Uncountable sets and an infinite real number game

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The game. Alice and Bob decide to play the following infinite game on the real number line. A subset S of the unit interval $[0, 1]$ is fixed, and then Alice and Bob alternate playing real numbers. Alice moves first, choosing any real number a_1 strictly between 0 and 1. Bob then chooses any real number b_1 strictly between a_1 and 1. On each subsequent turn, the players must choose a point strictly between the previous two choices. Equivalently, if we let $a_0 = 0$ and $b_0 = 1$, then in round $n \geq 1$, Alice chooses a real number a_n with $a_{n-1} < a_n < b_{n-1}$, and then Bob chooses a real number b_n with $a_n < b_n < b_{n-1}$. Since a monotonically increasing sequence of real numbers which is bounded above has a limit (see [8, Theorem 3.14]), $\alpha = \lim_{n\to\infty} a_n$ is a well-defined real number between 0 and 1. Alice wins the game if $\alpha \in S$, and Bob wins if $\alpha \notin S$.

Countable and uncountable sets. An set X is called *countable* if it is possible to list the elements of X in a (possibly repeating) infinite sequence $x_1, x_2, x_3, \ldots$ Equivalently, X is countable if there is a function from the set $\{1, 2, 3, \ldots\}$ of natural numbers to X which is *onto*. For example, every finite set is countable, and the set of natural numbers is countable. A set which is not countable is called *uncountable*. Cantor proved using his famous *diagonalization argument* that the real interval $[0, 1]$ is uncountable. We will give a different proof of this fact based on Alice and Bob's game.

PROPOSITION 1. *If S is countable, then Bob has a winning strategy.*

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Proof. Since S is countable, one can enumerate the elements of S as $s_1, s_2, s_3, \ldots$ Consider the following strategy for Bob. On move $n \geq 1$, he chooses $b_n = s_n$ if this is a legal move, and otherwise he randomly chooses any allowable number for b_n. Since $\alpha < b_n$ for all n, it follows that $\alpha \neq b_n$ for any $n \geq 1$, and thus $\alpha \notin S$. This means that Bob always wins with this strategy!

If $S = [0, 1]$, then clearly Alice wins no matter what either player does. Therefore we deduce:

COROLLARY 1. *The interval* $[0, 1] \subset \mathbb{R}$ *is uncountable.*

This argument is in many ways much simpler than Cantor's original proof!

Perfect sets. We now prove a generalization of the fact that $[0, 1]$ is uncountable. This will also follow from an analysis of our game, but the analysis is somewhat more complicated. Given a subset X of $[0, 1]$, we make the following definitions:

- A *limit point* of X is a point $x \in [0, 1]$ such that for every $\epsilon > 0$, the open interval $(x - \epsilon, x + \epsilon)$ contains an element of X other than x.
- X is *closed* if every limit point of X belongs to X.
- X is *perfect* if it is non-empty [2], closed, and if every element of X is a limit point of X.

For example, the famous middle-third *Cantor set* is perfect (see [8, §2.44]). If $L(X)$ denotes the set of limit points of X, then a nonempty set X is closed $\Leftrightarrow L(X) \subseteq X$, and is perfect $\Leftrightarrow L(X) = X$. It is a well-known fact that every perfect set is uncountable (see [8, Theorem 2.43]). Using our infinite game, we will give a different proof of this fact. We recall the following basic property of the interval $[0, 1]$:

[2] Some authors consider the empty set to be perfect.

($\star$) Every non-empty subset $X \subseteq [0,1]$ has an *infimum* (or *greatest lower bound*), meaning that there exists a real number $\gamma \in [0,1]$ such that $\gamma \leq x$ for every $x \in X$, and if $\gamma' \in [0,1]$ is any real number with $\gamma' \leq x$ for every $x \in X$, then $\gamma' \leq \gamma$.

The infimum γ of X is denoted by $\inf(X)$.

Let's say that a point $x \in [0,1]$ is *approachable from the right*, denoted $x \in X^+$, if for every $\epsilon > 0$, the open interval $(x, x+\epsilon)$ contains an element of X. We can define *approachable from the left* (written $x \in X^-$) similarly using the interval $(x-\epsilon, x)$. It is easy to see that $L(X) = X^+ \cup X^-$, so that a non-empty set X is perfect $\Leftrightarrow X = X^+ \cup X^-$.

The following two lemmas tell us about approachability in perfect sets.

LEMMA 1. *If S is perfect, then* $\inf(S) \in S^+$*.*

Proof. The definition of the infimum in ($\star$) implies that $\inf(S)$ cannot be approachable from the left, so, being a limit point of S, it must be approachable from the right.

LEMMA 2. *If S is perfect and $a \in S^+$, then for any $\epsilon > 0$, the open interval $(a, a+\epsilon)$ also contains an element of S^+.*

Proof. Since $a \in S^+$, we can choose three points $x, y, z \in S$ with $a < x < y < z < a+\epsilon$. Since $(x,z) \cap S$ contains y, the real number $\gamma = \inf((x,z) \cap S)$ satisfies $x \leq \gamma \leq y$. If $\gamma = x$, then by definition ($\star$) we have $\gamma \in S^+$. If $\gamma > x$, then definition ($\star$) implies that $\gamma \in L(X)$ and $(x, \gamma) \cap S = \emptyset$, so that $\gamma \notin S^-$ and therefore $\gamma \in S^+$.

