An Introduction to Veech Surfaces

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We give a gentle introduction to the basics of Veech surfaces, with an emphasis on the Veech Dichotomy, followed by a sketch of the present state of the literature. These notes arose from lectures for a summer school held at the Institute de Mathématiques de Luminy in June 2003. We thank the participants, especially Jayadev Athreya who prepared an initial set of notes, and other speakers for various comments.

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Chapter 1

Introduction to Veech Surfaces

1.1 From Billiards to Flat Surfaces

1.1.1 Billiards

A seemingly innocuous problem is to analyze the billiard flow on rational-angle Euclidean polygons. That is, given a polygon whose angles are rational multiples of π, consider the trajectories of an ideal point mass, that moves at a constant speed without friction in the interior of the polygon and enjoys elastic collisions with the boundary — angles of incidence and reflection are equal.

For more on billiards and related matters, see [T] and [MT] as well as the sections of Eskin, Forni, Masur and Zorich.

1.1.2 Unfolding

We now describe the unfolding process for rational billiards. Given a billiard trajectory (that avoids the vertices) beginning at a side of a rational angle polygon, this yields a surface. The process has arisen in various guises, see in particular Katok and Zemlyakov, [KZ].

Given a collision with a side we reflect the *polygon* along the side, obtaining a mirror image of the original polygon, on which the billiard now continues in its original direction, instead of reflecting off the side. Continuing this process *ad infinitum*, we would obtain a laundry line (a ray in the plane), along which various copies of the polygon are strung. But, since our

polygon has rational angles, there are only finitely many possible angles of incidence of our chosen trajectory with these copies. Thus, the billiard eventually exits a copy of the polygon in a side that is parallel with the initial side. We now identify these sides by translation; we continue this process, considering any unpaired side that the billiards meets as the new initial side. The result is a new polygon with various 'opposite' sides identified; on this 'flat surface', the billiard moves along straight line segments, up to translation.

The 1-form dz on the complex plane induces a 1-form on our surface. There is a unique complex structure on the surface such that this 1-form is holomorphic. The process thus results in a Riemann surface with a distinguished abelian differential (that is, holomorphic 1-form). There is a close relationship between the flows on the flat surface and various properties of the 1-form.

Unfolding: Two Examples

First, let us consider billiards in the unit square, see Figure 1.1.2. Suppose our billiard trajectory starts near the bottom left corner (the origin) and has slope $1 > s > 0$. Thus it collides initially with the right side. We reflect about this side to get a mirror image of the square upon which our trajectory continues with this slope. The next side it hits is the top of the new (right) square; reflecting about that side we get a third square that sits above the second (bottom right) square. Continuing this procedure, we eventually end up with four copies of our original square; we can appropriately translate one of the copies so as to form a larger square. As an exercise, the reader should now check that we can follow all billiard paths within this larger square, if we identify opposite sides by translation. Thus, a torus is formed. Each trajectory of the billiard flow is mapped to a trajectory for the linear flow in the same direction on the torus.

If we now take the isosceles triangle with angles $(\pi/5, \pi/5, 3\pi/5)$ as our initial table, the unfolding process yields a star-shaped polygon with opposite sides identified, see Figure 1.1.2.

(The reader should note that differing billiard trajectories give apparently different polygons, but should show that these differences are accounted for
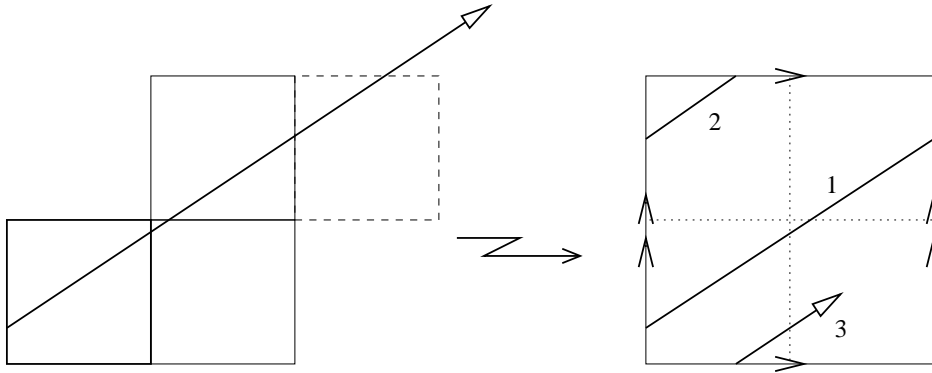
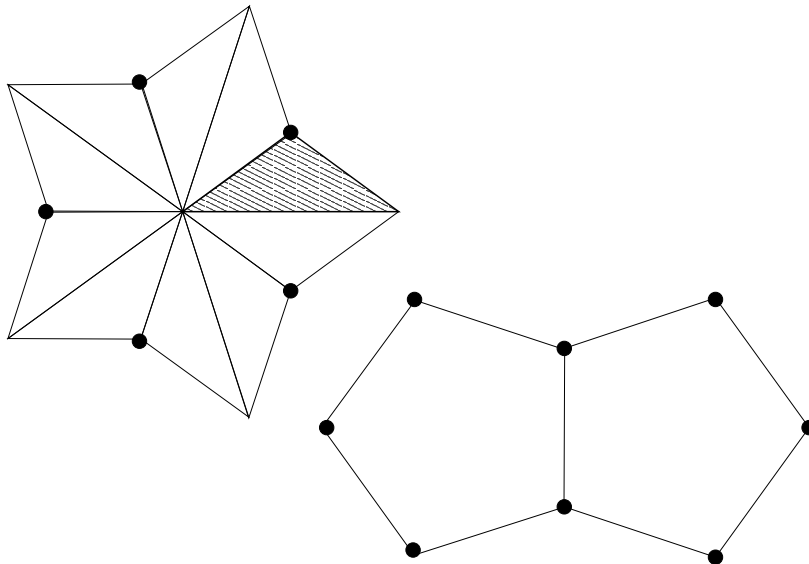Figure 1.1: Unfolding; square table to torus surface.

Figure 1.2: Surface from triangle; same translation surface. (Identify parallel sides by translation.)

by the translations of the various identified sides!) This is a compact, oriented topological surface. An easy Euler characteristic calculation shows that it has genus two.

The identifications of the sides lead to interesting identifications of the vertices. While the "outside" vertices of the stellated pentagon collapse to a point with angle 2π, the "inside" vertices yield a point with total angle 6π! (This phenomenon did not arise in our first example — the large square with its sides identified — as there the vertices are identified to a single point of angle 2π.) Indeed, a Gauss-Bonnet calculation will now confirm that our surface is of genus two.

This difference between our genus two and genus one examples reflects the fact that while the torus is naturally flat (its universal cover is the Euclidean plane $\mathbb{R}^2$), a genus 2 surface is naturally hyperbolic (universal cover $\mathbb{H}^2$), and *cannot be forced to be flat.*

1.1.3   From 1-forms to Surfaces

Now consider a pair (X, ω), a Riemann surface X with a holomorphic 1-form ω. Locally (i.e., in each coordinate patch) $\omega = f(w)dw$. Given a point $p_0 \in X$, we define new coordinates by the map

$$z(p) = \int_{p_0}^{p} \omega \,.$$

In these coordinates, $\omega = dz$ locally.

