

Deducing the Lenz vector of the hydrogen atom from supersymmetry

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Abstract. The ideas of supersymmetric quantum mechanics are applied to the non-relativistic hydrogen atom to give a first principles derivation of its Lenz vector.

It is well known that the spectrum of the hydrogen atom possesses degeneracies beyond what one would expect from rotational invariance alone, and that the additional degeneracies are fully accounted for by an extra constant of motion in the problem, the so-called Laplace-Runge-Lenz vector. Knowledge of the Lenz vector permits a purely algebraic solution to the Coulomb problem that was first laid out in the classic works of Pauli, Fock and Bargmann and later elaborated on by many other authors. There is now a large literature [1, 2] on the group theoretical approach to the Coulomb, oscillator and allied problems and its applications to various fields of physics.

Recently an alternative approach to the degeneracies in the hydrogen spectrum based on the notion of supersymmetry [3–5] was explored by several authors [4–8]. For our purposes, the key idea of this approach is the following. The radial part of the hydrogen atom wavefunction is determined by an effective potential that is the sum of the true (Coulomb) potential and a centrifugal barrier term. It turns out that the effective potentials for two neighbouring values of l (the orbital quantum number) are supersymmetric partners of each other, and that this fact can be used to account completely for the Coulomb degeneracy. The two alternative explanations of the Coulomb degeneracy—based either on the Lenz vector or on the idea of supersymmetry—appear to have little in common with each other. One naturally wonders what, if any, the connection between these two approaches is. The purpose of this paper is to answer this question. We will show that if the supersymmetric approach is pushed far enough it will yield the explicit form of the Lenz vector, thereby making contact with the alternative approach. This point does not seem to have been demonstrated earlier, to the best of our knowledge.

Our derivation is carried out in the context of the 2D hydrogen atom rather than the 3D hydrogen atom, because the former has a smaller symmetry group and is easier to treat. The specific task before us is to use supersymmetry to deduce the extra constants of motion of the 2D hydrogen atom (besides the orbital angular momentum component L_z) and to relate these to the full three-dimensional Lenz vector. Before proceeding to this task, we summarize the main results from supersymmetric quantum mechanics that we need for this purpose.

Consider a particle of mass m moving in a one-dimensional potential $V(x)$. If the ground state energy of the particle is made zero (by a suitable redefinition of the energy

scale), the potential $V(x)$ can be expressed in terms of the ground state wavefunction ψ_0 and its second derivative. When this is done the Hamiltonian of the system takes the form

$$H = (\hbar^2/2m)[-d^2/dx^2 + \psi_0''/\psi_0]. \quad (1)$$

If we define the operators

$$A = \hbar(2m)^{-1/2}[d/dx + \psi_0'/\psi_0] \quad A^+ = \hbar(2m)^{-1/2}[-d/dx + \psi_0'/\psi_0] \quad (2)$$

(1) can be re-expressed as $H = AA^+$. We next introduce another Hamiltonian $H_S = A^+A$ which is conventionally termed the 'supersymmetric partner' of H . The relationship of H and H_S to each other and also the physical meanings of the operators A and A^+ are illustrated in figure 1. One sees from the figure that H and H_S have the same energy eigenvalues except that the lowest eigenvalue of H is missing in H_S . The operator A^+ has the property that it transforms any eigenstate of H (except the ground state) into an eigenstate of H_S with the same energy. The operator A does the reverse, namely, it transforms any eigenstate of H_S back into an eigenstate of H with the same energy. The proofs of these statements may be found in any of the tutorial articles [3-5] on supersymmetric quantum mechanics mentioned earlier.

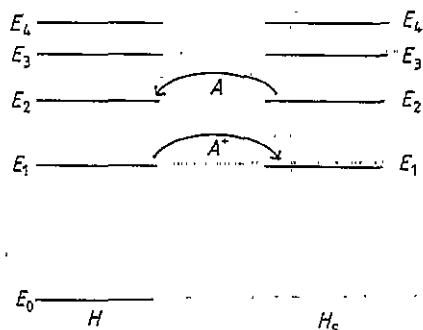


Figure 1. Spectrum of the Hamiltonian H (at left) and of its supersymmetric partner H_S (on right). Both spectra are identical except that the lowest energy level of H is missing in H_S . The operator A^+ transforms any eigenstate of H into an eigenstate of H_S with the same energy, while A transforms any eigenstate of H_S into an eigenstate of H with the same energy.

