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Pioneers of representation theory, by Charles W. Curtis, Amer. Math. Soc., Providence, RI, 1999, xvi + 287 pp., $49.00, ISBN 0-8218-9002-6

The theory of linear representations of finite groups emerged in a series of papers by Frobenius appearing in 1896–97. This was at first couched in the language of characters but soon evolved into the formulation now considered standard, in which characters give the traces of representing linear transformations. There were of course antecedents in the number-theoretic work of Lagrange, Gauss, and others—especially Dedekind, whose correspondence with Frobenius suggested a way to move from characters of abelian groups to characters of arbitrary finite groups.

In the past century this theory has developed in many interesting directions. Besides being a natural tool in the study of the structure of finite groups, it has turned up in many branches of mathematics and has found extensive applications in chemistry and physics. Marking the end of the first century of the subject, the book under review offers a somewhat unusual blend of history, biography, and mathematical exposition.

Before discussing the book itself, it may be worthwhile to pose a general question: Does one need to know anything about the history of mathematics (or the lives of individual mathematicians) in order to appreciate the subject matter? Most of us are complacent about quoting the usual sloppy misattributions of famous theorems, even if we are finicky about the details of proofs.

There seems to be a recent trend in undergraduate textbooks (especially in subjects like abstract algebra and number theory) to include snippets of history and biography. This is certainly a harmless way to add human interest to what might otherwise seem dry axiomatics, but may not by itself make the subject matter more understandable. It is much easier to convey the facts of Emmy Noether's life than to explain to undergraduates what she accomplished mathematically.

Aside from the human interest involved in biographical studies, there may be some intellectual value in retracing the way mathematical ideas have developed. This development is often messy, however. Occasionally good ideas emerge prematurely in obscure places and are forgotten for a time, only to be rediscovered independently. Sometimes the original motivation for an investigation looks a bit eccentric to later generations, as in the case of Hamilton's approach to quaternions. But, in the end, one is often just curious to know where the currently accepted ideas came from.

Whatever one's view may be on the role of the history of mathematics in teaching or research, probably most people will agree that it is more challenging to deal with the twentieth century than with the immediately preceding centuries. Mathematics tends to be hierarchical, making it difficult to appreciate later work without a substantial foundation in earlier work.

Even with the best of efforts, there will always remain some unknowns. It is true that many aspects of documentation and communication have improved in the last century. But this may only complicate the task of the historian. As

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mathematics is developed more rapidly and in more places by more people, tracing the development of an idea does not become easier. The era of electronic mail also threatens to deprive future historians of the type of written documentation which exists for example in the correspondence between Dedekind and Frobenius.

Which brings us back to the book under review. Curtis writes an expert account of the genesis of the theory of finite group representations, based on a lifetime of involvement with the subject. He focuses on the work of four pioneers, who are pictured on the cover of the book:

Ferdinand Georg Frobenius (1849–1917)
William Burnside (1852–1927)
Issai Schur (1875–1941)
Richard Brauer (1901–1977)

The biographical sketches in the book are worth reading even apart from the more technical material. The treatment here is far removed from the romantic storytelling of E.T. Bell. While some of the outlines are familiar, new details emerge from the author's search of archives and his consultations with experts.

The story Curtis tells involves many other significant players as well, including Tadasi Nakayama, Emmy Noether, and Alfred Young. In tracing the development of the main ideas, he takes advantage of the groundwork laid by mathematical historians (notably Thomas Hawkins), while adding further interpretive insights. For example, he explains the work of the Belgian mathematician Jacques Deruyts (1862–1945), which anticipated some of Schur's work on polynomial representations of the general linear group. (This was only recently explicated by J.A. Green.) The chapter titles give a good idea of the overall coverage:

I. Some 19th-Century Algebra and Number Theory
II. Frobenius and the Invention of Character Theory
III. Burnside: Representations and Structure of Finite Groups
IV. Schur: A New Beginning
V. Polynomial Representations of $GL_n(\mathbb{C})$
VI. Richard Brauer and Emmy Noether: 1926–1933
VII. Modular Representation Theory

A starting point for Frobenius was the notion of "group determinant" formulated by Dedekind for a finite group G of order n. Assign the group elements (in some order) to the integers $1, \ldots, n$, and let $x_1, \ldots, x_n$ be corresponding indeterminates. Indicate the inverse of the group element assigned to k by k'. Then form the determinant having as (k, l)-entry $x_{lk'}$. The problem posed by Dedekind (and solved by him in special cases) is to factor this polynomial explicitly.

