A Reader's Guide to Euler's Introductio

V. Frederick Rickey

Department of Mathematical Sciences
United States Military Academy
West Point, NY 10996

Email: fred-rickey@usma.edu

Abstract: Leonhard Euler's *Introductio in analysin infinitorum* is surely one of his most famous works. For a century after its publication in 1748 it was widely read by aspiring mathematicians. Today, thanks to John Blanton, it is available in English translation. To encourage aspiring historians to read this famous work, a reader's guide will be distributed. It will summarize the contents of the individual chapters of the *Introductio*, explain points that the reader might miss, point out antecedents of the work, and detail how the work influenced later mathematics.

Prepared for the *Euler 2K + 2 Conference. Countdown to the Tercentenary*, Rumford Maine, 4-7 August 2002.

The *Introductio in analysin infinitorum* (1748), by Leonhard Euler (1707–1783) is one of the most famous and important mathematical books ever written. Euclid's *Elements* (300 BC) ranks supreme, and Newton's *Principia* (1687) is the most famous scientific book ever written, but only partly deals with mathematics. Placing the *Introductio* second on the all time list in mathematics, may engender some arguments, but they will be interesting. That there have been 21 printings/editions in half a dozen languages certainly speaks highly of the importance of the *Introductio*.

For many years I have been involved in seminars that took as their goal the reading of original mathematical texts. This has always been a rewarding experience and a synergistic one, for I have always learned more in the seminar setting than I ever could have learned on my own. One advantage of the group is that it was easier to compare different editions, and especially translations, of a work.

The purpose of this document is to try something a bit different. When I participated in reading groups I always took along some notes about points that I wanted to make about the text. What follows is a much more detailed set of notes of that type. As I reread Euler's *Introductio* I made notes about what I found important, interesting, incorrect, curious, historically important, etc. My hope is that these notes will help others who have an interest in reading the *Introductio*. As often as I could, I gave pointers to earlier work and to later work, but, I confess, these references are fairly sparce at this time. Suggestions would be most welcome. I have also given references to the relevant historical literature.

I have tried to give a synopsis of what is in the work, so these notes should help people find the portions of the book that most interest them. These notes may not be much fun to read (although I have tried make them interesting), for they deal with the details of the text. But I do hope you will find them useful.

These notes should be considered a first draft. As you will not, I have not given a discussion of each chapter. More importantly, much more work needs to be done in showing the antecedents of the *Introductio* both in Euler's own work and in the work of others. To follow up on how the ideas introduced here were developed by Euler and other mathematicians is an even bigger task, but I plan to continue to work on it. Any contributions you have will be most welcome.

Question: Is there interest in setting up an email reading group to read the *Introductio*? One idea would be to model it on The Free Lance Academy, the Home of Slow Readers: http://www.freelance-academy.org/

Printing History of Euler's Introductio.

The following intends to be a complete list of all editions of Euler's *Introductio*. It was compiled by consulting various on-line library catalogues, the *National Union Catalog of Pre-1956 Imprints*, and MathSciNet (the on line version of *Mathematical Reviews*). Additions and corrections would be welcome. Each edition is denoted by the year of publication and a letter to indicate the language used (Latin, French, German, Russian, English, or Spanish)

1748*l* *Introductio in analysin infinitorum.* Lausannae: Apud Marcum-Michaelem Bousquet & socios., 1748. 2 v., front., 40 folded leaves of plates, 28 cm. Eneström number 101. Volume two is E102.
Also available in micro-opaque format. New York: Readex Microprint, 1968. 10 cards; 23 x 15 cm. in the *Landmarks of Science* series.
The frontispiece, bearing the title "Analyse des Infiniment-Petits," was ingraved by Soubeyron after work of Dela Monce. The title page is printed in red and black. The portrait of Jean Jacques Dortours de Mairan, to whom the book was dedicated by Bousquet with Euler's permission [*Introductio* **1922**, p. ix], was engraved by Etienne Ficquet (1719–1794) after work of J. Toquet. Jean Jacques Dortous de Mairan (1678-1771) was secretary of the Académie Royale des Sciences, geologist, astronomer, and physicist. The ornamental strip (bandeau) is by Papilon, but I can't read his name on my copy. Information from the Catalogue Collectif de France (http://ccfr.bnf.fr).

1783*l* *Introductio in analysin infinitorum.* Reprint (?) of Latin original. Did Euler play any role in revising his work for this edition, or is it merely a reprint?

1786*f* *Introduction à l'analyse des infiniment petits par M. Euler. Premiere partie. De la nature des fonctions des quantités variables, traduite du latin par M. Pezzi; précédée de l'éloge de M. Euler prononcé à la rentrée de l'Académie royale des sciences, le 6 février 1785 par M. le Marquis de Condorcet.* Strasbourg: aux dépens de la Librairie Académique, 1786.

1788*g* *Leonhard Eulers Einleitung in die Analysis des Unendlichen. Aus dem Lateinischen übersetzt und mit Anmerkungen und Zusätzen begleitet von Johann Andreas Christian Michelsen.* Berlin: S. F. Hesse and C. Matzdorff (vol. 3 only), 1788–1791. 3 v. (1788–1796), illus. 21 cm. USMA: QA35 .E87 1788. Not listed in WorldCat (OCLC).

1796*f* *Introduction à l'analyse infinitésimale par Léonard Euler; traduite du latin en français, avec des notes et des éclaircissements, par J. B. Labey.* Paris: Barrois, 1796–1797. Vol. 1: XVI + 364 p. Vol. 2: 424 p. + 149 diagrs on 16 folded plates, 27 cm. Jean Baptiste Labey (1750?-1825) is the editor and translator.
Also available in micro-opaque format. New York: Readex Microprint, 1968. 9 cards; 23 x 15 cm. in the *Landmarks of Science* series.
On the web at http://gallica.bnf.fr/scripts/ConsultationTout.exe?E=0&O=N003884 .

1797*l* *Introductio in analysin infinitorum*, Editio nova. Lugduni: Apud Bernuset, Delamolliere, Falque & soc., 1797. Vol. 1: xvi + 320 p., 149 diagrs. on 16 plates, fold. table. Vol. 2: [2] + 398 p., 16 leaves of plates. 27cm. University of Michigan: QA 35 .E88in 1797.

1835*l* *Introductio in analysin infinitorum.* Another Latin printing. Boyer 1951, p. 226, does not refer to a Latin edition of this date, but of a German one.

1835*f* *Introduction à l'analyse infinitésimale par Léonard Euler; traduite du latin en français, avec des notes et des éclaircissements, par J. B. Labey.* Paris: Chez Bachelier. 2v. [NUC, v. 163, p. 297]. Can't find it in http://www.ccfr.bnf.fr/

1835*g* *Leonhard Eulers Einleitung in die Analysis des Unendlichen. Aus dem Lateinischen übersetzt und mit Anmerkungen und Zusätzen begleitet von Johann Andreas Christian Michelsen.* Neue unveränderte berichtigte Aufl. Berlin: G. Reimer. 2 vols. (v1: xvi + 456; v2: viii + 392), tables, one folding.

