Section Three

Other Games

While we have so far concerned ourselves solely with casino games, some of the principles we have used are equally applicable to other gambling situations. In this section, we apply mathematical theory to horse race betting and backgammon.

Horse racing is the number one spectator sport in America and a large amount of its success in this regard can be attributed to the wagering opportunities. The racegoer becomes a participant in the spectacle. While we offer no surefire system, we do suggest an approach that shows promise for the gambler-investor.

Backgammon is an exceedingly complicated game from a mathematical point of view. Because of the possibility of repeated restarts by the counters, the game is potentially infinite. This impedes analysis, but we offer several insights into the end game and the doubling cube, parts of the game where optimal strategies can be computed.

Chapter 6

Horse Racing

While so far I have limited myself to discussing casino games, the concepts presented apply equally to other gambling games, such as horse racing. At the racetrack, one is offered a variety of different wagering possibilities. The player can wager on one or more horses to win, place, or show, as well as combine any number of horses in the various exotic bets (daily double, exactas, quinellas, etc.) The goal of the gambler at the racetrack is to isolate in each race those bets that yield a positive return, after considering the pari-mutuel takeout and breakage.

One approach with applications in horse racing involves the use of a technique called hedging. Hedging, often used in the securities and financial markets, involves taking two or more investment positions simultaneously. The risks should cancel out and an excess rate of expected return should remain.

Why "Hedge?"

In the securities and finance markets, to hedge is to take two or more investment positions simultaneously. The risks should cancel out and an excess rate of expected return should remain.

In a real horse race (or any pari-mutuel contest for that matter), the true probabilities are not known. If we knew the true probabilities or had better estimates than the pari-mutuel pool offers, we might find horses with a positive expectation. Then we could simply bet directly on those horses instead of developing the following method for the daily double.

There is a plausible argument which upholds the pari-mutuel pool's estimate of the true horse winning probabilities: "If there were a method of predicting horse winning probabilities, and these probabilities differed enough from the pari-mutuel pool's estimate to give the predictor an advantage, then he would place bets and by so doing would cause the pari-mutuel pool odds to shift in such a way as to reduce that advantage. With many bettors and much information and available computing power the overall effect is to reduce such advantages so they are small or even become disadvantages."

In other words, "If you could beat the casinos at blackjack, then they would change the game so you couldn't. Thus, there isn't any system for beating them."

If we assume that pari-mutuel pool probabilities are true probabilities then the horse hedge system does not improve our edge over the track take! You might think that makes the horse hedge idea useless, but this is not true. Consider the daily double pool: The payoffs should be consistent with the probabilities in the individual race win pool; but in general, they aren't consistent. Thus, we have a chance to use the probabilities based on the individual race win pools.

The Daily Double

Let's apply the horse hedge idea to the daily double bet. The same idea, with some modifications, also applies to exactas, pick six and similar bets and to exactas, quinellas and trifectas in jai alai.

For a little background on the daily double, I quote from the book *Science in Betting: The Players and the Horses*, by E. R. Da Silva, and Roy M. Dorcus:

In daily double betting, any horse in the first race can be combined with any horse in the second race, and to win the bettor must successfully select the winners of both races. Some bettors combine all of the horses in the first two races. If there are ten horses in each race, in order to cover all possible combinations of horses, one would have to buy one hundred tickets at \$2.00 each. If by chance long-odds horses won both races, it would be possible to make a profit on that single daily double. However, such a situation is not common throughout a week or a season. One daily double \$2.00 ticket at Del Mar recently paid \$2,878.60, another paid \$685 and there were in addition three others during this season which paid over \$200—yet actual returns for this season were only \$6,808. The average number of horses was eleven in the first race and ten in the second. To have combined all these horses in all the daily doubles for this season would have cost \$200 per race, and since there were forty-two days in this season, the total cost would have been \$9,240, producing a loss of several thousand dollars.

Notice that they consider betting equal amounts on each horse. From one season at Del Mar, they found that \$9,240 in total bets were returned; \$6,808 for a payback fraction of 0.74 or a loss of 26% of the amount bet. Thus, betting equal amounts on each combination did not work.

Illustrated in Table 6-1 is the horse hedge method for daily doubles in a real race. The Table lists the winning probabilities based on odds for the first race at Del Mar on August 13, 1980. The horses are listed according to post position in the first column. The second column has the handicapping odds given in the *L.A. Times* on the morning of race day. The third column is obtained from these odds by taking the right hand number in the second column and dividing by the sum of the two numbers.

For example, 30-1 gives a probability of $1/31 = 0.0323$. For the horse in the 13th post position, 7-2 gives a probability of $2/9 = 0.2222$. When there is no track take, the probabilities calculated this way must add up to 1.00.

Table 6-1

First Race

Horse [p.p.]	A.M. Odds [h.c.]	Prelim. Probs.	Corrected Probs.	Final Odds: 1	Prelim. Probs.	Corrected Probs.
1	30-1	0.0323	.0246	114.80	.0086	.0068
2*	4-1	0.2000	.1526	3.10	.2439	.1922
3	2-1	0.3333	.2544	0.70	.5882	.4635
4	20-1	0.0476	s	s	s	s
5	5-1	0.1667	.1272	7.30	.1205	.0949
6	6-1	0.1429	.1090	12.10	.0763	.0601
7	10-1	0.0909	.0694	24.00	.0400	.0315
8	30-1	0.0323	.0246	72.60	.0136	.0107
9	30-1	0.0323	s	s	s	s
10	8-1	0.1111	.0848	6.40	.1351	.1065
11	15-1	0.0625	.0477	52.50	.0187	.0147
12	20-1	0.0476	.0363	82.50	.0120	.0094
13	7-2	0.2222	s	s	s	
14	8-1	0.1111	s	s	s	
15	10-1	0.0909	.0694	81.20	.0122	.0096
	sum	1.7237				
After scratches		1.3105	1.0000	1.2691	1.0000	
		Estimated track take			21.20%	

When there is a track take, the probabilities calculated from the final payoff odds at race time will equal more than 1.00. In fact, they add to 1/K, where K is the fraction of the pool, which is returned to the bettors. This rule is not quite exact due to the irregular effects of breakage, but the effects are generally small and not worth discussing.

In order to correct for probabilities that do not add up to 1.00, we add them, deducting horses which may have been scratched. We then use the final total and divide it into the preliminary probabilities so that it equals 1.00. (Corrected probabilities appear in column four.)

Column five gives the final odds on various horses. Column six has corresponding uncorrected probabilities and column seven lists corrected probabilities. Notice that column six adds to 1.2691; by dividing this into 1.00 we get 0.7880 which corresponds to a track take of 21.30% for this particular race.

Column three equaled 1.7237 before deducting the horses which were later scratched, making the track take too large. The sum for Del Mar is typically about 1.20; therefore the handicapper's setting of the odds was not consistent. On average, the odds were set too low in this race for the horses. When four horses were scratched, the odds on the remaining horses gave probabilities which equaled 1.3105. That is the typical sum at Del Mar.

Table 6-2 presents the probability calculations for the second race. The fourth column appears to equal 0.9999, but shows 1.0000, because the entries have been rounded off to four places.

The final outcome of the daily double: horse 2 won the first race; horse 1 won the second race; and a winning $2 ticket paid back $38.60 or $19.30 per unit bet. The amount bet on each of the 15 × 8 or 120 combinations is proportional to the product of the corresponding probabilities.

For example, if we use the corrected probabilities based on the morning odds, we have .1526 for horse 2 in the first race and .1725 for horse 1 in the second race. The product of these two numbers is .0263. That means we bet .0263 of our total unit bet on the combination which actually won the daily double.

