NATIONAL ACADEMY OF SCIENCES

GRIFFITH CONRAD EVANS

1887—1973

A Biographical Memoir by
CHARLES B. MORREY

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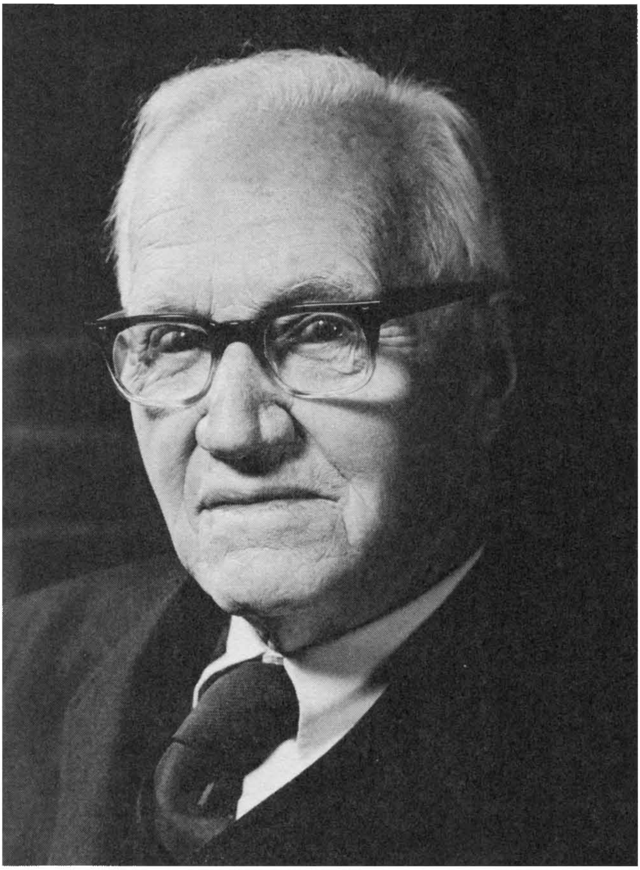

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Griffith C Evans

GRIFFITH CONRAD EVANS

May 11, 1887–December 8, 1973

BY CHARLES B. MORREY

GRIFFITH CONRAD EVANS was born in Boston, Massachusetts on May 11, 1887 and died on December 8, 1973. He received his A.B. degree in 1907, his M.A. in 1908, and his Ph.D. in 1910, all from Harvard University. After receiving his Ph.D., he studied from 1910 through 1912 at the University of Rome on a Sheldon Traveling Fellowship from Harvard. He began his teaching career in 1912 as assistant professor of mathematics at the newly established Rice Institute, now Rice University, in Houston, Texas. He became professor there in 1916 and remained with the Institute until 1934. While he was at Rice, he was able to attract outstanding mathematicians, such as Professor Mandelbrojt of the University of Paris, and young mathematicians, such as Tibor Rado and Carl Menger, to Rice as visiting professors. Long before Evans left Rice it was internationally known as a center of mathematical research.

Evans was brought to the University of California at Berkeley in 1934 as a result of a nationwide search; he arrived with a mandate to build up the Department of Mathematics in the same way that Gilbert Lewis had already built the chemistry faculty. Evans struggled with himself to effect the necessary changes with justice. His innate sense of fairness, modesty, and tact, as well as his stature as a scientist, brought eminent success. By the time he retired in 1954, he had had the satisfaction of seeing the department evolve into one of the country's major centers of mathematical activity. His retirement did not diminish his interest in science nor subtract from his pleasure at seeing others achieve goals he cherished.

A few years before World War II, Professor Evans and others on the Berkeley campus recognized the importance of the fields of probability and statistics, and Professor Jerzy Neyman was brought to that campus by Evans in 1939 to organize the Statistical Laboratory. A period of rapid growth followed; by the close of World War II the Laboratory had transformed Berkeley into one of the three principal centers of probability and statistics in the country. The size and importance of the Laboratory continued to grow, and a separate Department of Statistics was established in 1955.

Shortly after coming to Berkeley, Professor Evans inaugurated a seminar in mathematical economics, which he graciously held in his home once a week. This seminar became internationally known, providing an inspirational educational activity and establishing a tradition of mathematical economics on the Berkeley campus that continues to the present. The seminar was attended by both students and faculty and promoted a friendly atmosphere in the department.