If we change base points in some small patch, then our coordinates change by a translation:

$$c := \int_{p_0}^{p} \omega - \int_{p_1}^{p} \omega = \int_{p_0}^{p_1} \omega \,.$$

Since c does not depend on p, our transition maps are of the form $z \mapsto z + c$. Thus the pair (X, ω) gives a structure which is reasonably called a *translation surface.*

We need to take care in the above discussion. At a zero of multiplicity k, locally we have $\omega = z^k dz$, hence $\omega = d(z^{k+1}/(k + 1))$. That is, instead of

the surface locally resembling the complex plane $\mathbb{C}$ (as it does away from the zeros), at a zero the surface instead locally resembles the $(k + 1)$-fold cover of $\mathbb{C}$ via the map $z \mapsto z^{k+1}$. Thus, the total angle around the zero is $2\pi(k+1)$.

By your favorite general theorem about Riemann surfaces (either Gauss-Bonnet or Riemann-Roch), the total number of zeros (counting multiplicity) of the abelian differential ω is $2g - 2$, where g is the genus of the surface X.

Fixing the orders of all zeros, we call the associated subset of translation surfaces a *stratum*. Thus, we have a stratum for each integer partition of $2g - 2$. See [M] for more discussion of these matters.

1.1.4 SL$(2, \mathbb{R})$-action and Veech Groups

The group $\mathrm{SL}(2, \mathbb{R})$ acts on the space of translation surfaces: a pair (X, ω) is given by its charts, with coordinate functions to the complex plane (and all transition maps are translations). We'll now consider $\mathbb{C}$ with its natural structure as the real plane. Given a matrix $A \in \mathrm{SL}(2, \mathbb{R})$, the new point $A \circ (X, \omega)$ is the surface whose charts are the charts for (X, ω), with coordinate functions post-composed with the linear action of A on $\mathbb{R}^2$. This action preserves orders of zeros, it thus preserves each stratum. Note that an element of $\mathrm{SO}(2, \mathbb{R})$ acts on a translation surface as a (piecewise) rotation; this action corresponds to multiplying ω by a nonzero complex number of norm one.

We denote the stabilizer of (X, ω) under the action of $\mathrm{SL}(2, \mathbb{R})$ by $\mathrm{SL}(X, \omega)$. Recall that $\mathrm{SL}(2, \mathbb{R})$ does not act faithfully on the upper half-plane; it is the projective group $\mathrm{PSL}(2, \mathbb{R})$ that does so. We define the *Veech Group*, $\mathrm{PSL}(X, \omega)$, to be the image of $\mathrm{SL}(X, \omega)$ in $\mathrm{PSL}(2, \mathbb{R})$.

Examples Revisited

For the torus, we consider the maps

$$(x, y) \mapsto (\, x, \, x + y \bmod 1 \,)$$

and

$$(x, y) \mapsto (\, x + y \bmod 1 \,, \, y \,).$$

These are *Dehn twists* about the curves corresponding to the x- and y-axes respectively. Their derivatives are given by the matrices $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ respectively. We have that $A_i \in \mathrm{SL}(\mathbb{C}/\mathbb{Z}^2, dz) = \mathrm{SL}(2, \mathbb{Z})$. The reader should verify this last equality!

For our genus two example, we can decompose the surface into two vertical cylinders of height and width (h_1, w_1) and (h_2, w_2), see Figure 1.1.4. On each cylinder we can define a Dehn twist via

$$(x, y) \mapsto (\, x, \ y + \mu^{-1}x \bmod h \,),$$

where following tradition, the *modulus* of the cylinder is $\mu = w/h$. Note that each Dehn twist is constant on the vertical sides of the corresponding cylinder; we can certainly glue them together to get a globally defined function. But, in order to preserve our flat structure, a diffeomorphism must have its derivative (off of the singularities) constant in our coordinates. We call such maps *affine diffeomorphisms*, and denote the group that they form by $\mathrm{Aff}(X, \omega)$.

Thus, in order to construct an affine diffeomorphism of the surface from these Dehn twists we must be able to take some power of each twist so that the resulting derivatives agree. For this, we must have $r\mu_1 = s\mu_2$ for some integers r and s; in words: *the moduli of the cylinders must be rationally related*. In this example, we get very lucky and the moduli are in fact the same. The reader is encouraged to check this trigonometry!

This stellated pentagon has its Veech group generated by an element of order five — the obvious rotation — and an element of order two. Can you find 'the' element of order two? On a related surface — the Golden Cross, see say [HS2] or [Mc2] — it acts as a square root of the famous "hyperelliptic involution" of the surface.

We must emphasize that it is very rare that the Dehn twists on cylinders match up to give a global affine diffeomorphism!
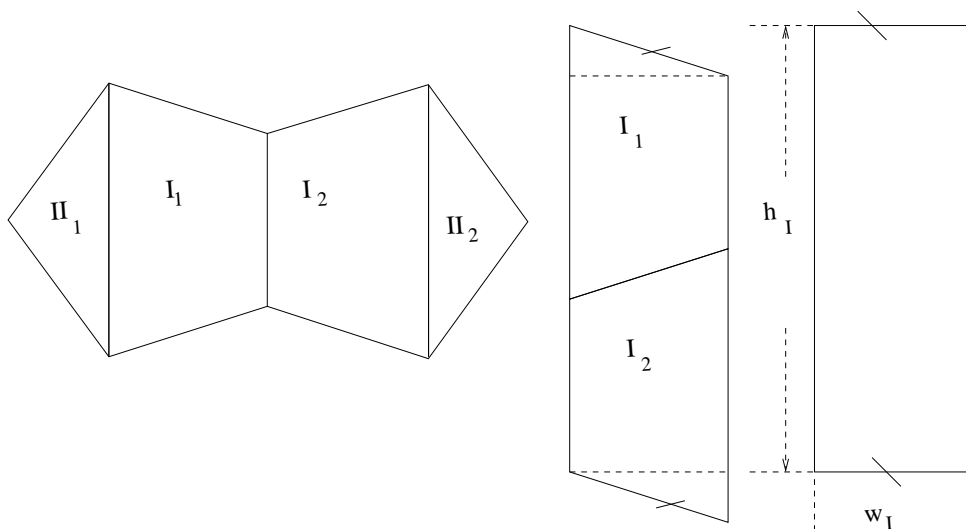
Figure 1.3: Vertical cylinders.

1.2 The Veech Dichotomy

Recall the theorem of Weyl for geodesic flow on the torus: in any rational direction θ, all orbits are closed, whereas the flow in any irrational direction is uniquely ergodic: it is ergodic with respect to a unique non-atomic measure, which is (induced by) Lebesgue measure. Veech proved an analogous result for a class of particularly nice surfaces.

We can define directions θ of flow on a given translation surface (X, ω): use the coordinate charts to pull-back from the real plane the straight lines of direction θ. The directional flow F_θ is the map from $X \times \mathbb{R}^+$ to X sending pairs (x, t) to x', where x' is length t from x along a line segment in the direction θ. Of course, the true definition of F_θ recognizes that the translation surface has singularities! It is a theorem of Kerckhoff-Masur-Smillie [KMS] that for a fixed translation surface (X, ω), for almost every direction θ the flow F_θ is uniquely ergodic. See [M] for related discussion.