Table 6-2

Second Race

Horse [p.p.]	A.M. odds [h.c.]	Prelim. Probs.	Corrected Probs.	Final Odds: 1	Prelim. Probs.	Corrected Probs.
1*	7-2	.2222	.1725	2.60	.2778	.2313
2	15-1	.0625	.0485	18.80	.0505	.0421
3	3-1	.2500	.1941	5.90	.1449	.1207
4	6-1	.1429	.1109	13.30	.0699	.0582
5	5-2	.2857	.2218	3.80	.2083	.1735
6	6-1	.1429	.1109	15.80	.0595	.0496
7	10-1	.0909	.0706	2.80	.2632	.2192
8	10-1	.0909	.0706	6.90	.1266	.1054
	SUM	1.2880	1.0000		1.2007	1.0000
		Estimated track take			16.72%	

Therefore, we have a return of \$19.30 × this probablity or .5080 of a unit which means we lost 49.2% of our bet. If we had used the final odds, the probabilities are .1922 and .2313. Their product is .0445 and we would receive this amount × \$19.30 or .8580 of a unit, or a loss of 14.2%.

On page 127, Da Silva and Dorcus warn you that:

> In doing any statistical work on daily doubles, the reader must be careful not to use the actual closing odds of the horses, as listed the day following the races in result charts from newspapers or from the Form or the Telegraph, since these last-minute odds are not available to the daily double bettor for either the first or the second races. The bettor must rely only upon the probable odds for statistical study of daily double betting, odds which are given in the Morning Line at the tracks, in the Form or Telegraph under different handicappers such as Sweep, Analyst, Trackman, or given in the track programs.
>
> Furthermore, in dealing with these probable odds, the bettor must remember that they may or may not correspond to the last-minute closing odds on the toteboard.

(For the first race only, the actual odds that we would use in practice may be fairly close to these final odds if we were actually at the track watching the toteboard.) At this point, we can see difficulties with the horse hedge idea as it relates to the daily double.

For example, there is a minimum \$2 bet. In order to approximate the various probabilities of the typical one hundred or so combinations, we have to make several hundred \$2 bets which requires a substantial bankroll. Another problem is that the final pari-mutuel pool odds are unknown. Even if we did know the odds on the individual races, the true probabilities of the individual horses winning in their respective races would still be unknown. Therefore, we don't know if the horse hedge method will give us an advantage over the track take.

Even if it does give us an advantage, we don't know if we can gain enough to overcome the track take for an overall advantage.

This is the reason why this system needs further development.

One way to get around the difficulties is to keep a record of the final odds and the corresponding probabilities and bet accordingly. If pari-mutuel odds are a fair estimate of the true odds, then this indicates the sort of gain to be had from horse hedging. If the gain is large enough to produce a substantial advantage, then there might still be an advantage if we use good odds that are available to us at the time we place our bets.

To show you how to keep this sort of record, I will use one average figure to correct of the track take. Table 6-3 shows the sum of the uncorrected probabilities for the first five races on three consecutive days. The days are August 13, 14 and 15, 1980 at Del Mar. The two entries followed by question marks suggest that there may be data errors or newspaper misprints. Except for the two questionable figures, the uncorrected probability sums are close to 1.20. The average, of the 13 remaining races in Table 6-3 works out to be 1.2033.

Table 6-3

Uncorrected Probability Sums

Race #	8/13/80	8/14/80	8/15/80
1	1.2691?	1.1970	1.2177
2	1.2007	1.1968	1.2104
3	1.2064	1.1973	1.2026
4	1.1960	1.2049	1.2059
5	1.2082	0.9966?	1.1986

To simplify, I shall use 1.20 in my computations in Table 6-4. The fractions estimate the investment returned for each day the horse hedge system is used at Del Mar. If you want to construct a similar table, get extensive racing records from your track, and determine whether the method works over a past sample.

Table 6-4

Race Date	Race 1 Winner Odds: 1	Prelim. Prob.	Corrected Prob.	Race 2 Winner Odds: 1	Prelim. Prob.	Corrected Prob.	Product of Probs.	Daily Double Payback for 1	Fraction Returned
				Del Mar					
8/13/80	3.10	.2439	.2033	2.60	.2778	.2315	.0470	19.30	.9080
8/14/80	2.50	.2857	.2381	.50	.6667	.5556	.1323	6.50	.8600
8/15/80	0.70	.5882	.4902	1.90	.3448	.2874	.1409	5.50	.7748
8/16/80	1.80	.3571	.2976	2.30	.3030	.2525	.0752	12.90	.9695
8/17/80	5.50	.1538	.1282	6.70	.1299	.1082	.0139	50.30	.6979
8/18/80	1.60	.3846	.3205	9.50	.0952	.0794	.0254	41.60	1.0582
8/20/80	0.90	.5263	.4386	5.30	.1587	.1323	.0580	17.80	1.0327
8/21/80	2.20	.3125	.2604	6.40	.1351	.1126	.0293	45.50	1.3343
8/22/80	1.70	.3704	.3086	11.60	.0794	.0661	.0204	36.90	.7532
8/23/80	7.40	.1190	.0992	1.10	.4762	.3968	.0394	27.60	1.0865

Ten races: estimated average fraction returned 0.9476

Table 6-4 shows the idea at Del Mar. The second, third, and fourth columns list the corrected probabilities based on the final odds of 2-1 for the winning horse in the first race. The fifth, sixth, and seventh columns do the same thing for the second race. The eighth column is the product of these probabilities (the pari-mutuel estimate of the probability of a pair of horses winning the daily double). For the last column, multiply the payback on $1 which is the fraction of the unit be returned to us.

In our sample of ten races, we get an estimated payback of 94.76%, or a loss of 5.24%. We are estimating the average effective track take as 16.67% so the system does better than average but still does not win.

For a clear explanation of daily double betting, exactor or exacta betting, odds, and trifecta betting, I refer you to the appendix of *Harness Racing Gold,* by Prof. Igor Kusyshyn, published by International Gaming Inc., 1979 ($14.95).

The New York Racing Association takeout is currently 17% although it has been 14%. The California takeout is 15.75%. Of course the effect of breakage is to increase the average takeout somewhat beyond these figures.

Readers who want to know more about the calculation of winning probabilities based on the pari-mutuel odds should read Chapter 3 in *Horse Sense,* by Burton P. Fabricand, published by David McKay and Co., 1965. The book is hard to obtain, but I believe you can find it in the larger libraries.

Fabricand takes a sample of 10,000 races, with 93,011 horses and 10,035 winners (some dead heats). He finds that the average loss, from betting on the favorites (high pari-mutuel probability of winning), is considerably smaller than the average loss from betting the long shots (low pari-mutuel probability of winning).

For extreme favorites, the sample showed a profit and for horses with a pari-mutuel win probability of 30% or more the average loss was just a few percent. It ranged gradually higher as the odds lenghtened for horses with odds of 20 to 1 or more and pari-mutuel probabilities averaging about .025; the average loss to the bettors was 54 percent.

This indicates that the odds, from the pari-mutuel pool for winners, are systematically biased; they can be improved by incorporating a correction factor based on a data sample similar to Fabricand's. The correction would increase the probabilities assigned to the favorites and decrease the probabilities assigned to the long shots systematically.

A more readily available source for the same information is Fabricand's latest book *The Science of Winning*, published by Van Nostrand Reinhold in 1979. On page 37, a table shows how a player's expectaton varies with the odds. The sample has 10,000 races with 10,035 winners because of dead heats.

In 1984 a book by Ziemba and Hausch, *Beat the Racetrack: A Scientific Betting System*, appeared with a practical winning method at the track. I went to Hollywood Park with the author William T. Ziemba, Ph.D., and used the system successfully. The idea of true win probabilities discussed in this chapter is used by them to check the place and how pari-mutual pools. When horses in these pools are significantly under bet, they offer positive expectations. Good bets appear on average about once per two races.

