Philip LaFleur
Expository Paper

Napoleon's Theorem

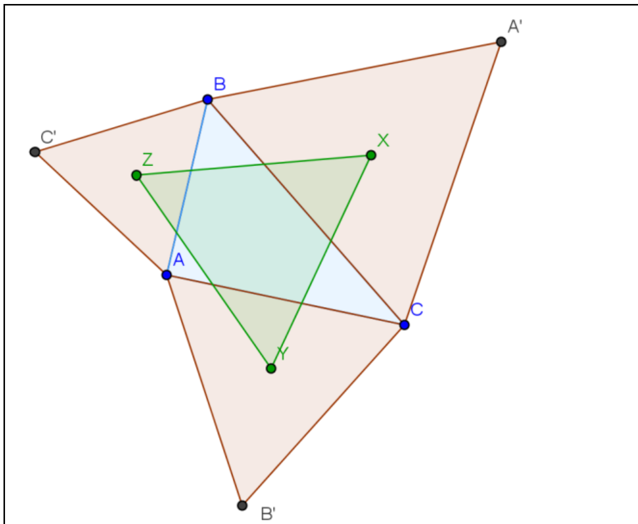
Napoleon

Most people have heard of Napoleon Bonaparte, the infamous emperor of France. He was well known for being vertically challenged (short), and for his failed attempt to take over the world. What most people do not know is that Napoleon was a talented student and had a great understanding of Euclidean geometry. Napoleon had weekly meetings with well-known mathematicians of his time to discuss mathematics. Napoleon had such a good understanding of geometry that he discovered and proved the following theorem.

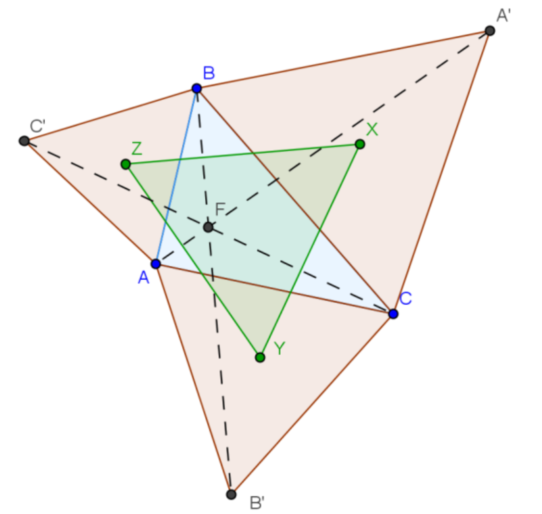
> If equilateral triangles are constructed on the sides of any triangle, then the centroids of the three equilateral triangles will themselves form an equilateral triangle.

Even though Napoleon was credited with this theorem, there are many who are skeptical that he discovered it.

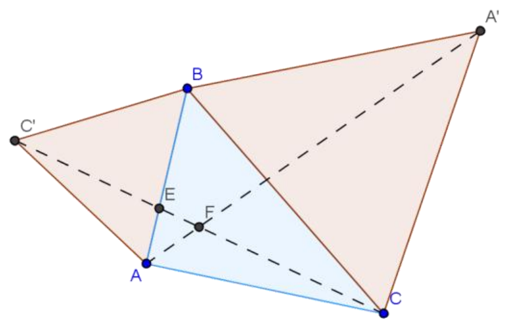
Napoleon's proof involved the use of trigonometry to show that sides of the triangle formed by the centroids were congruent. However, I chose to find a proof that shows the angles formed by this triangle are all 60^{o}, and therefore is an equilateral triangle. A proof that I have found is as follows:

	Given triangle ABC, with side lengths of a, b, and c. Construct equilateral triangles BCA', ACB', and ABC', and cetroids X, Y, and Z respectively. Connect X, Y, and Z to form triangle XYZ.

