell University, the ability of a film to keep our attention span is closely linked to the distribution of individual-shot lengths. Films whose distributions best mirrors the $\frac{1}{f}$ fluctuations found in chaos theory are those we find most natural to watch. According to a study of 150 popular films, action films in general perform better than dramas and rom-coms under the analysis, and new films in general outdo old films. *A Perfect Storm* was the film which best matched natural fluctuations.

Group theory could save your life

Introduction

Imagine you are walking down the road, not a care in the world. It is a sunny day – no clouds or thunderstorms to be seen. Out of nowhere, a masked bandit appears with a gun. Like any normal person you are shocked and scared. You offer your wallet, thinking that giving money will prevent you from getting hurt, but the bandit refuses. He wants information...

Specifically, he wants to know what the *Fisher-Griess Monster* looks like. 'What!?' You were expecting questions about your bank account, and about your private keys (for RSA decryption purposes), but no, it is a question about a monster. And this monster is not the sort that lurks under your bed or in the closet, unless you happen to have a book or two about group theory in those places.

Groups

If you have already taken *620-297: Group Theory & Linear Algebra*, an equivalent subject, or taught yourself group theory, you can safely skip ahead to the section about the monster. Otherwise read on for a basic introduction to groups that should provide the necessary framework for beginning to understanding the Monster.

A group is a mathematical object, consisting of a set and an operation, with a few properties imposed on it. It might sound abstract, and indeed it is, but we have all encountered groups since primary school. We all learn to count, $1, 2, 3 \ldots$, but $(\mathbb{N}, +)$, the natural numbers with addition is not a group. If we add the negative numbers too, we get a group $(\mathbb{Z}, +)$, the integers under addition. Likewise, $(\mathbb{R} \setminus \{0\}, \times)$, the non-zero real numbers under multiplication is a group.

In order to qualify to be a group, G, a few properties must be satisfied. For $g_1, g_2 \in G$, we demand that $g_1 + g_2 \in G$. We require the existence of an identity element $e \in G$ which has the property $g + e = e + g = g$. We also desire that for all $g \in G$, a unique inverse $g^{-1} \in G$ exists which has the property that $g + g^{-1} = g^{-1} + g = e$.

It should be evident, with a little thought, that the above examples of $(\mathbb{R} \setminus$

$\{0\}, \times)$ and $(\mathbb{Z}, +)$ fulfil these requirements and so are groups. Another use of groups is in describing *symmetries* of objects. A symmetry can be thought of in almost the most obvious way, as a manipulation of an object that returns it unchanged. In order to make the definition consistent with the structure of the group, we will also include doing nothing as a symmetry (this is the identity object).

If we have a square, the symmetries consist of a rotation of: $0, 90, 180$ and 270 degrees, together with reflections in: the horizontal plane, vertical plane and two diagonal planes. It should be clear to see that any combination of symmetries gives another symmetry, and that there is exactly one way to 'undo' each symmetry, satisfying the group properties. There are a total of 8 symmetries. This and other groups that describe the symmetries of regular polygons are known as *dihedral* groups and identified by D_n, where n is the number of sides. Groups describing symmetries can also be made for the polyhedra, or objects such as a Rubik's Cube.

With this short introduction into what groups are and some of their properties, I can now introduce the *Monster*.

The monster

The history is as follows. The *Fischer-Griess Monster* was constructed in 1982 by Robert Griess. It was first proposed as being a group nine years beforehand, in 1973 by Griess and Bernd Fischer. John Conway nicknamed it the *Monster*.

Now, as far as group sizes go (for finite groups), the Monster is *very very* large. It is in fact the largest of the sporadic groups. The number of elements in it is equal to:

$$808017424794512875886459904961710757005754368000000000 =$$

$$2^{46} \times 3^{20} \times 5^9 \times 7^6 \times 11^2 \times 13^3 \times 17 \times 19 \times 23 \times 29 \times 31 \times 41 \times 47 \times 59 \times 71$$

In the same way as D_4 was a description of the symmetries (rotations and reflections) of a square, the Monster group can be thought of as the group of rotations in 196883-dimensional space. Contained within the Monster are many subgroups, which as the name indicates, are subsections of the group which are themselves groups, just as $(2\mathbb{Z}, +)$, the even numbers under addition, are a subgroup of $\mathbb{Z}$. Through knowledge of these subgroups (which are smaller in size and easier to comprehend), some properties of the Monster can be identified more easily than if the whole structure were observed directly.