Chapter 7

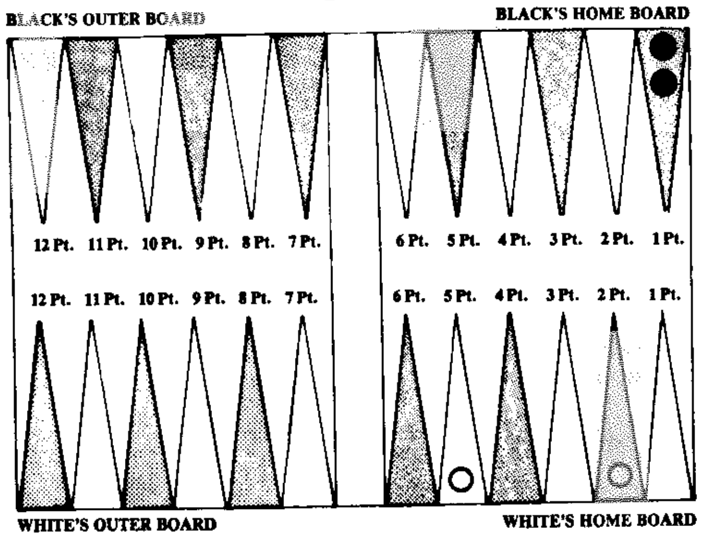
Backgammon

Backgammon has taken its place alongside bridge as a favorite pastime of sophisticated gamers. It is essentially a racing game, where each player tries to get his pieces off the board first. But, thanks to the doubling cube, it's also a gambling game, played for high stakes in clubs across the country. The basics of this intriguing board game are really very easy to master. (See *The Rules of Backgammon*, pp. 84-85.)

This chapter will focus on several aspects of backgammon that can be solved mathematically. You will learn useful but simple odds for bearing off with only two men left. Most good players already know this. But don't go away, good players. Later you will learn facts about backgammon that few in the world are aware of.

As an introduction to end positions, suppose you are White and it is your turn to roll in the position of Diagram 1. The doubling cube is in the middle.

Diagram 1



Questions:

1. What is your chance to win?
2. Should you double?
3. How much do you gain or lose by doubling?
4. If you double, should Black accept?
5. How much does Black gain or lose by accepting your double?

White wins only if he bears off on his next roll. So to help us solve end positions of this type, we calculate a table of chances to take off men in one roll. The exact result is given in Table 7-1, and the chances to the nearest percent are given in Table 7-2.

As you can see from Table 7-1, the exact chances of winning if you have a man on the five point and a man on the two point are 19 in 36 or .5277. . . Table 7-2 gives your chances to the nearest percentage, or 53%. Now you have the answer to question 1.

To see how Table 7-1 is calculated, recall that there are 36 *equally likely* outcomes for the roll of two dice. These are listed in Table 7-3.

Table 7-1

a man on the	0 pt	1 pt	2 pt	3 pt	4 pt	5 pt	6 pt
0 pt	off	36	36	36	34	31	27
1 pt	36	36	36	34	29	23	15
2 pt	36	36	26	25	23	19	13
3 pt	36	34	25	17	17	14	10
4 pt	34	29	23	17	11	10	8
5 pt	31	23	19	14	10	6	6
6 pt	27	15	13	10	8	6	4

Table 7-2

a man on the	0 pt	1 pt	2 pt	3 pt	4 pt	5 pt	6 pt
0 pt	off	100%	100%	100%	94%	86%	75%
1 pt	100%	100%	100%	94%	81%	64%	42%
2 pt	100%	100%	72%	69%	64%	53%	36%
3 pt	100%	94%	69%	47%	47%	39%	28%
4 pt	94%	81%	64%	47%	31%	28%	22%
5 pt	86%	64%	53%	39%	28%	17%	17%
6 pt	75%	42%	36%	28%	22%	17%	11%

Table 7-3

second die shows► ▼ first die shows	1	2	3	4	5	6
1	1-1	1-2	1-3	1-4	1-5	1-6
2	2-1	2-2	2-3	2-4	2-5	2-6
3	3-1	3-2	3-3	3-4	3-5	3-6
4	4-1	4-2	4-3	4-4	4-5	4-6
5	5-1	5-2	5-3	5-4	5-5	5-6
6	6-1	6-2	6-3	6-4	6-5	6-6

Think of the two dice as labelled "first" and "second." It might help to use a red die for the "first" die and a white one for the "second" die. Then if the red (first) die shows 5 and the white (second) die shows 2, we call the outcome 5-2. If instead the first die shows 2 and the second die shows 5, this is a different one of the 36 rolls and we call it 2-5. Outcomes are named x-y where x is the number the first die shows and y is the number the second die shows.

To see that White has 19 chances in 36 to win in the situation presented in Diagram 1, we simply count winning rolls in Table 7-3. If either die shows at least 2 and the other shows at least 5, White wins. He also wins with 2-2, 3-3, and 4-4. This gives the 19 (shaded) winning outcomes in Table 7-3.

As another example, suppose the two men to bear off are both on the six point. Then if two different numbers are rolled, White can't come off in one turn. Of the six doubles, only 3-3 or higher works. This gives four ways in 36 or 11%, in agreement with Tables 7-2 and 7-3. This simple counting method produces all the numbers in Table 7-1.

Now we are ready to answer question 2. Should White double, in Diagram 1? The answer is yes, and here's why. We have seen

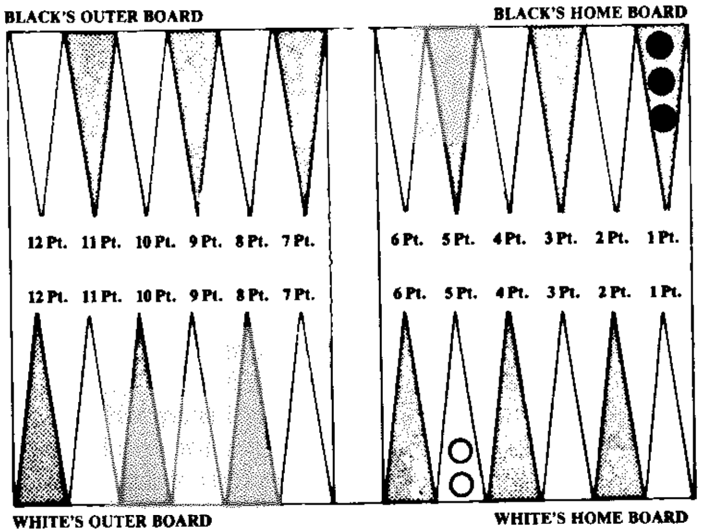
that White wins, on average, 19 times in 36. If we say the stake is one unit, then if he does *not* double, in 36 times he wins one unit 19 times and loses 1 unit 17 times for a gain of two units/36 times = 1/18 = 0.055. . . If White *does* double, Black can either accept or fold. Suppose Black accepts. Then the stakes are two units and a calculation like the previous one shows White gains an average of four units /36 times = 1/9 = 0.111. . . unit per time. White gains twice as much by doubling as by not doubling. If Black folds instead, then White wins one unit at once, which is even better.

This also answers the rest of the questions. In answer to question 3, White gains an extra 5.55% of a unit, on average, by doubling. Answer to question 4: Black should accept. He loses 1/9 unit on average by accepting and one unit for sure by folding. This answers question 5: if Black makes the error of folding, he loses an extra 8/9 unit or 89%.

The usefulness of Table 7-2 is generally limited to situations where you have just one or two rolls left before the game ends. But it is surprising how often the Table is valuable. Here are some more examples to help alert you to these situations. In Diagram 2, Black has the doubling cube. White has just rolled 2-1. How does he play it?