Philip LaFleur
Expository Paper

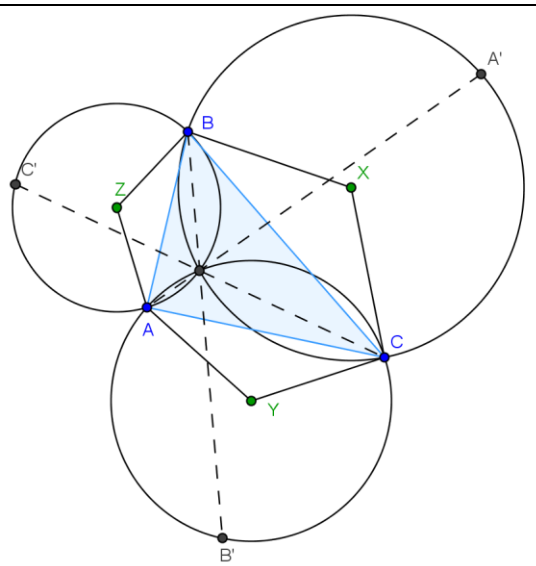


Next, I will connect AA', BB', and CC'. It is known that these 3 lines intersect at a common point F, called Fermat's Point. Fermat's Point is the location where the sum of the distances from the vertices is the smallest. So Fermat's Point is the point F that yields the smallest distance for AF+BF+CF.



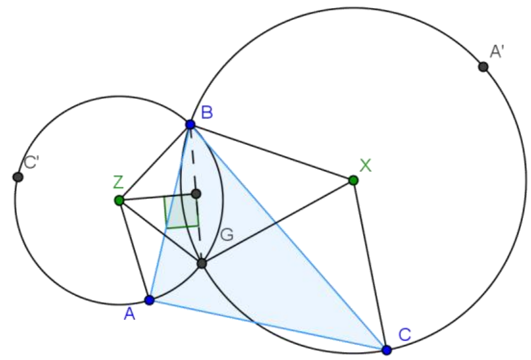
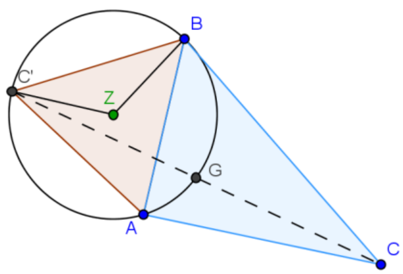
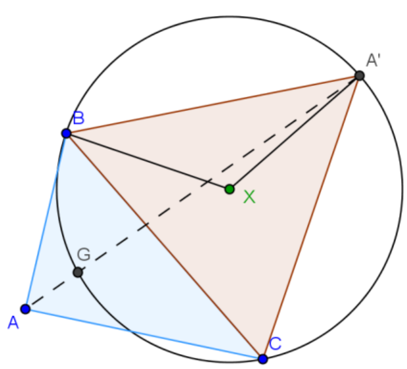
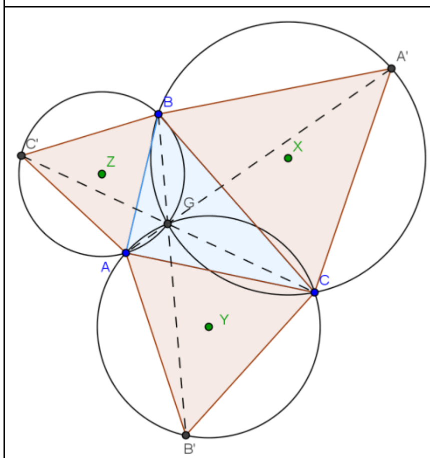
In the picture at the left, I have isolated the pieces of the diagram that I need to show that triangle C'BC is congruent to triangle ABA'. I know that AB=BC', BA'=BC, and C'BC=60^{o}+ x = ABA'. Thus, by side angle side, the two triangles are congruent.