We say that F_θ is periodic if the surface decomposes into a finite number of cylinders in the direction θ, and furthermore these cylinders have pairwise commensurable moduli: $\mu_i/\mu_j \in \mathbb{Q}$. Note that it is not necessary that the

actual period lengths of the cylinders be the same, nor even commensurable — as the vertical flow on our genus two example already shows!

Recall that the Veech group of (X, ω) is defined such that it acts on the hyperbolic plane. We say that such a group is a *lattice* if the quotient space under this action has finite (induced) hyperbolic area. In this setting, we also say that $\mathrm{SL}(X, \omega)$ is a lattice. (There are several ways of defining the term lattice; this definition works in our setting.)

Theorem 1. *Veech Dichotomy:* [1] *Let (X, ω) be a translation surface. Suppose $SL(X, \omega)$ is a lattice in $SL(2, \mathbb{R})$. Then for each direction θ, the flow F_θ is either periodic or uniquely ergodic.*

If $\mathrm{SL}(X, \omega)$ is a lattice, then (X, ω) is called a Veech surface. The theorem states that a Veech surface has dynamical properties similar to the touchstone surface, the square torus. In what follows, we'll sketch a proof — coming from Veech's original proof [Vch2], especially as adapted by Vorobets [Vor].

1.3   Structure of Veech Groups

A *separatrix* is a geodesic line emanating from a singularity, a *saddle connection* is a separatrix connecting singularities (with no singularities on its interior). To each saddle connection we can associate a *holonomy vector*: we 'develop' the saddle connection to the plane by using local coordinates, the difference vector defined by the planar line segment is the holonomy vector.

1.3.1   Discreteness

The following theorem seems to be in the folklore of the subject, our proof is modeled on that of Proposition 3.1 of [Vor]. See [M] for a second proof of this fundamental result.

Proposition 1. *Let (X, ω) be a translation surface. Then the set of holonomy vectors of saddle connections, $V_{sc}(X, \omega)$, is discrete in $\mathbb{R}^2$.*

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[1]The authors of [MT] have asked us to point out that this clarifies their statement of the Veech Dichotomy.

Sketch of Proof: We assume that the surface does admit singularities. Since there are only finitely many of these singularities, it is clear that every point p of the surface admits some positive $\epsilon(p)$ such that there is a punctured disk of radius $\epsilon(p)$ centered at p that is void of singularities.

Choose any vector $v \in \mathbb{R}^2$. At each singularity, form every geodesic ray of holonomy v. Each ray is in general a sequence of saddle connections followed by a separatrix. Since there are only finitely many singularities and the total angle at any of these is finite, there are only finitely many of these geodesic rays. Let $\epsilon = \min(\epsilon(p))$, where p runs over the endpoints of the paths of these geodesic rays.

Clearly, there is no saddle connection ending within the punctured ϵ-disk about the end point of any of our geodesic rays. But, this means that v cannot be the limit of holonomy vectors of saddle connections. Since v was arbitrary, we find that $V_{\mathrm{sc}}(X, \omega)$ is discrete. $\qquad\square$

1.3.2 Non-cocompactness

Again following Vorobets, one has an easy proof of the following result, originally due to Veech [Vch2].

Lemma 2. *Let (X, ω) be a translation surface. Then the group $SL(X, \omega)$ is a discrete subgroup of $SL(2, \mathbb{R})$*

Sketch of Proof: Any $A \in \mathrm{SL}(2, \mathbb{R})$ acts so as to send saddle connections of (X, ω) to saddle connections of $A \circ (X, \omega)$. Let $\{A_n\} \subset \mathrm{SL}(X, \omega)$ be a sequence approaching the identity (where $\mathrm{SL}(2, \mathbb{R})$ has its usual topology), $A_n \to I$. Let $v, w, \in V_{\mathrm{sc}}(X, \omega)$ be linearly independent. Then $A_n v \to v$ and $A_n w \to w$. By discreteness of $V_{\mathrm{sc}}(X, \omega)$, for n sufficiently large, $A_n v = v$ and $A_n w = w$. But v and w are linearly independent; they form a basis for $\mathbb{R}^2$. Hence, for all large n we have that $A_n = I$. We conclude that $\mathrm{SL}(X, \omega)$ is discrete. $\qquad\square$

Standard terminology: a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$ is a *Fuchsian group*.

Similarly, $\mathrm{SL}(X, \omega)$ is never cocompact: $\mathrm{SL}(X, \omega)$ being *cocompact* would simply mean that in the natural quotient topology $\mathrm{SL}(X, \mathbb{R})/\mathrm{SL}(X, \omega)$ is compact. We disprove this by finding a continuous (nonnegative) real valued function on $\mathrm{SL}(2, \mathbb{R})$ that is constant on cosets, but has no minimum value.

Consider the function $\Lambda : \mathrm{SL}(2, \mathbb{R}) \to \mathbb{R}^+$, given by $A \mapsto l(A \circ (X, \omega))$, where $l(X, \omega)$ denotes the length of the shortest saddle connection. If $\mathrm{SL}(X, \omega)$ were cocompact, the function Λ would have a minimum, say $\alpha > 0$.

But, take any saddle connection. We can normalize by rotating (X, ω) so that this saddle connection is in the vertical direction; we can send the length to zero via the *geodesic flow*: $g_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$. Since both rotation and geodesic flow are realized in $\mathrm{SL}(2, \mathbb{R})$, we clearly have a contradiction to the minimality of α. We conclude that $\mathrm{SL}(X, \omega)$ is not cocompact.

1.3.3   Parabolic Elements

It is a well-known fact for Fuchsian groups that any non-cocompact lattice must have a parabolic element; see, say, [K]. Conjugating the group, the parabolic fixed point may be taken to be infinity, the parabolic then acts as a translation; the quotient can be informally envisioned as having a cone with missing point at infinity, a *cusp*.

The following is a restatement of Lemma 3.7 of [Vor].

Lemma 3. *Let $\Gamma \subset SL(2, \mathbb{R})$ be a non-cocompact lattice, such that $g_t\Gamma$ is divergent (i.e., leaves every compact set) in $SL(2, \mathbb{R})/\Gamma$. Then there is a $\alpha \neq 0$ with $h_\alpha = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \in \Gamma$.*

Thus, if Γ is a lattice, the only way a trajectory of the geodesic flow on $\mathrm{SL}(2, \mathbb{R})/\Gamma$ can escape to infinity is via a cusp.

1.3.4   Affine Diffeomorphisms and Veech Groups

In fact, $\mathrm{SL}(X, \omega)$ is the group of derivatives of orientation-preserving affine diffeomorphisms. To sketch a proof of this, we take (X, ω) normalized such that X has area one with respect to the area form, $d\lambda$, induced by ω. Let ϕ be an orientation-preserving affine diffeomorphism of (X, ω). The derivative of ϕ is its Jacobian derivative in the usual sense. With the real structure of the translation surface, this derivative is a constant (off of the singularities)

2×2 real matrix. Thus

$$1 = \int_X d\lambda = \int_{\phi^{-1}(X)} |\mathrm{Jac}(\phi)| \, d\lambda = |\mathrm{Jac}(\phi)|.$$

Thus, the derivative of ϕ is of determinant one. In brief: Area preserving implies determinant one. (By the way, it is a significant fact that the "derivative" map has finite kernel in $\mathrm{Aff}(X, \omega)$, [Vch2]: any ϕ whose derivative is the identity is certainly an automorphism of the complex structure of X, in genus greater than one, there are only finitely many of these.)