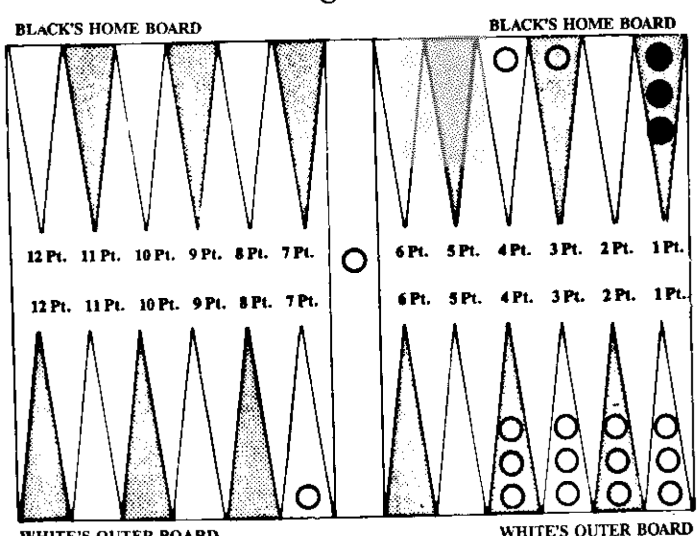
If Black rolls double on the next turn, he wins at once and it won't matter what White did. So White only needs to consider the case where Black does not roll doubles. Then White will have one more turn, and he wants to leave himself with the greatest chance to bear off on that turn. White can move one man from the 5 point to the 4 point and one man from the 5 point to the 3 point. By Table 7-2, this gives him a 47% chance to win if Black does not roll doubles. Or, White can move one man from the 5 point to the 2 point, leaving the other man on the 5 point. This gives him a 53% chance to win if Black does not roll doubles, so this is the best way to play the 2-1.

Diagram 2



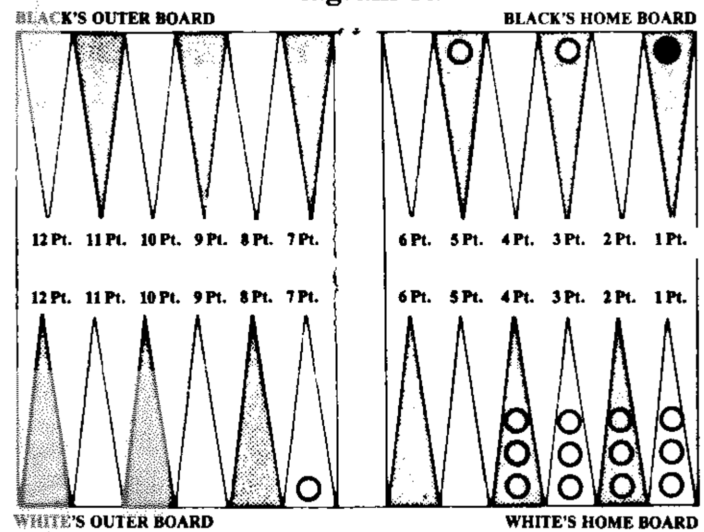
In Diagram 3, White's problem is to avoid a backgammon: if Black wins before the White men escape from Black's home board, Black will win 3 units. Otherwise, he will only gammon White for two units.

Diagram 3



White has just rolled 4-1. He must use the 4 to move the man on the bar to the Black 4 point (dotted circle). White can then move this man on to the Black 5 point, in which case, if Black does not roll doubles, White's situation on his last turn is shown in Diagram 3a.

Diagram 3a



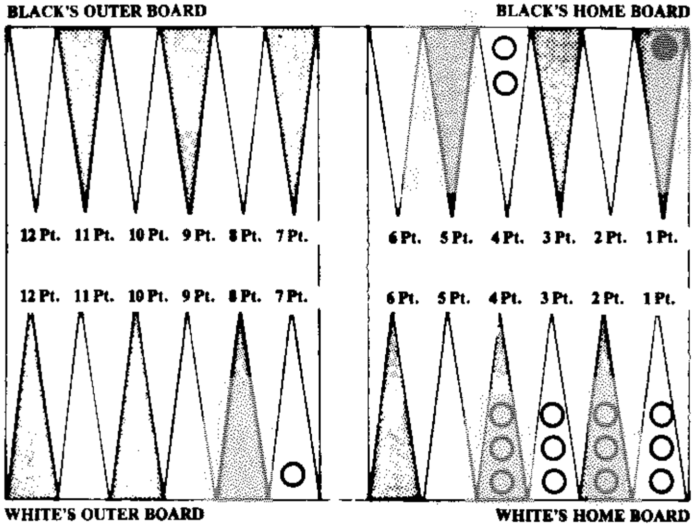
The chance for White to remove both men from Black's home board on the next roll is the same as the chance to bear off both men when one is on the 4 point and the other is on the 2 point. According to Table 7-2, this is 64%.

Suppose instead White plays both men to the Black 4 point. Then Diagram 3b shows the board if he survives Black's next roll.

His chance to save himself from backgammon is the same as bearing off two men from the 3 point in one roll. Table 7-2 gives 47%. Therefore, the play in Diagram 3a is best.

If instead White rolled 4-2 in Diagram 3, he could enter on 2 and move his other man to the 7 point, giving an 86% chance (Table 7-2, man on 5 point and man on 0 point) to escape Black's home board on the next roll. Or he could play to leave his two

Diagram 3b



back men on the Black 5 and 4 points. This gives only a 69% chance and is the inferior choice.

An outstanding reference work is *Backgammon* by Paul Magriel, The New York Times Publishing Company, 1977, $20. Most of Table 7-1 appears there on page 404. A convenient reference for practical play is the "Backgammon Calculator," Doubleday, 1974, $1.95. This handy cardboard wheel has most of Table 7-2 on the back.

Here are some questions to check your understanding. Refer to Diagram 2, assuming Black has the doubling cube and White has just rolled 2-1.

1. Should Black double after White makes the best move?

2. How much would Black gain or lose by so doubling?

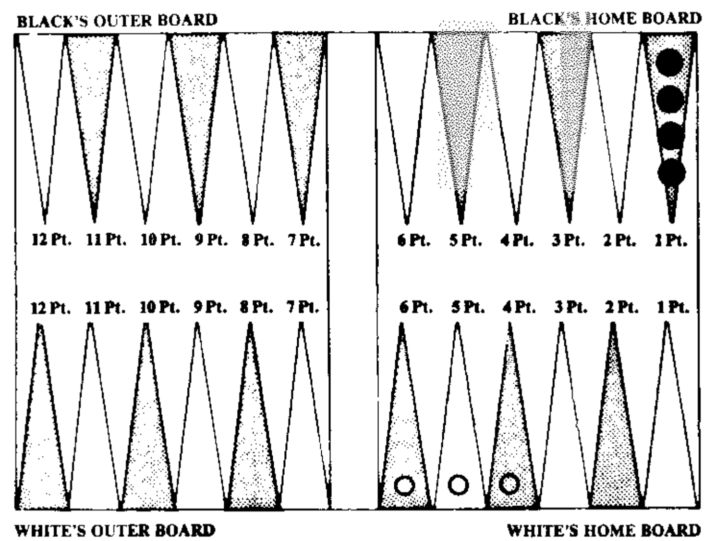
3. Should White accept a Black double? If he does, instead of folding, how much does he gain or lose?

4. What is the best way for White to play 3-2 in Diagram 2?

We will now present the complete, exact solution to all backgammon positions when each player has only one or two men left in his own home board. Don Smolen and I calculated it in 1975 and kept it to ourselves for several years.

I realize that it is often not practical or desirable to use the tables I provided during the game. Fortunately, many of these situations are covered by a handy rule that appeared in a "Sheinwold on Backgammon" column in the *Los Angeles Times*. Sheinwold considers the situation in Diagram 4. The problem is whether White, having rolled 6-2, should play the 2 so that he leaves his two men on 5 and 2 or 4 and 3.

Diagram 4



We solved this same problem earlier when discussing Diagram 2. We saw then from Table 7-2 that leaving men on 5 and 2, is best because it gives White a 53% chance to get off on the next turn, whereas leaving men on 4 and 3 gives only a 47% chance. Now consider the general question: If you have to leave one or two men after your turn, what is the best "leave"? Assuming that the positions between which you must choose have the *same pip count*, the correct rules, which Sheinwold gives, is:

Rules for Leaving One or Two Men

(1) If possible, leave one man rather than two.
(2) If you must leave two men, leave them on different points, if possible.
(3) If you still have a choice, move off the 6 point.
(4) If you are already off the 6 point, move the man on the lower point.

It is easy to prove these rules correct by using Table 7-2. This is shown again here in condensed form as Table 7-4.