1885_g'_ _Einleitung in die Analysis des Unendlichen von Leonhard Euler. Erster Teil. Ins Deutsche übertragen von H. Maser._ Berlin: Julius Springer, 1885. x + [2] + 319 + [1].
Reprinted in 1983.

1922_l_ _Leonhardi Euleri introductio in analysin infinitorum; adiecta est Euleri effigies ad imaginem ab E. Handmann pictam expressa. Tomus primus. Ediderunt Adolf Krazer et Ferdinand Rudio._ This is _Leonhardi Euleri Opera Omnia. Series prima. Volumen VIII._ Originally by Societas Scientiarum Naturalium Helveticae, Geneva. Turici, Lipsiae: Orell Füssli; Lipsiae, Berolini: B. G. Teubner. Now available through Birkhäuser: http://www.birkhauser.ch/books/math/euler/index.html for 150 euros. ISBN 3-7643-1407-9. Available on the web at http://gallica.bnf.fr/document?O=N006958 .

1938_r_ Russian edition. Probably this is the first edition of _Introductio_ **1961**. Mentioned on p. 110 of _Leonhard Euler 1707–1783. Beiträge zu Leben und Werk_ (1983). Boyer 1951, p. 225, says "a partial Russian translation apparently has not been published."

1961_r_ _Vvedenie v analiz beskonechnykh. Tom I, II._ (Russian) Vol. 1: Second edition, translated from the Latin by E. L. Pacanovskiĭ, with an introductory essay by A. Špaĭzer. Vol. 2: translated by V. S. Gohman, with an introductory essay and a commentary by I. B. Pogrebysskiĭ. Both volumes edited by I. B. Pogrebysskiĭ. Moscow: Gosudarstv. Izdat. Fiz.-Mat. Lit., 1961. Vol. 1: 315 pp. Vol. 2: 391 pp. MR 24 #A3054.

1967_l_ _Introductio in analysin infinitorum._ Bruxelles: Culture et Civilisation, 1967. 2 v., front., diagrs., 26 cm.
Photographic reprint of the 1748 edition.
Available on the web at http://gallica.bnf.fr/scripts/ConsultationTout.exe?E=0&O=N003351 .

1983_g_ _Einleitung in die Analysis des Unendlichen. Erster Teil. Mit einer Einführung zur Reprintausgabe._ Berlin, Heidelberg, New York: Springer-Verlag, 1983. 21+xi+319 pp. ISBN 3-540-12218-4. MR 85d:01030 by C. J. Scriba contains useful information.
This photoreproduction of _Introductio_ **1885** contains a useful new introduction by Wolfgang Walter of the Matehmatisches Institut I der Universität Karlsruhe, pp. 5-21.

1987_f_ _Introduction á l'analyse infinitésimale_, Paris: ACL-éditions, 1987-1988. 2 vol. (XIV, 364 p.; 424 p., 16 f. de pl.) ; 25 cm + 2 p. Reprint of _Introductio_ **1796**. ISBN 2876940043 (t.1) ISBN 2876940078 (t.2) [Information from the Swiss National Library: http://www.vtls.snl.ch/]

1988_e_ _Introduction to analysis of the infinite. Book I._ Translated by John D. Blanton. New York: Springer-Verlag, 1988. xvi+327 pp. ISBN 0-387-96824-5. MR 89g:01067. Book II appeared in 1990. xii+504 pp. ISBN 0-387-97132-7. MR 91i:01143 by Doru Ştefănescu.

1995_l_ _Introductio in analysin infinitorum._ Apparently a reprint of _Introductio_ **1967**. Known only from a vague record in the Catalogue Collectif de France (http://ccfr.bnf.fr).

2000_s_ _Introducción al análisis de los infinitos_, Spanish translation from the Latin by José Luis Arantegui Tamayo. Annotated by Antonio José José Durán Guardeño. With introductory material by Javier Ordóñez, Mariano Martnez Pérez and Durán Guardeño. Edited by Antonio José Durán Guardeño and Francisco Javier Pérez Fernández. _Sociedad Andaluza de Educación Matemática "Thales", Seville: Real Sociedad Matemática Española, Madrid_, 2000. lx+407 pp. ISBN 84-923760-3-1. MR 2002d:01014b. The first volume of this boxed set is a facsimile reprint of the Latin original of Volume I. ISBN 84-923760-2-3. MR 2002d:01014a. The ISBN for the set is 84-923760-4-X. For additional information see http://thales.cica.es/thales/euler.html and http://www.arrakis.es/∼mcj/libros.htm .
Note: Harvard has a manuscript of the second volume of the _Introductio_ **i**n French [NUC v. 163, p. 297].

In preparing these notes I have primarily used the English translation of John D. Blanton (*Introductio* **1988**), but frequently consulted the Culture et Civilization edition (*Introductio* **1967**) which is a photographic reprint of the original 1748 edition, and which is also available on the web. The German translation by Wolfgang Walter, *Introductio* **1983**, has been consulted, for that is the first edition I read and it has my notes in it. Finally, I have used the edition from the *Opera omnia* of 1922, for it contains useful footnotes about errors in the original and references to other works pertaining to the text (it is also available on the web). When I have noted significant differences in the texts I have made note of them.

Subject Classifications

The subject classification varies from library to library, but the following have been used:

Mathematics – Early works to 1800
Functions – Early works to 1800
Trigonometry – Early works to 1800
Series, Infinite – Early works to 1800
Products, Infinite – Early works to 1800
Continued fractions – Early works to 1800

The copies at The Ohio State University and at the University of Cincinnati carry the additional subject classifications

Bookplates – Examples (Bernoulli family)
Bookplates – Examples (Shtab, Bulgaria)
Old Book List – Switzerland – Lausanne – 1748
Autographs – Examples (Bernoulli, J.)

but I must check what this means.

Caput Primum. De Functionibus in genere, pp. 3–15, §§1–26.

The modern reader is likely to miss the import of this chapter, "On Functions in General," for its main idea has been so totally incorporated into mathematics that we think nothing of it. This chapter is about functions. It is not about curves. This change of viewpoint represents a benchmark in the history of the calculus. Newton and Leibniz studied curves. The title of the first calculus book, *Analyse des infiniment petits, pour l'intelligence des lignes courbes* (L'Hospital 1696), reflects this early point of view. Agnesi's book, published the same year as the *Introductio*, also studies curves.

Henk J. M. Bos, "The fundamental concepts of the Leibnizian calculus," pp. 83–99 in *Lectures in the History of Mathematics* (AMS, 1993) has a somewhat different view on this point. He maintains that the 'variable' was the underlying entity. But that still supports the position that Euler introduced a fundamental change into the basic concepts of the calculus.

