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On asymptotics of the solution of the moving oscillator problem

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Abstract

Asymptotic behavior of the solution of the moving oscillator problem is examined for large and small values of the spring stiffness for the general case of non-zero beam initial conditions. In the limiting case of infinite spring stiffness, it is shown that the moving oscillator problem for a simply supported beam is not equivalent, in a strict sense, to the moving mass problem. The two problems are shown to be equivalent in terms of the beam displacements but are not equivalent in terms of stresses (the higher order derivatives of the two solutions differ). In the general case, the force acting on the beam from the oscillator is shown to contain a high-frequency component , which does not vanish and can even grow when the spring coefficient tends to infinity. The magnitude of this force and its dependence on the oscillator parameters can be estimated by considering the asymptotics of the solution for the initial stage of the oscillator motion. It is shown that, for the case of a simply supported beam, the magnitude of the high-frequency force depends linearly on the oscillator eigenfrequency and velocity. The deficiency of the moving mass model is principally that it fails to predict stresses in the supporting structure. For small values of the spring stiffness, the moving oscillator problem is shown to be equivalent to the moving force problem. The discussion is amply illustrated by results of numerical experiments.

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1. Introduction

The calculation of the dynamic response of a distributed parameter system carrying one or more travelling subsystems is very important in many engineering applications related, for example, to the analysis and design of highway and railway bridges, cable-railways, and the like. Two simple models of moving subsystems are generally accepted in studies of this subject, where the emphasis is placed on the dynamics of the distributed parameter system rather than on that of the moving subsystem: *moving mass* and *moving oscillator* models. The difference between the two models is that the stiffness of the coupling between the moving subsystem and the continuum in the former model is assumed infinite. In the following, the problem of the vibration of the distributed parameter system due to the moving mass or oscillator will be referred to as the moving mass or moving oscillator problem, respectively. If the velocity of the moving subsystem is low, its inertia can be neglected, and one arrives at the *moving force* problem, which is considerably simpler than either of these two problems. There is a large body of literature devoted to all three problems, and a number of methods for solving them have been developed during the past several decades. We refer the interested reader to the ample lists of references [1,2], as well to those in other works cited throughout this paper. Some discussion of papers on the moving mass problem related to the main subject discussed in this study is given in Section 4.

The purpose of this study is to examine the asymptotic behavior of the solution of the moving oscillator problem for large and small values of the spring stiffness and to establish the relationship between the moving oscillator problem and the other two. The emphasis is put on the asymptotics for large values of the spring stiffness, and our reasons for carefully examining this case are as follows.

It is commonly accepted that the moving oscillator problem, in the limit of infinite coupling stiffness, is equivalent to the moving mass problem (see, e.g. Refs. [3,4]). One can also find statements in some papers to the effect that their authors used large values of the spring stiffness in their numerical experiments, thus modeling the moving mass problem. At first glance, the assumption about the equivalence of the two problems seems to be valid taking into account the fact that the amplitude of the oscillator vibration vanishes when the spring stiffness goes to infinity. It is also substantiated by numerous results of numerical experiments presented in the literature, which show the convergence of the response solution of the moving oscillator problem as the spring stiffness grows. To the authors' best knowledge, however, the validity of this assertion has never been proved in the literature but was taken for granted.

On the other hand, when modeling the multiple moving oscillator problem, one can observe that the force acting on the beam from the second oscillator is very different from that of the first. Fig. 1 illustrates this; it shows the time history of the forces acting on the unit dimensionless simply supported beam with zero initial conditions traversed by two identical high-frequency oscillators of unit weight moving with the velocity $v = \pi/2$ (more details about this example, as well as about the subsequent illustrations, are given in Section 6). The second oscillator enters the left end of the beam at the moment when the first oscillator leaves the beam. Both oscillators are assumed to have zero initial conditions at the moment when they enter the beam. Thus, the problem is decomposed into two problems of one moving oscillator (in the intervals $[0,2/\pi]$ and $[2/\pi, 4/\pi]$, respectively). The only difference between these two problems is that the beam initial conditions in the second problem are non-zero. The difference in the two forces is easily seen:

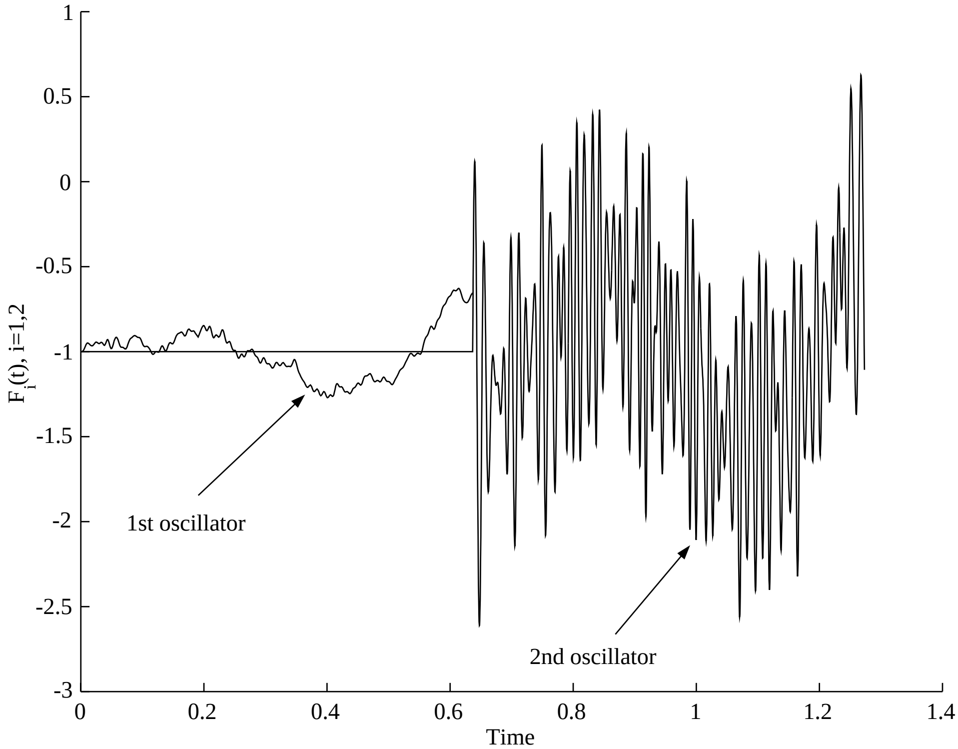

Fig. 1. Forces acting on the SS beam from two oscillators traversing it one after another.

whereas the force from the first oscillator may be associated with the inertia of the moving mass, the force from the second oscillator contains a high-frequency component of large magnitude, which is not typical of a moving mass solution. This phenomenon implies that the solution to the problem considered cannot adequately be approximated by any kind of moving mass solution.