Table 7-4

a man on the	1 pt	2 pt	3 pt	4 pt	5 pt	6 pt
0 pt	100% 1 pip	100% 2 pips	100% 3 pips	94% 4 pips	86% 5 pips	75% 6 pips
1 pt	100% 2 pips	100% 3 pips	94% 4 pips	81% 5 pips	64% 6 pips	42% 7 pips
2 pt		72% 4 pips	69% 5 pips	64% 6 pips	53% 7 pips	36% 8 pips
3 pt			47% 6 pips	47% 7 pips	39% 8 pips	28% 9 pips
4 pt				31% 8 pips	28% 9 pips	22% 10 pips
5 pt					17% 10 pips	17% 11 pips
6 pt						11% 12 pips

To check the rules, we simply check Table 7-4 for each pip count to see if it always tells us which of two "leaves" to pick. For example, with a pip count of 6, part (1) of the rule says correctly that 0 pt. -6 pt. is best. Then (2) says correctly that among the three remaining two-men positions, 3 pt. -3 pt. is worst. In a similar way the rule is verified in turn for positions with pip counts of 4, 5, 6, 7, 8, and 10. There's nothing to check for pip counts of 1, 2, 3, and 9 because the choices are equally good for these pip counts. There's nothing to check for counts of 11 and 12 because for these pip counts there is only one choice of position.

More examples illustrating the rule appear in *How Good are You at Backgammon: 75 Challenging Test Situations* by Nicolaos and Vassilios Tzannes, Simon and Shuster, 1974. You can use these rules to solve at once test situations 40, 41, 42, and 43. The authors give a rule (page 94) but it is neither as clear nor as simple as ours.

We proved the rule for leaving one or two men just for the case where you will have at most one more turn to play. In that case,the percentages in Table 7-4 let us compare two positions to see which is better. What if there is a chance that you'll have more than one turn? This could happen, for instance, if we change Diagram 4 so that Black has five men on the one point instead of four. Then Black could roll non-doubles on his next turn, leaving three men on the 1 point; White could roll 1-2 on his next turn, reducing his 5 pt. -2 pt. position to one man on the 4 point: Black could roll non-doubles again, leaving one man on the 1 point; and White then gets a second turn. It turns out that the rule gives the best choice against *all* possible positions of the opponent, not just those where you will have at most one more turn to play. (Note: There is one possible, unimportant exception that might arise, but the error is at most a small fraction of a percent.)

Now we return to the Thorp-Smolen solution of all end games with just one or two men in each home board. We will label home board positions as follows: 5 + 3 where there is a man on the 5 point and a man on the 3 point, with the largest number first. With both men on, say, the 4 point, we call the position 4 + 4.

With only one man on the 5-point we write 5 + 0. Think of the 0 as indicating that the second man is on the 0 = "off" point.

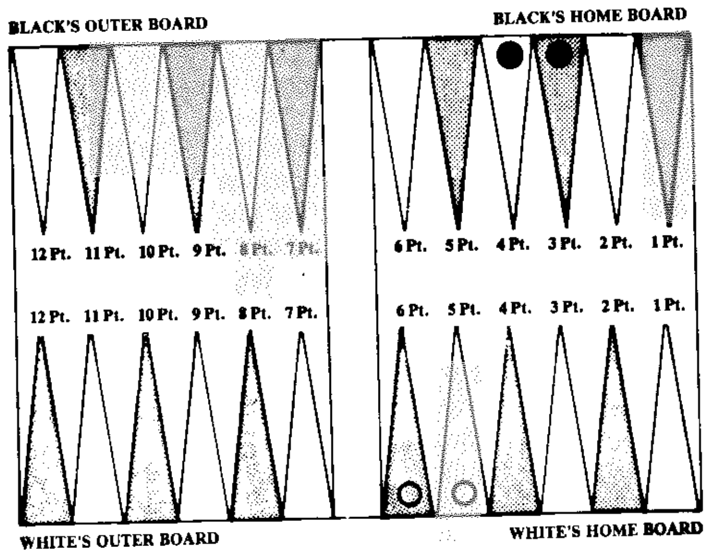
There are six home board positions with one man, namely 1 + 0, 2 + 0, . . . 6 + 0. There are 21 home board positions with two men. Thus there are 27 one- or two-man positions for each player.*

Table 7-5 gives the first part of our solution. It tells Player One's "expectation," rounded to the nearest percent, if One has the move and Two owns the cube. By One's expectation we mean the average number of units One can expect to win if the current stake is "one unit" and if both players follow the best strategy. Of course, if a player doesn't follow the best strategy, his opponent can expect on average to do better than Table 7-5 indicates.

The A above 6 + 0 means this column also applies to any count of up to 3 pips: 1 + 0, 2 + 0, 1 + 1, 3 + 0, or 2 + 1. The C above 6 + 0 means that this column also applies to 4 + 0, 3 + 1, 5 + 0, or 4 + 1. The A for Player One means the same as for Player Two.

We will illustrate the use of the Table with Diagram 5.

Diagram 5



*See Appendix H, pg. 140.

Table 7-5

Two has → / One has ↓	A,C 6+0	2+2	3+2	4+2 5+1	5+2	3+3 4+3	6+1	5+3	6+2	4+4	6+3	5+4	6+4	5+5	6+5	6+6
2+1 A	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
3+1, 4+0	89	90	90	91	94	95	95	96	96	97	97	97	98	98	98	99
5+0	72	74	75	78	85	87	88	89	90	92	92	92	94	95	95	97
4+1	61	63	65	70	78	82	84	85	86	88	89	89	91	94	94	96
6+0	50	53	56	61	72	76	79	81	82	85	86	86	89	92	92	94
2+2	44	48	51	57	69	74	77	78	80	83	85	85	88	91	91	94
3+2	39	42	46	52	66	71	75	76	78	81	83	83	86	90	90	93
4+2, 5+1	28	32	36	44	60	66	70	72	74	78	80	80	34	88	88	92
5+2	06	11	16	27	48	55	60	63	66	71	73	73	79	84	84	89
3+3	-06	00	06	18	41	50	56	59	62	68	71	71	77	82	82	88
4+3	-06	00	06	18	41	50	56	59	61	67	70	70	76	82	82	88
6+1	-17	-10	-04	09	35	45	51	54	57	64	67	67	74	80	80	87
5+3	-22	-15	-09	05	32	41	48	51	54	61	64	64	71	78	78	85
6+2	-28	-21	-14	01	29	38	45	49	52	59	63	63	70	77	77	84
4+4	-39	-31	-23	-08	22	33	40	44	48	55	59	59	67	74	75	82
6+3	-44	-36	-28	-12	18	28	36	40	44	51	55	55	63	71	71	79
5+4	-44	-37	-30	-14	17	28	35	39	43	51	54	55	62	70	70	78
6+4	-56	-48	-40	-23	06	17	25	29	33	42	46	46	55	63	63	72
5+5	-67	-59	-51	-35	-02	10	19	24	28	37	41	42	51	59	60	70
6+5	-67	-59	-51	-36	-07	03	11	15	19	28	32	33	43	51	53	63
6+6	-78	-71	-64	-50	-25	-17	-09	-05	-01	07	12	12	23	32	36	48

It is White's turn to move so he becomes Player One. Player Two, or Black, has the cube. We look along the row 6 + 5 and the column 4 + 3. Table 7-5 shows Player One's (White's) expectation as 03, so White has a 3% advantage. He expects to win on average 3% (more exactly, 2.54%) of the current stake. If the current stake is $1,000, White should accept a Black offer to "settle" the game if Black offers more than $25.40. If Black offers less, White should refuse.

Table 7-6 gives the expected gain or loss (to the nearest percent) for Player One when he has the move and the doubling cube is in the middle.

Unlike Table 7-5, in this case One has the option of doubling before he moves. If One does not double, Two will be able to double on his turn. If One doubles, Two then has the choice of accepting the double or folding. If Two accepts, play continues with doubled stakes and Two gets the cube. If Two folds, he loses the current (undoubled) stakes and the game ends.