Lifesaving?

So back to the story. You have just been held at gunpoint, and asked to describe the Fischer-Griess Monster. You tell him that it has a finite number of members. You tell him that it is the largest of the sporadic groups. But when the bandit asks you to describe what it looks like, you fail to respond adequately. As a result, you are shot. As blood pools around you, you reach for your phone and dial 000. You tell the operator that you have been shot, but unfortunately they too want to know the size of the Monster Group. It seems that 'this line is reserved for people who know elementary group theory'.

The scenario is clearly preposterous. One would hope that emergency numbers do not require knowledge of maths. But it does illustrate at least two points. One: there is lots of maths that is unknown in the wider population. Two: maths can allow us easily (and sometimes not-so-easily) to create objects that we can't comprehend. Elementary group theory can be a very visual and understandable field of maths, one that can work with little more than school-level knowledge of what symmetries are. But using it creatures can be created which can't actually be visualised, even though their construction is known, that is, unless you have an idea of what 196883-dimensional space looks like.

Post-script

Some readers may note the situation that motivated this article is based on a cartoon from *Abstruse Goose*,[1] a little-known webcomic with a similar simple-drawn style and geekish humour to the ever prevalent *xkcd*.[2] If you're interested, *Abstruse Goose* draws lots of its ideas from maths, so the cartoons can help keep you occupied when you should be studying for vector calculus. Have fun!

— Mark Kowarsky

[1] http://www.abstrusegoose.com/96

[2] Which, we might add, is also similar to the Paradox comics – see page 12

The Calculus of Variations

Anyone reading this article has probably come across a differential equation on more than a few occasions. You're probably also aware how useful a tool they can be. Solutions to differential equations are some of the most widely used advanced[1] results in mathematics – especially from a scientific perspective. Sending a person into space, splitting the atom, or even building the computer that I'm writing this article on, are all developments that owe a large deal to the solution of 'DEs'.

Nevertheless, it is perhaps surprising that so many areas of modern development are founded on this one method. Without questioning the utility or soundness of DEs and their solutions, we may validly ask whether DEs are only so widely used because there is no other similar tool. Perhaps this is a form of Kaplan's 'law of the instrument': if all you have is a hammer, everything looks like a nail.[2] To skip to the chase, there is in fact an alternative way to doing many DE-based problems, and just like any change in perspective it makes some problems easier, some harder and provides new insights. As the title to this article hints, this method is called the calculus of variations.

Motivation: The shortest distance between two points

What is the shortest distance between two points? Don't scoff too quickly, as it isn't always a straight line. For instance, the shortest distance between two points on a spherical surface like the Earth clearly isn't a straight line. Even if the two points are on a flat Euclidean space, how do you show the shortest distance is a line? Well let's have a try. Working in Cartesian coordinates let the two points be $A = (x_1, y_1)$ and $B = (x_2, y_2)$. Then the distance, d, between them is

$$d = \int_A^B ds.$$

[1] I use 'advanced' to distinguish them from ideas like arithmetic, which are obviously more widely used. That said, given the number of addition/multiplication errors I still make, perhaps this distinction is somewhat inappropriate!

[2] This is the colloquial form; the original phrase is: 'Give a small boy a hammer, and he will find that everything he encounters needs pounding'. Abraham Kaplan, *The Conduct of Inquiry: Methodology for Behavioural Science* (1964) p 28.

In Euclidean space we can rewrite ds using Pythagoras' theorem:

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (\frac{dy}{dx})^2}dx.$$

Subbing this in gives an integral which we want to minimise in order to minimise the distance:

$$d = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}dx, \qquad y' = \frac{dy}{dx}.$$

Our problem has been rephrased in terms of finding a curve y which minimises the above integral. In fact there are a large number of problems that can be phrased in terms of making an integral equation stationary; for example, I'm sure you can think of how to turn a volume, time or mass into an integral, much like what was done for distance above. The calculus of variations gives a way of determining what curve makes these integrals stationary.

