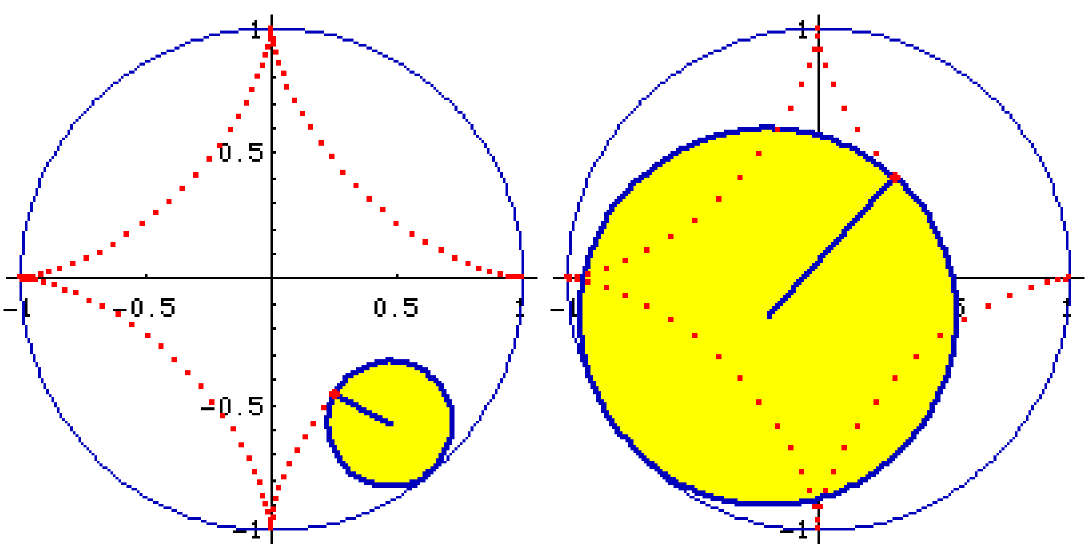
Astroid [1]

1 Description

An Astroid is a curve traced out by a point on the circumference of one circle (of radius r) as that circle rolls without slipping on the inside of a second circle having four times or four-thirds times the radius of the first. The latter is known as *double generation*. The Astroid is thus a special kind of a hypocycloid—the family of analogous curves one gets if one allows the ratios of the radii to be arbitrary. In 3D-XplorMath, the radius r is represented by the parameter aa. A nice geometric property of the Astroid is that its tangents, when extended until they cut the x-axis and the y-axis, all have the same length. This means, if one leans a ladder (say of length L) against a wall at all possible angles, then the envelope of the ladder's positions is part of an Astroid. Since (by symmetry) the tangent to the Astroid at a point p closest to the origin has a slope of plus or minus one, it follows that the distance of p from the origin is $L/2$, and so L is the "waist-diameter" of the Astroid, i.e., the distance from p to $-p$. Since the diagonal of the Astroid clearly has length $2L$, it is twice as long as the waist-diameter.



[1] This file is from the 3D-XploreMath project.
Please see http://rsp.math.brandeis.edu/3D-XplorMath/index.html

It can be shown that the normals of an Astroid envelope an Astroid of twice the size. (To see a visual demonstration of this fact, in 3D-XplorMath, select Show Osculating Circles and Normals from the Action menu.) If you think about what this means, you should see that it gives a ruler construction for the Astroid: Intersect each ladder (between the x-axis and the y-axis) for the smaller Astroid with the orthogonal and twice as long ladder (between the 45-degree lines) for the larger Astroid.