FUNCTIONAL ANALYSIS

In the first decade of the century, while Evans was a student, functional analysis was beginning to attract the interest of the mathematical community. Classical analysis was concerned with functions of real and complex variables, while functional analysis was concerned with functionals, that is, functions of "variables" that may themselves be ordinary functions or other mathematical entities. For example, if f

denotes any ordinary function continuous for $0 \leq x \leq 1$, we may define a functional F by the equation

$$F(f) = \int_0^1 f(x)dx.$$

Evans began his career as a research scientist before he received the Ph.D. degree. He published his first paper in 1909. During the ensuing ten years, he contributed a great deal to the development of the general field of integral equations and more general functional equations. His principal results concerned certain integro-differential equations and integral equations with singular kernels. His interest in this field had been greatly stimulated by his contact with Professor Vito Volterra at the University of Rome. He received early recognition for this important work in 1916 when he was invited to deliver the prestigious Colloquium Lectures before the American Mathematical Society on the subject "Functionals and their Applications" (see bibliography, 1918).

POTENTIAL THEORY IN TWO DIMENSIONS

In 1920 Professor Evans published the first of his famous research papers on potential theory. He was among the first to apply the new general notions of measure and integration to the study of classical problems. In the course of this research, he introduced many ideas and tools that have proven to be of the utmost importance in other branches of mathematics, such as the calculus of variations, partial differential equations, and differential geometry; for example, he used certain classes of functions that are now known as "Sobolev spaces."

Introduction to Potential Theory. The central idea in potential theory is the notion of the potential of a distribution in R_3. Given a distribution of mass g, we define its potential U by the equation

$$(1) \qquad U(M) = \int_W |MP|^{-1} g(P)\, dP$$

$$(W = R_3, M = (x,y,z), P = (\xi,\eta,\zeta))$$

whenever this is defined. In case g is Hoelder continuous[1] for all P and vanishes outside a compact set, then U is of class C^2 and its second derivatives are Hoelder continuous.[2] In this case:

$$(2) \qquad \Delta U(M) \equiv U_{xx}(x,y,z) + U_{yy}(x,y,z) + U_{zz}(x,y,z)$$
$$= -4\pi g(M), \qquad M = (x,y,z).$$

A solution that satisfies (2) with $\Delta U(M) = 0$ on some domain is said to be "harmonic" on that domain. Such a function has derivatives of all orders.

The fundamental problem in potential theory is the Dirichlet problem. Roughly speaking, this consists in proving the existence and uniqueness of the function U that satisfies Laplaces equation on a given domain G, is continuous on $\overline{G}$ (the closure of G), and takes on given continuous boundary values on the boundary ∂G of G.

Another problem, the Neumann problem, is to show the existence (and uniqueness except for an arbitrary additive constant) of a function V that satisfies Laplaces equation on G, is continuously differentiable on $\overline{G}$, and for which the

[1] A function g is Hoelder continuous on a set S if, and only if,

$$|\phi(P) - \phi(Q)| \leq L \cdot |PQ|^{\mu}$$

for some constants L and μ, with $0 < \mu < 1$ and all P and Q are both on S.

[2] See Oliver Dimon Kellogg, *Foundations of Potential Theory* (New York: Dover, 1929), p. 38 or 152, for instance: "This could be called a 'classical result.'"

outer normal derivative $\partial V/\partial n$ takes on given continuous values on ∂G.

The function $u(r,\theta)$, defined by[3]

$$(3)\quad u(r,\theta) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \dfrac{(1-r^2)}{1+r^2-2\quad\cos(\phi-\theta)} f(\phi)d\phi, & r<1 \\ f(\theta), & \textit{if } r=1, \end{cases}$$

is the solution of the Dirichlet problem in the case where G is the unit circular disc in R_2. In case

$$(4)\qquad \int_{\partial G} g(\phi)\, d\phi = 0 \quad\text{and}\quad \int_{\partial G} V(1,\theta)\, d\theta = 0,$$

the solution of the Neumann problem with boundary values $g(\theta)$ on ∂G proceeds as follows. Let

$$(5)\quad v(r,\theta) = -\frac{1}{2\pi}\int_0^{2\pi} \log\left[1+r^2-2r\cos(\phi-\theta)\right] g(\phi)\, d\phi.$$