The General Problem and the Euler Equation [3]

The following derivation of the general solution makes use of integration by parts. If you don't know about integration by parts or don't care how the result was derived, feel free to skip to the solution – or perhaps refer to old Wiki for a refresher on integration by parts.[4] In general the problem is to find a curve $y(x)$ that makes the following integral stationary:

$$I = \int_{x_1}^{x_2} f(x, y, y')dx.$$

In order to find a curve that minimises this, it seems reasonable to see how the integral changes if $y(x)$ changes a small amount. In order to do this we

[3] Sometimes called the Euler-Lagrange Equation.

[4] This derivation was based on that in M. Boas, *Mathematical Methods in the Physical Sciences* (2006 3rd ed), Ch 9. For a more elegant and rigorous derivation see L Landau and E Lifshitz, *Mechanics* (1976 3rd ed), 2.

introduce an arbitrary continuous function $\eta(x)$ such that $\eta(x_1) = \eta(x_2) = 0$, a small parameter ε and a new function $Y(x)$:

$$Y(x) = y(x) + \varepsilon\eta(x) \implies \frac{dY}{d\varepsilon}$$
$$\therefore Y' = y' + \varepsilon\eta \implies \frac{dY'}{d\varepsilon} = \eta'.$$

These last properties will be used shortly. This new function Y has the same values as y at the endpoints by construction and differs from y only by a small amount in between. Thus by changing ε the behaviour of the integral for different y values can be determined. Using this logic, we proceed as follows with a new integral:

$$I(\varepsilon) = \int_{x_1}^{x_2} f(x, Y, Y')dx$$
$$\therefore \frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial Y}\frac{\partial Y}{\partial \varepsilon} + \frac{\partial f}{\partial Y'}\frac{\partial Y'}{\partial \varepsilon}\right)dx$$
$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial Y}\eta(x) + \frac{\partial f}{\partial Y'}\eta'(x)\right)dx.$$

Now in order to make the integral stationary, from elementary calculus this means setting $\frac{dI}{d\varepsilon} = 0$. To get $y(x)$ and not $Y(x)$ requires setting $\varepsilon = 0$ (and hence $Y = y$). Thus this becomes

$$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x)\right)dx = 0.$$

Applying integration by parts to the second half of this integral and remembering that $\eta(x_1) = \eta(x_2) = 0$ yields

$$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta(x)dx = 0.$$

But as $\eta(x)$ was a completely arbitrary function, the only way to ensure that this is true (i.e. the integral is stationary in general) is to require the term in brackets to vanish:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.$$

This is the famous Euler equation. On a historical note the calculus of variations was first motivated by the Brachistochrone[5] problem. Galileo was one of the first to consider it, but he arrived at the wrong solution. Later on, in solving the Brachistochrone problem, Jacob Bernoulli created a method that was refined by Euler into the calculus of variations.

Finding the Shortest Distance

Let's try applying this solution to the integral for the shortest distance between two points, written above. Here

$$f(x, y, y') = \sqrt{1 + (y')^2}.$$

Subbing this into the Euler equation:

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right) = 0.$$

Now

$$\frac{y'}{\sqrt{1 + (y')^2}} = c,$$

a constant, so

$$y' = \pm\sqrt{\frac{c^2}{1 - c^2}} = m,$$

another constant. Finally,

$$\frac{dy}{dx} = m \implies y(x) = mx + b,$$

for constants m, b.

[5]This ungainly word has its origins in Greek: *brachistos* (shortest), and *chronos* (time); or so Wikipedia tells me.