2 Formulas

Parametric equations for an Astroid are:
$x(t) := r\cos(t)^3$,
$y(t) := r\sin(t)^3$,
so the Astroid can also be described by the implicit equation:
$|x|^{2/3} + |y|^{2/3} = r^{2/3}$.
This formula gives an astroid centered at the origin with its four cusps lying on the axes at unit distance from the origin. To get rid of the fractional power, cube both sides:

$$\begin{aligned} x^2 + y^2 + 3x^{4/3}y^{2/3} + 3x^{2/3}y^{4/3} &= 1 \\ 3(x^{4/3}y^{2/3} + x^{2/3}y^{4/3}) &= 1 - x^2 - y^2 \end{aligned}$$

and replace $x^{2/3}$ by $1 - y^{2/3}$ and $y^{2/3}$ by $1 - x^{2/3}$ to obtain:

$$\begin{aligned} 3(x^{4/3}(1 - x^{2/3}) + (1 - y^{2/3})y^{4/3}) &= 1 - x^2 - y^2 \\ 3(x^{4/3} + y^{4/3}) &= 1 + 2(x^2 + y^2) \end{aligned}$$

If we next square both sides of the equality $x^{2/3} + y^{2/3} = 1$ and simplify, we find $x^{4/3} + y^{4/3} = 1 - 2x^{2/3}y^{2/3}$, and if we use this to substitute for $x^{4/3} + y^{4/3}$ in the above equation to obtain:

$$\begin{aligned} 3(1 - 2x^{2/3}y^{2/3}) &= 1 + 2(x^2 + y^2) \\ -6x^{2/3}y^{2/3} &= 1 + 2(x^2 + y^2) - 3. \end{aligned}$$

Finally, we cube both sides to find:

$$-6^3x^2y^2 = (2(x^2 + y^2) - 2)^3$$

and divide by 2^3 to reach our desired implicit equation.

$$(x^2 + y^2 - 1)^3 + 27x^2y^2 = 0.$$

3 History

Quote from Robert C. Yates, 1952:

> The cycloidal curves, including the astroid, were discovered by Roemer (1674) in his search for the best form for gear teeth. Double generation was first noticed by Daniel Bernoulli in 1725.

Quote from E. H. Lockwood, 1961:

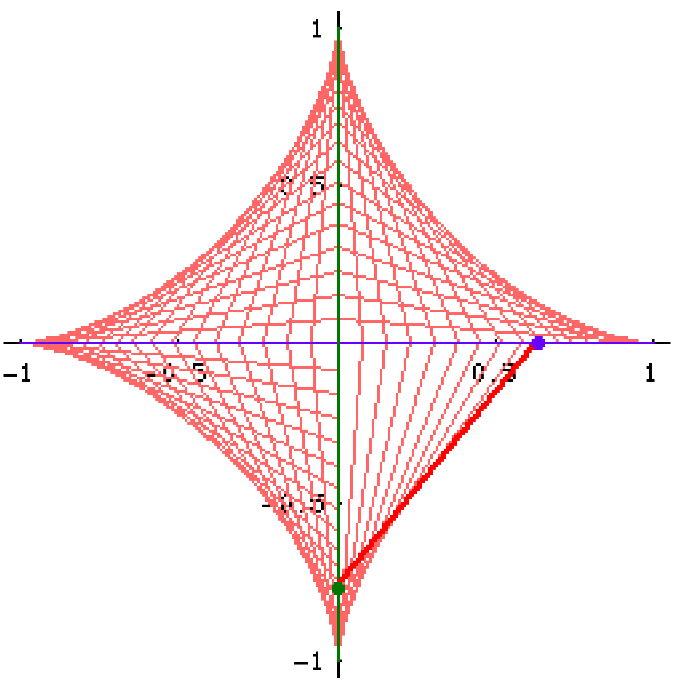
> The astroid seems to have acquired its present name only in 1838, in a book published in Vienna; it went, even after that time, under various other names, such as cubocycloid, paracycle, four-cusp-curve, and so on. The equation $x^{2/3} + y^{2/3} = a^{2/3}$ can, however, be found in Leibniz's correspondence as early as 1715.