It is easy to see that rv_r is harmonic on G, and

$$(6)\qquad rv_r(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)}{1+r^2-2r\cos(\phi-\theta)}\, g(\phi)\, d\phi$$

$$-\frac{1}{2\pi}\int_0^{2\pi} g(\phi)\, d\phi.$$

The first term on the right in (6) is the solution of the Dirichlet problem with boundary values $g(\theta)$. If (4) holds, the second term is zero and V is one of the desired solutions.

Among Evans' first results were those concerning the function

[3]This is Poisson's Integral Formula.

$$(7)\quad u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}(1-r^2)\,[1+r^2-2r\cos(\phi-\theta)]^{-1}\,dF(\phi),$$

where $F(\phi)$ is of bounded variation and periodic. Evans proved the following:

- the function $u(r,\theta)$ is harmonic in G, the unit disc in R_2;
- $\int_0^{2\pi} |u(r,\theta)|\, d\theta$ is bounded for $r < 1$;
- $u(r,\theta) = u_1(r,\theta) - u_2(r,\theta)$ each u_i being harmonic and non-negative on ∂G;
- if $P = (1,\phi)$ is a point on ∂G, where $F(\phi)$ is continuous and $F'(\phi)$ exists and $F'(\phi) = f(\phi)$, then $u(r,\theta) \to f(\phi)$ as $(r,\theta) \to (1,\phi)$ "in the wide sense"; i.e., $(r,\theta) \to (1,\phi)$ remaining in any angle with vèrtex at $(1,\phi)$.
- If F and F' are continuous, then (7) reduces to the solution of the Dirichlet problem with continuous boundary values $f(\phi)$.

Conversely, if we assume that $u = u_1 - u_2$ where each $u_i \geq 0$ and is harmonic on G, then u is given by (7).

Early Discussion of the Dirichlet Problem. The first attempt to solve the Dirichlet problem was made by Green in 1828.[4] His method was to show the existence of a Green's function of the form

$$G(Q,P) = \frac{1}{r} + V(Q,P), \qquad r = (P,Q).$$

This function is the Green's function for the region R and the pole P. In terms of this Green's function we have

$$U(P) = -\frac{1}{4\pi}\iint_S U(Q)\,\frac{\partial}{\partial n}G(Q,P)\,dS,$$

[4]See Kellogg, *Foundations of Potential Theory*, p. 38.

where S is the boundary of R. This development is based, however, on the existence and differentiability of $G(Q,P)$, which is obtained using physical considerations and so is not logically suitable for a mathematical derivation.

In 1913 Lebesgue gave an example of the impossibility of the solution of the Dirichlet problem.[5] The region R can be obtained by revolving about the x-axis the area bounded by the curves

$$y = e^{-1/x}, \qquad y = 0 \quad \text{and} \quad x = 1.$$

This type of region is called a Lebesgue spine. It can be shown that the region obtained by revolving about the x-axis the area bounded by the curves

$$y = x^n, \qquad y = 0, \qquad x = 1, \qquad n > 1,$$

is a regular region; i.e., the Dirichlet problem is always solvable.

The Logarithmic Potential Function. A similar theory holds for the two-dimensional situations. One considers the logarithmic potential function in R_2, defined by

(8) $$U(M) = \int_W \left[\log\left(\frac{1}{MP}\right)\right] g(P)\,dP, \qquad M = (x,y), \qquad W = R_2$$

whenever this is defined. If g is Hoelder-continuous for all P and vanishes outside a compact set, then U is of class C^2, and its second derivatives are Hoelder-continuous everywhere. In this case,

(9) $$\Delta U(M) = U_{xx}(x,y) + U_{yy}(x,y) = -2g(M), \qquad M = (x,y).$$

A solution of (9) that satisfies $\Delta U(M) = 0$ on some domain is

[5] *Ibid.*, p. 285, 334.

said to be harmonic on that domain; such a function has derivatives of all orders.