Table 7-7 gives the expected gain or loss for Player One when he has the move and the doubling cube. The columns for 6 + 4, 5 + 5, 6 + 5, and 6 + 6 are the same as for Table 7-6 so they have been omitted.

In this case, One has the option of doubling before he moves. However, in contrast to Table 7-6, if One does not double, he keeps the cube so Two cannot double on his next turn. If One does double, Two can accept or fold. If he accepts, the stakes are doubled, play continues, and Two gets the cube. If instead Two folds, he loses the current (undoubled) stake and the game ends. Table 7-8 also tells whether One should double and whether Two should accept when One has the cube.

Doubling strategy is the same whether One has the cube or it is in the middle, except for the shaded region. If he makes the mistake of doubling, Two should accept. When the cube is in the middle, One should double for positions in the shaded regions and Two should accept.

We will now show how to use the tables to play *perfectly* in any of the 27 × 27 = 729 end positions covered by the tables. We'll

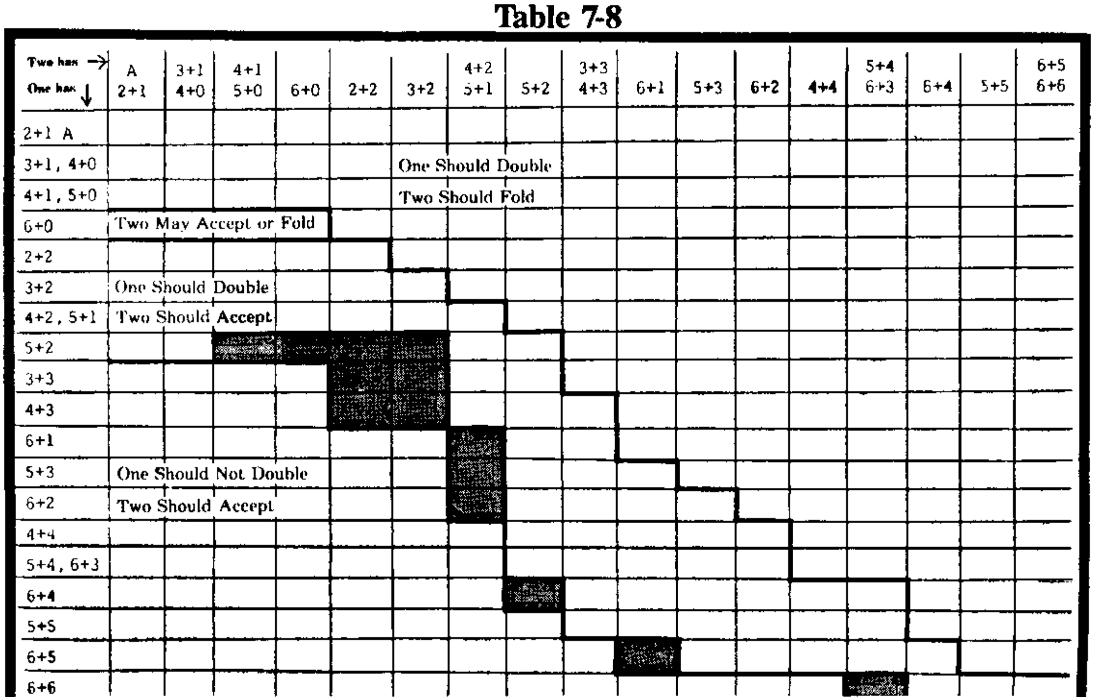
Table 7-6

Two has → / One has ↓	A,C 6+0	2+2	3+2	4+2 5+1	5+2	3+3 4+3	6+1	5+3	6+2	4+4	6+3	5+4	6+4	5+5	6+5	6+6
6+0 A,C	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
2+2	89	95	100	100	100	100	100	100	100	100	100	100	100	100	100	100
3+2	78	85	91	100	100	100	100	100	100	100	100	100	100	100	100	100
4+2, 5+1	56	64	72	88	100	100	100	100	100	100	100	100	100	100	100	100
5+2	11	22	32	53	95	100	100	100	100	100	100	100	100	100	100	100
3+3, 4+3	-06	01	12	36	83	100	100	100	100	100	100	100	100	100	100	100
6+1	-17	-10	-04	19	70	89	100	100	100	100	100	100	100	100	100	100
5+3	-22	-15	-09	10	63	82	95	100	100	100	100	100	100	100	100	100
6+2	-28	-21	-14	01	57	77	91	98	100	100	100	100	100	100	100	100
4+4	-39	-31	-23	-08	44	66	81	88	96	100	100	100	100	100	100	100
6+3	-44	-36	-28	-12	36	57	72	80	88	100	100	100	100	100	100	100
5+4	-44	-37	-30	-14	34	55	71	78	86	100	100	100	100	100	100	100
6+4	-56	-48	-40	-23	13	34	50	58	67	83	91	92	100	100	100	100
5+5	-67	-59	-51	-35	-01	21	39	47	56	74	83	83	100	100	100	100
6+5	-67	-59	-51	-36	-05	08	21	30	38	56	65	66	86	100	100	100
6+6	-78	-71	-64	-50	-22	-10	-02	03	07	16	24	25	46	65	71	96

Table 7-7

Two has → / One has ↓	A 2+1	3+1 4+0	5+0	4+1	6+0	2+2	3+2	4+2 5+1	5+2	3+3 4+3	6+1	5+3	6+2	4+4	6+3	5+4
6+0 A,C	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
2+2	89	89	89	89	89	95	100	100	100	100	100	100	100	100	100	100
3+2	78	78	78	78	78	85	91	100	100	100	100	100	100	100	100	100
4+2, 5+1	56	56	56	56	56	64	72	88	100	100	100	100	100	100	100	100
5+2	11	11	19	24	29	32	34	53	95	100	100	100	100	100	100	100
3+3, 4+3	-06	00	09	15	21	24	27	36	83	100	100	100	100	100	100	100
6+1	-17	-10	-00	06	13	16	19	25	70	89	100	100	100	100	100	100
5+3	-22	-15	-05	02	08	12	15	22	63	82	95	100	100	100	100	100
6+2	-28	-21	-10	-03	04	08	11	18	57	77	91	98	100	100	100	100
4+4	-39	-31	-20	-12	-04	-00	04	11	44	66	81	88	96	100	100	100
6+3	-44	-36	-24	-16	-08	-04	00	08	36	57	72	80	88	100	100	100
5+4	-44	-37	-25	-17	-09	-05	-01	07	34	55	71	78	86	100	100	100
6+4	-56	-47	-35	-26	-18	-14	-10	-01	16	34	50	58	67	83	91	92
5+5	-67	-58	-45	-36	-27	-23	-18	-10	08	21	39	47	56	74	83	83
6+5	-67	-58	-45	-37	-29	-24	-20	-12	05	14	22	30	38	56	65	66
6+6	-78	-70	-57	-49	-41	-37	-33	-25	-08	00	08	13	17	25	30	30

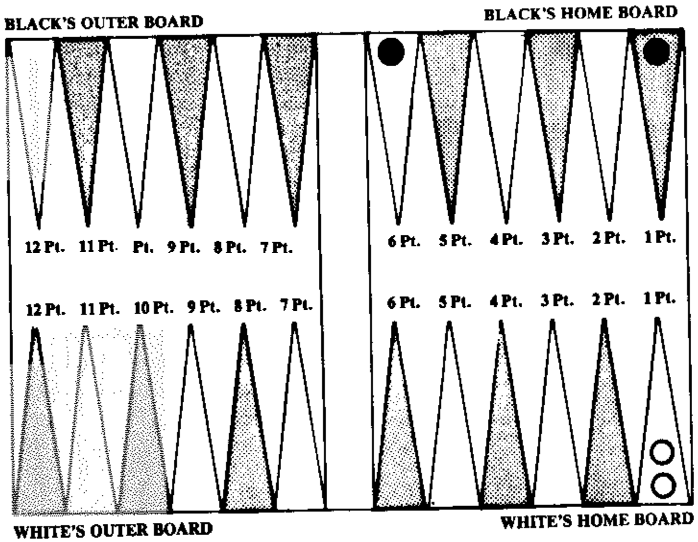
Table 7-8



run through sample end games step by step, showing player expectation, doubling strategy, and the best way to play each roll.