From these lemmas, we deduce:

PROPOSITION 2. *If S is perfect, then Alice has a winning strategy.*

Proof. Alice's only constraint on her nth move is that $a_{n-1} < a_n < b_{n-1}$. By induction, it follows from Lemmas 1 and 2 that Alice can always choose a_n to be an element of $S^+ \subseteq S$. Since S is closed, $\alpha = \lim a_n \in S$, so Alice wins!

From Propositions 1 and 2, we deduce:

COROLLARY 2. *Every perfect set is uncountable.*

Further analysis of the game. We know from Proposition 1 that Bob has a winning strategy if S is countable, and it follows from Proposition 2 that Alice has a winning strategy if S contains a perfect set. (Alice just chooses all of her numbers from the perfect subset.) What can one say in general? A well-known result from set theory [1, §6.2, Exercise 5] says that every uncountable *Borel set* [3] contains a perfect subset. Thus we have completely analyzed the game when S is a Borel set: Alice wins if S is uncountable, and Bob wins if S is countable. However, there do exist non-Borel uncountable subsets of $[0, 1]$ which do not contain a perfect subset [1, Theorem 6.3.7]. So we leave the reader with the following problem:

Problem: Do there exist uncountable subsets of $[0, 1]$ for which: (a) Bob has a winning strategy; (b) Alice does not have a winning strategy; or (c) neither Alice nor Bob has a winning strategy?

Related games. Our infinite game is a slight variant of the one proposed by Jerrold Grossman and Barry Turett in [2] (see also [6]). Propositions 1 and 2 above were motivated by parts (a) and (c), respectively, of their problem. The author originally posed Propositions 1 and 2 as challenge problems for the students in his Math 25 class at Harvard University in Fall 2000.

A related game (the "Choquet game") can be used to prove the Baire category theorem (see §8.C of [5] and [3]). In Choquet's game, played in a given metric space X, Pierre moves first by choosing a non-empty open set $U_1 \subseteq X$. Then Paul moves by choosing a non-empty open set $V_1 \subseteq U_1$. Pierre then chooses a non-empty open set $U_2 \subseteq V_1$, etc., yielding two decreasing sequences U_n and V_n of non-empty open sets with $U_n \supseteq V_n \supseteq U_{n+1}$ for all n, and $\cap U_n = \cap V_n$. Pierre wins if $\cap U_n = \emptyset$, and Paul wins if $\cap U_n \neq \emptyset$. One can show that if X is complete, then Paul has a winning strategy, and if X contains a non-empty open set O which is a countable union of closed sets

[3] A Borel set is, roughly speaking, any subset of $[0, 1]$ that can be constructed by taking countably many unions, intersections, and complements of open intervals; see [8, §11.11] for a formal definition.

having empty interior, then Pierre has a winning strategy. As a consequence, one obtains the *Baire category theorem:* If X is a complete metric space, then no open subset of X can be a countable union of closed sets having empty interior.

Another related game is the Banach-Mazur game (see §6 of [7] and §8.H of [5]). A subset S of the unit interval $[0, 1]$ is fixed, and then Anna and Bartek alternate play. First Anna chooses a closed interval $I_1 \subseteq [0, 1]$, and then Bartek chooses a closed interval $I_2 \subseteq I_1$. Next, Anna chooses a closed interval $I_3 \subseteq I_2$, and so on. Together the players' moves determine a nested sequence I_n of closed intervals. Anna wins if $\cap I_n$ has at least one point in common with S, otherwise Bartek wins. It can be shown that Bartek has a winning strategy if and only if S is meagre (see Theorem 6.1 of [7]). (A subset of X is called *nowhere dense* if the interior of its closure is empty, and is called *meagre*, or of the *first category*, if it is a countable union of nowhere dense sets.) It can also be shown, using the axiom of choice, that there exist sets S for which the Banach-Mazur game is undetermined (neither player has a winning strategy).

For a more thorough discussion of these and many other *topological games*, we refer the reader to the survey article [9], which contains an extensive bibliography. Many of the games discussed in [9] are not yet completely understood.

Games like the ones we have been discussing play a prominent role in the modern field of *descriptive set theory*, most notably in connection with the *axiom of determinacy* (AD). (See Chapter 6 of [4] for a more detailed discussion.) Let X be a given subset of the space ω^ω of infinite sequences of natural numbers, and consider the following game between Alice and Bob. Alice begins by playing a natural number, then Bob plays another (possibly the same) natural number, then Alice again plays a natural number, and so on. The resulting sequence of moves determines an element $x \in \omega^\omega$. Alice wins if $x \in X$, and Bob wins otherwise. The axiom of determinacy states that this game is determined (i.e., one of the players has a winning strategy) for *every* choice of X.

A simple construction shows that the axiom of determinacy is inconsistent with the axiom of choice. On the other hand, with Zermelo-Fraenkel set theory plus the axiom of determinacy (ZF+AD), one can prove many non-trivial theorems about the real numbers, including: (i) every subset of $\mathbb{R}$ is Lebesgue measurable; and (ii) every uncountable subset of $\mathbb{R}$ contains a perfect subset. Although ZF+AD is not considered a "realistic" alternative to ZFC (Zermelo-Fraenkel + axiom of choice), it has stimulated a lot of mathematical research, and certain variants of AD are taken rather seriously. For example, the axiom of *projective determinacy* is intimately connected with the continuum hypothesis and the existence of large cardinals (see [10] for details).

References

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