A variable quantity can be determined by substituting a number for the variable. This is also an ontological change from earlier work. For Newton, Leibniz, and the Bernoullis, the variables represented line segments, areas, etc.

§§1–3 introduce Euler's concepts of constants and variables. This is something that is not discussed today, but probably should be. Variable quantities "take on all possible numbers." He enumerates what he means by numbers at the end of §3, but as we read this text we should strive to get a better understanding of Euler's concept of number.

Do note that imaginary numbers are used (Blanton in *Introductio* **1988** translates this as complex numbers, a term that was later introduced by Gauß).

§4 gives the main definition: "A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities." [When quoting in English, I use the translation of Blanton unless otherwise indicated.] Precisely what he means by this will have to be inferred from what he writes but note that in §6 he allows transcendental operations which the integral calculus "supplies in abundance." We shall see that this certainly includes infinite series, infinite products, and infinite continued fractions. Needless to say he cannot be too precise here in the opening pages of his treatise. However, you should note that the concept introduced here is not as general as the modern concept introduced by Dirichlet a century later.

The word "function" — in a mathematical sense — appears for the first time in a manuscript written by Leibniz in 1673 entitled "Methodus tangentium inversa seu de functionibus,"

The development of the concept of function is a very complicated one and there is a good deal of secondary literature. The classic paper on the topic is A. P. Youschkevitch, "The concept of function up to the middle of the 19th century," *Archiv for History of the Exact Sciences*, 16 (1976/77), 37–85. MR 58 #15925. If you are interested in original sources, and you should be, a paper by Dieter Rüthing gives 21 definitions of function ("Some definitions of the concept of function from Joh. Bernoulli to N. Bourbaki," *The Mathematical Intelligencer*, 6 (1984), no. 4, 72–77; MR 86a:01003). He gives no commentary so you have the opportunity to think through the changes in this concept for yourself. Israel Kleiner, "Evolution of the Function Concept: A Brief Survey," *College Mathematics Journal*, 20 (1989), 282-300 won the MAA's George Pólya Award.

§9 defines polynomials and rational functions. The general form for a polynomial is $a+bz+cz^2+dz^3+ez^4+fz^5+\&c$, but Euler does not specify if the coefficients are real or imaginary. The meaning of the "$+\ \&c$" (et cetera) is also unclear. Blanton, *Introductio* **1988** renders it as an ellipsis, but that makes it look like an infinite polynomial. We will need to try to clarify this through later usage. This is the first — of many, many — instance where the notation would be simplified by the use of subscripts, but Euler does not have them (who created them?).

The remainder of the chapter deals with multiple valued function and odd and even functions. The impatient reader should move on to the next chapter.

Caput II. De transformatione Functionum, pp. 15–35, §§27–45.

"On the Transformation of Functions" describes how to rewrite functions algebraically. Euler considers this topic because sometimes one form of a function is easier to work with than another.

In §27 his fourth example, $\sqrt{1+z^2}+z = 1/[\sqrt{1+z^2}-z]$, is chosen to avoid division by 0, yet it seems to contradict the statement in §5 that a function takes on every value since complex arguments are allowed.

§28 deals with factoring. He distinguishes between roots of equations and zeros of functions, but, of course, without using this modern vocabulary. Blanton, in *Introductio* **1988**, uses the term "linear factors" whereas Euler uses "Factores simplices." This modern terminology connontes analytic geometry in a way that Euler's term does not. Note that in the next section, Blanton slips back to Euler's terminology.

Note also that §28 ends with a clear statement of the Fundamental Theorem of Arithmetic. See the Pólya Prize winning paper of William Dunham, "Euler and the Fundamental Theorem of Algebra," *College Mathematics Journal*, 22 (1991), 282-293.

§29 shows that he is aware of the Factor Theorem: If $z = f$ is a root of the equation $Z = 0$ then $z - f$ is a factor of the function Z.

§31 is a response to a claim of Leibniz thet $1/(x^4 + a^4)$ could not be integrated by partial fractions (reference?). He wants to show that any quartic (presumably with real coefficients) can be factored into two real quadratics. He uses an indirect proof, though a proof by cases would do as well. However when he assumes that the two imaginary quadratic factors have the form $zz-2(p\pm q\sqrt{-1})z+r\pm s\sqrt{-1}$, he assumes too much. There is no justification for the complex conjugates appearing here, at least not without further proof. Two notational points: (1) Euler writes "zz" and "z^2" interchangeably, (2) Euler consistently' writes "$\sqrt{-1}$" throughout the text; he did not introduce "i" until 1777. Blanton, however, in *Introductio* **1988** consistently uses "z^2" and "i". In checking the details, it might help to know that Euler expressed the solutions of $x^2 = -px+q$ (the quadratic formula) in the form $x = -\frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2+q}$ [Euler, *Elements of Algebra*, §642].

§32. The phrase "Although the same method of proof is not valid for higher powers," makes me wonder if he has some doubts about his proof of the Fundamental Theorem of Algebra. Need to check this out.

§33 begins with a clear statement of the Fundamental Theorem of Algebra, but he only gives the geometric intuition and does not attempt a proof (Bolzano does this in 1817 in his "Rein analytische Beweiss"; for an English translation, see Russ, HM 7 (1980), 156-185). The argument in §34 that an odd degree polynomial has at least one real root is pure murk by modern standards, but, I believe, was quite acceptable in the seventeenth century. Read it with an open mind, ignoring the rigor mortis you learned in school.

§35 has a nice inductive argument.

§§38-46. The rest of the chapter deals with partial fractions. The computational details of this material are well known from a standard calculus course (rather they were a few years ago), but the proofs are seldom presented (see Chrystal's *Algebra*). The main difference is that he factors over the complex numbers and so does not have to consider the irreducible quadratic case. According to a note in the *Opera omnia*, Euler comes back to the material in §41 in papers of 1780 and 1809. Note that Euler frequently does examples several ways; this is good pedagogy.

Caput III. De transformatione Functionum per substitutionem, pp. 36–45, §§46 [bis] –58.

Euler's incredible cleverness becomes clear in this chapter "On the Transformation of Functions by Substitution." After hinting at composition of functions, he is very explicit about the content of this chapter:

> A new variable may be introduced for either of two reasons: if the expression of y by z contains a radical, it may be removed, or if the relationship between y and z is given by an equation of higher order so that y cannot be expressed explicitly in terms of z, then a new variable x is introduced in terms of which both y and z can be conveniently defined.

Section 47–51 deal with the removal of radicals by introducing a new variable and §§52–58 deal with equations.

After getting warmed up in §§47–49, his cleverness becomes clear in §50. Suppose $y = \sqrt{(a+bz)(c+dz)}$. He wants to introduce a new variable x by which both y and z can be expressed without radicals. Having read the previous sections you might think he would put $\sqrt{(a+bz)(c+dz)} = x$. But then when you square both sides to eliminate the radical, you would have to use the quadratic formula to solve for z. That is too messy. Insead, Euler lets $\sqrt{(a+bz)(c+dz)} = (a+bz)x$. Then when both sides are squared, the factor $a+bz$ cancelles, and one has a linear equation in z. The method introduced in §50 is used in his *Institutiones Calculi Integralis*, §88.