I earlier referred to a book entitled *How Good Are You at Backgammon: 75 Challenging Test Situations* by Nicolaos and Vassilios Tzannes, Simon and Shuster, 1974. Consider first Situation 74 from the Tzannes' book. This is shown in Diagram 6.

Diagram 6



It is Black's turn so he is Player One. Black doubles. Should he? If he does, should White accept? The cube is in the middle. We look in Table 7-8, row 6+1, column 1+1. Black should not double. If he does, White should accept. (This is correctly recommended by Tzannes' book.) Table 7-6 shows that Black's expectation under best play, which means not doubling, is −17%. If instead Black has the cube, we use Tables 7-7 and 7-8. In this example we get exactly the same answer. This isn't always the case, though, as we will see.

This example is also easy to analyze directly. If Black bears off in his next turn he will win. The chances are 15/36 (Table 7-1). If he does not bear off at once, White will win and Black will lose. So if the current stake is 1 unit, and Black does not double, Black's expected gain is +1 unit X 15/36 − 1 unit X 21/36 = −6/36 = −16 2/3%. Now suppose Black doubles and White accepts. Then Black's expected gain is +2 units X 15/36 − 2 units X 21/36 = −12/36 = −33%. On average Black will lose an extra 16 2/3% of a unit if he makes the mistake of doubling and White accepts.

It's easy to see from this type of reasoning that if Player One has any two-man position and Player Two will bear off on the next turn, then Player One should not double (if he can) when his chance to bear off in one roll is less than 50%. If his chance to bear off is more than 50%, he should double. Referring to the same Table 7-1 proves this rule which the Tzannes cite for these special situations:

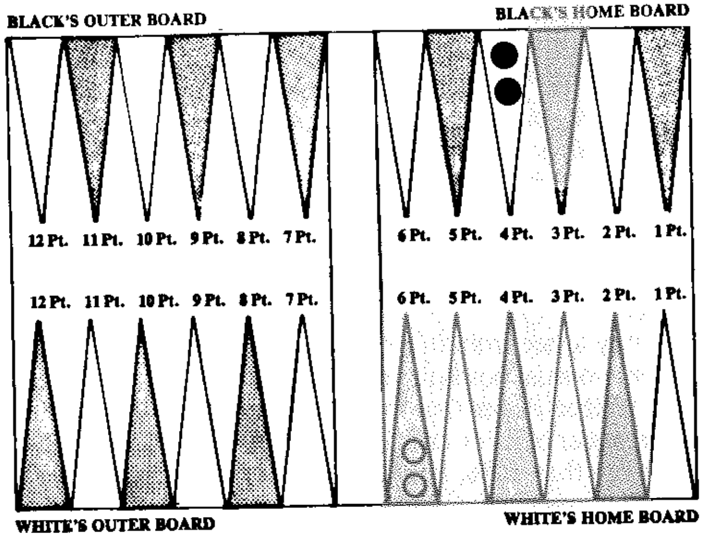
> *With double three, six-one, six-two (or*
> *anything worse)*
> *Keep dumb, hope for the best. Anything*
> *better, don't delay,*
> *Double the stakes with zest.*

The Tzannes' Situation 73 is similiar.

Here is a trickier situation that I don't think you could figure out without help from the tables. Suppose White has 6+6, Black has 4+4, White is to roll and the doubling cube is in the middle. This is shown in Diagram 7. Should White double? How does the game proceed for various rolls?

White is Player One. He consults Table 7-6 and sees his expectation is 16%. But Table 7-8 tells White not to double. We now show how to use that table to play optimally for a sample series of rolls. Suppose White rolls 3-1. How does he play it? He can end up with 6+2 or with 5+3.

Diagram 7



The rule stated earlier says that 5 + 3 looks better because it gives him a greater chance to bear off on the next turn. This is proven by the tables as follows: after White plays, it will be Black's turn. Black will be Player One with 4 + 4, White will be Player Two with either 5 + 3 or 6 + 2. The cube will be in the middle. Which is best for White? Consult Table 7-6. We find Player One (Black) has an expectation of 88% if White has 5 + 3 whereas Black has 96% if White has 6 + 2. White wants to keep Black's expectation down so he plays to leave 5 + 3.

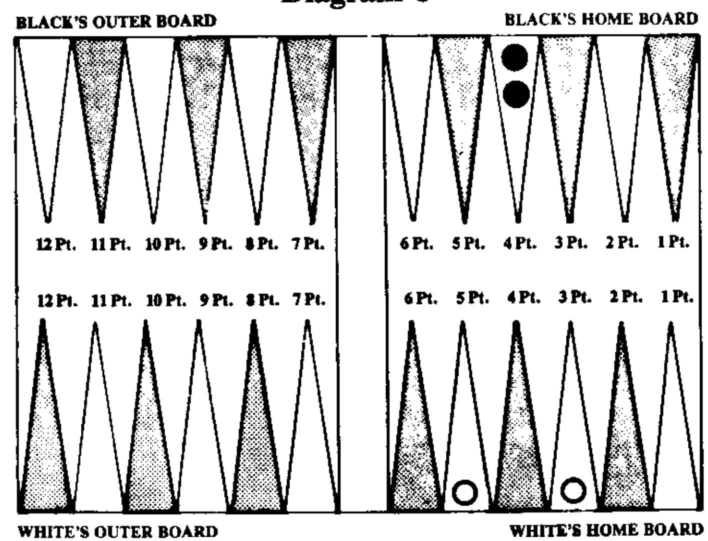
The situation after White makes this move is shown in Diagram 8.

Black is to roll and the cube is in the middle. Should Black double? Should White accept? Table 7-8 says Black should double and White should accept. Table 7-6 says Black's expected gain is 88% of the one-unit stake.

Next Black rolls 2-1. He can leave 4 + 1 or 3 + 2. The rule mentioned says 4 + 1 is better. To confirm this, note that after Black moves, White will be Player One with 5 + 3, Black will be Player

Two with either 4 + 1, or 3 + 2 and White will have the cube. Therefore we consult Table 7-7, not Table 7-6. If Black leaves 4 + 1, White's expectation is 2% of the current two-unit stake. If Black leaves 3 + 2, White's expectation is 15%. Therefore Black leaves 4 + 1.

Diagram 8



It is now White's turn. The situation is shown in Diagram 9.

The stake is 2 units, White's expectation is 2% of 2 units or .04 unit and White has the cube. What should he do? Table 7-8 tells us White should not double.

White now rolls 5-2, leaving 1 + 0. Black does not have the cube. Table 7-5 gives his expectation as 61% of 2 units or 1.22 units. He wins or loses on this next roll.

The tables show certain patterns that help you to understand them better. For instance, for a given position it is best for Player One to have the cube. It is next best for Player One if the cube is in the middle and it is worst for Player One for Player Two to have the cube. Therefore for a given position, Player One's expectation is greatest in Table 7-7, least in Table 7-5, and in between in

Diagram 9

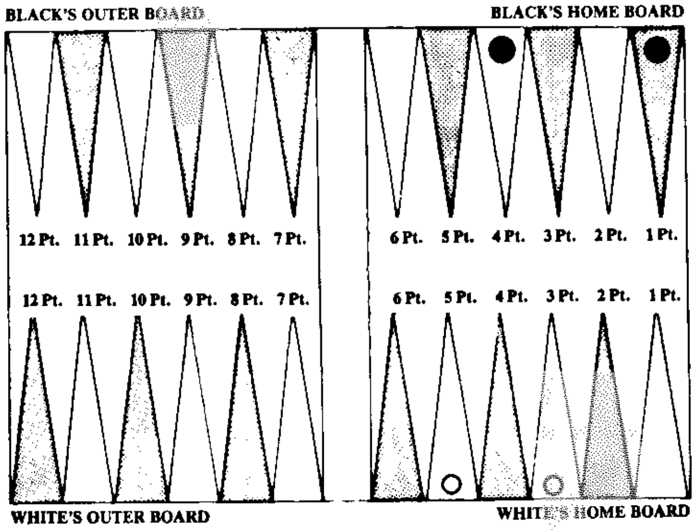


Table 7-6. For instance, with Player One having 6+6 and Player Two having 4+4, Player One's expectation is 25% if he has the cube, 16% if it is in the middle, and 7% if player Two has the cube.