We turn now to the two-dimensional hydrogen atom (with $1/r$ potential between the electron and proton). The eigenvalues and eigenfunctions of this system were obtained a long time back by Zaslav and Zander [9]. The polar eigenfunctions are labelled by an energy quantum number n and an angular momentum quantum number l . The energy quantum number takes on all integer values from one up while, for a given n , l takes on all values from $-(n-1)$ to $(n-1)$ in integer steps; thus the degeneracy associated with a particular n value is $2n-1$. Figure 2 shows the spectrum of the 2D hydrogen atom, with degenerate states placed alongside each other in horizontal rows: the ground state ($n=1$) is non-degenerate, the first excited state ($n=2$) is threefold degenerate, the second excited state ($n=3$) is fivefold degenerate, and so on. The states also form vertical towers, with each tower being characterized by a particular value of l . Two neighbouring towers with l values differing by unity have identical

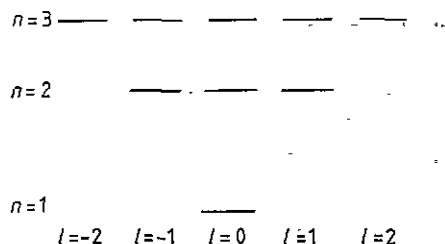


Figure 2. Spectrum of the two-dimensional hydrogen atom. Each level is labelled by two quantum numbers, the energy quantum number n and the orbital quantum number l . All levels in the same horizontal row share the same energy quantum number n , which is indicated to the left of the row. All levels in the same column (or tower) share the same orbital quantum number l , which is indicated at the bottom of the tower.

energy levels except that the tower with the higher l has one level (the lowest) missing. This is reminiscent of figure 1 and suggests that the effective potentials that give rise to these towers are supersymmetric partners of each other. From what was said in the previous paragraph, we can anticipate the existence of supersymmetry operators A and A^+ that allow us to pass horizontally between towers. However we know that any operators that transform degenerate eigenstates into each other must commute with the Hamiltonian. Thus we are led to suspect that the operators A and A^+ must somehow be related to components of the Lenz vector. We proceed to supply the analytical detail needed to refine and complete this argument.

The first step in applying supersymmetry to the problem is to reduce the Schrödinger equation to a one-dimensional form. If we write the hydrogen atom wavefunction as $\psi(r, \varphi) = r^{-1/2} u(r) \exp(i l \varphi)$, we find that the reduced radial wavefunction $u(r)$ obeys a one-dimensional Schrödinger equation containing the effective potential

$$V_l(r) = -e^2/r + (\hbar^2/2m)(l^2 - \frac{1}{4})/r^2. \quad (3)$$

For a particular l this effective potential generates one of the vertical towers shown in figure 2. To construct the supersymmetry operators that connect towers with quantum numbers l and $l+1$, we need the ground state eigenfunction of the effective potential V_l . This is found to be $\psi_0 = r^{l+1/2} \exp[-(me^2/\hbar^2)(l+\frac{1}{2})^{-1}r]$ up to an unimportant normalization factor (we have specialized to the case $l > 0$ for definiteness). At this point we can verify that the effective potentials V_l and V_{l+1} indeed satisfy the relation [3]

$$V_{l+1}(r) = -V_l(r) + (\hbar^2/m)(\psi'_0/\psi_0)^2 \quad (4)$$

required of a potential and its supersymmetric partner. On substituting the expression for ψ_0 into (2) we obtain the supersymmetry operators as

$$A = \hbar(2m)^{-1/2} [d/dr + (l+\frac{1}{2})/r - (me^2/\hbar^2)/(l+\frac{1}{2})] \quad (5a)$$

and

$$A^+ = \hbar(2m)^{-1/2} [-d/dr + (l+\frac{1}{2})/r - (me^2/\hbar^2)/(l+\frac{1}{2})]. \quad (5b)$$

These operators connect the reduced radial wavefunctions $u_{n,l}(r)$ and $u_{n,l+1}(r)$, i.e. A^+ transforms $u_{n,l}(r)$ into $u_{n,l+1}(r)$ and A does the reverse. We want to construct the generalizations of (5a) and (5b) that transform the 2D eigenfunctions $\psi_{n,l}(r, \varphi)$ and $\psi_{n,l+1}(r, \varphi)$ into each other.