§51 deals with the case of an irreducible quadratic under the radical sign and he considers a number of different cases.

§52 deals with the second problem of the chapter. He remarks that there is no "general solution" for $ay^\alpha + bz^\beta = cy^\gamma z^\delta = 0$ in terms of either y or z (What did Abel think when he read this?). Then he provides three different solutions to the problem and these are illustrated with an example. Pedagogically, it would provide more reinforcement if this example was done using his three methods rather than proceeding formally as he does. I encourage you to do so.

At a first reading, I would be inclined to skip this chapter.

Caput IV. De explicatione Functionum per series infinitas, pp. 45–60, §§59–76.

This chapter, "On the development of functions in infinite series," is based on the principal that it is easier to do calculus with polynomials than with rational functions or other types of functions. So if we could just reduce those other functions to infinite series — long polynomials — then our life would be easier. Euler clains in §59 that every function can be expressed as a power series, and is analytic in our sense (of course, he does not use this terminology).

As usual, Euler begins with easy cases and moves on to more complicated ones.

§60 converts $1/(\alpha + \beta z)$ to an infinite series, first by repeated long division (Newton also did this) and then by the technique that is to be the mainstay of the chapter. Set this fraction equal to an infinite series with unknown coefficients, cross multiply, and solve for the unknown coefficients. This gives him a recurrence relation for the coefficients, but he also finds a formula for the coefficients.

§61 does the next most general case, $(a+bz)/(\alpha+\beta z+\gamma z^2)$. He admits that repeated division "becomes tedious" and so *supposed* that this fraction can be expressed as an infinite series and then proceeds as before, but this time gets a recurrence that determines the n^{th} term of the series in terms of the previous two. At this stage we really do wish that he knew about subscripts and I encourage you to rewrite this in modern notation — then the recurrecnce becomes absolutely transparent.

§62. Two exaples were enough for Euler. He sees the pattern and we should too. If the denominator of the fraction is a polynomial of degree n then the recurrence depends on the previous n coefficients. He ends this section with the kind of etymological remark that is most helpful: the word 'recurrent' "comes from the fact that if we wish to investigate subsequent terms we have to "run back", to previous terms." [Remember that your *Curriculum vitae* runs through your life.] Euler attributes DeMoivre with the introduction of the word and the editor of the *Opera omnia* tells us where: *Philosophical Transactions of the Royal Society of London*, 32 (1722/3), 1724, no. 373, p. 162 (Early volumes are available on the web at http://www.bodley.ox.ac.uk/ilej/journals/).

§63. This is the general case again, except, in order to avoid fractions in the coefficients, he takes the constant term of the denominator to be 1. Also, all other terms in the denominator are negative — again, this clever choice simplifies the result. Euler shows that the coefficients of the infinite series are 'weighted sums' of the coefficients of the numerator and denominator of the fraction. This must be an early use of such a phrase. The result of this section allows him to easily convert rational functions to power series and he often uses it in the future without any mention, e.g., in §215. This is a common habit of Euler, so watch for it.

§§64–70 deal with the case where the denominator is a power. No new techniques are involved so this could be omitted on a first reading.

The mention of second differences in §65 at first seems out of place and he turns in into a general result at the beginning of §66. This topic is treated in the first two chapters of his *Foundations of Differential Calculus*, which has been translated by Blanton (Springer, 2000), but I did not notice him using the results of this chapter.

§71 introduduces the Binomial Theorem in the form that Newton used. It is stated in more familiar form near the end of §72. Note his tendency to state several particular cases of the general result. He does not give a proof, but the editor of the *Opera omnia* refers us to his paper in volume 15 of series I and also to the *Institutiones calculi differentialis, partis posterioris*, chapter IV = *OO* I, X, 276 (this is not in the English translation of the first part of the *ICD* that Blanton has published. Abel was the first to provide a good proof of the Binomial Theorem for all exponents.

Caput V. De Functionibus durrum pluriumve variabilium, pp. 60–69, §§77–95.

This chapter deals with functions of two or more variables, i.e., of expressions made up of two or more variables. In §78 he notes that each variable can be determined in an infinite number of ways. Consequently a function of two variables admits "an infinity of infinite determinations," and when there are three variables "the determinations will be greater by infinity." This is an interesting hint about different orders of infinity, but will have to wait more than a century for Georg Cantor to develop it. Indeed, in this case a function of several real or complex variables has no more "determinations" than a function of one variable.

§79 divides functions of several variables into algebraic and transcendental and he even speaks of a function being "less transcendental" than another. He remarks that he will not pursue this distingtion further here, but I do not know if he does so elsewhere.

§80 introduces explicit and implicit functions. He notes that $V^5 = (ayz + z^3)V^2 + (y^4 + z^4)V + y^5 + 2ayz^3 + x^5$ is an implicitly irrational function of y and z. I wonder what Abel thought when he read this? Might comments like this have motivated his interest in the solution of fifth degree equations?

§81 introduces the notion of a homogeneous function (every term has the same degree) and is the most important part of this chapter. The concept is due to Viete in 1646 (according to the *Opera omnia* edition), but the term was introduced by Johann Bernoulli in 1726 in his "De integrationibus aequationum differentialium," (*Commentarii Academiae Petropolitanae*, vol. 1 = *Opera* III, 108–124; reference from Cantor 1910, p. 704). He first remarks that constant functions are not given a degree and then changes his mind and says they have degree zero.

§90 contains a nice proof that a homogeneous polynomial of degree n can be factored into n terms of the form $\alpha y + \beta z$.

The chapter ends with the classification of polynomials into reducible and irreducible.

Caput VI. De Quantitatibus exponentialibus ac Logarithmis, pp. 69–85, §§96–113.

The modern reader should consider starting his reading of the *Introductio* with this chapter which introduces the exponential and logarithm functions. This will avoid some troubling distinctions made earlier and also avoid material that does not quite seem relevant to the modern reader.

Euler begins with a very nice introduction to the exponential function a^z. He notes that when z is a fraction then one needs to take the 'primary value' of the function for the exponential is to be a single valued function. Also, the exponent z is restricted to real values. Ordinarily it is either unclear what values z can assume or it is allowed to take on all complex values. This is the first case where an explicit restriction is made.

In §99 we see the first example of a function with a 'jump,' namely

$$0^z = \begin{cases} 1, & \text{if } z > 0 \\ 0, & \text{if } z = 0 \\ \infty, & \text{if } z < 0. \end{cases}$$

Note that Euler knows that $0^0 = 0$, an expression which many people think is undefined. He also shows that using a negative value for a in the function a^z is very inconvenient. Thus he concludes that a should be a positive number larger than 1. The reasons for this choice are seldom made clear, but they should be, especially in elementary texts. This is one place where Euler's explanatory skills should be emulated in textbooks.