1.4   Proof of the Veech Dichotomy

Rotations leave the underlying structure unchanged, we can thus suppose that the vertical direction is non-uniquely ergodic. This is only possible if $g_t\omega$ is divergent, that is if $g_t\mathrm{SL}(X, \omega)$ leaves every compact set of the quotient $\mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(X, \omega)$; this follows from *Masur's Criterion* , see Theorem 3 of [M] and the sketch of its proof, given in §3 there. This criterion is key to the proof; it is closely related to a combinatorial criterion of Boshernitzan for non-unique ergodicity of an interval exchange transformation [B], [Vch] and the discussion in [M].

By hypothesis, $\mathrm{SL}(X, \omega)$ is a lattice; by our basic facts, it has a parabolic element. In fact, since the vertical direction is divergent, there is a parabolic element of the type given in Lemma 3. The next lemma shows that the existence of a parabolic element implies important geometric information about the translation surface (X, ω).

Lemma 4. *Let h_α be as above. If $h_\alpha \in SL(X, \omega)$, then X decomposes into a finite number of vertical cylinders of moduli $\mu_i = \frac{p_i}{q_i}\alpha$, $p_i, q_i \in \mathbb{Z}$.*

Proof. Denote the affine map with derivative h_α by ϕ. Let Σ be the set of singular points on (X, ω). Then, ϕ acts by permutation on Σ. At each $p_i \in \Sigma$, we have outgoing *separatrices* — geodesics emanating from the singularities, see Figure 1.4. Let $\{L_1, L_2, \ldots, L_k\}$ denote the set of outgoing separatrices in the vertical direction. Then ϕ also acts on this set by permutation; by passing to a power $\psi = \phi^n$, we can assume that ψ fixes both every singularity and each of the L_i.
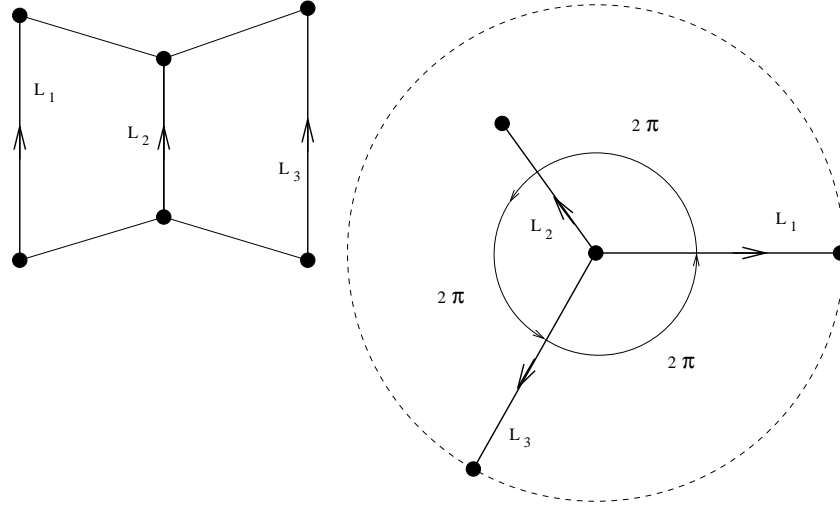
Figure 1.4: Vertical saddle connections. (Three outgoing, giving also three incoming.)

The affine diffeomorphism ψ acts up to translation exactly as its derivative; the derivative fixes the vertical direction, and hence ψ restricted to any L_i acts as a pure translation. Since a translation with a fixed point can only be the identity, we conclude that ψ fixes each vertical separatrix L_i pointwise.

We claim that each L_i is in fact an outgoing saddle connection. Indeed, if a separatrix L is not a saddle connection, then it must in fact be dense in some open subset U of X. But if L_i is dense in some U, then ψ is identity on U; since $h_\alpha \neq I$, this leads to a contradiction.

Next, we claim that ALL vertical leaves are closed. Consider an arbitrary point $p \in X$ not lying on any of our L_i. Let $\mathcal{F}_t$ denote the vertical flow on X. If $\mathcal{F}_t(p)$ is not closed, then it is dense in some minimal component — see the proof of Theorem 1.8 of [MT]. On the other hand, $\mathcal{F}_t(p)$ does not encounter any singularity, as we have assumed that p is not on any of the L_i. Hence, p flows in parallel with the L_i; in particular, the distance of any $\mathcal{F}_t(p)$ to the L_i cannot be made arbitrarily small. Thus, $\mathcal{F}_t(p)$ is certainly not dense; it must be closed.

We now have a cylinder decomposition of (X, ω) in the vertical direction. The powers of the affine Dehn twist of a given vertical cylinder are of derivative $\begin{pmatrix} 1 & 0 \\ k\mu & 1 \end{pmatrix}$ where μ is the modulus. Since $d\psi = \begin{pmatrix} 1 & 0 \\ n\alpha & 1 \end{pmatrix}$ is constant, the moduli of the various vertical cylinders are all rational multiples of α. $\quad\square$

So we have the Veech Dichotomy: if the flow is not uniquely ergodic, it gives a divergent trajectory in $\mathbb{H}/\mathrm{PSL}(X, \omega)$, thus there is a parabolic element in $\mathrm{SL}(X, \omega)$, and we can then decompose our surface into cylinders with commensurable moduli.

Remark 1. Note that the Theorem leads to a simple necessary condition for a surface to be Veech: in each direction with a cylinder decomposition, the moduli of the cylinders must be commensurable. That is, if there are two cylinders with moduli $\mu_1, \mu_2, \mu_1/\mu_2 \notin \mathbb{Q}$, we are not on a Veech surface. In fact, a Veech surface has a cylinder decomposition in the direction of any of its saddle connections.

Consider our basic example, the square torus. In this case, $\mathrm{SL}(X, \omega) = \mathrm{SL}(2, \mathbb{Z})$; it is thus a lattice, and Veech's result recovers the result we mentioned as a theorem of Weyl.

1.5 Arithmeticity

1.5.1 Theorem of Gutkin and Judge

For surfaces that can be tiled by squares — called, most simply, *square-tiled surfaces* —, we have that $\mathrm{SL}(X, \omega)$ is *commensurate* to $\mathrm{SL}(2, \mathbb{Z})$ (the groups share a common finite index subgroup) and thus (X, ω) is a Veech surface. Any lattice that has a $\mathrm{SL}(2, \mathbb{R})$-conjugate commensurate to $\mathrm{SL}(2, \mathbb{Z})$ is called *arithmetic*. (This weaker type of relationship between groups is called *commensurability*.) Let us say that a surface (X, ω) is *tiled by parallelograms* if it is in the $\mathrm{SL}(2, \mathbb{R})$ orbit of a square-tiled surface.

One has the following theorem of Gutkin-Judge, for a simple proof see [HL].