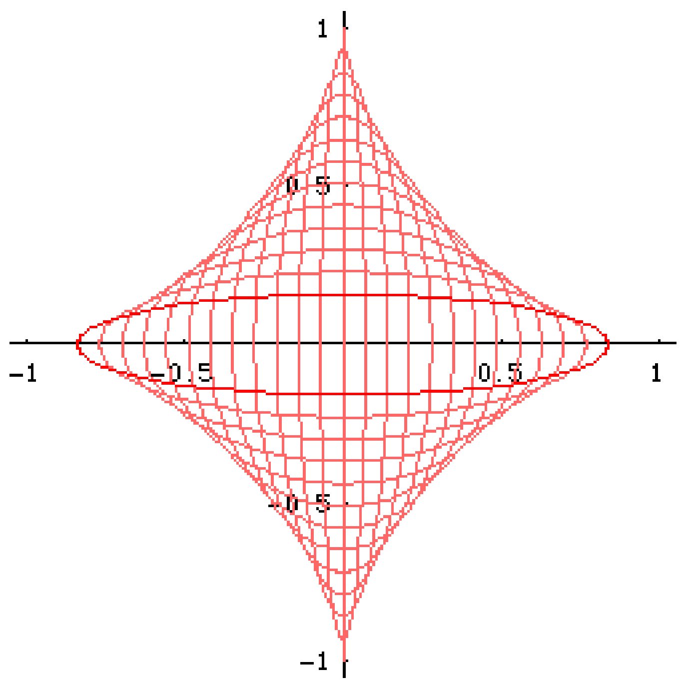
4 Properties

4.1 Trammel of Archimedes and Envelope of Ellipses

Define the *axes* of the astroid to be the two perpendicular lines passing through pairs of alternate cusps. A fundamental property of the Astroid is that the length of the segment of a tangent between these two axes is a constant. The Trammel of Archimedes is a mechanical device that is based on this property: it has a fixed bar whose ends slide on two perpendicular tracks. The envelope of the moving bar is then the Astroid, while any particular point on the bar will trace out an ellipse.

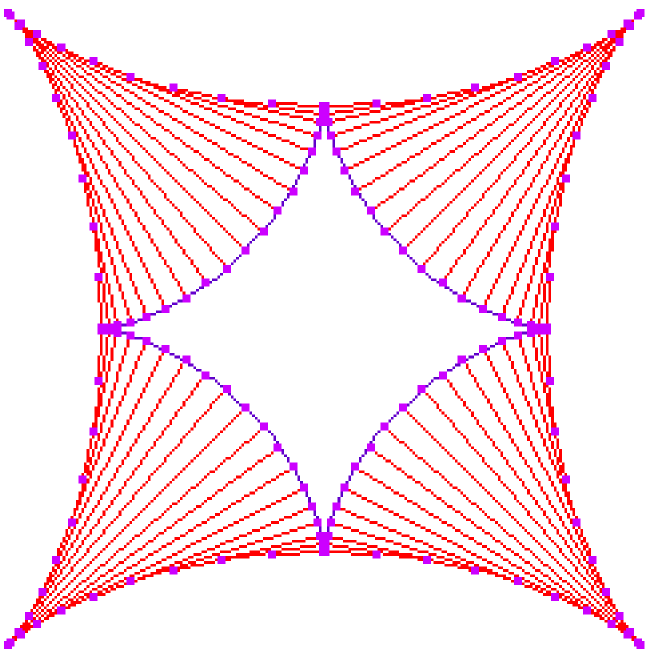
The Astroid is also the envelope of co-axial ellipses whose sum of major and minor axes is contsant.