§102 introduces the logarithm as the inverse of the exponential function. He intoduceds the concept of a base to the logarithms, but makes it clearer in the next section when he expresses this in words: "the number whose logarithm is equal to 1 is the logarithmic base." He notes that the logarithms of negative numbers will be complex.

§105 explains the use of the word transcendental.

§106 explains how to approximate logarithms using only square roots: Bound the number whose logarithm you want between two numbers whose logarithms you know and then take the log of the geometric mean of these two numbers. I.e., suppose you want the logarithm of w, where $y < w < v$, and that you know $\log y = z$ and $\log v = x$. Then $\log \sqrt{vy} = (x + z)/2$ is also known. Now w is either between y and $\sqrt{vy}$ or it is between $\sqrt{vy}$ and v. Now iterate the process. As an example he computes $\log_{10} 5$ correctly. He notes that this process of repeatedly taking logarithms of geometric means converges. Luckily, it converges when he gets to the Z^{th} term. He is aware that there are better ways of computing logarithms. It seems to me that this might make a nice student exercise.

The "golden rule of logarithms" for converting from one base to another is discussed in §107. §109 notes that if we know the logarithms of primes then we can compute the logarithms of all numbers. The Fundamental Theorem of Arithmetic is implicitly used.

§110 shows, with four examples, how logarithms can be used to carry out computations. Euler begins with a problem that has its background in music theory when he computes $2^{7/12}$ using the trick that $7/12 = 1/2 + 1/12$. Such tricks were quite useful in simplifying computations in the days before calculators. The third example considers the repoputation of the earth by Noah and his family after the flood. If the population increases by one-sixteenth each year then after 400 years "the whole earth would never be able to sustain that population." He does have a dry sense of humor.

§111 discusses the most important application of logarithms, namely when the unknown variable is in the exponent. He gives two nice examples. I find it significant that his few examples are much more informative and interesting than the dozens that are found in textbooks today.

Common logarithms are introduced in §112. On the use of the word 'mantissa' see http://members.aol.com/jeff570/m.html .

This really is a nice chapter to read.

Caput VII. De quantitatum exponentialium ac Logarithmorum per Series explicatione, pp. 85–93, §§114–125.

In this chapter Euler really gets going. He expresses the exponential and logarithmic functions as infinite series. He does not lay out a plan of what he intends to do (although the chapter title says it pretty clearly) but just starts going. This really is a fascinating chapter, but quite hard to interpret because he uses infinitely small numbers. But that is part of what makes it so interesting. My advice is that you read this with an open mind; don't pay too much attention to the rigor of what he does, just enjoy his creativity. My experiece is that a bit of knowledge of the non-standard anaylsis of Abraham Robinson really helps one to enjoy these manipulations.

In §114 he notes that if ω is infinitely small ("Sit ω numerus infinite parvus") then $a^\omega = 1 + \psi$ where again ψ is infinitely small. He chooses k so that $a^\omega = 1 + k\omega$ and shows that k depends on the base a. In §115 he raises both sides of this expression to the i-th power (remember he always writes $\sqrt{-1}$ for he has not yet introduced i to abbreviate it, so this i is just an exponent; Blanton uses j and that probably is less confusing for the modern reader, so I will use it). He then expands $(1 + k\omega)^j$ into an infinite series using the Binomial Theorem.

The clever maneuver comes in §115 where Euler lets $j = z/w$, where z is a finite number. Then ω is infinitely small and j is infinitely large. Because of this $(j-1)/j$, $(j-2)/j$, $(j-3)/j$, etc., are all equal to 1. Thus

$$a^z = 1 + \frac{kz}{1} + \frac{k^2z^2}{1\cdot 2} + \frac{k^3z^3}{1\cdot 2\cdot 3} + \frac{k^4z^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$$

When we let $z = 1$ we see the relationship between a and k:

$$a = 1 + \frac{k}{1} + \frac{k^2}{1\cdot 2} + \frac{k^3}{1\cdot 2\cdot 3} + \frac{k^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$$

He notes that if $a = 10$ then $k = 2.30258$ (as computed in §114, but of course he is not yet in a position to note that this is $\ln 10$).

In §119 he derives an infinite series for $\log(1 + x)$. In the next section he develops the relationship between k and the base and obtains $2.30258 = 9/1 - 9^2/2 + 9^3/3 - 9^4/4 + \cdots$. He realizes that this is paradoxical. This is the sort of thing that Euler does that brings condemnation from sticklers for rigor. But the important thing to note is that Euler knows this expression is senseless. He has a much better sense of convergence than he is usually given credit for. In the next section he obtains a different formula for k where the terms "decrease in a reasonable way" and so k can be approximated. In §123 he calls this series 'strongly convergent,' although I much prefer the Latin: quae Series vehementer convergunt.

Finally in §122 Euler does what we have been anticipating. He chooses the base of his logarithmic system so that $k = 1$. With this base he obtains the system of "natural or hyperbolic" logarithms (the two adjectives are italicized in the original but not in Blanton). He explains the later name: "since the quadrature of the hyperbola can be expressed through these logarithms," but does not mention Gregory of St. Vincent or his student de Sarassa.

§122 is where Euler introduces the symbol e. He first used the symbol in a manuscript, "Meditatio in Experimenta explosione tormentorum nuper instituta" (Meditation on experiments made recently on the firing of cannon), written at the end of 1727 or the beginning of 1728 but not printed until 1862 in Euler's *Opera postuma mathematica et physica*, Petropoli, edited by P. H. Fuss and N. Fuss (vol ii, pp. 800–804). The manuscript describes seven experiments performed between August 21 and September 2, 1727. Euler next used e in a letter addressed to Goldbach on November 25, 1731, writing that it "denotes that number whose hyperbolic logarithm is = 1," which is a common expression for Euler for years to come. He used e again in 1736 in his *Mechanica*, and this was its first appearance in print with this meaning [http://www.veling.nl/anne/templars/constants.html].

In §123 he uses his series for $\log((1 + x)/(1 - x))$ to compute the logarithms of numbers between 1 and 9, except for 7. His derivation of $\log 7$ is rather clever.

Finally, in the last section of the chapter (§125) he obtains the famous relationsip

$$e^z = \left(1 + \frac{z}{j}\right)^j,$$

where j is an infinitely large number. Note that there is no limit in this expression.

Caput VIII. De quantitatibus transcendentibus ex Circulo ortis,
pp. 93–107, §§126–142.

This chapter, on the transcendental quantities which arise from the circle, or the trigonometric functions, is my favorite, for it is laced with interesting ideas. Dirk Struik, *Source Book*, pp. 235–351, has an English translation (different from Blanton's) of part of this chapter as well as an interesting introduction.