Theorem 5. (Gutkin–Judge) *The surface (X, ω) is tiled by parallelograms if and only if $SL(X, \omega)$ is arithmetic.*

In particular this theorem proves that all square-tiled surfaces are Veech, since any arithmetic group is a lattice. This implies that any square-tiled surface satisfies the Veech alternative; this difficult result had previously been shown by Veech [Vch] using Boshernitzan's criterion.

1.5.2   Consequences and Examples

Note that an arithmetic group need not be contained in $\mathrm{SL}(2, \mathbb{Z})$. For example, consider the surface given by two unit volume squares placed one on top of the other. This is a degree 2 cover of the torus, with a one-cylinder decomposition, of modulus $1/2$. Thus, in $\mathrm{SL}(X, \omega)$ we have the element $\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$, that is obviously not in $\mathrm{SL}(2, \mathbb{Z})$.

Another square-tiled surface provides a cautionary example. There exist oriented affine diffeomorphisms of parabolic derivative that are not formed by taking powers of Dehn twists in the cylinder decomposition of the corresponding fixed direction. (However, as Veech [Vch] showed, some finite power of such an affine diffeomorphism is given in such a manner.) Consider the genus two square-tiled surface formed by 3 squares stacked one on top of the other, with top and bottom identified, and side segments identified such that there is a single singularity of total angle 6π. Then one can show that there is an affine diffeomorphism of derivative $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$; however, it is the cube of this matrix that corresponds to the fundamental vertical Dehn twist here. For more on this, see [HL].

The Gutkin-Judge result implies that any surface of arithmetic Veech group is a branched cover of the torus, with branching above one sole point. In general there are surfaces that have the same (or commensurate) Veech group, but are not related by any tree of finite covers that are "balanced", see [HS2].

The group $\Gamma = \left< \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right>$ is not commensurable to any Veech group [GHS]. Indeed, it is known that any Veech group with a hyperbolic element of trace in $\mathbb{Q}$ must be arithmetic [KS, Mc], and in particular a lattice. The group Γ however, is not a lattice, but possesses hyperbolic ele-

ments. Note that any finite-index subgroup H of Γ then includes hyperbolic elements with rational trace. The same is thus true for any group commensurable to Γ, and our result follows.

In any fixed stratum, the set of square-tiled surfaces of that stratum is dense. Indeed, integration of ω along its periods relative to the singularities provides local coordinates for the stratum, see [E]; these coordinates are contained in $\mathbb{Q} + i\mathbb{Q}$ exactly when (X, ω) is square-tiled. Thus, density of $\mathbb{Q} + i\mathbb{Q}$ in $\mathbb{C}$ gives the result. On the other hand, Gutkin and Judge gave an argument showing that in any stratum the set of Veech surfaces is of measure zero (if $g \geq 2$) — see [M] for the definition of this measure. This is loosely analogous to the fact that the rationals are of measure zero in the real numbers.

1.5.3   Non-arithmetic Surfaces Exist

Non-arithmetic lattice Veech groups exist. In fact, our other favorite example — the surface arising from the $(\pi/5, \pi/5, 3\pi/5)$-triangle —, has Veech group that contains $< S, R >$, where S is the aforementioned diffeomorphism that induces the Dehn twist on each of the two vertical cylinders, and R the order five rotation. In fact, this is the entire Veech group. This group is a lattice; moreover, it is non-arithmetic.

This Veech group is (conjugate to) a well-known group, a so-called *Hecke group*. The Hecke group of index n is $\Gamma_n = < z \to -1/z, z \to z + 2\cos(\pi/n) >$. The group above is conjugate to Γ_5. In fact, Veech showed that each Hecke group of odd index n, as well as a subgroup of index two in each even index case, is also realized as a Veech group. All but three of these are non-arithmetic groups, and are known to be pairwise incommensurable [L].

Chapter 2

State of the Art

In this new century, two perspectives on Veech groups have been fruitful. The first, of a longer tradition, employs so-called scissor invariants of linear flows on the translation surface (X, ω). The second, pioneered by McMullen [Mc], emphasizes the algebro-geometric aspects of the Riemann surface X imposed by characteristics of $\mathrm{SL}(X, \omega)$.

2.1 Background: Scissor Invariants

Kenyon and Smillie [KS] introduced an invariant for translation surfaces, called the *J-invariant*; this invariant is an extension of the Sah-Arnoux-Fathi invariant used for the study of interval exchange transformations. Calta [Ca] has recently used the *J*-invariant to characterize the Veech surfaces in the stratum of genus 2 surfaces with a single singularity; this stratum is denoted $\mathcal{H}(2)$, see §2 of [M].

Definition 1. Let P be a planar polygon of vertices $v_1, \ldots, v_n$. We define $J(P)$ as $v_1 \wedge v_2 + v_2 \wedge v_3 + \cdots + v_{n-1} \wedge v_n + v_n \wedge v_1 \in \mathbb{R}^2 \wedge_{\mathbb{Q}} \mathbb{R}^2$.

This is indeed a scissors invariant, in the following sense.

Proposition 2. *Suppose that $P = P_1 \cup \cdots \cup P_k$ is a cellular decomposition of P into polygons P_i. Then $J(P) = J(P_1) + \cdots + J(P_k)$.*

Now, any translation surface can be given as a finite union of polygons, with appropriate side identification; indeed, some authors define the notion of translation surface in this way, see Definition 4 of [M]. If (X, ω) is a

translation surface, and $(X, \omega) = P_1 \cup \cdots \cup P_k$ is a cellular decomposition of Σ into polygons P_i, then we define $J(X, \omega)$ as the sum of the $J(P_i)$.

Theorem 6. (Kenyon-Smillie) *The value $J(X, \omega)$ is independent of choice of polygonal cellular decomposition of (X, ω).*

One has the possibility of studying various projections of the J-invariant. In particular, the Sah-Arnoux-Fathi invariant can be recovered in this manner. Consider

$$\pi_{xx} : \mathbb{R}^2 \wedge \mathbb{R}^2 \quad \to \mathbb{R} \wedge \mathbb{R}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \wedge \begin{pmatrix} c \\ d \end{pmatrix} \quad \mapsto a \wedge c \, .$$

We define J_{xx} as $\pi_{xx}(J)$ and J_{yy} analogously. Let $T : I \to I$ be an interval exchange transformation on a real interval I, with the lengths of the ith subinterval denoted by λ_i, $1 \le i \le n$. For $i \in \{1, \ldots, n\}$, let $t_i \in \mathbb{R}$ denote the translation applied to the ith subinterval. The Sah-Arnoux-Fathi invariant is defined as

$$\mathrm{SAF}(T) = \sum_{j=1}^{n} \lambda_j \wedge t_j \in \mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R} \, .$$

The set of all interval exchange transformations on I forms a group under composition of functions; Arnoux [A2], see also [A], showed that the SAF-invariant defines a group homomorphism to $\mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$. Furthermore, since the commutator subgroup of the group of interval exchange transformations is a simple group, the SAF-invariant gives what is essentially the only non-trivial homomorphism defined on the group.

The fundamental property of the SAF-invariant is its invariance under induction:

Proposition 3. (Arnoux) *Let T be an interval exchange transformation on an interval I, and suppose that $K \subset I$ is a subinterval that meets every orbit of T. Let S denote the interval exchange transformation induced on K by T. Then $SAF(S) = SAF(T)$.*

The following is crucial in the work of Calta.

