Oblique Shock Waves

In reality normal shock waves don't often occur. Oblique shock waves are more common. We would like to know what causes them, and how we can calculate flow properties around them.

1 Shock Wave Angles

When an aircraft is flying, it creates disturbances in the flow. These disturbance spread around with the speed of sound a. Figure 1 visualizes these disturbances for an airplane traveling from point A to point B.

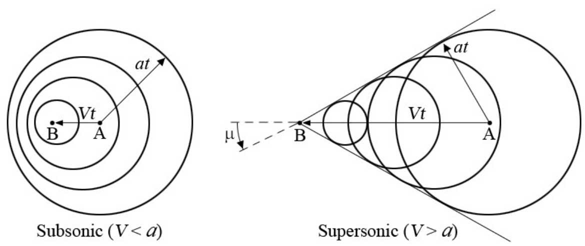

Figure 1: Visualization of the disturbances in a flow.

When the airplane flies at a subsonic velocity ($V < a$), the disturbances can move upstream. If the airplane, however, flies at a supersonic speed ($V > a$), the disturbances can not. In fact, they all stay within a cone and stack up at the edge, forming a so-called **Mach wave**. This cone has an angle μ, where μ is called the **Mach angle**. From figure 1 it can be derived that

$$\sin \mu = \frac{at}{Vt} = \frac{a}{V} = \frac{1}{M}. \tag{1.1}$$

The above relation is, however, only theoretical. In practice the shock wave doesn't have an angle μ but an angle β, called the **wave angle**. For shock waves we always have $\beta > \mu$. Finally there is the special case with $\beta = 90°$, at which we once more have a normal shock wave. So a normal shock wave is just a special case of the oblique shock wave.

2 Oblique Shock Wave Relations

We will try to derive some relations for oblique shock waves. But before we can do that, we need to make some definitions.

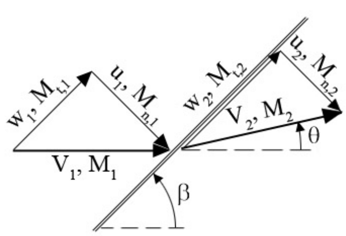

Figure 2: Properties of the oblique shock wave.

We know that the velocity V_1 before the shock wave is directed horizontally. We examine two components of this velocity: The component normal to the shock wave u_1 and the component tangential to the shock wave w_1. Corresponding are the Mach number normal to the shock wave $M_{n,1}$ and the Mach number tangential to the shock wave $M_{t,1}$. We can do the same for the velocities after the shock wave (but now with subscript 2). All the properties have been visualized in figure 2. Also note the **deflection angle** θ.

Using the variables described above, we can derive some relations. It turns out that these relations are virtually the same as for a normal shock wave. There's only one fundamental difference. Instead of using the total velocity, we only need to consider the component of the velocity normal to the shock wave (being u). We then get

$$\rho_1 u_1 = \rho_2 u_2, \tag{2.1}$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2, \tag{2.2}$$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}. \tag{2.3}$$

But what about the tangential component of the velocity w? Well, using the momentum equation, we can derive the simple relation

$$w_1 = w_2. \tag{2.4}$$

So now we have used the continuity equation, the momentum equation and the energy equation. In the previous chapter we now continued to express ratios like p_2/p_1 as a function of the Mach number. We can do the same again. However, this time we express everything in the component of the Mach number normal to the flow, being

$$M_{n,1} = M_1 \sin\beta. \tag{2.5}$$

Going through a lot of derivations, we can find that

$$M_{n,2}^2 = \frac{2 + (\gamma - 1) M_{n,1}^2}{2\gamma M_{n,1}^2 - (\gamma - 1)} \qquad \text{with} \qquad M_2 = \frac{M_{n,2}}{\sin(\beta - \theta)}, \tag{2.6}$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1) M_{n,1}^2}{2 + (\gamma - 1) M_{n,1}^2}, \tag{2.7}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_{n,1}^2 - 1\right), \tag{2.8}$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} = \left(1 + \frac{2\gamma}{\gamma + 1}\left(M_{n,1}^2 - 1\right)\right)\left(\frac{2 + (\gamma - 1) M_{n,1}^2}{(\gamma + 1) M_{n,1}^2}\right). \tag{2.9}$$

We can once more see that these equations are virtually the same as for a normal shock wave. The only difference is that we now need to take the component of the Mach number normal to the flow.

3 The Deflection Angle

There is one last variable for which we can derive an equation. That variable is the deflection angle θ. This angle is usually determined by the shape of the object causing the shock waves. We can find that

$$\tan\theta = \frac{2}{\tan\beta}\frac{M_1^2 \sin^2\beta - 1}{M_1^2 (\gamma + \cos(2\beta)) + 2}. \tag{3.1}$$

This equation is called the **θ-β-M relation**. Many important things can be derived from it.

Let's suppose we know θ and M_1. We can then find the corresponding values of the wave angle β. For relatively low values of θ you will find two solutions for β. There are thus two possible shock waves. The shock wave with the higher angle of β is called the strong shock wave, while the one with the lower angle is called the weak shock wave. In nature, the weak shock wave is almost always present. So usually the smallest of the two solutions can simply be used.

But what happens if θ gets bigger? Soon θ will reach a maximum value θ_{max}. At this point there is only one solution for β. If θ gets even bigger, no solutions exist anymore. So an oblique shock wave is not possible then. Instead, the shock wave will detach and get a curved shape. We will briefly examine the detached shock wave later in this chapter.