The second paragraph of §126 is loaded with ideas that have been adopted by all mathematicians and represent a watershed in the development of trigonometry. He begins "Ponamus ergo Radium Circuli seu Sinum totum esse = 1". Hitherto the radius of the circle used in trigonometry was huge so that decimals were avoided, but Euler introduces the unit circle for doing trigonometry. Euler's approach has become the standard approach today. Also, this is essentially the last use of the phrase 'total sine' for the largest sine.

The second clause of the sentence is equally amazing, for he claims that it is clear (liquet) that the circumfrence of a unit circle is irrational. I doubt that this is 'clear' to many today, but Euler knows that this is true. His continued fraction in §369 explains this.

Next he gives the value of π that F. de Lagny (1660-1734) computed in 1719 and published in 1720. The error in the 113-th decimal place (the 7 should be an 8) was corrected by G. de Vega (1756–1802) in 1794.

It is well known today that the first person to use π to represent the ratio of the circumference of a circle to its diameter was William Jones (1675–1749) in his *Synopsis palmariorum mathesios* (1706, p. 263). See http://www.veling.nl/anne/templars/constants.html, which quotes the passage where it is introduced.

When Euler introduced the symbol π here he uses the word "scribam," "I write" (in *Introductio* **1988**, p. 101, Blanton incorectly translates this as "we will use the symbol π"). The use of the first person certainly suggests that he things he is introducing the symbol for the first time, and that was widely believed until the passage in the Jones text was discovered in the 1890s (I need to find the reference to this paper). No doubt the popularity of Euler's *Introductio* popularized the use of the symbol π.

§127 introduces the trigonometric functions. He lets the variable z be an arc, not an angle. He denotes the sine of this arc by "*sin. A. z.*" In some ways this is a very nice notation. The first period indicates that 'sin' is an abbreviation for 'sine' and the explicit use of 'A' emphasizes that we are taking sines of arcs.

Next he develops a wealth of trigonometric fourmulae. That he does so indicates that trigonometry was not a well known subject in his day. Also, the fact that he is doing all of this on the unit circle is important. A picture at this point really would help, but there are no diagrams in this volume (there are lots in the second volume of the *Introductio*).

In §128 Euler begins by stating the formula $\sin(y+z) = \sin y \cos z + \cos y \sin z$ and then derives a number of consequences. It is unfortunate that he does not derive this formula, for then the development would be self contained.

These sections are a nice development of many standard trigonometric formulae. At the end of §129 he derives a sequence of formulae, but does not introduce a variable n for the integer in them; had he, he would have $\sin(ky + z) = 2\cos y \cdot \sin((k-1)y + z) - \sin((k-2)y + z)$. It seems out of character for Euler not to give this general formula.

Observe that Euler writes $(\sin z)^2$ rather than our modern $\sin^2 z$. This is one place where it is too bad that we have not followed his notation, for it is far less confusing to students.

After all these mundane formulae, Euler gets serious again in §132. He observes that $(\sin z)^2 + (\cos z)^2$ — I wish he had written the more natural $(\cos z)^2 + (\sin z)^2$ — factors over the complexes as $(\cos z + \sqrt{-1}\sin z)(\cos z - \sqrt{-1}\sin z)$. Then — and this is really Euler in action — he changes variables and considers $(\cos z + \sqrt{-1}\sin z)(\cos y - \sqrt{-1}\sin y)$. He multiplies this out and uses the formulas for the sum of two angles. Then he introduces a third factor, works out the product, and then does an "Eulerian induction" to concludes, in §133, DeMoivre's formula (without mentioning it by name), $(\cos z + \sqrt{-1}\sin z)^n = \cos nz + \sqrt{-1}\sin nz$. This is used in §147. He uses this to derive formulas for $\cos nz$ and $\sin nz$, which are finite series (Blanton's use of an ellipsis belies this; the original has '&c'). [Need to check if Abel cites these formulae, for this was a problem of considerable interest to him.]

In §134 he lets z be infinitely small, so that $\sin z = z$ and $\cos z = 1$, and then uses the expansions for $\sin z$ and $\cos z$ to obtain the well known infinite series for $\sin z$ and $\cos z$. In this derivation he says "If n

is an infinitely large number, so than nz is a finite number, say $nz = v$" so (as the commas indicate) he is apparently not concerned about n being so large that v is infinite also. An orgy of computation follows where he expresses $sin\frac{m}{n}\frac{\pi}{2}$ as an infinite series up to the 29-th power of m/n. He comments that this series converges quickly (Series exhibitae maxime convegent). This is more evidence for my thesis that Euler has a well honed concept of convergence.

In §136, for the first time, Euler talks of angles. He explains that if we know the sines and cosines of angles less than 30 degrees then we can compute the sines and cosines of all angles.

In §137 Euler develops the formula for $\tan(a + b)$. Sanford, in her *Short History*, indicates that this is the first occurrance of such fomulae.

§138 develops the famous Euler formula $e^{iv} = \cos v + i \sin v$.

In §139 Euler refers to §130. This is a typo for §133 which is silently corrected in the *Opera omnia*.

§140 develops the series for arctangent and the famous formula of Leibnitz (as it is spelled here): $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots$ which Euler remarks in §134 is hardly convergent (vix convergentum). He develops a series which is "much more rapidly" converging (multo magna).

http://www.math.niu.edu/ rusin/known-math/99/zeta2 Quotes Weil.

Caput IX. De investigatione Factorum trinomialium, pp. 107–128, §§143–165.

This chapter investigates trinomial factors in polynomials. A clever factorization trick using trigonometry is used to confirm the Fundamental Theorem of Algebra. He uses these factors on infinite series and expresses them as infinite products. The highpoint of the chapter is his infinite product for the sine. Euler makes several clever uses of infinites and infinitesimals. The formulas developed here will be used in the next chapter to sum some infinite series.

After discussing the Factor Theorem in §143, he notes that complex linear factors of polynomials can be hard to find. Since they occur in pairs, he considers the trinomial $p - qx + rx^2$ which is irreducible over the reals, i.e., $4pr > q^2$ or $q/2\sqrt{pr} < 1$. Then he rechristens this quotient which is less than one $\cos\phi$. Thus the trinomial becomes $p - 2\sqrt{pr}\cos\phi + rz^2$. "Lest some irrationality cause problems" — and I don't understand why he says this — he decides to deal with $p^2 - 2pqz\cos\phi + p^2q^2$.

In §146 Euler notes that the complex factors of this trinomial are $(p/q)(\cos\phi \pm i\sin\phi)$ and so if this is substituted in the original polynomial, one gets a pair of equations in p and q. This looks messy, but in §167, DeMoivre comes to the rescue. He illustrates these methods in §150–1 using $a^n \pm z^n$. In the next few sections he deals with more complicated examples.