This was our motivation to more closely examine the effect of non-zero beam initial conditions on the moving oscillator solution and to compare it with the corresponding moving mass solution. In Section 3, the problem of equivalence of the moving mass and moving oscillator solutions is examined, and the explanation of the behavior of the curve depicted in Fig. 1 is given. It will be shown that the moving oscillator solution does not tend to the corresponding moving mass solution in a strict sense and that the force acting on the beam from the oscillator contains a high-frequency component which not only does not vanish but also can grow when the spring stiffness increases. In spite of this, the two solutions are still equivalent in terms of the beam displacement (which is further referred to as *weak equivalence*).

The examination of the equivalence problem brought us to the conclusion concerning *deficiency of the moving mass model*, which is discussed in Section 4. While the moving mass model can be used to accurately approximate the displacement of a long bridge due to a real vehicle with a stiff suspension, it fails to predict stresses. The issue of deficiency of the moving mass solution comes into play when the beam initial conditions are allowed to be non-zero and originates from the fact that the moving mass problem statement is physically incorrect in this case. Note that, in most publications on the moving mass problem, the initial conditions for the beam either are not discussed at all or are assumed to be zero. Even when the governing equations are written for arbitrary initial conditions, the discussions are usually reduced to zero initial conditions by means of the ubiquitous phrase "without loss of generality." Indeed, in many cases, the presumption of zero initial conditions simplifies calculations and causes no loss of generality. However, this expedient does not work in the case of the moving mass problem, which is an idealization obtained by assuming infinitely large stiffness of the coupling between the subsystems.

The asymptotics of the moving oscillator solution for small values of the spring stiffness are discussed in Section 5.

2. Problem statement

The vibration of a uniform beam traversed by an oscillator of mass m_0 attached to the beam through a spring of stiffness k moving with a constant velocity v is governed by the equations

$$\rho\frac{\partial^2}{\partial t^2}w + EI\frac{\partial^2}{\partial x^4}w = -(m_0 g + k(w(vt,t) - z(t)))\delta(x - vt), \tag{1}$$

$$m_0\ddot{z} = k(w(vt,t) - z(t)), \quad 0 \leqslant x \leqslant L, \quad 0 \leqslant t \leqslant L/v. \tag{2}$$

subject to given boundary and initial conditions, where $z(t)$ is the absolute displacement of the lumped mass. The beam ends are assumed to be fixed (simply supported or clamped).

The equations governing the moving mass and moving force solutions, $w_{mm}(x,t)$ and $w_{mf}(x,t)$, are well known to be

$$\rho\frac{\partial^2}{\partial t^2}w_{mm} + EI\frac{\partial^2}{\partial x^2}w_{mm} = -\left[m_0 g + m_0\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)^2 w_{mm}(x,t)\,|_{x=vt}\right]\delta(x - vt), \tag{3}$$

and

$$\rho\frac{\partial}{\partial t^2}w_{mf} + EI\frac{\partial}{\partial x^4}w_{mf} = -m_0 g\delta(x - vt), \tag{4}$$

respectively, subject to given boundary and initial conditions.

The basic purpose of this study is to examine the asymptotics of the solution of Eqs. (1) and (2) for large values of the spring stiffness k, to find out what new phenomena are associated with non-zero beam initial conditions, and to establish the conditions under which the moving mass and moving oscillator problems are equivalent in the limit of infinite spring stiffness.

We will also formally prove an intuitively clear fact that, if the spring stiffness tends to zero, the solution of the moving oscillator problem (1), (2) reduces to that of the moving force Eq. (4).

3. Asymptotics for large values of spring stiffness

3.1. Equation for the concentrated elastic force

Let us rewrite Eq. (2) in the form

$$\ddot{z} + \omega_0^2 z = \omega_0^2 w(vt, t), \tag{5}$$

where $\omega_0 = \sqrt{k/m_0}$ is the eigenfrequency of the oscillator. The solution to Eq. (5) is the sum of the solution to the homogeneous Eq. (5) satisfying given initial conditions and the particular solution satisfying zero initial conditions and is given by

$$z(t) = z(0)\cos\omega_0 \mathrm{t} + \frac{1}{\omega_0}\dot{z}(0)\sin\omega_0 t + \omega_0 \int_0^t w(v\tau, \tau)\sin\omega_0(t-\tau)\,\mathrm{d}\tau. \tag{6}$$

Let ω_0 be large, and let beam initial conditions be arbitrary. Taking the integral on the right-hand of Eq. (6) by parts four times with regard to the condition $w(0,0) = 0$ and dropping the terms of order less than $1/\omega_0^2$, we get

$$\begin{aligned} z(t) = {} & z(0)\cos\omega_0 t + \frac{1}{\omega_0}\dot{z}(0)\sin\omega_0 t \\ & + w(vt,t) - \frac{1}{\omega_0}\dot{w}(0,0)\sin\omega_0 t - \frac{1}{\omega_0^2}\ddot{w}(vt,t) + \frac{1}{\omega_0^2}\ddot{w}(0,0)\cos\omega_0 t + \mathrm{o}\left(\frac{1}{\omega_0^2}\right), \end{aligned} \tag{7}$$

where $\dot{w}(vt, t)$ and $\dot{w}(vt, t)$ are the convective derivatives,

$$\dot{w}(vt,t) \equiv \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) w(x,t)|_{x=v_t}, \quad \ddot{w}(vt,t) \equiv \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)^2 w(x,t)|_{x=vt}.$$

Note that, for $t = 0$, we have

$$\dot{w}(0,0) = vw_x(0,0), \ddot{w}(0,0) = v^2 w_{xx}(0,0) + 2vw_{xt}(0,0), \tag{8}$$

where w_x, w_{xx} and w_{xt} are the partial derivatives.

It is evident from physical considerations that the oscillator initial conditions cannot be taken as arbitrary but, rather, must be consistent with large spring stiffness. Under the assumption that the oscillator initial conditions are due to external forces of finite magnitude acting on the oscillator before it enters the beam, it can easily be shown that we should confine our consideration to the case where

$$z(0) = \mathrm{O}\left(\frac{1}{\omega_0^2}\right), \quad \dot{z}(0) = \mathrm{O}\left(\frac{1}{\omega_0}\right). \tag{9}$$

Then, it follows immediately from Eqs. (7) and (9) that the relative oscillator displacement $z(t) - w(vt, t)$ vanishes when the spring stiffness goes to infinity.