4 Multiple Shock Waves

There are many cases in which multiple shock waves occur. We will examine a few. First let's consider a single shock wave with wave angle β_1, colliding with a wall parallel to the free stream. What happens to this shock wave?

To answer this question, we look at the flow after the shock wave. This flow has been deflected towards the wall by an angle θ. Since the flow can't go through the wall, it needs to be deflected the other way, by the same angle θ. To accomplish this, there will be a new shock wave.

You may initially think that this new shock wave has the same wave angle β_1. This is, however, not true. To see why, we need to look at the Mach numbers. Before the first shock wave, the flow had a Mach number M_1. After the first shock wave (and before the second), the flow has a lower Mach number M_2. By combining this new Mach number with the deflection angle θ, the new wave angle β_2 can be found. So the second shock wave will have a wave angle β_2.

Now let's look at another situation: the case where two shock waves A and B intersect each other. At the point of intersection, two new shock waves C and D will appear, each with different wave angles β_C and β_D. What information can we use to determine these wave angles?

Experiments have shown that, after the two new waves, the flows from both waves travel in the same direction. In between these two flows is the so-called **slip line**. This line is called the slip line, because the two flows "slip" with respect to each other – they usually have a different velocity.

So the final directions of both flows are the same. However, because there is a straight line between the two flows, their pressures must be equal as well. So $p_{C,2} = p_{D,2}$. Using these two boundary conditions the wave angles β_C and β_D can be determined.

5 The Detached Shock Wave

If we put a rather blunt body in a supersonic flow, we won't get an (attached) oblique shock wave. Instead, we will get a **detached shock wave**. The properties of this shock wave vary along the shock wave. At the front of the shock wave, the wave angle β is $90°$. So we have a normal shock wave there. Behind this shock wave, the flow is subsonic.

As we go further from the shock wave, the wave angle β decreases. As β decreases, the deflection angle θ initially increases. It soon reaches its maximum, after which it once more starts to decrease.

Not much after we reached θ_{max}, we find the **sonic line**. At this line the Mach number of the flow behind the shock wave is $M_2 = 1$. As we continue our travel along the shock wave, the shock wave loses strength. It's not longer able to slow down the flow to subsonic velocities. So the Mach number behind the shock wave M_2 will be above 1.

As we go even further away from our blunt body, the shock wave will continue to lose strength. Eventually, when $\theta = 0$ again, its strength will have disappeared entirely.

Performing calculations on a blunt shock wave is very difficult. It is therefore not part of this course.

6 Expansion Waves

Suppose we have an airflow moving along a wall, which suddenly makes an angle θ away from the flow. We then get an **expansion wave**. In this expansion wave, the airflow "bends" around the wall edge. While the airflow changes direction, its velocity also changes. This happens according to

$$d\theta = \sqrt{M^2 - 1}\frac{dV}{V}. \tag{6.1}$$

Now how can we find the Mach number after the expansion wave? For that, we first have to rewrite dV/V to

$$\frac{dV}{V} = \frac{2}{2 + (\gamma - 1) M^2}\frac{dM}{M}. \tag{6.2}$$

Using this, we can find that θ is equal to

$$\theta = \int_{M_1}^{M_2} \frac{2\sqrt{M^2 - 1}}{2 + (\gamma - 1) M^2}\frac{dM}{M}. \tag{6.3}$$

The integral is kind of complex, but it can be solved. Because of its importance, it has gotten its own symbol and name. This integral is named the **Prandtl-Meyer function** $\nu(M)$, defined as

$$\nu(M) = \int \frac{2\sqrt{M^2 - 1}}{2 + (\gamma - 1) M^2}\frac{dM}{M} = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \arctan\left(\sqrt{\frac{\gamma - 1}{\gamma + 1}\left(M^2 - 1\right)}\right) - \arctan\left(\sqrt{M^2 - 1}\right). \tag{6.4}$$

Using this function, we can derive an expression for θ, being

$$\theta = \nu(M_2) - \nu(M_1). \tag{6.5}$$

However, we usually don't need to calculate θ. Usually we know θ and M_1 and we need to know M_2. How do we find M_2 then? Well, we first use M_1 to find $\nu(M_1)$. We then add this result up to θ to find $\nu(M_2)$. From this we can derive M_2 (often using tables). In general we can say that $M_2 > M_1$.

How do the various flow properties behave during expansion waves? It can be shown that the flow is isentropic, so the entropy s stays constant. Therefore also T_t and p_t stay constant. From this we can derive that

$$\frac{T_2}{T_1} = \frac{2 + (\gamma - 1) M_1^2}{2 + (\gamma - 1) M_2^2}, \tag{6.6}$$

$$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\gamma - 1}} = \left(\frac{2 + (\gamma - 1) M_1^2}{2 + (\gamma - 1) M_2^2}\right)^{\frac{1}{\gamma - 1}}, \tag{6.7}$$

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma - 1}} = \left(\frac{2 + (\gamma - 1) M_1^2}{2 + (\gamma - 1) M_2^2}\right)^{\frac{\gamma}{\gamma - 1}}. \tag{6.8}$$

In a shock wave the pressure, density and temperature increase. In an expansion wave it is exactly opposite: they all decrease.