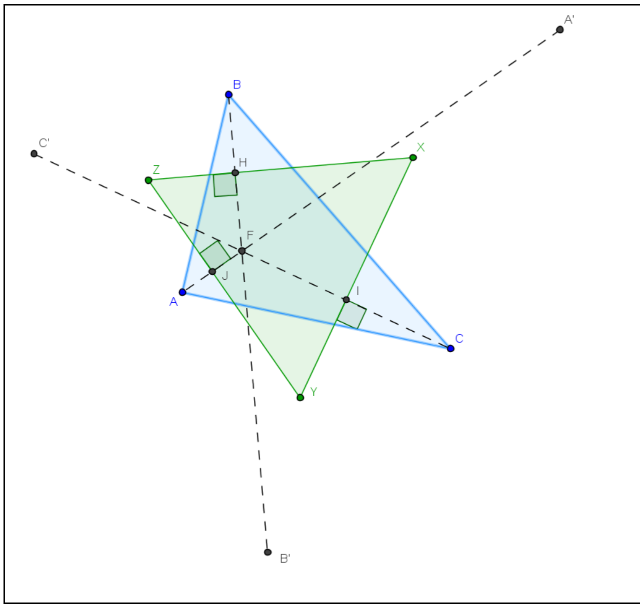
Using the fact that triangle C'FA' is congruent to triangle ABA', I can discover the measures of the angles formed by Fermat's Point. I know that BAA' = BC'C since they are corresponding parts of congruent triangles and C'EB = AEF since they are vertical angles. Thus, triangle C'EB is similar to triangle AEF. This means that C'BE = AFE. Since C'BE = 60^{o}, then I know that AFE = 60^{o}. This same approach can be done to determine that
C'FB= BFA'= A'FC= CFB'= B'FA= AFC'=60^{o}.



Next I will construct circumcircles for equilateral triangles BCA', ACB' and ABC' with centers X, Y, and Z respectively. Points X, Y, and Z are the centers for both their respective triangle and circumcircles. Since triangles ABC', BCA', and ACB' are equilateral, then triangles AZB, BXC, and CYA are isosceles with their largest angle being 120^{o}.

Philip LaFleur
Expository Paper

	The circles centered at points Z and X intersect at points B and G. Triangle GZB is isosceles because and are both the radius of the circle with center Z and are therefore congruent. If a line were drawn from point Z to bisect chord BG, it would split triangle GZB into two congruent triangles. Since triangle GZB is isosceles, and the line bisects chord BG forming two congruent triangles, then this line must also be perpendicular to chord BG. It follows that if you were to do the same from point X, that this line too would be perpendicular to the common chord BG. Thus, the perpendicular bisector of the common chord BG must pass through points Z and X. So is perpendicular to chord BG.
	At this point, I must show that point G is in fact the same as point F (Fermat's Point). It is known that the Fermat Point is the common intersection of segment AA', CC', and BB'. So we need only show that G is on the segment AA'. A similar argument will imply that G is also on segment CC' and therefore that G must be the Fermat Point. To carry this out, notice that the vertices B, C', A are equally spaced around the circle, since BC'A is an equilateral triangle. Thus, BZC' = 120^{o} = arc BC'. Since point G lies on the same circumcircle, I can use a theorem and say that BGC' is going to be half of the measure of arc BC'. Thus, BGC' = 60^{o}. By the same process, it follows that arc A'B = 120^{o} and BGA' = 60^{o}.
	I will use the same theorem as above to say that CGA, AGB, and BGC are all equal to – . Since AGB = 120 and AGB' BGA' = 60^{o}, then B'GB A'GA = 180^{o} and is therefore a straight line. The same can be done with C', G, C and B, G, B'. Since all three lines intersect at point G, then G=F, the Fermat point.



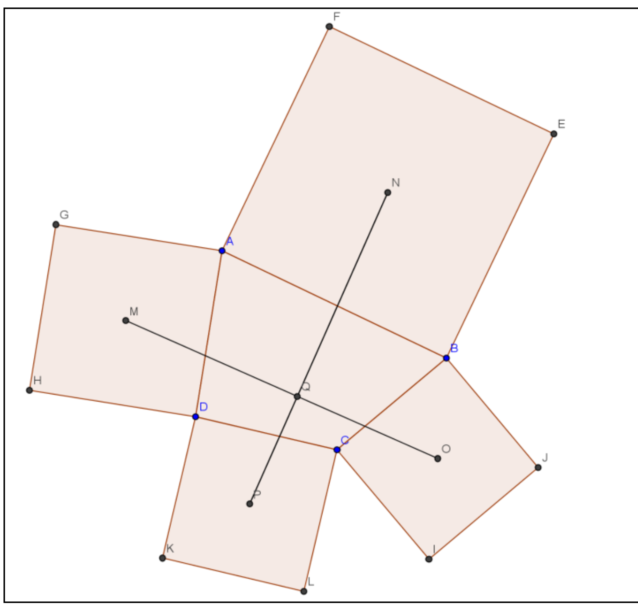
I have shown that AFB = 120°, is perpendicular to , and is perpendicular to . I know that the sum of the angles in quadrilateral ZHFJ is 360°. So 360° = 90° + 120° + 90° + Z. Thus Z = 60°. The same can be done for quadrilaterals XIFH and YJFI. Since Z, Y, and X are all = 60°, triangle XYZ is equilateral. Thus, proving Napoleon's Theorem.