Further, multiplying both sides of Eq. (7) by the spring stiffness k, we find the elastic force of interaction between the beam and oscillator,

$$\begin{aligned} f(t) \equiv -k(w(vt,t) - z(t)) = {} & -m_0\omega_0(\dot{w}(0,0) - \dot{z}(0))\sin\omega_0 \mathrm{t} \\ & + m_0(\ddot{w}(0,0) + \omega_0^2 z(0))\cos w_0 \mathrm{t} - m_0\ddot{w}(vt,t) + \mathrm{o}(1). \end{aligned} \tag{10}$$

As can be seen, in the general case of oscillator and beam initial conditions, the elastic interaction force contains two harmonic components with frequency ω_0 due to the eigenvibration of the oscillator, which do not vanish when k goes to infinity. Moreover, the amplitude of the first of them grows infinitely as the spring stiffness tends to infinity. This means that the *moving mass and moving oscillator problems are generally not equivalent* in the limit of infinite spring stiffness since the force on the right-hand side of Eq. (1) does not tend to that of Eq. (3).

The oscillating character of the interaction force in the moving oscillator problem implies, in particular, that the picture of the shear force distribution changes rapidly in time, especially in the vicinity of the moving oscillator attachment point, since the concentrated force acting on the beam is equal to the jump in the shear force at that point. This phenomenon is illustrated in Fig. 2, which shows the shear force distributions at three close instants, $t = 0.8$, 0.805, and 0.81, for the example discussed above (see explanations to Fig. 1). It demonstrates that the jump in the shear force distribution changes considerably in a short time interval, which may imply that the beam is subject to (high-frequency) damaging stresses.

3.2. Conditions for the equivalence of the moving oscillator and moving mass problems

Since the operators on the left sides of Eqs. (1) and (3) are exactly the same, the solution of the moving oscillator problem, together with all its derivatives, tends to that of the moving mass problem if the right-hand of Eq. (1) tends to that of Eq. (3) as $\omega_0 \to \infty$. It follows from Eq. (10)

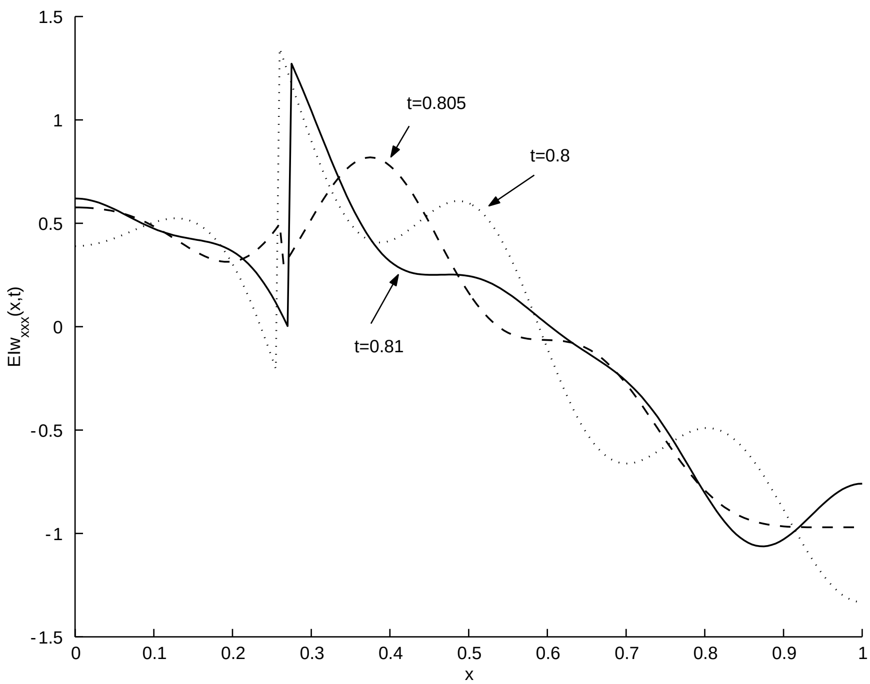

Fig. 2. Shear force distributions at three close instants in the two-oscillator problem (the first oscillator is outside the beam): $t = 0.8$ (· · · · ·), $t = 0.805$(- - - - -), and $t = 0.81$ (——).

that the latter condition holds only if oscillator initial conditions satisfy the relations

$$z(0) = -\frac{\ddot{w}(0,0)}{\omega_0^2}, \quad \dot{z}(0) = \dot{w}(0,0) \equiv v w_x(0,0). \tag{11}$$

The second condition of Eq. (11) implies that the velocity of the oscillator at $t = 0$ is directed along the tangent line to the beam at $x = 0$, and the first condition of Eq. (11) implies that the spring is prestressed to make the force acting on the beam at $t = 0$ equal to the initial inertia force inherent in the moving mass problem.

In the case of zero beam initial conditions, Eq. (11) are satisfied by taking zero oscillator initial conditions. Now, let the beam initial conditions be non-zero. If the left end of the beam is simply supported, the initial slope is generally a certain finite number not depending on the oscillator eigenfrequency, and the second condition of Eq. (11) cannot be satisfied since, by virtue of Eq. (9), $\dot{z}(0)$ must vanish when $\omega_0 \to \infty$. This implies that the moving oscillator problem in the limit of infinite spring stiffness *is not equivalent in the strict sense* to the moving mass problem. In the case of the clamped left end, conditions (11) are fulfilled if

$$z(0) = -\frac{v^2}{\omega_0^2} w_{xx}(0,0), \quad \dot{z}(0) = 0. \tag{12}$$

Thus, we may conclude that, except for the case of zero initial conditions, there is no strict equivalence between the two problems in the limit of infinite spring stiffness if the left end of the beam is simply supported. If the left end of the beam is clamped, it is always possible to choose oscillator initial conditions such that the moving oscillator problem is strictly equivalent to the moving mass problem in the limit of infinite spring stiffness. If the oscillator initial conditions do not match the beam initial conditions as described by Eq. (12), the two problems are not equivalent.

Remark 1. Consider a simply supported beam with non-zero initial conditions. When solving the moving oscillator problem numerically for a set of increasing finite values of the oscillator eigenfrequency, we can always force the oscillator initial conditions to satisfy conditions (11), which implies that the force on the right-hand side of Eq. (1) tends to that of Eq. (3) and, accordingly, the corresponding solutions tend to the moving mass solution. It may seem that this reasoning contradicts the above conclusion about non-equivalence of the two problems. The contradiction is explained if we note that the requirement of the fixed value of the oscillator velocity $\dot{z}(0)$ implicitly suggests infinitely growing forces acting on the oscillator before it enters the beam. Hence, the limit procedure cannot correspond to any physically correct process. The other side of this conclusion is that the moving mass problem statement is physically incorrect in the case of non-zero beam initial conditions. This is further discussed in Section 4.