Sometimes two or even all of the expectations are the same. For instance, if Player One has 6+6 and Player Two has 6+5, Player One's expectation is 71% if he has the cube or if it's in the middle. If Player Two has the cube Player One's expectation drops to 36%.

Examination of the doubling strategies in Table 7-8 shows that the positions where Player One should double and Player Two should fold are the same whether Player One has the cube or the cube is in the middle. Although this happens for the two-man end positions we are analyzing here, it is not always true in backgammon. The positions where Player One should double and it doesn't matter if Player Two accepts or folds also are the same in Table 7-8. But some of the positions where Player One should

double and Player Two should accept are different. If Player One has the cube, Table 7-8 shows that he should be more conservative. Intuitively, this is because if he has the cube and does not double, he prevents Player Two from doubling, whereas if the cube is in the middle, Player Two cannot be prevented from doubling.

Table 7-8 leads to an example that will confound the intuition of almost all players. Suppose Player One has 5+2 and has the cube. Consider two cases: (a) Player Two has 1+0 and (b) Player Two has 6+0. In which of these cases should Player One double? Clearly 6+0 is a worse position than 1+0. And the worse the position the more likely we are to double, right? So of the four possible answers (double 1+0 and 6+0, double 1+0 but not 6+0, double 6+0 but not 1+0, don't double 1+0 or 6+0) we "know" we can eliminate "double 1+0, don't double 6+0," right? WRONG. The only correct answer, from Table 7-8 is: double 1+0 but don't double 6+0. Try this on your expert friends. They will almost always be wrong. If they do get it right they probably were either "lucky" or read this chapter. In that case if you ask them to explain why their answer is correct, they probably won't be able to.

You may think that the loss would be slight by doubling 6+0 erroneously. But you have an expected gain of 29% by not doubling (Table 7-7) whereas by doubling it can be shown that your expectation drops to only 11%.

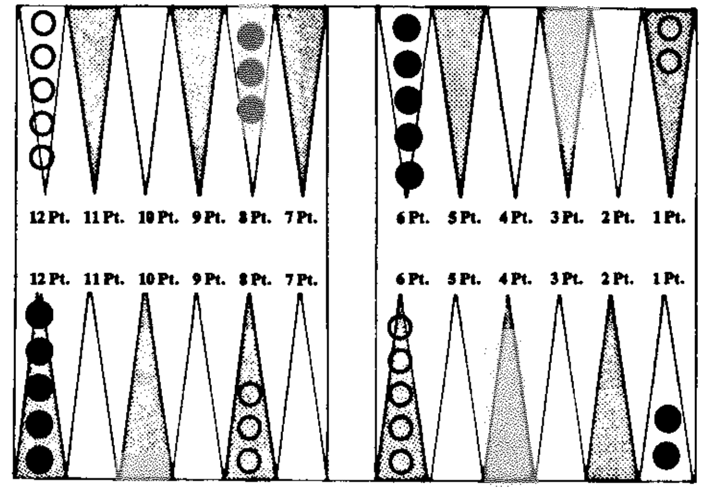
The exact explanation is complex. The basic idea, though, is that if Player One doubles Player Two, Player Two accepts, and Player One doesn't win at once, Player Two can use the cube against Player One with great effect at Players Two's next turn.

Jacoby and Crawford discuss what is essentially the same example (they give Player Two 4+1 instead of 6+0) on pages 116-117 of their excellent *The Backgammon Book,* Viking Press, New York, 1970. Table 7-8 shows that essentially the same situation occurs when Player One has 5+2 and Player Two has 4+1, 5+0, 6+0, 2+2 or 3+2 and for no other two-man end positions.

Tables 7-5, 7-6, 7-7, and 7-8 present the complete, exact solutions to two-man end games in backgammon. The tables were calculated by a general method I have discovered for getting the complete exact solution to all backgammon positions that are pure races (i.e. the two sides are permanently out of contact). The intricate and difficult computer programs for computing Tables 7-5 through 7-8 were written by Don Smolen so Tables 7-5 through 7-8 are our joint work. Don was a computer scientist at Temple University. He is now trading stock options on the floor of the American Stock Exchange. A skilled backgammon player, he won the 1977 American Stock Exchange tournament.

* * *

The Rules of Backgammon



The backgammon board is divided into two rectangles by the *bar*. Each side of the board contains six *points* of alternating colors. The game is played with light and dark pieces called *stones*, with the lighter color designated "White" and the darker color "Black." Each player has fifteen stones.

The stones are initially arranged as shown in the diagram above. In this case, the Black player would be seated at the bottom of the board, while White would be at the top. The points are numbered on the diagram for the sake of clarity; no numbers appear on an actual backgammon board. The six points in the upper righthand corner constitute Black's *inner table*. The six points at the lower right are White's *inner table*. The object of the game is to move your stones around the board until they are all in your inner table. In this case, White would move his stones counterclockwise and Black would progress clockwise. Once your stones are all in your inner table, you begin to *bear them off*, and the first player to remove all of his stones from the board wins the game.

Backgammon is played with two dice, which are shaken in and rolled from a cup. To begin play, each player rolls one die, and the high roll gets the first turn. The first move is determined by the two numbers which the opponents rolled. From then on, each player rolls two dice when it is his turn to move. The two numbers rolled dictate the players' moves as follows. Suppose the numbers on the dice are 5 and 3. The player may a.) move one stone five points and then three more, b.) move one stone three points then five more, or c.) move one stone five points and another stone three. When a double number is rolled, the player may make four moves. A roll of 3-3 would allow moving one stone 12 points, four stones 3 points each, or any other pattern involving groups of three.

When moving his stones around the board, no player may land on a point occupied by two or more *opposing* stones. Such a point is considered *made* by the opponent and often interferes with the way in which a player intended to take his turn. On the other hand, a point occupied by a single stone is a *blot*. This point is vulnerable to being *hit* by an opposing stone that lands on it. When a stone is hit, it is sent off the board and onto the bar, where it must remain until it can be entered on the opponent's inner table. The player whose stone has been hit is forbidden to make any other moves until he has entered his stone from the bar.

In order to enter, the player must roll a number which allows him to move to a point on his opponent's inner table which is not *made*. For example, say White has hit one of Black's blots, sending it to the bar. White has made points 3, 5, and 6 on his inner table. In this situation, Black must roll a 1, 2, or 4 in order to enter his stone.

Once a player has succeeded in moving all fifteen of his stones into his inner table, he may begin *bearing off*. This consists of removing stones from the points and off the board according to the numbers rolled. For example, a roll of 3-4 means that a player may remove stones from positions 3 and 4. If he has no stones on one or both of these points, he bears off from the next lowest point. The "race" to bear off continues until one player has taken

all his stones off the board. He, of course, is the winner.

A player is credited with having won a double game, or a *gammon*, if he bears off all his stones before his opponent has borne off any. If a player wins a game while his opponent still has a stone on the winner's inner table or on the bar, he has made a *backgammon* and wins triple the stakes.

If you play backgammon for money, a *doubling cube* is used. This cube bears the numbers 2, 4, 8, 16, 32, and 64. At any point during the game, one player may double the stakes. His opponent must either accept the double or forfeit the game. If he accepts, the opponent gains possession of the cube and may "turn the cube" back at *his* opponent whenever he feels he has the upper hand in the game. It's not hard to see that high stakes game can result very easily from a game in which the lead changes hands frequently.