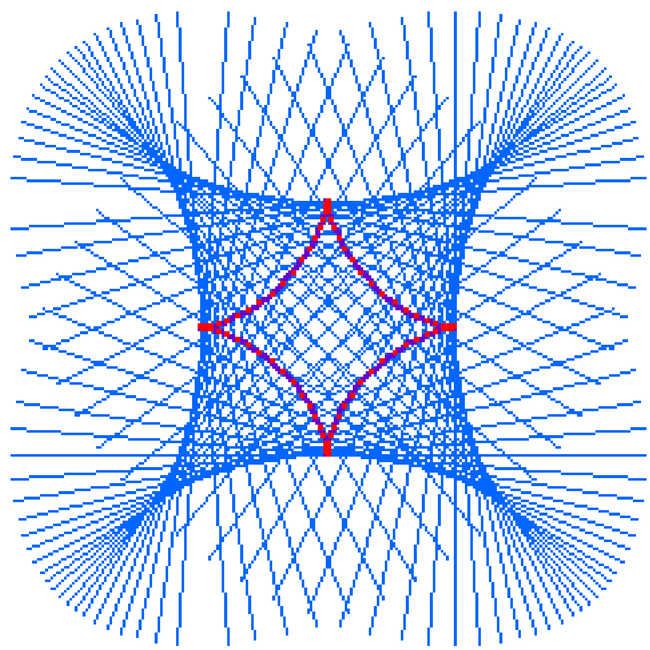




4.2 The Evolute of the Astroid

The evolute of an astroid is another astroid. (In fact, the evolute of any epi- or hypo- cycloid is a scaled version of itself.) In the first figure below, each point on the curve is connected to the center of its osculating circle, while in the second, the evolute is seen as the envelope of normals.