The above analysis shows that high-frequency oscillations of the elastic (dynamic) force appear when a stiff oscillator enters an already vibrating beam. In the general case, the oscillator initial conditions cannot be adjusted to satisfy Eq. (11), and we need to examine the effect of the high-frequency component of the dynamic force on the beam vibration. We will show that the beam displacement is not sensitive to the oscillator initial conditions as long as they are consistent with large spring stiffness (i.e., satisfy Eq. (9)), and that, in spite of the "stress" nonequivalence, the two problems are still equivalent in terms of the beam displacements (i.e., the response of the beam due to the moving oscillator tends to that due to the moving mass as $k \to \infty$). We will call this *weak equivalence*.

3.3. Weak equivalence of the moving mass and moving oscillator problems

Let the oscillator initial conditions not satisfy Eq. (11). Eq. (10) can be written in the form

$$f(t) = -c_1 m_0 \omega_0 \sin \omega_0 t + c_2 m_0 \cos \omega_0 t - m_o \ddot{w}(vt, t) + \mathrm{o}(1), \tag{13}$$

where c_1 and c_2 are determined by the beam and oscillator initial conditions and do not depend on ω_0. Note that it may seem, from Eq. (10), that c_2 also depends on the oscillator eigenfrequency, but, by virtue of condition (9), it does not (if ω_0 increases, the initial oscillator displacement inevitably vanishes). Let us represent the solution to Eq. (1) in the form

$$w(x, t) = w_{mm}(x, t) + \tilde{w}(x, t), \tag{14}$$

where $w_{mm}(x, t)$ is the solution to the corresponding moving mass problem (3) satisfying the given initial conditions

$$w_{mm}(x, 0) = w(x, 0), \quad \frac{\partial}{\partial t} w_{mm}(x, t)\,|_{\,t=0} = \frac{\partial}{\partial t} w(x, t)|_{\,t=0},$$

and

$$\tilde{w}(x, 0) = \frac{\partial}{\partial t}\tilde{w}(x, t)\,|_{\,t=0} = 0.$$

Substituting Eq. (14) into Eq. (1) and taking Eq. (13) into account, we get

$$\begin{aligned} \rho\frac{\partial^2}{\partial t^2} w_{mm} + EI\frac{\partial^4}{\partial x^4} w_{mm} + \rho\frac{\partial^2}{\partial t^2}\tilde{w} + EI\frac{\partial^4}{\partial x^4}\tilde{w} &= -(m_0 g + m_0 \ddot{w}_{mm}(vt, t) \\ &+ c_1 m_0 \omega_0 \sin \omega_0 t - c_2 m_0 \cos \omega_0 t + m_0 \ddot{\tilde{w}}(vt, t) + \mathrm{o}(1))\delta(x - vt). \end{aligned} \tag{15}$$

Comparing the right sides of Eqs. (3) and (15), we find that $\tilde{w}(x, t)$ satisfies the equation

$$\rho\frac{\partial^2}{\partial t^2}\tilde{w} + EI\frac{\partial^4}{\partial x^4}\tilde{w} = -(m_0 \ddot{\tilde{w}}(vt, t) + c_1 m_0 \omega_0 \sin \omega_0 t - c_2 m_0 \cos \omega_0 t + \mathrm{o}(1))\delta(x - vt)$$

subject to the given boundary conditions and zero initial conditions. Dropping the small order term and rewriting the last equation in the form

$$\begin{aligned} &\rho\frac{\partial^2}{\partial t^2}\tilde{w} + EI\frac{\partial^4}{\partial x^4}\tilde{w} + m_0 \ddot{\tilde{w}}(vt, t)\delta(x - vt) \\ &= -(c_1 m_0 \omega_0 \sin \omega_0 t - c_2 m_0 \cos \omega_0 t)\delta(x - vt), \end{aligned} \tag{16}$$

we find that it governs the vibration of the beam with the rigidly attached weightless mass m_0 that moves along the beam with the velocity v (or, in other words, the beam with the mass distribution given by $\rho + m_0\delta(x - vt)$) excited by the moving harmonic force.

It is evident that the second harmonic force on the right-hand side of Eq. (16) can be dropped for sufficiently large values of ω_0 (it remains constant when $\omega_0 \to \infty$, whereas the first force increases infinitely). Thus, for simplicity of the notation, it is sufficient to prove that the solution to the equation

$$\rho\frac{\partial^2}{\partial t^2}\tilde{w} + EI\frac{\partial^4}{\partial x^4}\tilde{w} + m_0\ddot{\tilde{w}}(vt,t)\delta(x-vt) = (c_1 m_0\omega_0 \sin(\omega_0 t)\delta(x-vt), \tag{17}$$

satisfying the given boundary and zero initial conditions tends to zero when $\omega_0 \to \infty$. The solution to Eq. (17) is given by

$$\begin{aligned}\tilde{w}(x,t) &= \int_0^t \int_0^L g(x,\xi;t-\tau)c_1 m_0\omega_0 \sin(\omega_0\tau)\delta(\xi - v\tau)\,\mathrm{d}\xi\mathrm{d}\tau \\ &= c_1 m_0\omega_0 \int_0^t g(x,v\tau;t-\tau)\sin\omega_0\tau\,\mathrm{d}\tau,\end{aligned}$$

where $g(x,\xi;t)$ is the dynamic Green's function of the system governed by the left-hand side of Eq. (17) with regard to the boundary conditions. Although its closed-form representation is not available (and is unlikely to be found), it is sufficient for our purposes that such a function exists (which follows from physical considerations). Taking the last integral by parts twice, we get

$$\begin{aligned}\tilde{w}(x,t) =& -c_1 m_0 \int_0^t g(x,v\tau;t-\tau)\mathrm{d}\cos w_0\tau = -c_1 m_0 g(x,v\tau;t-\tau)\cos\omega_0\tau|_{\tau=0}^t \\ &+ \frac{c_1 m_0}{\omega_0}\frac{\mathrm{d}}{\mathrm{d}\tau}g(x,v\tau;t-\tau)\sin\omega_0\tau|_{\tau=0}^t - \frac{c_1 m_0}{\omega_0}\int_0^t \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}g(x,v\tau;t-\tau)\sin\omega_0\tau\,\mathrm{d}\tau.\end{aligned}$$