Napoleon's Theorem does not specify whether the equilateral triangles constructed need to be outward or not. Unfortunately my proof will not hold true if the triangles are constructed inwards. This is due to the fact that they must be outward in order for Fermat's point to be located. Thus, Napoleon's proof must be used for inward equilateral triangles.

van Aubel

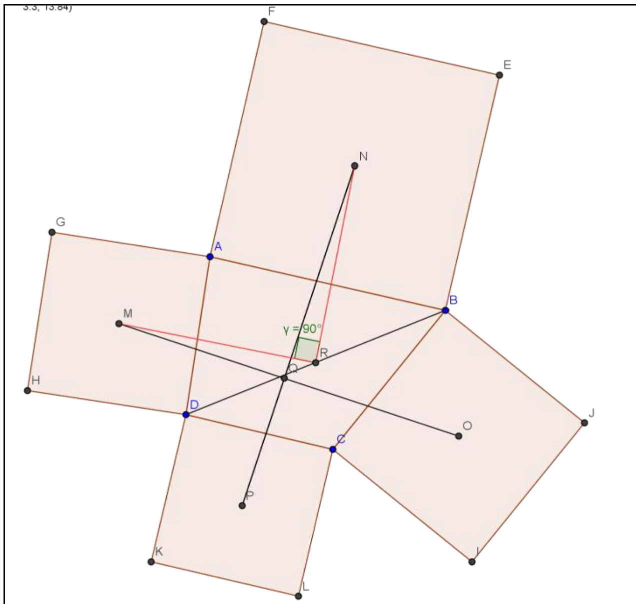
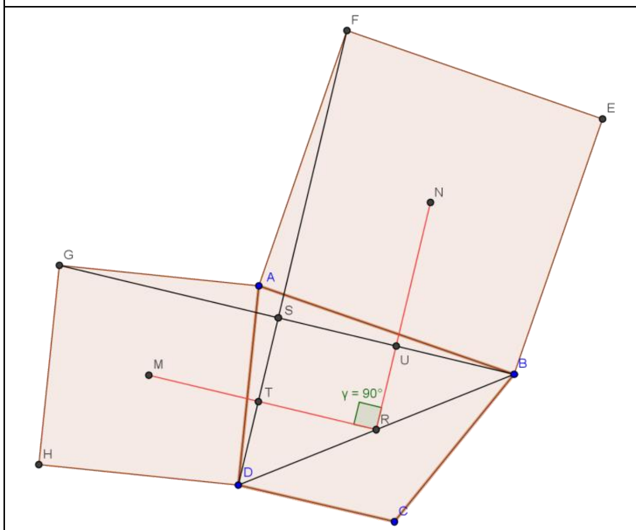
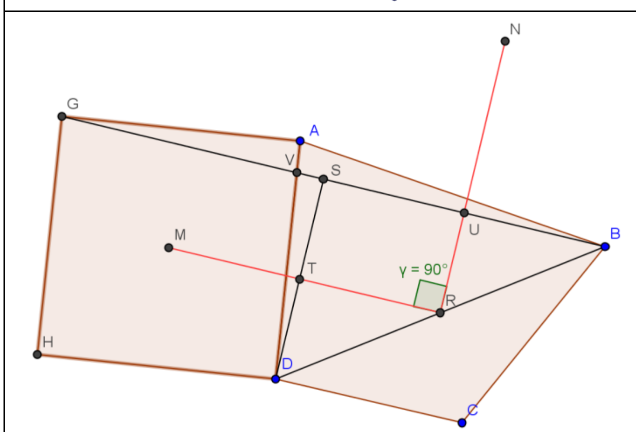
Napoleon's Theorem can be related to van Aubel's Theorem. I was unable to find any information about van Aubel, even a first name, on the internet. The theorem is as follows:

> Given a quadrilateral, place a square outwardly on each side, and connect the centers of squares on opposite sides. The two lines formed have equal length and are perpendicular to one another.