The Dirichlet and Neumann Problems in Space. The solution of the Dirichlet problem in the unit sphere S is given by[6]

$$(10)\quad u(M) = \begin{cases} (4\pi)^{-1} \iint_S (1-r^2)(MP)^{-3} f(P)\, dS, & 0 \le r \le 1, \\ & (r = OM), \quad M = (x,y,z) \\ f(M) & , r = 1 \quad (S = S_1, \quad S_r = \partial B(0,r)). \end{cases}$$

The solution of the Neumann problem with given values $g(M)$ of the normal derivative is obtained as in the case of the unit circle as follows. Let

$$(11)\qquad v(M) = -(4\pi)^{-1} \iint_S (MP)^{-1} g(P)\, dS.$$

Then it is easy to see that rv_r is harmonic and

$$(12)\qquad rv_r(M) = \frac{1}{4\pi} \iint_S (1-r^2)(MP)^{-3} g(P)\, dS - \frac{1}{4\pi} \iint_S g(P)\, dS.$$

The first term on the right is just the solution of the Dirichlet problem with the boundary values $g(M)$. If $\iint_S g(M)\, dS = 0$, then v is a desired function.

Evans and his colleague H. E. Bray proved a necessary and sufficient condition that a function u, harmonic on the

[6]This is Poisson's Integral Formula for three dimensions.

unit ball, be given by the formula

$$(13) \qquad u(M) = (4\pi)^{-1} \iint_S (1 - r^2)(MP)^{-3}\, dG(P)$$

for some distribution $G(e)$ on S, is that

$$\iint_S |u(M)|\, dS \quad \text{be bounded for} \quad 0 \leq r \leq 1,$$

or that $u = u_1 - u_2$ where u_1 and u_2 are non-negative and harmonic on $B(0,1)$. If $F(e)$ is a distribution on S, and if $\lim_{p \to 0} (4\pi\rho^2)^{-1} \cdot |F[B(P,\rho)]| = f(P)$, then $u(M) \to f(P)$ as $M \to P$ in the wide sense (i.e., M remains in a cone with vertex at P).

The Riesz Theorem. A function V is said to be "superharmonic" on a domain Ω if, and only if, (i) it is lower semicontinuous and $\not\equiv +\infty$ on Ω, and (ii) $V|M| \geq$ its mean value over the surface of any sphere with center M that lies with its interior in Ω.

Professor Evans proved that any potential function of a positive mass is superharmonic on any domain on which it is defined. Evans also gave the simplest proof of the following theorem due to F. Riesz:

Suppose u is superharmonic on a domain Ω, and D is any domain, the closure of which is compact and lies in Ω. Then

$$u(M) = U(M) + v(M), M \epsilon D,$$

where U is the potential of a positive mass on D and v is harmonic on D.[7]

[7] F. Riesz, "Sur des fonctions superharmoniques et leur rappaport à la theorie du potential," *Acta Math*, 48 (1926):329–43; 54 (1930):321–60.

Connection with Sobolev Spaces. In addition, Evans proved the following important theorems: Suppose U is superharmonic on a domain Ω. Then $U(x,y,z)$ is absolutely continuous in each variable for almost all pairs of values of the other two and retains this property under one-to-one changes in variables of class C^1.[8]

Finally Professor Evans proved the following theorems: Suppose U is superharmonic on some domain and $U_\rho(M)$ denotes the average of U over the surface $\partial B(M,\rho)$; then $U_\rho(M)$ is continuously differentiable over any domain $\Omega\rho_0$ (which consists of all M such that $B(M,\rho_0) \subset \Omega$) and $\nabla U_0 \rightarrow \nabla U$ in L_2 on any such domain. A necessary and sufficient condition for a potential U of $f(e)$ to have a finite Dirichlet integral is that $\int_W U(M)\, df(e)$ exist. In this case U must belong to the Sobolev space H_1^1 on interior domains. Evans proved many more similar theorems.

A Sequence of Potentials (A Sweeping Out Process). Evans gave a simple proof that the limit of a non-decreasing bounded sequence of potential functions of positive mass each distributed on a fixed bounded closed set F is itself a potential of positive mass F. The limit of a non-increasing sequence of such functions, however, is not necessarily superharmonic (since the limit of a non-increasing sequence of lower-semicontinuous functions is not necessarily lower-semicontinuous).