The first term on the right-hand of the equation vanishes since $g(x,0;t)=0$ (fixed left end) and $g(x,\xi;0)=0$ (the deflection of the system at $t=0$ due to the unit impulse applied at $t=0$ is zero), and we finally arrive at the equation

$$\begin{aligned}\tilde{w}(x,t) =& \frac{c_1 m_0}{\omega_0}\frac{\mathrm{d}}{\mathrm{d}\tau}g(x,v\tau;t-\tau)\,|_{\tau=t}\sin\omega_0 \mathrm{t} \\ &- \frac{c_1 m_0}{\omega_0}\int_0^t \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}g(x,v\tau;t-\tau)\sin\omega_0\tau\,\mathrm{d}\tau.\end{aligned} \tag{18}$$

It follows from the last equation that, in view of the finiteness of the first and second partial derivatives of the Green's function, the solution to Eq. (17) tends to zero as $\omega_0 \to \infty$,

$$\tilde{w}(x,t) = \mathrm{O}\left(\frac{1}{\omega_0}\right),$$

which proves the weak equivalence of the moving mass and moving oscillator problems in the limit of infinite spring stiffness.

From a physical standpoint, the small effect of the additional harmonic forces on the beam vibration is explained by the fact that these forces have high frequency and excite only high order eigenvibrations of the beam, the contribution of which in the beam response is negligible. Moreover, no resonance phenomena can take place because of the finiteness of the passage time and since the time-varying system governed by the left-hand of Eq. (17) has no fixed resonance frequencies.

Remark 2. The incorporation of damping into the oscillator model complicates all calculations and makes the analysis more involved. It is for this reason that we consider here the undamped oscillator. It can be shown, however, that all basic findings of this study remain valid for the damped case. The elastic force in the damped case is described by an equation similar to Eq. (10), with the functions $\sin \omega_0 t$ and $\cos \omega_0 t$ being replaced by the $e^{-\alpha t} \sin \sqrt{\omega_0^2 - \alpha^2}$ and $e^{-at} \cos \sqrt{\omega_0^2 - \alpha^2}$, respectively, where α is the damping coefficient. This implies, in particular, that the conclusion about nonequivalence of the moving mass and moving oscillator problems for the general case of initial conditions remains valid. The coefficients of the latter functions are more complicated functions of the initial conditions and the spring and damper coefficients. It can be proved, however, that these coefficients vanish when the oscillator initial conditions satisfy the same Eq. (11).

Although the high-frequency component of the dynamic interaction force does not result in any increase of the beam displacement, the magnitude of this force (or, to be more specific, the magnitude of the total dynamic force) is of great significance, in particular to pavement wear [5]. The magnitude of the additional high-frequency force is determined by the oscillator parameters and the beam and oscillator initial conditions and may be considerable. Thus, it is important to establish the dependence of this force on the oscillator parameters and initial conditions and to find a priori estimates of the peak values of the concentrated force acting on the beam. In the next section, we will show that the magnitude of the dynamic force $f(t)$ can be estimated by considering its asymptotics for small values of time.

3.4. Asymptotics of the dynamic force at the initial stage of oscillator motion

As was shown above, when a stiff oscillator enters the already vibrating beam, there appears a harmonic high-frequency component in the elastic interaction force with frequency ω_0 and the accompanying force associated with the high-frequency low-amplitude vibration $\tilde{w}(x,t)$ of the beam excited by the former force. Although, as was shown in Section 3.3, $\tilde{w}(x,t)$ tends to zero as $\omega_0 \to \infty$, the force $m_0\ddot{\tilde{w}}(x,t)$ associated with this vibration does not (note that this is clearly seen in Fig. 2). These forces are added to the force associated with the inertia of the moving mass such that the resulting dynamic force acting on the beam from the oscillator can be represented as

$$f(t) = -m_0\ddot{w}_{mm}(vt,t) + f_{ad}(t). \tag{19}$$

By virtue of Eqs. (10) and (14), the equation for the additional force is given by

$$f_{ad}(t) = -m_0\omega_0 c_1 \sin \omega_0 t + m_0 c_2 \cos \omega_0 t - m_0\ddot{\tilde{w}}(vt,t), \tag{20}$$

where c_1 and c_2 are determined by the oscillator and beam initial conditions and do not depend on ω_0 (see also the remark after Eq. (13)).

If $c_1 \neq 0$ and ω_0 is sufficiently large, the amplitude of the harmonic force can be very well approximated by the first term only. The inertia term $m_0\ddot{w}(vt,t)$ associated with the moving mass solution can also be neglected for large ω_0. The last term in Eq. (20) is a rather complicated function and can only be accurately calculated numerically. As can be seen from Eq. (18), its magnitude depends linearly on the amplitude of the exciting harmonic force (and, hence, on ω_0), and, thus, it generally cannot be neglected. Note, however, that, due to the damping inherent in any real problem, the high-frequency vibrations of the beam and oscillator decrease rapidly. Thus, from a practical standpoint, it is sufficient to be able to estimate the peak values of the dynamic force at the initial stage of the oscillator motion. For small t such that $vt \ll L$ (the oscillator is close to the left end of the beam), the third term on the right-hand of Eq. (20) can be dropped. This can formally be proved if we take into account that the harmonic force with frequency ω_0 excites mainly the beam eigenvibrations at the eigenfrequencies ω_n close to ω_0 and that the wave numbers λ_n are proportional to the square roots of the eigenfrequencies, $\sqrt{\omega_n}$. Then it follows that the maximum magnitude of the dynamic force $f(t)$ can be very well approximated by that for small t which is given by

$$\max_t |f(t)| \approx m_0\omega_0 v\left(w_x(0,0) - \frac{\dot{z}(0)}{v}\right), \quad vt \ll L. \tag{21}$$

Note that the expression in parentheses is the difference between the initial slope of the beam at $x = 0$ and the direction of the oscillator velocity. As can be seen from Eq. (21), the magnitude of the dynamic force is proportional to the oscillator eigenfrequency and velocity.