As a first step in this direction, we modify (5a) and (5b) so that they transform the full radial functions $R_{n,l}(r) = r^{-1/2}u_{n,l}(r)$ and $R_{n,l+1}(r)$ into each other. This merely involves changing $(l+\frac{1}{2})$ to $(l+1)$ in the middle term of (5a) and $(l+\frac{1}{2})$ to l in the middle term of (5b). Next we multiply the modified forms of (5a) and (5b) by $\hbar(2m)^{-1/2}$ (to give them dimensions of energy-length, the dimensions of the sought after Lenz vector) and pull out a factor of $(2l+1)^{-1}$ in front to obtain

$$A = (\hbar^2/2m)(2l+1)^{-1}[(2l+1) d/dr + (2l+1)(l+1)/r - 2(me^2/\hbar^2)] \quad (6a)$$

and

$$A^+ = (\hbar^2/2m)(2l+1)^{-1}[-(2l+1) d/dr + l(2l+1)/r - 2(me^2/\hbar^2)]. \quad (6b)$$

To complete the transformation of (6a) and (6b) we (1) replace l everywhere in A^+ by $-i\partial/\partial\varphi$ and $(l+1)$ everywhere in A by $-i\partial/\partial\varphi$ (because these derivatives, acting on the angular functions $e^{i\varphi}$ or $e^{i(l+1)\varphi}$, give back l or $l+1$, respectively); and (2) affix an overall factor of $e^{i\varphi}$ to A^+ or $e^{-i\varphi}$ to A (because the full 2D forms of A^+ or A must increase or decrease l by one unit in the angular part of the wavefunction). On incorporating these changes into (6a) and (6b) and dropping the overall factor of $(2l+1)^{-1}$ we obtain

$$A = e^{-i\varphi}(\hbar^2/2m)[-2i\partial^2/\partial r\partial\varphi - \partial/\partial r - (2/r)\partial^2/\partial\varphi^2 + (i/r)\partial/\partial\varphi] - e^{-i\varphi}e^2 \quad (7a)$$

and

$$A^+ = e^{i\varphi}(\hbar^2/2m)[2i\partial^2/\partial r\partial\varphi - \partial/\partial r - (2/r)\partial^2/\partial\varphi^2 - (i/r)\partial/\partial\varphi] - e^{i\varphi}e^2. \quad (7b)$$

Now we introduce the Hermitian operators $A_x = \frac{1}{2}(A^+ + A)$ and $A_y = -\frac{1}{2}i(A^+ - A)$. The expressions for these operators in 2D cartesian (x, y) coordinates are found to be

$$A_x = (2m)^{-1}(p_y L_z + L_z p_y) - e^2 x/r \quad (8a)$$

and

$$A_y = (2m)^{-1}(-p_x L_z - L_z p_x) - e^2 y/r \quad (8b)$$

where p and L are the momentum and angular momentum operators, respectively. A_x and A_y as given in (8), together with L_z , constitute the three constants of motion of the 2D hydrogen atom.

It is easily verified that (8a) and (8b) are just the x - and y -components of the Lenz vector-operator [10]

$$A = (2m)^{-1}(p \times L - L \times p) - e^2 r/r \quad (9)$$

but with all dependences on the z -coordinate suppressed. If one insists that there be total symmetry between the coordinates x , y and z (as we know must be the case for the 3D hydrogen atom), then (9) is easily seen to be the simplest vector operator consistent with (8). This completes our derivation of the Lenz vector.

Recently several authors [5, 7, 8] have shown how the supersymmetry of both the non-relativistic and relativistic Coulomb problems can be exploited to calculate their energy eigenvalues and eigenfunctions. The present paper demonstrates yet another use of supersymmetry—namely, the deduction of the Lenz vector. We should remark

that an alternative quantum mechanical derivation of the Lenz vector was given many years back by Bander and Itzykson [11], based on the earlier work of Bargmann and Fock. In this method [11, 12] one begins from the Schrödinger equation in momentum space and stereographically maps the momentum variables onto the surface of a four-dimensional sphere (of suitable energy-dependent radius) in order to bring out the full $SO(4)$ symmetry of the problem. The six generators of the rotations that take the four-dimensional sphere into itself then yield the three components of the orbital angular momentum and the three components of the Lenz vector [13].

Perhaps the most striking implication of the supersymmetry of the hydrogen atom is that there are connections between different atoms/ions in the Periodic Table. This connection was first pointed out by Kosteletsky and Nieto [6] and later analysed from a somewhat different point of view by Haymaker and Rau [4].

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