Nevertheless, Evans showed how to associate a particular type of positive mass distribution with a particular type of non-increasing sequence of potential functions on a bounded, closed set F. To do this, Evans let $U_1, U_2, \ldots,$ be a non-increasing sequence of potentials of positive mass distributions $f_1, f_2, \ldots,$ respectively on F. Let U_0 be the limit

[8] See bibliography entries of 1935 for the three-dimensional case and those in 1920 for the two-dimensional case.

function. Clearly $U_0(M) \geq 0$ but is not necessarily superharmonic, although it is harmonic on T where T is the infinite domain lying in the complement of F, whose boundary $t \subset F$. The f_i are uniformly bounded. Hence there is a subsequence $\{i_n\}$ such that $\{f_{i_n}\}$ converges weakly to a positive mass function f on F (or a subset of F). Thus,

$$\lim_{n \to \infty} \int_W \phi(M)\, df_{i_n}(e_p) = \int_W \phi(M)\, df(e_p) \quad (W = 3 \text{ space})$$

for every bounded continuous function ϕ. Also, f is independent of the subsequence. Let U be the potential of f, then $f(e) = 0$ for all Borel sets $e \subset T$. Thus we may associate the positive mass function f with the non-increasing sequence $U_1, U_2, \ldots$, all the mass having been swept out of T. Since $f(e) = 0$ for all $e \subset T$, U must satisfy the Laplace's equation on T.

Professor Evans discovered a great variety of similar sweeping out processes.[9] He applied this type of process to sweep out a unit mass at a point Q in a domain T containing Q. This led to a number of interesting results and to a formula for the Green's function for T with pole at Q.

Capacity. The notion of the capacity of a set arises in the applications and was used by Evans and was developed at some length in the second part of his paper "Potentials of Positive Mass."[10] Evans also defined the idea of a regular boundary point. It turns out that a boundary point Q of a domain Σ is regular if, and only if, a barrier $V(M,Q)$ can be constructed at Q. Such a function $V(M,Q)$ is continuous and superharmonic in Σ, which approaches Σ at Q, and has a positive lower bound in Σ outside any sphere with center at

[9] See *Transactions of the American Mathematical Society*, 38 (1935):205–13.
[10] *Ibid.*, 218–26.

Q. If every boundary point is regular then the Dirichlet problem is solvable.

Multiple Valued Harmonic Functions in Space. In 1896[11] Sommerfelt developed a method of using multivalued harmonic functions in three space to solve certain problems in potential theory, particularly the diffraction problem for a straight line. In 1900[12] Hobson used a combination of double-valued harmonic functions to obtain the conductor potential for a circular disc.

Evans showed that if s is a simple closed curve of 0 capacity (any curve with a continuously turning tangent has capacity 0), there exists a unique surface S bounded by s that has a minimum capacity among all such surfaces. If the part of S outside a neighborhood of s is composed of a finite number of sufficiently smooth pieces and $V(M)$ is the conductor potential for S, then Evans showed that V must satisfy

$$\frac{\partial V(Q)}{\partial n^+} = \frac{\partial V(Q)}{\partial n^-}, \qquad Q \subset S$$

where n^+ and n^- are the normals to S at Q. Moreover, if this holds on a smooth part of S, then S is analytic on that part. The proof of this involves "double valued" functions, the tract by Evans, "Lectures on Multiple-Valued Harmonic Functions in Space" (see bibliography, 1951), presents an extensive systematic development of a part of the theory of such functions.

A simple example of a multiple space of a type used by Evans is the double space H, which consists of all ordered

[11] Mathematische Theorie der Diffraction, *Math Annalen*, 47(1896):317–74 and Über verzweigte Potential wie Rauma, *Proceedings of the London Mathematical Society*, 28(1897):395–429.

[12] E. W. Hobson, "On Green's Function for a Circular Disc," *Cambridge Philosophical Transactions*, 18(1900):277–91.

pairs (M,m) where $m = 0$ or $+1$ or -1, and

$$\text{if } m = -1, \text{ then } M \in R_3 - s; \quad \text{if } m = +1, \text{ then } M \in R_3 - s;$$
$$\text{if } m = 0, \text{ then } M = s, \quad s = \{(x,y,z): x^2 + y^2 = 1, \text{and } z = 0\}.$$

Geometrically, we may think of H as consisting of two infinite, flat, rigid, 3-dimensional sheets of the form $R_3 - s$ joined together along s.