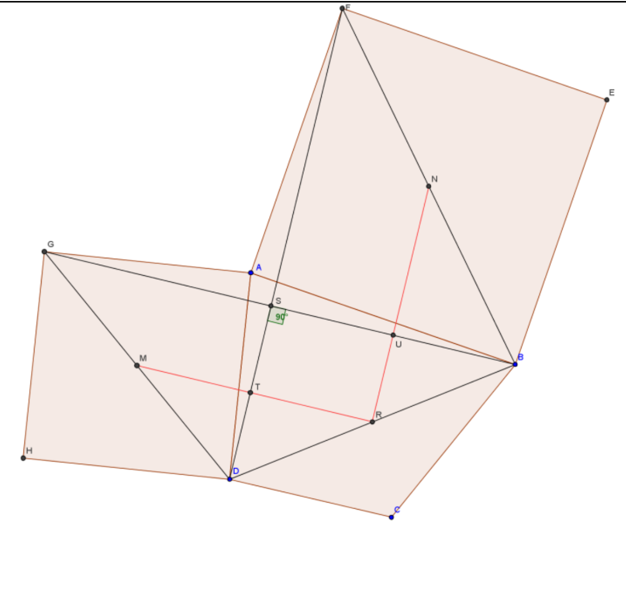


Given quadrilateral ABCD, construct outward squares on each side. Find the centers of squares ABEF, BCJI, DCLK, and ADHG, and label them N, O, P and M respectively. Connect the centers of the opposite squares, forming and . These segments intersect at point Q.

Philip LaFleur
Expository Paper

	Draw a diagonal segment from D to B, forming . Find the midpoint of , and call it R. Connect R to the midpoints of two squares on adjacent sides, forming and . Then connect points F and D to form triangle FAD. Next connect points G and B to form triangle GAB.
	= since they are sides of square FABE. = since they are sides of square GADH. Since and , then . Then by SAS, triangles FAD and GAB are congruent.
	This means that . Since are vertical angles, then they too are congruent. I then know that and the latter is 90⁰. Therefore, . Since is a line and , then .

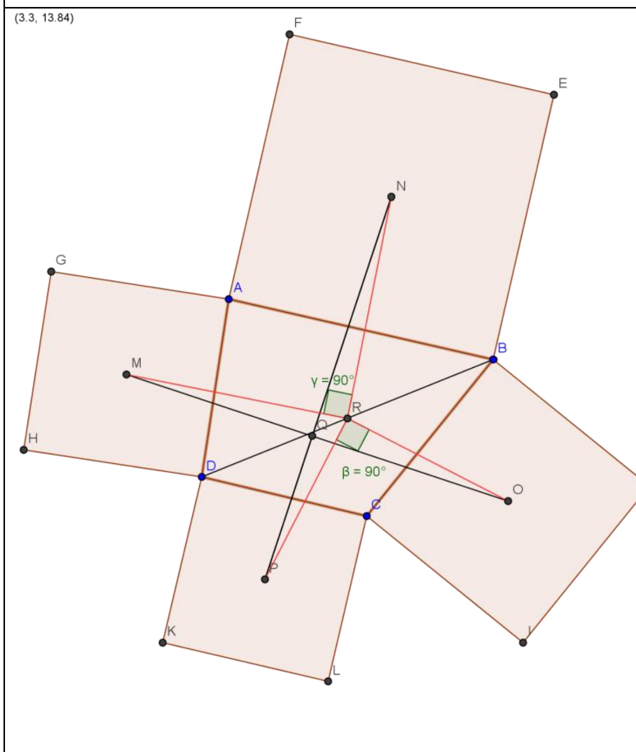
Philip LaFleur
Expository Paper



Since N is the midpoint of and point R is the midpoint of , then by the *Midpoint Theorem*, is half the length of and parallel to . This same process can be applied to show that is half the length of and parallel to . Since , it follows that .