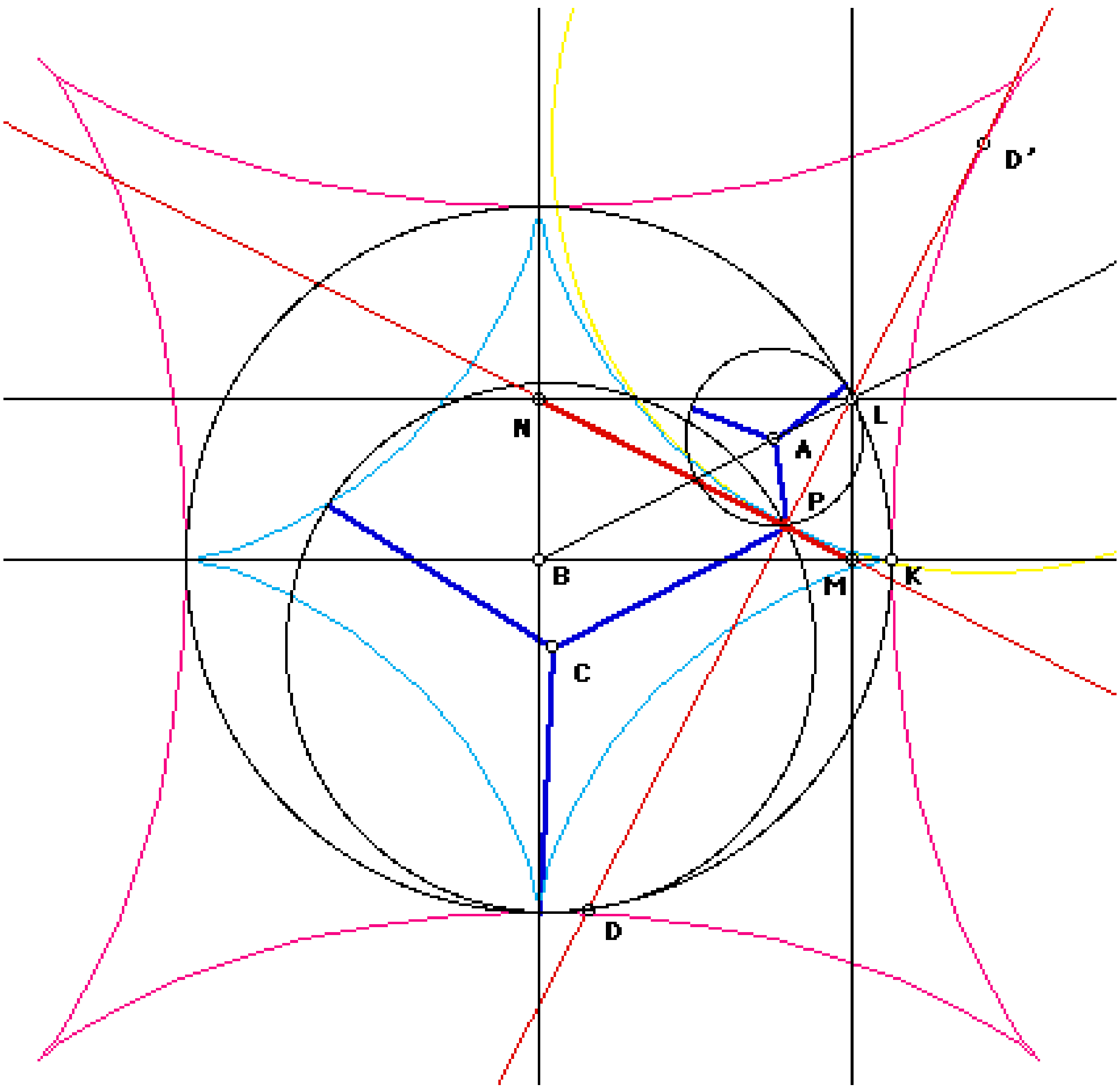


4.3 Curve Construction

The Astroid is rich in properties that can be used to devise other mechanical means to generate the curve and to construct its tangents, and the centers of its osculating circles.

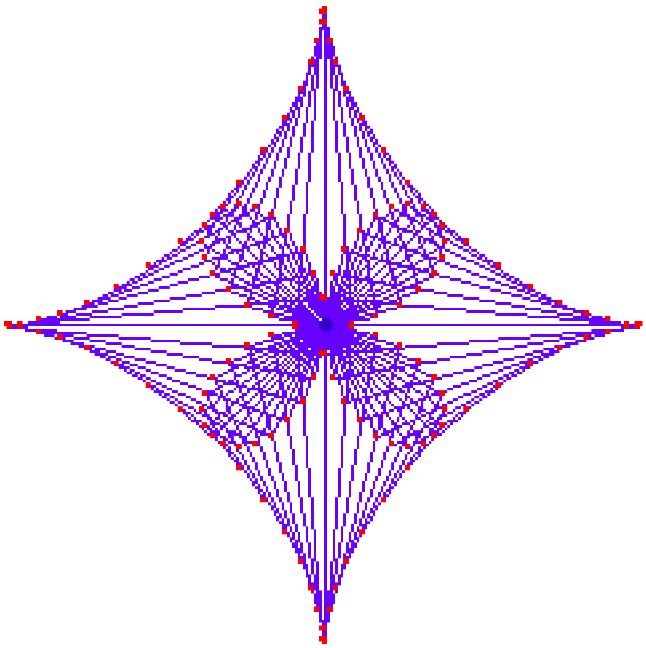
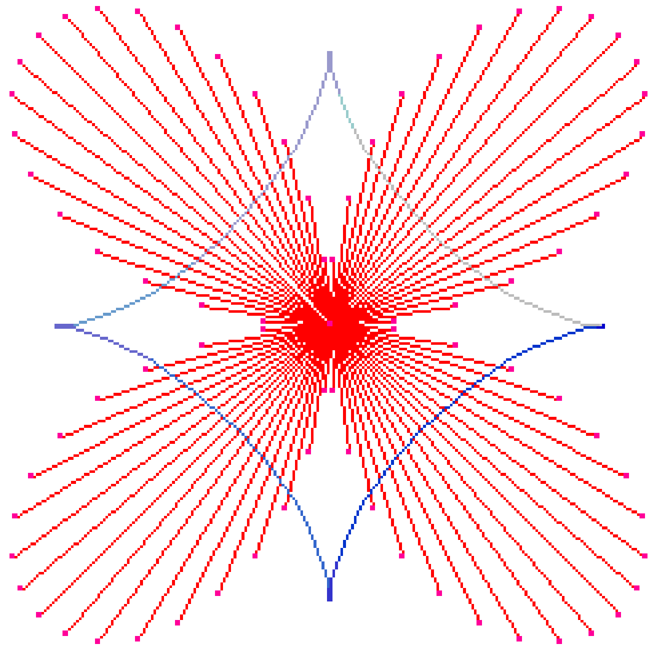
Suppose we have a circle C centered at B and passing through some point K. We will construct an Astroid that is also centered at B and that has one of its cusps at K.

Choose the origin of a cartesian coordinate system at B, and take the point $(1,0)$ at K. Given a point L on the circle C, drop a perpendicular from L to the x-axis, and let M be their intersection. Similarly drop a perpendicular from L to the y-axis and call the intersection N. Let P be the point on MN such that LP and MN are perpendicular. Then P is a point of the Astroid, MN is the tangent to the Astroid at P, and LP the normal at P. If D is the intersection of LP and the circle C, and D' is the reflection of D thru MN, then D' is the center of osculating circle at P.