Such a space is a three-dimensional analog of a Riemann surface in the complex plane, and Evans' result led him into his extensive research on multiple-valued functions, his principal interest during his later years.

Since any multiple space is, by definition, a topological space, open, closed, and connected subsets of such spaces are defined and the usual theorems hold. Also harmonic, superharmonic, and subharmonic functions can be defined on domains $T \subset$ multiple spaces. Much of the existence and uniqueness theory for harmonic and potential functions is carried over by Evans to the case of multiple-valued functions, that is, single valued functions defined on multiple spaces. For example, Evans showed that there is a unique harmonic function that takes on given continuous boundary values at all regular points. Moreover the definitions and theorems about barriers carry over.

But there are many new results for infinite domains (on multiple spaces). For example, Evans proved that there is a unique function $\lambda(M)$ bounded and harmonic in $T_1 \cup T_2 \cup \ldots \cup T_n$ that takes on the values 1 at infinity on the leaf T_1 and approaches 0 at infinity on the other leaves. ($T_i = T \cap H_i$ where H_i is the i-th leaf of H.) Let T be a bounded domain $\subset H$, a particular space, and let A be a fixed point in T. Evans showed that there exists a unique Green's function with pole at A that has the following properties:

As a function of M:

(i) $\gamma(A,M)$ is harmonic in T except at A.

(ii) $\gamma(A,M)$ is bounded except in a neighborhood of A and $\gamma(A,M) - 1/AM$ remains bounded near A.
(iii) $\gamma(A,M)$ vanishes at all regular points of the boundary t of T.

In addition, $\gamma(B,A) = \gamma(A,B)$ for A and B where A and $B \epsilon T$. There is a unique function $K(A,T)$ with the same properties, with A and M on $A = T_1 \cup \ldots \cup T_n$. Also, $K(A,M) \rightarrow 0$ as $M \rightarrow \infty$ on any leaf. Finally Evans proved that there is a unique surface of minimum capacity that spans a given space curve s or a set of space curves $\{s\}$. It is the locus of the equation $\lambda(M) = 1/2$ where H is taken as a two-leaved space and the s_i are chosen as branch curves in H.

Finally, a version of Green's theorem holds for domains T on multiple spaces whose boundary consists of several branch curves $s_1, \ldots, s_r$ of zero capacity and several smooth surfaces.[13]

MATHEMATICAL ECONOMICS

Evans' work in mathematical economics was that of a pioneer. At a time when most economists in this country disdained to consider mathematical treatments of economic questions, he boldly formulated several mathematical models of the total economy in terms of a few variables and drew conclusions about these variables. Some of these expositions were based on the theories of Cournot (1837) and some are found in the book *Mathematical Introduction to Economics* by Evans.

The simplest theory is the following: It is envisaged that there is only one commodity being manufactured by one producer, and one consumer. The cost of manufacturing and marketing u units of the commodity per unit time is $q(u)$; this

[13] For a full discussion, see Griffith Conrad Evans, "Multiply Valued Harmonic Functions. Green's Theorem," *Proceedings of the National Academy of Sciences of the United States of America*, 33(1947):270–75.

is called the cost function. The consumer will buy y units of the commodity (per unit time) if the price is p per unit; thus $y = \phi(p)$ is the demand function. The market is in equilibrium if $y = u$, that is, if all the commodity is sold. Clearly the profit π made by the producer is given by

$$\pi = pu - q(u) = p\phi(p) - q[\phi(p)].$$

The producer is a monopolist if he can sell all he produces at any given price. In this case it would be reasonable to assume that the producer would set the price to maximize his profits. This leads to the equation:

$$\frac{d}{dp}\{p\,\phi(p) - q\,[\phi(p)]\} = \frac{d\pi}{dp} = 0.$$

In order to get a solution, we must know the functions $q(u)$ and $\phi(p)$. The simplified form for $q(u)$ is $Au^2 + Bu + C$. C represents the overhead and should be > 0. The average cost per unit is