If c_1 (the difference between the initial beam slope and the direction of the oscillator velocity) is small or the spring stiffness is not sufficiently large, the estimate can be improved by taking the second term on the right-hand of Eq. (20) into account. Moreover, we can also take the first term in Eq. (19) into account by noting that, for $vt \ll L$, it can be considered constant, $\ddot{w}_{mm}(vt,t) \approx \ddot{w}(0,0)$. Thus, for a simply supported beam and zero oscillator initial conditions, we get the approximate formula for the concentrated force acting on the beam for $vt \ll L$,

$$F(t) \approx -m_0 g - m_0\omega_0 v w_x(0,0)\sin\omega_0 t - 2m_0 v w_{xt}(0,0)(1 - \cos\omega_0 t). \tag{22}$$

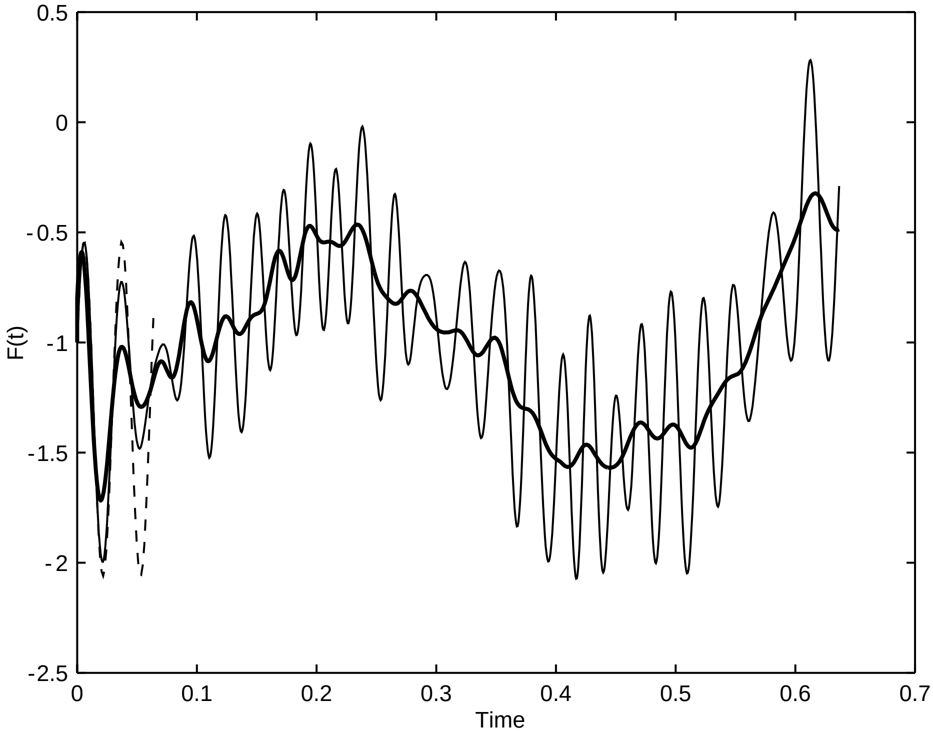

Fig. 3. Dynamic force $F(t)$ acting on the SS beam from the undamped (——) and damped (▬▬) oscillators with $\omega_0 = 200$ and $v = \pi/2$, and its approximation (- - - - -) by Eq. (22).

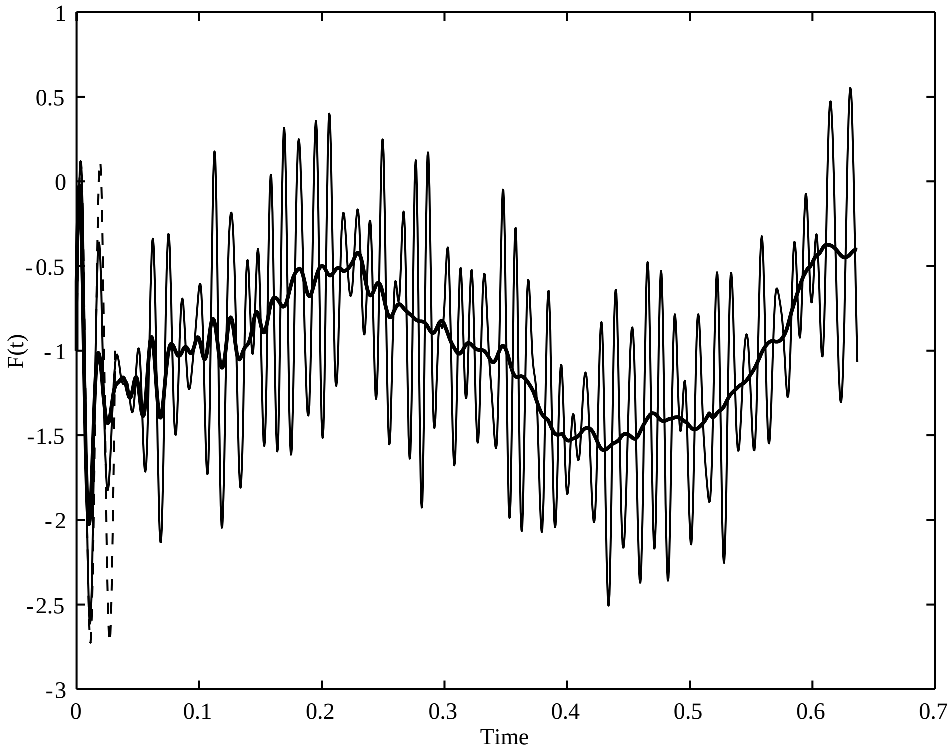

Fig. 4. Dynamic force $F(t)$ acting on the SS beam from the undamped (——) and damped (▬▬) oscillators with $\omega_0 = 400$ and $v = \pi/2$, and its approximation (-----) by Eq. (22).

Figs. 3–5 illustrate this. The concentrated forces $F(t)$ acting on the unit, simply supported beam from the undamped oscillators are depicted by the thin solid lines. Figs. 3 ($\omega_0 = 200$, $v = \pi/2$) and 4 ($\omega_0 = 400$, $v = \pi/2$) demonstrate that the magnitude of the elastic force grows with the increase of the oscillator eigenfrequency. Figs. 4 and 5 ($\omega = 400$, $v = \pi$) show that it is a linear function of the oscillator velocity. Approximations of the concentrated force for small values of time by means of Eq. (22) are shown in Figs. 3–5 by the dashed lines. The bold lines show the forces acting on the beam in the case of the damped oscillators with the damping coefficients $c_0 = 6$ (Fig. 3) and $c_0 = 12$ (Figs. 4 and 5) (for both oscillators, damping is about 15%of critical). These figures clearly demonstrate that (1) the high-frequency oscillations of the elastic force reduce rapidly and (2) the approximate Eq. (22) can be used to adequately estimate upper bounds of the peak values of the elastic force. If c_1 is zero, the dynamic force does not depend on ω_0 in this case, the inertia force cannot generally be neglected, and estimates (21) and (22) are not applicable.