Remark 2. One easily shows that if T is periodic, then $\mathrm{SAF}(T) = 0$. Furthermore, an interval exchange transformation T of *three* subintervals is periodic if and only if $\mathrm{SAF}(T) = 0$. This last is directly related to rotations: let R_α denote the rotation of angle $\alpha \in \mathbb{R}$; this map of the circle to itself is periodic if and only if $\alpha \in \mathbb{Q}$.

Note, however, Arnoux and Yoccoz [AY] constructed an interval exchange transformation T of 7 subintervals with $\mathrm{SAF}(T) = 0$, but such that T is minimal, and in fact uniquely ergodic. The geometry of this interval exchange transformation is extremely interesting, see [A3].

The invariance under induction of interval exchange transformation of the SAF-invariant affords the possibility of defining an SAF-invariant for a measured foliation $\mathcal{F}$ of a surface: Choose a normalized full transversal I for $\mathcal{F}$, thus in particular this interval I meets all leaves of $\mathcal{F}$, and define $\mathrm{SAF}(\mathcal{F}) = \mathrm{SAF}(T)$, where T is the interval exchange transformation defined on I by the first return map along leaves of $\mathcal{F}$. This invariant is independent of choice of I.

Kenyon and Smillie easily show the following.

Proposition 4. *Let (X, ω) be a translation surface. Then $J_{xx}(X, \omega)$ equals the SAF-invariant for the vertical foliation of (X, ω); similarly, $J_{yy}(X, \omega)$ equals the SAF-invariant for the horizontal foliation of (X, ω).*

It is deft use of the J-invariant that allows Kenyon-Smillie to reach the main result of [KS], that in turn lead to the following sobering result.

Theorem 7. (Kenyon–Smillie, Puchta) *Suppose that T is an acute, non-isosceles, rational-angled triangle, and that (X, ω) is the translation surface associated to T by the usual unfolding process. Then (X, ω) is a Veech surface if and only if T has angles:*

(a) $(\pi/4, \pi/3, 5\pi/12)$, (b) $(\pi/5, \pi/3, 7\pi/15)$, or (c) $(2\pi/9, \pi/3, 4\pi/9)$.

Kenyon and Smillie also show that an acute, isosceles, rational-angled triangle gives a Veech surface if and only if the smallest angle is of the form π/n.

2.2 Results of Calta

A translation surface (X, ω) is said to be *completely periodic* if for every direction whose linear flow admits a periodic orbit, and hence a cylinder, (X, ω) admits a decomposition into cylinders in this direction. Clearly, Veech surfaces are completely periodic. The converse is in general false; consider the slit torus examples of [M], see also [HS3], [Mc2]. However, one has the following.

Theorem 8. (Calta) *A translation surface belonging to $\mathcal{H}(2)$ is completely periodic if and only if it is a Veech surface.*

Furthermore, in this stratum, every non-arithmetic Veech surface is "quadratic" in the sense that up a change within the $\mathrm{SL}(2,\mathbb{R})$-orbit, all of its (absolute) periods are contained in some real quadratic field. Here, the *absolute periods* of (X,ω) are the periods of ω: $p(\gamma) = \int_\gamma \omega$ with $\gamma \in \mathrm{H}_1(X,\mathbb{Z})$; thus the result is that $p(\mathrm{H}_1(X,\mathbb{Z})) \subset \mathbb{Q}(\sqrt{d}) \times \mathbb{Q}(\sqrt{d})$, with $d > 0$ a non-square integer. Amongst all quadratic translation surfaces, Calta gives equations distinguishing the Veech surfaces.

The main idea of the proof is to introduce the following intermediate property. Here, given a direction v, the projection J_{vv} is defined analogously to J_{xx} and J_{yy}.

Definition 2. A direction is called a *homological direction* for (X,ω) if it is the direction of some absolute period of ω. A translation surface has *Property X* if for every homological direction v one has $J_{vv} = 0$.

Every periodic direction of course has a representative in $p(\mathrm{H}_1(X,\mathbb{Z}))$; Property X may be thought of as being "virtually" completely periodic — every direction that is a candidate to be completely periodic passes the test of vanishing of the corresponding projection of the J-invariant.

Calta's proof of Theorem 8 consists of showing that for translation surfaces of $\mathcal{H}(2)$ the three properties are equivalent: Property X, completely periodic, Veech. One easily shows that Property X does imply complete periodicity here — this is an application of Remark 2, and strongly depends on the genus being 2. The converse is significantly more complicated, and Calta uses explicit quadratic equations. A number theoretic argument shows that the $\mathrm{SL}(2,\mathbb{R})$-orbit of a translation surface with Property X is closed in $\mathcal{H}(2)$; by *Smillie's Theorem*, announced in [Vch3], the surface must then be Veech.

An analogous discussion allows Calta to show that the completely periodic surfaces of the remaining stratum of genus 2 translation surfaces, $\mathcal{H}(1,1)$, are also quadratic, and to again give explicit equations.

One can give a geometric interpretation of Calta's work, that can be compared to the appearance of Hilbert modular surfaces in the work of Mc-Mullen, see below. Beginning with a completely periodic surface in $\mathcal{H}(1,1)$, consider the $\mathrm{SL}(2,\mathbb{R})$-orbits of the surface found by fixing the absolute periods and deforming the relative periods; here "relative" means relative to the

singularities. Thus, one considers the $SL(2, \mathbb{R})$-orbits of the various surfaces found by varying the position of the zeros of ω. The result, $\mathcal{M}$, is a *closed* sub-manifold of $\mathcal{H}(1,1) \cup \mathcal{H}(2)$ of real dimension 5. The intersection of $\mathcal{M}$ with $\mathcal{H}(2)$ is a finite union of $SL(2, \mathbb{R})$-orbits of Veech surfaces.

2.3 McMullen's Approach

The approach emphasized by McMullen [Mc] studies properties of the Riemann surface X implied by hypotheses on the group $SL(X, \omega)$. Any affine diffeomorphism ϕ of (X, ω) is such that the pull-back map ϕ^* acts on $H^1(X, \mathbb{R})$ so as to preserve the two dimensional real subspace V generated by the real and imaginary parts of ω. If ϕ has derivative $D\phi$ hyperbolic of trace t, then $T^* := \phi^* + (\phi^*)^{-1}$ acts on V as multiplication by t. McMullen relates this to the structure of the endomorphism ring of the Jacobian of X.

2.3.1 Algebro-Geometric Background

We briefly recall some standard terminology and results from algebraic geometry, see the textbooks [Ha], [GrHa], [FK]; the classic reference on abelian varieties is [Mu]; for a constructive treatment of real multiplication see [BL], as well as [R]. See [Hi] or [vdG] for an introduction to the study by the school of F. Hirzebruch of the geometry and arithmetic of Hilbert modular surfaces. Our discussion closely follows §4 of [Mc3].

The Jacobian

Key to our discussion is the g-complex dimensional vector space $\Omega(X)$ of 1-forms on a Riemann surface X of genus g. Indeed, whereas the results discussed so far are related to the flat structure induced on X by integration of a single 1-form, we now fix a base point and consider integration of a vector whose entries form a basis for $\Omega(X)$. This gives a map to $\mathbb{C}^g$ that is only well-defined after dividing by the lattice formed by the integrals along closed curves. The result is the famed *Abel-Jacobi map* from X to the complex torus defined as the *Jacobian variety* of X, $\mathrm{Jac}(X)$.