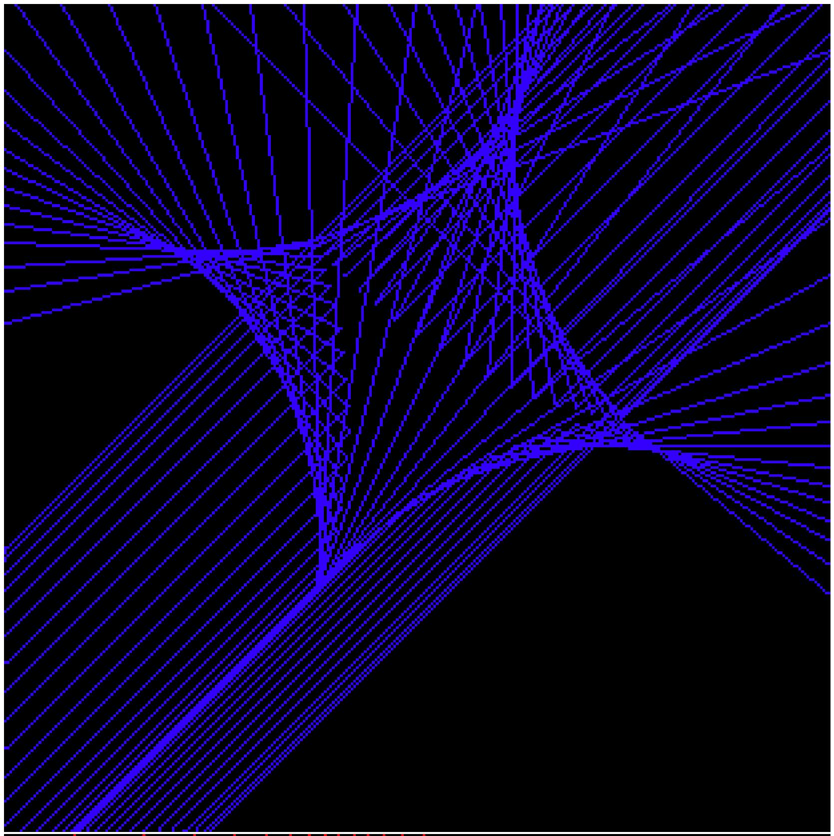
4.4 Pedal, Radial, and Rose

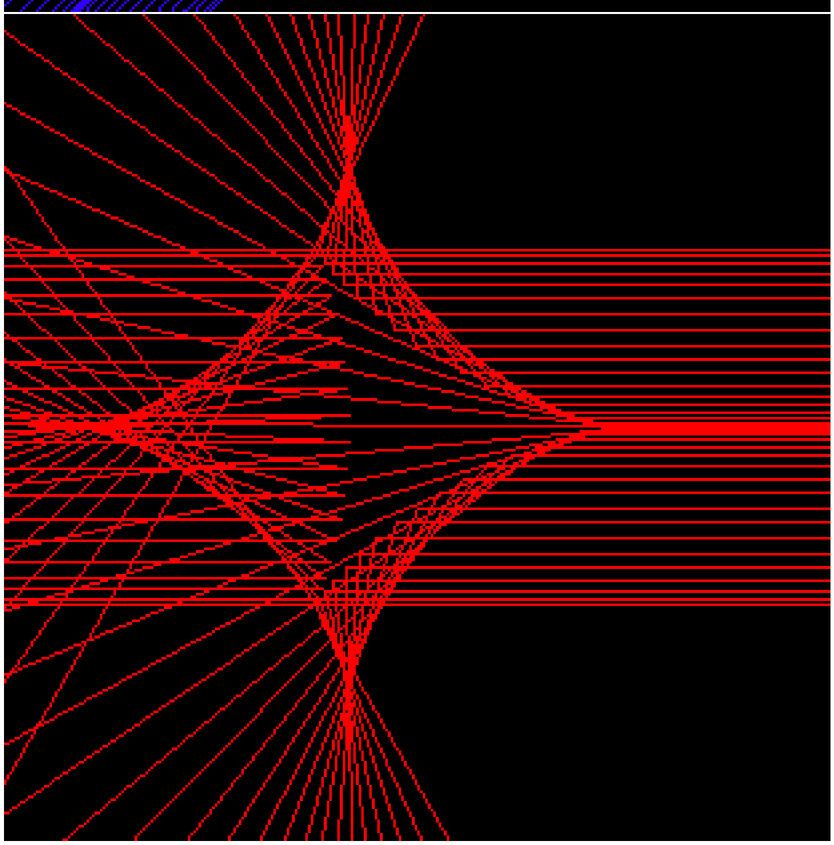
The pedal of an Astroid with respect to its center is a 4- petaled rose, called a quadrifolium. The Astroid's radial is also a quadrifolium. (For any epi- or hypo- cycloid, the pedal and radial are equal, and is a rose.)