The pace quickens at §155 when Euler extends these methods to infinite series. Using the methods of §151 he derives

$$e^x - 1 = x \cdot \prod_{n=1}^{\infty} (1 + \frac{x}{j} + \frac{x^2}{(2n)^2\pi^2}),$$

where j is infinite. The next section makes it very clear that he is skilled in dealing with infinites and infinitesimals. He derives the more intresting formula

$$\frac{e^x - e^{-x}}{2} = x \cdot \prod_{n=1}^{\infty} (1 + \frac{x^2}{n^2\pi^2}).$$

Caput X. De usu factorum inventorum in definiendis summis Serierum infinitarum, pp. 128–145, §§165–183.

§168. Euler writes down the formula

$$\sum_{i=0}^{\infty} \frac{1}{i^{2n}} = \frac{2^{2n-2}}{(2n+1)!} K_n \pi^{2n}, \quad n = 1, 2, \cdots 13,$$

but was unable to recognize the coefficients K_n even though his comment about the 'extraordinary usefulness' of this 'quite irregular' sequence makes one believe that he is holding back. It was not until his *Institutions calculi differentialis* (1755) was published that he saw the connection: The sum is $2^{2n-1}\pi^{2n}B_{2n}/(2n)!$, where B_k is the k^{th} Bernoulli number [§§124, 125, 151; Hairer and Wanner 1996, p. 67].

This chapter is reprinted in *Pi: A Source Book* by L. B. Berggren, J. M. Borwein, and P. B. Borwein (Eds.).

Caput XI. De aliis Arcuum atque Sinum expressionibus infinitis, pp. 145–161, §§184–198.

Caput XII. De reali Functionum fractarum evolutione, pp. 161–174, §§199–210.

Caput XIII. De Seriebus recurrentibus, pp. 175–197, §§211–233.

Caput XIV. De multiplicatione ac divisione Angulorum, pp. 198–220, §§234–263.

Caput XV. De Seriebus ex evolutione Fractorum ortis, pp. 221–252, §§264–296.

Caput XVI. De Partitione numerorum, pp. 253–275, §§297–331. "Tabula ad paginam 275 Tom. I." is inserted between pages 274 and 275, facing p. 274.

This chapter "On the Partition of Numbers" is the origin of a whole branch of number theory, the partition of positive integers into positive summands (Aubrey J. Kempner, "The Development of "Partitio Numerorum," with Particular Reference to the Work of Messrs. Hardy, Littlewood and Ramanujan," *American Mathematical Monthly*, 30 (1923), 354-369). The work began in 1740 when Professor Philipp Naude of Berlin wrote Euler a letter asking in how many ways a given sum of money b could be divided between c persons, provided each share is an integer [D. J. Struik reviewing a Russian paper by A. A. Kiselev and G. P. Matvievkaaja, MR 33 #7227].

Walter calls this chapter a highpoint of the book [*Introductio* **1983**, p. 19]. The research was done by Euler between 1740 and 1744 and contains the first systematic observations on additive number theory and is the first application of analytic tools in number theory.

The chapter ends with the well known problem of which items can be weighed with a given set of integral weights.

Caput XVII. De usu Serierum recurrentium in radicibus aequationum indagandis, pp. 276–295, §§332–355.

This chapter begins with a rather precise (for the time) reference to a paper of Daniel Bernoulli from Volume III of the *Commentaries of the Saint Petersburg Academy*. In the *Opera omnia*, Rudio gives the title of this paper as "Observationes de seriebus recurrentibus," but no such paper is listed in the Index of Daniel Bernoulli's Works: http://www.birkhauser.ch/books/math/bernoulli/daniel.html . The paper Euler must be referring to is "Observationes de Seriebus quae formantur ex additione vel subtractione quacunque terminorum se mutuo consequentium, ubi praesertim earundem insignis usus pro inveniendid radic[ibus] omnium Aequationum Algebraicarum ostenditur," which appears in the *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, III, 1728 (1732), pp. 85–100 = *Die Werke von Daniel Bernoulli. Vol. 2. Analysis, Wahrscheinlichkeitsrechnung*, edited by L. P. Bouckaert and B. L. van der Waerden, Birkhäuser,

1982. The discrepency must be due to the fact that the abstract of the paper carries an abbreviated French title. I need to check if Bernoulli uses the word "recurrent."

Caput XVIII. De fractionibus continuis, pp. 295–320, §§356–382.

§369 develops Brouncker's continued fraction for $\pi/4$. See J. Dutka, "Wallis's product, Brouncker's continued fraction, and Leibniz's series," *Archiv for History of the Exact Sciences*, 26 (1982), 115–126.

§381 uses "the usual method for finding the greatest common divisor" to convert any rational number into a finite continued fraction. He does not refer to the Euclidean algorithm by that name. He also makes no fuss over the fact that the continued fraction is finite.

His second example shows how to convert any decimal into a continued fraction. He deals specifically with $\sqrt{2}$ but the method is perfectly general, and easy to check today on a hand held calculator (note the integral part and then subtract it, take the reciprocal, and repeat). He obtains $\sqrt{2} = [1; 2, 2, 2, \cdots]$ but makes no comment about irrationality.

The third example deals with e and is significant. Using the method of Example II he begins with $e = 2.718281828459$, obtains $(e-1)/2 = [0; 1, 6, 10, 14, 18, 22, \cdots]$ and notes that the denominators form a geometric progression. Curiously, he does not make any remark about the irrationality of e, a result that is attributed to his colleague Lambert. Hermite in his 1873 proof of the transcendence of e needed to prove the correctness of this continued fraction; see C. D. Olds, "The Simple Continued Fraction Expansion of e," *American Mathematical Monthly*, 77 (1970), 968-974, which won the Chauvenet Prize.

§382 hints that the convergents of continued fractions are the best approximations of numbers that one can obtain without using larger denominators, but this is not said very clearly.

The first example deals with the convergents of π, noting that Archimedes had 22/7 and Metius had 333/106. He obtains this by starting with $\pi = 3.1415926535$ and computing $\pi = [3; 7, 15, 1, 292, 1, 1, \cdots]$. No regularity at all has been noted for the continued fraction for π. Lambert (1770) computed 27 denominators, Lochs (1963) computed 968 [Hairer and Wanner 1996, p. 69].

The second example is a lovely application which explains the number of leap years in the Gregorian calendar. I do not know where he got his value for the lenght of the year, but it is critical to the answer.

"An essay on continued fractions," *Mathematical Systems Theory*, 18 (1985), 293-328, is a transltion of "De Fractionibus Continuis Dissertatio" (1744, E 71, *Opera Omnia*, Series I, vol 14) Euler's first published work on the theory of continued fractions, by B. F. Wyman and his mother M. F. Wyman, with an introduction by Christoper Byrnes.

Individuals Mentioned in the Text.

It is amazing how few individuals are referred to by name in the *Introductio*. Thus it seems desirable to list all of them. The references are by section number so that you can use any edition of the text.