Thus the shortest distance in flat space really is a straight line and we've shown it via a somewhat long-winded derivation. Nevertheless the general solution we've obtained, namely the Euler equation, is a very powerful tool. If we had inserted the ds for a spherical surface,[6] we would have found that the shortest distance between two points is a great circle: a circle on the sphere whose centre coincides with the centre of the sphere. This is why when you look at the path an airplane takes on a flat map of the world, it appears to be curved and not straight. The route it's taking is actually the shortest distance.[7]

Conclusion

I can imagine some might complain that the calculus of variations is just another hammer, in that we've gone from making an integral stationary to solving a DE. But to see that this isn't the case, consider what information is put into the problem to get a solution. In a heuristic sense, for a DE we input some initial conditions and then the solution is created by infinitesimal iterations: the DE takes the input values a differential amount forward, which creates new values to put into the DE, and the process is repeated to get a solution. On the other hand, the calculus of variations asks for the start and endpoint of the problem (for example, the endpoints A and B above) and then fills in what happened in the middle.

In this sense you can see that the calculus of variations is indeed a different way of solving such problems. The fact the problem is converted into a DE is more of a pragmatic matter – it allows us to bring to bear all the tools we have developed for solving DEs.

Unfortunately there isn't room for me to explore many of the other problems that are simplified by the calculus of variations – to give an example, in some areas of modern theoretical physics it is used even more than DEs. But if you found the calculus of variations interesting and wonder where else it could lead then you are in good company. The last of David Hilbert's twenty-three open problems in mathematics of 1900 was: further development of the calculus of variations.[8]

— Nick Rodd

[6] ds depends on the shape of the surface under investigation. It is rarely as simple as the Euclidean space example given above and the spherical derivation is somewhat more involved.

[7] Unless they're making use of jet streams, but this is not something I want to go into here.

[8] See the Wikipedia article on Hilbert's problems for more information: http://en.wikipedia.org/wiki/Hilbert's_problems

Solutions to Problems from Last Edition

We had a number of correct solutions to the problems from last issue. Below are the prize winners. The prize money may be collected from the MUMS room (G24) in the Richard Berry Building.

Adrian Khoo solved problem 1 and may collect $1.

Carol Badre solved problem 4 and may collect $3.

Natalie Aisbett solved all the problems and may collect $15.

Christopher Chen solved all the problems and may collect $15.

1. What is the minimum number of *snaps* needed to break a nxm chocolate bar into individual pieces, assuming that a snap cannot act on two disconnected portions of the chocolate bar at the same time?

   Solution: Each snap increases the total number of pieces by exactly 1. Therefore exactly $nm - 1$ snaps are required.

2. Let a, b, c be distinct integers, and let P be a polynomial having integer coefficients. Show that it is impossible to have $P(a) = b$, $P(b) = c$, and $P(c) = a$.

   Solution: Since $x - y | x^n - y^n$ for all integers x, y, n, we have that $x - y | P(x) - P(y)$ for any polynomial P. Thus $a - b | P(a) - P(b) \implies a - b | b - c$. Similarly, $b - c | c - a$ and $c - a | a - b$. Therefore $|a - b| = |b - c| = |c - a|$. Without loss of generality we assume $a \geq b, a \geq c$. Then $a - b = a - c \implies b = c$, which is a contradiction.

3. Consider an nxnxn cube as built from n^3 basic 1x1x1 cubes. Let a 2x2x2 cube with one basic cube missing be called a *block*. Prove that a 2^nx2^nx2^n cube with any basic cube removed (including in the interior) can be constructed entirely from blocks.

   Solution: We prove by induction. For $n = 1$ the cube is by definition a *block*. Assume the claim is true for $n = k$. Consider a 2^{k+1}x2^{k+1}x2^{k+1} cube with a basic cube removed. This can be broken up into eight pieces by slicing the cube in half in each dimension. Seven of these pieces are 2^kx2^kx2^k cubes. The eighth piece is a 2^kx2^kx2^k cube with a basic cube removed, which by the inductive assumption can be constructed from *blocks*. For the remaining seven pieces, if we remove the corner block closest to the centre of the original cube these too, by the assumption, can be constructed from *blocks*. Lastly, the seven basic cubes around the centre form another *block*. Thus the claim is true for $n = k + 1$, and by induction is true for all n.