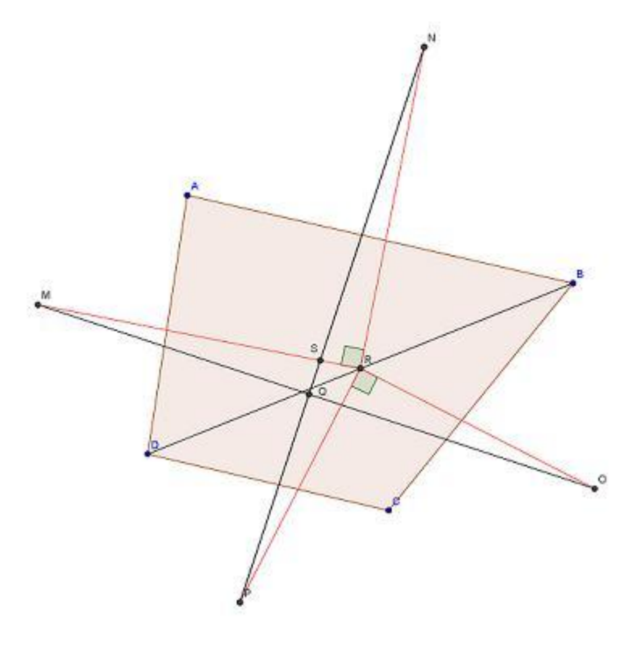
Since is parallel to , and is perpendicular to , then it follows that is perpendicular to . This means that This same reasoning can be applied to show that . Since three of the four angles in quadrilateral STRU are right angles, this means that .

Thus, and are equal in length and form right angle MRN.



Now connect the midpoints of the other two squares to point R. As with before, and have equal length and form right triangle ORP.

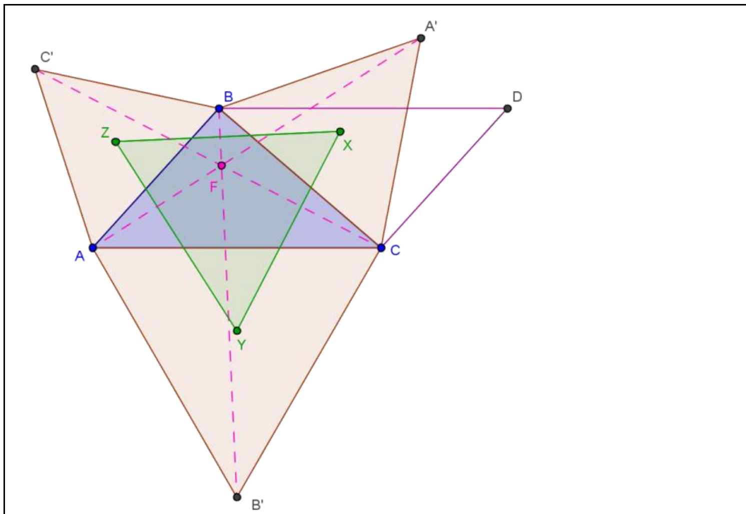
If you form triangles MRO and NRP, by side angle side, they are congruent to one another. Thus, is equal to .