The celebrated *Riemann Relations* show that $\mathrm{Jac}(X)$ is a *principally polarized abelian variety*: It is in particular a complex torus equipped with an embedding into complex projective space. Expressing $\mathrm{Jac}(X)$ as $\Omega^*(X)/\mathrm{H}_1(X, \mathbb{Z})$,

one avatar of the polarization is as a symplectic form on $H_1(X, \mathbb{Z})$. In fact, the intersection pairing on $H_1(X, \mathbb{Z})$ gives this symplectic form. Of course, as real vector spaces, $\Omega^*(X)$ and $H_1(X, \mathbb{R})$ are isomorphic; we can thus view $\Omega^*(X)$ as $H_1(X, \mathbb{R})$ with a complex structure. See chapter 4 of [Cl] for a discussion of related canonical isomorphisms.

Real Multiplication by a Field; Eigenforms

Given any principally polarized abelian variety $A \cong \mathbb{C}^g / \Lambda$, the polarization of A equips $\Lambda \cong H_1(A, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ with a symplectic form. The *endomorphism ring* $\mathrm{End}(A)$ consists of the Lie group homomorphisms of A; each endomorphism respects the Hodge decomposition $H^1(A, \mathbb{C}) \cong H^{(1,0)} \oplus H^{(0,1)}$ and induces an endomorphism of Λ.

   A field K is called *totally real* if it is a number field all of whose embeddings fixing $\mathbb{Q}$ have image in $\mathbb{R}$. Given a totally real field K with $[K : \mathbb{Q}] = g$, we say that A admits *real multiplication* by K if there is a faithful representation $\rho : K \to \mathrm{End}(A) \otimes \mathbb{Q}$ such that each $\rho(\kappa)$ is self-adjoint with respect to the induced symplectic form on $\Lambda \otimes \mathbb{Q}$. The holomorphic 1-forms on A form the g-dimensional $\mathbb{C}$-vector space $\Omega(A) \cong H^{(1,0)}$. Since $\rho(K)$ respects the Hodge decomposition, K acts on $\Omega(A)$ in a complex linear fashion. An eigenvector for this action is called an *eigenform* for the real multiplication of A. The action can always be diagonalized: $\Omega(A) = \oplus \mathbb{C}\omega_i$ for g eigenforms ω_i, thus there are eigenforms for any real multiplication.

   In the case that $A = \mathrm{Jac}(X)$, we can speak of $\omega \in \Omega(X)$ as being an eigenform. Indeed, given real multiplication on $\mathrm{Jac}(X) \cong \Omega(X)^* / H_1(X, \mathbb{Z})$, one finds that the eigenforms are exactly the eigenvectors for the dual action on $\Omega(X)$. The *eigenform locus* in $\Omega\mathcal{M}_g$ is the space of (X, ω) with ω an eigenform.

Remark 3. With only slight complication of the above, one can define real multiplication on an abelian variety of complex dimension g by a product K of totally real fields K_i, with $\sum [K_i : \mathbb{Q}] = g$.

Endomorphisms to Real Multiplication

The integral points $\mathfrak{o} = K \cap \mathrm{End}(A)$ of elements of K which act as endomorphisms of A form an *order* of K. That is, $\mathfrak{o}$ is a finite-index subring of $\mathcal{O}_K$, where $\mathcal{O}_K$ is the product of the rings of algebraic integers of the K_i. Of course, given an order $\mathfrak{o} \subset K$, and any faithful representation of $\mathfrak{o}$ as

self-adjoint endomorphisms of A, there is an induced real multiplication of A by K.

Indeed, suppose that some totally real algebraic integer t acts as an endomorphism T on an abelian variety A. Then one finds that $\mathbb{Z}[t] \subset \text{End}(A)$, by extending the map $t \mapsto T$ in the usual manner. Tensoring with $\mathbb{Q}$, one finds that A admits real multiplication by $\mathbb{Q}(t)$. Thus, a single endomorphism can induce real multiplication by a field.

Families with Real Multiplication by an Order

The appropriate level of abstraction is obtained by fixing a symplectic form on a lattice $L \cong \mathbb{Z}^{2g}$, and considering the injective homomorphisms ρ which send $\mathfrak{o}$ to $\text{End}(L)$ as self-adjoint endomorphisms. One then says that A admits *real multiplication by* $(\mathfrak{o}, \rho)$ if there is a symplectic isomorphism of L with $H_1(A, \mathbb{Z})$ such that $\rho(\mathfrak{o})$ coincides with the restriction of $\text{End}(A)$.

The space of all abelian varieties admitting real multiplication by some $(\rho, \mathfrak{o})$ can be determined in the following constructive manner. Tensoring the rank two $\mathfrak{o}$-module L with $\mathbb{R}$ allows us to find a decomposition into orthogonal eigenspaces, each of real dimension two: $L \otimes \mathbb{R} \cong \oplus_{i=1}^{g} S_i$. Fix i, and choose some positively ordered symplectic basis (a_i, b_i) for S_i; to each $\tau_i \in \mathbb{H}$, we have an $\mathbb{R}$-linear map from $\mathbb{C}$ to S_i induced by sending 1 to a_i and τ_i to b_i. Note that in particular this map respects the orientation of $\mathbb{R}^2 \cong S_i$.

Each $\tau := (\tau_1, \ldots, \tau_g) \in \mathbb{H}^g$ thus determines an isomorphism of real vector spaces that takes $L \otimes \mathbb{R}$ to $\mathbb{C}^g$ and thus induces a symplectic structure on $\mathbb{C}^g$; the image of $L \otimes 1$ is a lattice. The quotient, A_τ, of $\mathbb{C}^g$ by this lattice has real multiplication by $(\mathfrak{o}, \rho)$.

Every abelian variety admitting real multiplication by $(\mathfrak{o}, \rho)$ arises in this fashion. Indeed, given some $A = \mathbb{C}^g/\Lambda$, take Λ as L and use the symplectic form given by the principal polarization. Choose an integral basis for Λ and a compatible splitting of $\mathbb{C}^g$; we may assume that the basis of Λ is of the form $(1, b_i)$ with $b_i \in \mathbb{H}$. With $\tau = (b_1, \ldots, b_g)$, we find that $A_\tau = A$.