$$\frac{q(u)}{u} = Au + B + \frac{c}{u},$$

which may reasonably increase ultimately, so that $A > 0$. The "marginal unit cost" is

$$\frac{dq}{du} = 2Au + B.$$

If dq/du is ≥ 0 for $u \geq 0$, we must have $B \geq 0$; we may as well assume $B > 0$. Clearly $\phi(p)$ is *decreasing* and positive; the simplest form for $\phi(p)$ is $ap + b$ where $a < 0$ and $b > 0$. If the market is in equilibrium, we have

$$y = u = ap + b \qquad p = \frac{u-b}{a} \quad \text{or}$$

$$\pi = u \cdot \frac{u-b}{a} - Au^2 - Bu - C.$$

In order to maximize π, one must have $d\pi/du = 0$. This yields

$$\frac{2u}{a} - \frac{b}{a} - 2Au - B = 0$$

$$u = \frac{b + ba}{2 - 2Aa}, \qquad p = \frac{Ba + 2Aab - b}{2a(1 - Aa)}.$$

Of course a monopolist may choose u (or p) to satisfy some other condition.

As a second theory, Evans assumes that there are two producers manufacturing amounts u_1 and u_2 of the commodity (per unit time). Let us assume that the producers are subject to the same cost function $q(u_i) = A_i^2 + Bu_i + C$ and there is produced only what is sold; that is, the market is in equilibrium. If we assume the same demand function,

$$(1) \qquad y = u_1 + u_2 = ap + b,$$

then the selling values are pu_i and the profits are

$$\pi_i = pu_i - (Au_i^2 + Bu_i + C), \qquad i = 1,2.$$

Additional hypotheses are needed to find p and then u_i. Suppose each producer tries to determine u_i so as to maximize the total profit, still assuming equilibrium. In this case, we say that the producers are cooperating. Then the total profit $\pi = \pi_1 + \pi_2$ is

$$\pi = p(u_1 + u_2) - A(u_1^2 + u_2^2) - B(u_1 + u_2) - 2C,$$

$$\pi = \frac{u_1 + u_2 - b}{a}(u_1 + u_2) - A(u_1^2 + u_2^2) - B(u_1 + u_2) - 2C$$

using (1) to determine p. Assume u_1 and u_2 are chosen to maximize π. Then we must have

$$\frac{\partial \pi}{\partial u_1} = \frac{\partial \pi}{\partial u_2} = 0$$

or

$$\frac{2(u_1 + u_2) - b}{a} - 2Au_1 - B = 0,$$
$$\frac{2(u_1 + u_2) - b}{a} - 2Au_2 - B = 0,$$

which determine u_1 and u_2 uniquely.

As a third theory, let us suppose that a producer is subject to the same cost and demand functions but has no control over the price. Then he will choose u to maximize

$$\pi = pu - Au^2 - Bu - C$$

for the given price; this yields $d\pi/du = p - 2Au - B$ to determine

$$u = \frac{p - B}{2A}.$$

This theory can be generalized to the case where there are n producers, each subject to a different cost and demand function, but who all set the same price p. Then the i-th producer produces u_i units where the total profit is

$$\pi = \sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} (pu_i - A_i u_i^2 - B_i u_i - C_i).$$

This will be a maximum if

$$\frac{\partial \pi}{\partial u_i} = 0 \quad \text{or} \quad p - 2A_i u_i - B_i u_i, \qquad i = 1, \ldots, n,$$

which yields

$$u_i = \frac{p - B_i}{2A_i}, \qquad i = 1, \ldots, n.$$

Similar problems can be solved in some cases where the u_i and p depend on time. Evans solved such a problem in which there is one producer with

$$q = Au^2 + Bu + C, \quad y = ap + b + hp', \quad A > 0, \quad B > 0, \quad C > 0$$

$a < 0, b > 0, h > 0, p' = dp/dt$, and A, B, C, a, b, h

are all constants.