Archimedes of Syracuse (287 BC – 212 BC), §382.
Bernoulli, Daniel (1700–1782), §332.
Briggs, Henry (1561–1630), §106.
Brouncker, William (1620–1684), §369.
Caesar, Julius (100 BC – 44 BC), §382.
DeMoivre, Abraham (1667–1754), §§62, 211.
Diophantus of Alexandria (c. 200 – c. 284), §51.
Gregory XIII (Pope) (1502–1585), §382.
Leibniz, Gottfried Wilhelm (1646–1716), §140.
Metius, Adriaensz (1571–1635), §382.
Vlacq, Adriaan (1600–1667), §106.
Wallis, John (1616–1703), §§185, 286, 382.

Individuals Who Have Read Euler's Introductio

This is a topic that I think would be very interesting to develop, but I have not even begun.

Abel

Cramer, Gabriel (1704–1752). He claimed that had he know of the *Introductio* earlier he would have used it in this most famous work, *Introduction à l'analyse des lignes courbes algébraique* (1750), but it is unclear how much of the *Introductio* he read.

Lacroix

Peano, Guisseppe. Owned *Introductio* **1797**. http://www.dm.unito.it/biblioteca/Bmpeano.pdf .

References

Boyer 1951: C. B. Boyer, "The Foremost Textbook of Modern Times," *American Mathematical Monthly*, 58 (1951), 223–226.

Cantor 1901: Moritz Cantor (1829-1920), *Vorlesungen über Geschichte der Mathematik*. The third volume deals with the years from 1668 to 1758 and that is what interests us, for there is a chapter by chapter synopsis of the *Introductio* on pp. 699–721 (the second volume of the *Introductio* is treated on pp. 798–818). (I have also seen the reference "See Cantor III, 676–697 and 772–791," but that must be to the first edition of volume three). We are only interested in the third volume of this work, which was published in its second edition in 1901. The whole set of four volumes is Leipzig: Teubner, 1900-1908. Johnson Reprint, 1965. Micro-opaque. New York: Readex Microprint, 1968. 41 cards; 23 x 15 cm. (Landmarks of science).

Ebbinghaus 1991: H.-D. Ebbinghaus et al., *Numbers* with an introduction by K. Lamotke; translated by H.L.S. Orde; edited by J.H. Ewing. New York: Springer-Verlag, c1991.

Hairer and Wanner 1996: E. Hairer and G. Wanner, *Analysis by Its History*, New York, Berlin, Heidelberg: Springer-Verlag, second revised printing, 1997.

Additional References

James G. Simmonds, "A New Look at an Old Function, e," *Coll. Math. J.*, 26 (1995), 6–10. Pólya Prize.

Philip J. Davis, "Leonhard Euler's Integral: An Historical Profile of the Gamma Function," *Amer. Math. Monthly*, 66 (1959), 849–869. Chauvenet Prize.

Raymond Ayoub, "Euler and the zeta function," *Amer. Math. Monthly*, 81 (1974), 1067–1086. Lester R. Ford Award

Henry L. Alder, "Partition identities — from Euler to the present," *Amer. Math. Monthly*, 76 (1969), 733–746. Lester R. Ford Award

L A Golland and R W Golland, "Euler's troublesome series: an early example of the use of trigonometric series," *Historia Math.*, 20 (1993), 54–67.

J. A. van Maanen, "Leonhard Euler (1707-1783): man, worker, migrant, genius" (Dutch), *Nieuw Tijdschr. Wisk.*, 71 (1) (1983/84) 1–11.

A Weil, "Euler," *Amer. Math. Monthly*, 91 (1984), 537–542.

G. Chrystal, *Textbook of Algebra*, Volume 2, Chelsea Publishing (Now published by American Mathematical Society), 1964 (originally published in 1889).

Ivan Niven, "A Proof of the Divergence of [$\sum 1/p$]," *American Mathematical Monthly*, 78 (1971), 272–273.

Gelfond, A. O. "On some characteristic features of the ideas of L. Euler in the field of mathematical analysis and his "Introductio in analysin infinitorum"," (Russian) Uspekhi Matematicheskihk Nauk (N.S.) 12 (1957), no. 4(76), 29–39. MR 19,826h.

Quintero Zazueta, Ricardo, "A rereading of Euler's Introductio in analysin infinitorum." (Spanish. Spanish summary) *Miscelánea Mat. No.*, 28 (1999), 59–67. Listed MR 1 823 258 (2001:10).

Ferraro, Giovanni, "Functions, functional relations, and the laws of continuity in Euler," (English. English, Italian summary) *Historia Math.*, 27 (2000), 107–132. MR 2001b:01010 by Wilfried Schröder

Gutiérrez Dávila, Antonio, "Some methods used by Euler for power series expansions," (Spanish. English, Spanish summary) *Epsilon*, 15 (1999), no. 3(45), 311–338. MR 2001a:01021 quotes the author's summary.

Ferraro, Giovanni, "Some aspects of Euler's theory of series: inexplicable functions and the Euler-Maclaurin summation formula," (English. English, French, Italian summary) *Historia Math.*, 25 (1998), 290–317. MR 2000g:01021.

Laugwitz, Detlef, "Die Weiterentwicklung Leibnizscher Begriffe und Methoden durch Leonhard Euler," (German) [The further development of Leibniz' concepts and methods by Leonhard Euler], *300 Jahre Nova methodus von G. W. Leibniz (1684–1984)* (Noordwijkerhout, 1984), 232–238, *Studia Leibnitiana Sonderheft*, 14, Steiner, Wiesbaden, 1986. MR 90c:01020.

Dhombres, Jean, "Les présupposés d'Euler dans l'emploi de la méthode fonctionnelle," (French. English summary) [Euler's presuppositions about the use of the functional method] *Rev. Histoire Sci.*, 40 (1987), no. 2, 179–202. MR 89m:01044.

Dhombres, Jean, "Un texte d'Euler sur les fonctions continues et les fonctions discontinues, véritable programme d'organisation de l'analyse au 18e siècle," (French) [A text of Euler on continuous and discontinuous functions, a practical program for the organization of analysis in the 18th century] *Cahiers du séminaire d'histoire des mathématiques*, 9, 23–97, Univ. Paris VI, Paris, 1988. MR 89d:01029 by Jesper Lützen.

Lützen, Jesper, "The development of the concept of function from Euler to Dirichlet," (Danish. English summary), *Nordisk Mat. Tidskr.*, 25/26 (1978), no. 1, 5–32, 56. MR 81d:01015.

Laugwitz, Detlef, "Grundbegriffe der Infinitesimalmathematik bei Leonhard Euler," (German) [Fundamental concepts of infinitesimal mathematics in the work of Leonhard Euler], Mathemata, 459–483, Boethius Texte Abh. Gesch. Exakt. Wissensch., XII, Steiner, Wiesbaden, 1985. MR 86k:01024.

J. L. Coolidge, "The Number e," *American Mathematical Monthly*, 57 (1950), 591–602.