4.5 Catacaustic and Deltoid

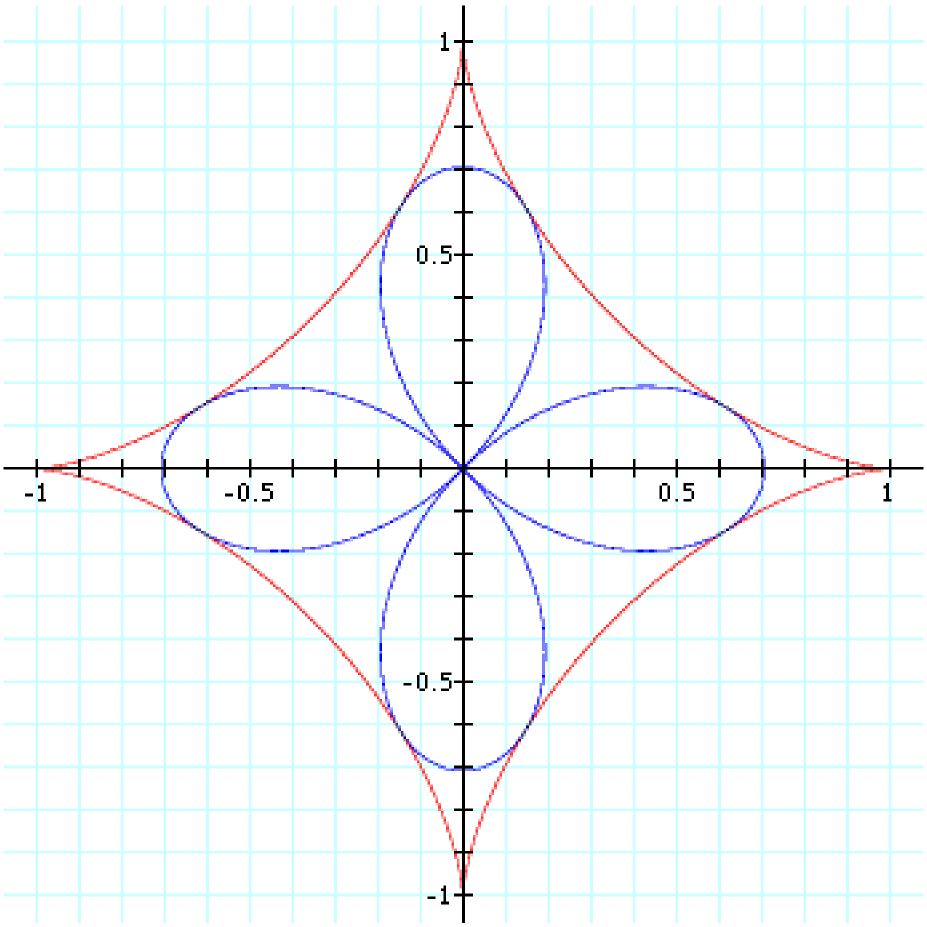
The catacaustic of a Deltoid with respect to parallel rays in any direction is an Astroid.





4.6 Orthoptic

We recall that the *orthoptic* of a curve C is the locus of points P where two tangents to C meet at right angles. The orthoptic of the Astroid is the quadriffolium $r^2 = (1/2)\cos(2\theta)^2$. [Robert C. Yates.]



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