4. On the deficiency of the moving mass model

The moving mass model is an idealization of the moving oscillator model obtained by assuming infinitely large stiffness of coupling between the subsystems. However, as was discussed in

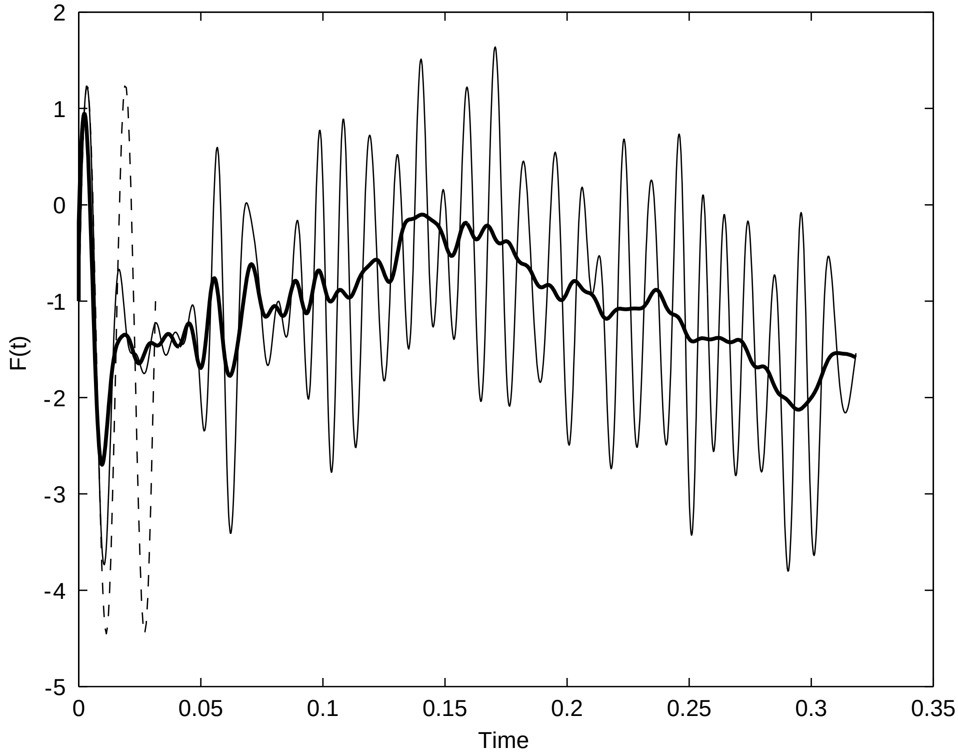

Fig. 5. Dynamic force $F(t)$ acting on the SS beam from the undamped (——) and damped (▬▬) oscillators with $\omega_0 = 400$ and $v = \pi$, and its approximation (- - - - -) by Eq. (22).

Remark 1, if the beam initial conditions are non-zero, it cannot be obtained from the moving oscillator model without assuming infinitely large forces acting on the mass. This implies that the moving mass model is *physically incorrect* if initial conditions for a simply supported beam are allowed to be non-zero.

As shown in Section 3.3, the use of the moving mass model is still justified when we need to calculate the beam displacement. The "cost" of this model incorrectness is that it fails to accurately calculate the concentrated force acting on the beam from the vehicle and, thus, to predict stresses in the beam. Indeed, when a vehicle with a stiff suspension enters an already vibrating bridge, its initial conditions are generally not "in agreement" with those of the bridge (i.e., do not satisfy Eq. (11)). As shown in Section 3, this results in the appearance of a high-frequency component in the dynamic interaction force. In certain circumstances, the magnitude of this force may be considerable and exceed that of the inertia force associated with the moving vehicle. Thus, neglecting this force, we are not able to accurately calculate stresses in the bridge. When using the moving mass model, the high-frequency component of the force is missing, which implies the *deficiency of the moving mass model* in that it fails to predict stresses in the bridge.

The deficiency of the moving mass model becomes even more evident when it is applied to solving the problem of several vehicles traversing a bridge represented by a simply supported beam. Assuming that a "rigid" vehicle approaches the beam moving along the rigid horizontal surface, we see that the vertical velocity of the vehicle at $x = -0$ is zero. When the vehicle enters the beam, we *must* admit that, in the framework of the moving mass model, its vertical velocity at $x = +0$ is $vw_x(0,0)$, which implies that the velocity has been instantly changed, which, in turn, suggests infinite force acting on the mass (beam).

In view of the above, it is not surprising that the multiple moving mass problem has been nearly neglected in the literature, and the problem of stress calculation has not been discussed at all. Moreover, in most publications on the moving mass problem, only zero initial conditions for the beam are considered (e.g., Refs. [6–8]). In [9], the governing equation is written for the case of several moving masses, and it is stated that the method is applicable to arbitrary initial conditions; however, by means of the *universal expedient* "without loss of generality", the analysis is reduced to one moving mass and zero initial conditions, and no numerical results related to several masses are presented. Moreover, the right-hand side of the governing equation in that paper suggests that all masses enter the beam at the same moment. In Ref. [10], all masses are also assumed to enter the beam at the same moment and the initial conditions are zero. Lee [11] examines the case of high velocities of the moving mass and considers the effect of the separation between the mass and beam. At the instant of recontact in his problem, the situation is the same as at the moment when a moving mass enters the already vibrating beam in the problem considered in this paper. For simplicity, the impact effect in that paper is neglected, and the concentrated force is assumed to have a jump at the instant of recontact. Since the paper examines only the beam displacement rather than stresses in the beam, such an approach seems to be justified in view of the analysis given in Section 3.3 of this paper. In Refs. [12–14], results of numerical experiments with several moving masses are presented; however, no discussions are given concerning what happens when a mass enters the already vibrating beam, the initial conditions are not presented, and the problem of shear stress calculation is not discussed.

5. Asymptotics for small values of spring stiffness

In order to prove that the solution of the moving oscillator problem tends to that of the moving force problem as the spring stiffness goes to zero, it is sufficient to show that the elastic force on the right-hand side of (1) vanishes, since, in this case, the right-hand side of (1) tends to that of Eq. (4). Introduce the notation $w_{max} = \max_{0<t<Tp}|w(vt,t)|$, were $T_p = L/v$ is the passage time. Let k be small such that $\omega_0 T_p \ll 1$. Consider first the case of zero oscillator initial conditions. By using Eq. (6), it can be easily shown that the maximum oscillator displacement satisfies the inequality

$$|z(t)| \leqslant w_{max}\frac{(\omega_0 T_p)^2}{2} = \mathrm{o}(1).$$

Then, the elastic force $f(t)$ tends to zero as $k \to 0$,

$$\max_{0<t<T_p} |f(t)| \approx k w_{\max} \to 0.$$

If the oscillator initial conditions are non-zero, the oscillator displacement can be represented as

$$z(t) = z(0) + \dot{z}(0)t + \mathrm{o}(1).$$

In this case,

$$\max_{0<t<T_p} |f(t)| \leqslant kw_{\max} + k|z(0)| + k|\dot{z}(0)|T_p \to 0.$$

Thus, we proved that for any fixed values of the oscillator initial conditions, the maximum magnitude of the elastic force acting on the beam vanishes as the spring stiffness goes to zero, which proves the equivalence of the two problems.