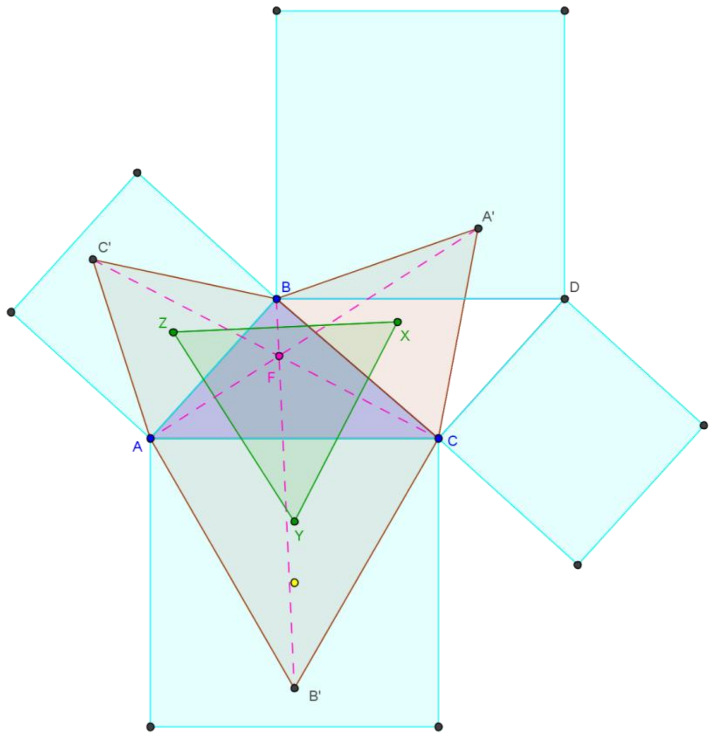
Since corresponding parts of congruent triangles are congruent, we know that SNR and QMR are congruent. By the sum of the interior angles of a triangle, I know that NRS is 90°, so the sum of RNS and RSN must be 90°. Since SNR and QMS are congruent, and MSQ and NSQ are vertical angles, I know that the sum of MSQ and QMS must also equal 90°. By the sum of the interior angles of a triangle, Since MSQ + QMS = 90°, then MQS = 90°. Thus, MQN = 90° and is perpendicular to . Therefore, the segments formed by connecting the midpoints of opposite squares will be both congruent and perpendicular.

Interesting Extension

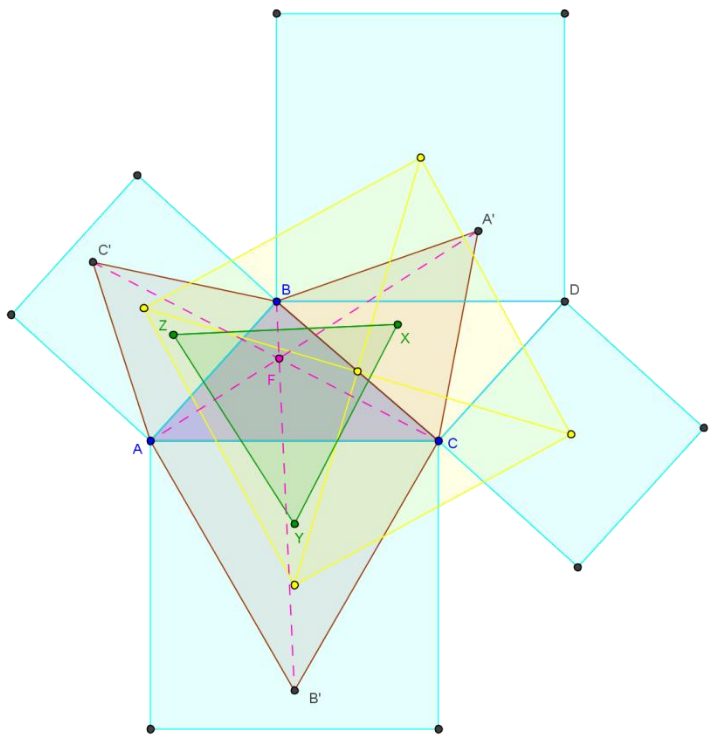
While playing around with these theorems, I wondered if there were any other polygons that this would work for. I tried pentagons and hexagons, but was unable to discover anything noteworthy. The one thing that I did find interesting I stumbled upon when trying to use van Aubel's theorem to help prove Napoleon's.



I was attempting to show that is perpendicular to , and thought that since van Aubel's theorem involves a right angle being formed, I thought that I might be able to manipulate things to work in my favor. I began by taking the picture from Napoleon, duplicating and rotating triangle ABC, and formed parallelogram ABDC.



Next, I drew the outward squares from the sides of parallelogram ABDC.



Then I found the midpoints of those squares and connected them to the opposite midpoints. From van Aubel's theorem, I know that these segments are congruent and are perpendicular to one another. I realized after looking at the picture that the segments connecting adjacent midpoints also form a square.

Using Geogebra, I manipulated the parallelogram to see if any change in size would yield something other than a square. However, I was unable to find any variances. This makes sense, since the midpoint of is also the midpoint of parallelogram ABDC. To prove the resulting figure is a square, one would need to show that the diagonals bisect each other and the result would follow. I will leave the statement as a conjecture and invite the reader to arrive at a proof.

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Expository Paper

References

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