Here is a concrete example involving the smallest nonabelian group $G = S_3$, the group of all permutations of $\{a, b, c\}$. The factorization of the group determinant is given below (as in Curtis, page 52).

$$\begin{vmatrix} x_1 & x_3 & x_2 & x_4 & x_5 & x_6 \\ x_2 & x_1 & x_3 & x_5 & x_6 & x_4 \\ x_3 & x_2 & x_1 & x_6 & x_4 & x_5 \\ x_4 & x_5 & x_6 & x_1 & x_3 & x_2 \\ x_5 & x_6 & x_4 & x_2 & x_1 & x_3 \\ x_6 & x_4 & x_5 & x_3 & x_2 & x_1 \end{vmatrix} = (u+v)(u-v)(u_1u_2 - v_1v_2)^2$$

Here the elements of G are ordered as $1, (abc), (acb), (bc), (ac), (ab)$. To explain the expression on the right, let ρ be a primitive cube root of 1, and set

$$\begin{array}{ll} u = x_1 + x_2 + x_3 & v = x_4 + x_5 + x_6 \\ u_1 = x_1 + \rho x_2 + \rho^2 x_3 & v_1 = x_4 + \rho x_5 + \rho^2 x_6 \\ u_2 = x_1 + \rho^2 x_2 + \rho x_3 & v_2 = x_4 + \rho^2 x_5 + \rho x_6 \end{array}$$

While a computer program such as Mathematica will readily verify the factorization of the determinant in this case, it cannot as readily explain what all this has to do with the representation theory of S_3. From a modern perspective, the factorization exhibits the decomposition of the regular representation of G. Each irreducible constituent occurs as often as its degree, the sum of squares of the degrees equalling the group order n. For S_3 there are two representations of degree 1 (the trivial and the sign representation), along with a representation of degree 2.

The early papers of Frobenius achieved a (complicated) definition of characters for arbitrary finite groups, yielding a rigorous treatment of the group determinant. He rapidly put in place the main results of the modern theory (including the definition of characters as traces of linear representations, orthogonality relations for characters, and Frobenius reciprocity for induced characters), together with explicit calculations of characters for symmetric groups and applications to the structure of finite groups. Burnside, Schur, and others simplified many of the proofs and extended the subject in new directions. Curtis explains all of this lucidly, taking the story somewhat beyond the middle of the twentieth century.

By the late 1930s, the pioneering work of Frobenius, Burnside, and Schur on representations over fields of characteristic 0 was being enriched by Brauer's deep study of representations over fields of prime characteristic. Even though this "modular" theory is less familiar to the general mathematical public, it has had a strong impact in areas such as algebraic topology and has generated powerful new conjectures (due to J.L. Alperin, M. Broué, and E. Dade in particular), insuring a vigorous life for the subject in the coming century. But this is for future historians to sort out.

What makes the book by Curtis especially attractive is the way it blends biography and the history of ideas with an explanation of the mathematics itself. The author writes in a careful but readable scholarly style, with judicious footnotes and full references to the primary literature. He goes to considerable pains to explain the sometimes opaque-looking early literature in modern language and notation. While it is quite possible to learn the basic facts about finite group representations from a wide variety of modern textbooks (including those written by Curtis and the late Irving Reiner), those who are at all attracted to the subject will certainly enjoy spending time with Curtis's account. The only prerequisite is a standard mathematical education.

The book is well-produced, with an interesting selection of photographs and only occasional misprints (as in the footnote on page 163). Once in a while the notation gets a bit out of control, for example in the varying use of G and H in the first section of Chapter IV. To obey the unwritten rule that a reviewer must say several critical things, I might also point out that the author uses commas more often than is strictly necessary.

All those with an interest in the representation theory of finite groups owe a debt of gratitude to Curtis for having written a thoughtful and informative account of this important chapter in twentieth-century mathematics.

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