Remark 3. It should be noted that the governing equations for the moving oscillator problem are sometimes written in a form different from Eqs. (1) and (2). The point is that the weight of the oscillator can be included either in Eq. (1) or in Eq. (2) depending on whether or not the initial sag in the spring due to the oscillator weight is taken into account. For example, in Ref. [1], the oscillator weight is assumed to be applied to the oscillator (the term $m_0 g$ is on the right-hand of Eq. (2)). Both formulations are mathematically equivalent and are obtained from each other by changing the variable $z(t)$. However, if the oscillator initial conditions are assumed to be the same (e.g., zero) in both formulations, the solutions for small values of the spring stiffness may considerably differ from each other. The conclusion drawn in this section about the equivalence of the moving force and moving oscillator problems in the limit of zero spring stiffness is valid only if the governing equation is written in form (1), (2) (i.e., if the initial sag in the spring due to the oscillator weight is taken into account).

6. Numerical examples

In our numerical experiments, we employed the unit dimensionless simply supported beam ($\rho = EI = L = 1$). The fundamental frequency of the beam is $\omega_1 = \pi^2$. The non-dimensionalization procedure is standard (see, e.g., reference [1]) and is not repeated here. The dimensionless oscillator velocity is given in terms of π in order to compare it with the "critical" velocity $v = \pi$ inherent in the moving force problem [15]. The oscillator weight was taken equal to one; i.e., $m_0 = 1/g \approx 0.1$, where g is the acceleration of gravity. All figures show the resulting concentrated force $F(t) = -m_0 g + f(t)$ acting on the beam from the oscillator.

The results depicted in Figs. 1 and 2 (the two moving oscillators problem) correspond to zero beam initial conditions and oscillator parameters $\omega_0 = 400$ and $v = \pi/2$. In all other experiments, the beam initial conditions were non-zero and remained the same to make it possible to examine the effect of variation of the oscillator parameters on the magnitude of the elastic force. To avoid the danger of choosing "unrealistic" initial conditions and the question of what initial functions should be considered as "appropriate," we did the following: First, we numerically solved the one moving oscillator problem for the beam with zero initial conditions and oscillator parameters $\omega_0 = 400$ and $v = \pi/2$. The functions $w(x,t)$ and $w_t(x,t)$ were calculated at the moment when the oscillator was at the right end of the beam and were then taken as the initial functions $w(x,0)$ and $w_t(x,0)$ for all following experiments.

To numerically solve the moving oscillator problem, the method described in Ref. [16] (one oscillator)and [2] (extension to the multiple moving oscillators problem) was used, which is based on the expansion of the solution in the series in terms of the beam eigenfunctions. The number of terms used in the series for all calculations was equal to eight, which is sufficient to make the results reliable (note that the maximum oscillator eigenfrequency considered, $\omega_0 = 400$, is less than the seventh beam eigenfrequency $\omega_7 \approx 484$).

The conventional series expansion is known to converge poorly when applied to calculation of the higher order derivatives of the response. To accurately calculate the shear force distributions depicted in Fig. 2, an improved series expansion suggested in Ref. [17] was employed, which makes use of the beam static Green's function and gives an exact value of the shear force jump.

7. Conclusions

Asymptotic behavior of the solution of the moving oscillator problem has been examined for large and small values of the spring stiffness for the general case of non-zero beam initial conditions.

1. It has been shown that, in the case of a simply supported beam with non-zero initial conditions, the moving oscillator problem is not mathematically equivalent to the moving mass problem in the limit of infinite spring stiffness. In the case of a clamped beam, the two problems are equivalent only under appropriate choice of the oscillator initial conditions. Nevertheless, when the spring stiffness goes to infinity, the beam displacement obtained by solving the moving oscillator problem tends to that due to the moving mass whatever the oscillator initial conditions consistent with the large spring stiffness. Thus, for sufficiently large spring stiffness, the beam displacement is a function of the beam initial conditions and the oscillator mass and velocity but is not sensitive to the spring coefficient and to the oscillator initial conditions. The two solutions differ by their higher-order derivatives and by the dynamic force acting on the beam from the mass.
2. The magnitude of the high-frequency component in the concentrated force has been shown to depend linearly on the oscillator eigenfrequency and velocity if the vector of the oscillator velocity is not directed along the tangent line to the beam at its left end. Asymptotic formulas (21) and (22) for the concentrated force acting on a simply supported beam at the initial stage of the oscillator motion have been derived, which provide a priori estimates for the maximum magnitude of the dynamic force acting on the beam for the damped case. Note that the high-frequency component of the dynamic force appears not only at the moment when the oscillator enters the beam but also any time the oscillator passes a point where the function describing the "road profile" is not smooth (the first derivative has a jump).
3. The existence of the high-frequency component of the interaction force in the moving oscillator problem results in a rapidly changing "picture of stresses" in the vicinity of the oscillator attachment point (Fig. 2). This effect may considerably affect pavement wear. If so, it follows from the results obtained that a vehicle with a softer suspension is more road-friendly than one with a stiffer suspension. This result agrees well with the conclusion made in Ref. [5, p.128] that "pavement profile deteriorates more rapidly under a steel suspension than under an air suspension carrying the same load". The report [5] also concludes that "the concentration of dynamic loads for air suspensions is only about half the magnitude of that for steel suspensions". Taking into account that the average eigenfrequency of steel suspensions

considered in that report is about twice as high as that of air suspensions, this observation perfectly agrees with the fact of linear dependence of the dynamic force on the oscillator eigenfrequency obtained in this study.
4. The adequacy of the moving mass model for modeling real vehicles and its physical incorrectness when applied to the case of a simply supported beam with nonzero initial conditions have been discussed. The deficiency of the model has been noted in that it fails to predict stresses in the bridge structure.
5. For small values of the spring stiffness, the solution of the moving oscillator problem has been shown to tend to that of the moving force problem.

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