4. Prove that $x^2 + 3 = 4y(y - 1)$ has no solutions in the integers.

   Solution: We rearrange the equation to produce $(2y - 1)^2 - x^2 = 4$. If x and y are both integers then so are x and $2y - 1$. Since the only squares that differ by four are 0 and 4, we have $x^2 = 0$ and $(2y - 1)^2 = 4$. Thus $2y - 1$ is either -2 or 2, which yield sa contradiction if y is an integer.

5. Each square of a 8x8 grid contains either a 1 or a 0. On this grid, you may choose any 3x3 or 4x4 subgrid to invert (swap all 0s to 1s, and all 1s to 0s). Using this operation repeatedly, can you always remove all the ones from the grid?

   Solution: The operation of 3x3 or 4x4 inversions are commutative, associative, and have order two (they square to produce the identity). Since there are 36 3x3 subgrids and 25 4x4 subgrids contained within an 8x8 grid, we have at most $2^{36+25} = 2^{61}$ distinct combinations of invesions (each base inversion is either switched *on* or *off*). Since there are 2^{64} possible arrangements of the 8x8 grid, there exist grids which cannot be *solved* by combining inversions.

6. Find the smallest n such that given any n distinct integers one can always find 4 different integers a, b, c, d such that $a + b - c - d$ is divisible by 20.

   Solution: We consider the problem modulo 20. Any collection of nine integers has either; a) four congruent integers $(a \equiv b \equiv c \equiv d)$, b) two sets of two congruent integers $(a \equiv c, b \equiv d)$, or c) ≥ 7 distinct integers. In both case a) and b) we have $a + b - c - d \equiv 0$. In case c), since $\binom{7}{2} = 21 > 20$, by the pigeon hole principle at least two of the pairs sum to the same value. Thus either $a + b \equiv c + d$ or $a + b \equiv a + c$. In the first case we have $a + b - c - d \equiv 0$, and in the latter we have $b \equiv c$, which is a contradiction. Thus any set of nine integers has the required property. Since the set $0, 0, 0, 1, 2, 4, 7, 12$ does not, $n = 9$ is the minimum such n.

7. Prove that $\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n}} < 2$.

   Solution: $\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n}} = \sum_{n=1}^{\infty} (\frac{\sqrt{n}}{n} - \frac{\sqrt{n}}{n+1}) = 1 + \sum_{n=2}^{\infty} \frac{\sqrt{n}-\sqrt{n-1}}{n} < 1 + \sum_{n=2}^{\infty} \frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n}\sqrt{n-1}} = 1 + \sum_{n=2}^{\infty} (\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}) = 1 + \frac{1}{\sqrt{2-1}} = 2 (telescoping)$.

Paradox Problems

Below are some puzzles and problems for which cash prizes are awarded. Anyone who submits a clear and elegant solution may claim the indicated amount (up to a maximum of four cash prizes per person). Either email the solution to the editor (see inside front cover for address) or drop a hard copy into the MUMS room (G24) in the Richard Berry Building; please include your name.

1. ($2) Prove that the sum of the 2009th powers of the first 2009 positive integers is divisible by 2009.
2. ($2) Two bikers, Sam and Steve, simultaneously set off from one end of a road, cycle back and forth along the road (turning instantaneously at the end-points), and stop when they arrive simultaneously at the end opposite from where they started. At this point Sam has travelled the length of the road nine times, and Stephen 13. How many times did they pass each other going in opposite directions?
3. ($3) Prove that a set of size n has no more that $n!$ partitions into disjoint subsets.
4. ($3) If you place twenty-one 3x1 blocks on a chessboard, so that there is one square not covered, what are the possible positions for this square?
5. ($4) If $n+1$ is a multiple of 24, show that the sum of divisors of n is also divisible by 24.
6. ($4) Prove that for every positive integer n, there is an integer x such that $x^2 - 17$ is divisible by 2^n.
7. ($6) For a 5x5 array of 1s and 0s, a *move* consists of choosing a square to change state (from 0 to 1 or 1 to 0), which causes each adjacent square in the same row or column to also change state. If the grid starts off containing 24 0s and a solitary 1, for which positions of this 1 can a combination of *moves* reduce the grid to all 0s? (Partial credit may be awarded.)

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