The term hp' is suggested by the consideration that the demand is greater when the price is going up than when the price is going down, other things being equal. This problem required more sophisticated mathematics. It is still assumed that $u = y$ for all t and that the rate of profit is $\pi = pu - q(u)$, and finally that the profit made during the interval (t_0, t_1) is

$$\pi = \int_{t_0}^{t_1} \pi(p,p')\, dt$$

is a maximum over any interval (t_0, t_1). This leads to the condition that the integral

$$(2) \quad \int_{t_0}^{t_1} \{p(ap + b + hp') - A(ap + b + hp')^2 - B(ap + b + hp') - C\}\, dt$$

is a maximum of any interval, where $\pi(p,p')$ is given by the integrand in (2). This is a standard problem in the calculus of variations.[14] From that theory we conclude that the Euler equation

$$(3) \quad \frac{d}{dt}(\pi_{p'}) = \pi_p$$

[14]See any book on the calculus of variations, for example, G. A. Bliss, *Calculus of Variations* (Washington, D.C.: Mathematical Association of America, 1925).

holds. Carrying out the differentiation with respect to t in (3), we get for Euler's equation

(4) $$\pi_{p'p'}\frac{d^2p}{dt^2}+\pi_{p'p}\frac{dp}{dt}=\pi_p,$$

where

(5) $\pi_{'p'}$ means $\dfrac{\partial^2\pi}{(\partial p')^2}$, $\pi_{p'p}$ means $\dfrac{\partial^2\pi}{\partial p'\partial p}$, π_p means $\dfrac{\partial\pi}{\partial p}$, etc.

$\pi(p,p')=p(ap+b+hp')-A(ap+b+hp')^2-B(ap+b+hp')-C$ and the derivatives in (5) are the indicated partial derivatives regarding p and p' as independent variables. Carrying out the differentiations in (5), we get

(6) $$\pi_{p'p'}=-2Ah^2,\ \pi_{p'p}=h(1-2aA),$$

$$\pi_p=a(2p-B)+hp'(1-2aA)+b(1-2aA)-3a^2Ap.$$

Setting $dp/dt=p'$ in (4) and using (5) and (6), Euler's equation becomes

$$-2Ah^2p''+h(1-2aA)p'=a(2p-B)+hp'(1-2aA)$$
$$+b(1-2aA)-2a^2Ap$$
$$<=> -2Ah^2p''=2a(1-aA)p+b(1-2aA)-aB$$
$$<=> p''=\frac{2a(1-aA)}{-2Ah^2}p+\frac{b(1-2aA)-aB}{-2Ah^2},\ \text{i.e., } p''$$
$$=M^2p-N^2,$$

which is reduced to the form

$$\frac{d^2p}{dt^2}=f(p).$$

This is solvable by standard methods in differential equations.

This is one of the simplest cases. More sophisticated theories involving such things as taxes, tariffs, rent, rates of change, transfer of credit, the theory of interest, utility, theories of production, and problems in economic dynamics were worked out by Evans.

Evans' scientific career resulted in over seventy substantial published articles, four books, and several classified reports. It should be added, since it is such a rare occurrence among mathematicians, that he continued his productive work for many years after his retirement. He gave a number of invited addresses in Italy and elsewhere during that period.

Professor Evans was elected to the National Academy of Sciences in 1933 and became a member of the American Academy of Arts and Sciences, the American Philosophical Society, the American Mathematical Society (vice president, 1924–26; president, 1938–40); the Mathematical Association of America (vice president, 1934), and the American Association for the Advancement of Science. He was a fellow of the Econometric Society.

Evans was invited to give addresses in connection with the Harvard Tercentenary and the Princeton Bicentennial Celebration. He was also asked to give the Roosevelt Lecture at Harvard in 1949 and was Faculty Research Lecturer in Berkeley in 1950 and was awarded an honorary degree by the University in 1956. The Griffith C. Evans Hall on the Berkeley campus was dedicated in 1971.

During World War I, Evans served as a captain in the Signal Corps of the U.S. Army. During World War II, he was a member of the Executive Board of the Applied Mathematics Panel and was part-time technical consultant, Ordnance, with the War Department. He received the Distinguished Assistance Award from the War Department in 1946 and received a Presidential Certificate of Merit in 1948.

The charming hospitality of the Evanses is remembered with pleasure by those fortunate enough to have been guests at their home. And Evans' own keen, dry sense of humor was much appreciated by his many friends and associates.

Professor Evans married Isabel Mary John in 1917. They had three children, Griffith C. Evans, Jr., George William Evans, and Robert John Evans and many grandchildren.

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