Hilbert Modular Varieties

Given L and $(\mathfrak{o}, \rho)$ as above, let $\text{Sp}(L \otimes \mathbb{R}) \cong \text{Sp}(2g, \mathbb{R})$ denote the $\mathbb{R}$-linear operators on $L \otimes \mathbb{R}$ which respect the symplectic form. Those symplectic automorphisms that commute with the action of $\mathfrak{o}$ preserve the splitting $L \otimes \mathbb{R} \cong \oplus_{i=1}^{g} S_i$. Therefore, each such automorphism acts on the set of

complex structures on $L \otimes \mathbb{R}$ that are compatible with the splitting. Since these complex structures are indexed by $\mathbb{H}^g$, one finds that the subgroup of symplectic automorphisms that commute with the action of $\mathfrak{o}$ is the image of an injective homomorphism $\iota : \mathrm{SL}(2, \mathbb{R})^g \to \mathrm{Sp}(L \otimes \mathbb{R})$. The integral points $\Gamma(\mathfrak{o}, \rho) := \iota(\mathrm{SL}(2, \mathbb{Z})^g)$ are exactly the automorphisms of the symplectic $\mathfrak{o}$-module L. The group $\Gamma(\mathfrak{o}, \rho)$ acts isometrically on $\mathbb{H}^g$ as elements of $\mathrm{SL}(2, \mathbb{Z})^g$, the finite volume quotient $X(\mathfrak{o}, \rho) := \Gamma(\mathfrak{o}, \rho) \backslash \mathbb{H}^g$, called the *Hilbert modular variety* of $(\mathfrak{o}, \rho)$, parametrizes pairs $(A, \mathfrak{o} \to \mathrm{End}(A))$ compatible with ρ. There is a natural forgetful map from $X(\mathfrak{o}, \rho)$ to $\mathcal{A}_g$, the coarse moduli space of principally polarized abelian varieties — one forgets the maps $\mathfrak{o} \to \mathrm{End}(A)$.

Multiplication by a Real Quadratic Order

When $g = 2$, there are two facts that simplify the above. First, it is well-known that the orders $\mathfrak{o}$ in real quadratic fields are uniquely determined by their discriminants $D = D(\mathfrak{o}) \in \mathbb{Z}$; we thus write $\mathfrak{o}_D$. Second, for each such $\mathfrak{o}_D$, there is essentially a unique representation $\rho_D : \mathfrak{o}_D \to \mathbb{Z}^4$ which respects the standard symplectic form on $\mathbb{Z}^4$; see say Theorem 2 of [R]. One thus finds a single Hilbert modular surface for each discriminant, $X_D := X(\mathfrak{o}_D, \rho_D)$.

Furthermore, one can give an explicit model for each of these. Let σ denote the non-trivial element in $\mathrm{Gal}(K/\mathbb{Q})$; for $M \in \mathrm{SL}(2, K)$, let M^σ denote the matrix whose elements are the images by σ of the corresponding elements of M. Then $\mathrm{SL}(2, K)$ acts on $\mathbb{H}^2$ by $M \circ (z_1, z_2) = (Mz_1, M^\sigma z_2)$, where elements of $\mathrm{SL}(2, \mathbb{R})$ act on $\mathbb{H}$ in the usual manner. One can show that $X_D \cong \mathrm{SL}(2, \mathfrak{o}_D) \backslash \mathbb{H}^2$.

For each of these X_D, the forgetful map to $\mathcal{A}_2$ is generically 2-to-1: homomorphisms from $\mathfrak{o}_D$ to $\mathrm{End}(A)$ are conflated with their compositions with σ. This forgetful map factors through the *symmetric Hilbert modular surface* formed as the quotient of X_D by the involution induced by the standard permutation on $\mathbb{H} \times \mathbb{H}$. The image variety in $\mathcal{A}_2$ is called a *Humbert surface*, after the work of G. Humbert in the late 19th century.

2.3.2   McMullen's Action by the Trace Field

With ϕ an affine diffeomorphism of hyperbolic derivative $D\phi$ having trace t, consider $T = \phi_* + (\phi_*)^{-1}$ acting on $\mathrm{H}_1(X, \mathbb{R})$. Since ϕ preserves intersections, it is easy to show that T is self-adjoint with respect to the corresponding

symplectic form. Since the pull-back of any affine diffeomorphism leaves $V \subset H^1(X, \mathbb{R})$ invariant, T leaves invariant the annihilator of V, defined as the space of cycles upon which all elements of V vanish.

In genus two, the annihilator and its orthogonal complement are both of real dimension two, giving thus complex lines in $\Omega^*(X)$. The self-adjoint T acts on each of these eigenspaces as multiplication by a real number. That is to say, T induces an endomorphism of $\mathrm{Jac}(X)$. When t is quadratic over $\mathbb{Q}$, the map $t \mapsto T$ as discussed in the treatment of real multiplication in §2.3.1 shows that $\mathrm{Jac}(X)$ admits real multiplication by $K = \mathbb{Q}(t)$.

The field K is independent of choice of hyperbolic element in $\mathrm{SL}(X, \omega)$; see the appendix of [KS] for the following: since those ϕ with $D\phi$ hyperbolic are in fact *pseudo-Anosov* maps, earlier results allow one to prove both that (1) K is the full *trace field* of $\mathrm{SL}(X, \omega)$, defined as the field generated by adjoining to $\mathbb{Q}$ the traces of all elements of the group; and, (2) $[K : \mathbb{Q}] \leq g$. Furthermore, see say Lemma 8 on p. 167 of [FLP], t is an algebraic integer.

2.3.3 Projecting Orbits to $\mathcal{M}_g$ and $\mathcal{A}_g$

The projection $\pi : \Omega\mathcal{M}_g \to \mathcal{M}_g$ is constant on orbits of $\mathrm{SO}(2, \mathbb{R})$. On the other hand, the stabilizer of $z = i$ under the transitive action of $\mathrm{SL}(2, \mathbb{R})$ by Möbius transformations on the Poincaré upper half-plane, $\mathbb{H}$, is $\mathrm{SO}(2, \mathbb{R})$. There is thus a map $\mathbb{H} \to \mathcal{M}_g$ that factors through $\mathrm{SL}(X, \omega)\backslash\mathbb{H}$. In fact, it is of great importance that this image in $\mathcal{M}_g$ is isometrically immersed with respect to the so-called Teichmüller metric, see [EG] for discussion of this metric in terms related to $\mathrm{SL}(2, \mathbb{R})$. The image in $\mathcal{M}_g$ is an algebraic curve if and only if $\mathrm{SL}(X, \omega)$ is a lattice, in which case this image is called a *Teichmüller curve* in $\mathcal{M}_g$.

The *Torelli map* $\tau : \mathcal{M}_g \to \mathcal{A}_g$ is defined by sending each X to $\mathrm{Jac}(X)$; for a discussion of the geometry of this map, see [Mu2]. In dimension $g = 2$, in fact $\mathcal{A}_2 = \tau(\mathcal{M}_2) \sqcup H_1$, where H_1 is the locus of abelian varieties that split as a product of two polarized elliptic curves. In particular, the Torelli map has dense open image in $\mathcal{A}_2$; there is thus a tendency in the literature to slur over the distinction of certain loci as being in one or the other of the spaces $\mathcal{M}_2$ and $\mathcal{A}_2$. For simplicity, call the map $\Omega\mathcal{M}_g \to \mathcal{A}_g$, given by composing the Torelli map with π, the projection to $\mathcal{A}_g$.

2.3.4   A Selection of Results

The fundamental observation of McMullen is that as soon as a translation surface (X, ω) with X of genus 2 admits a hyperbolic element in $\mathrm{SL}(X, \omega)$, then $\mathrm{Jac}(X)$ admits real multiplication by the trace field of $\mathrm{SL}(X, \omega)$, with ω an eigenform for this multiplication. The following result, false in higher genus, is crucial to McMullen's study in genus two.

Theorem 9. *(McMullen) The eigenform locus in $\Omega \mathcal{M}_2$ is $SL(2, \mathbb{R})$-invariant.*

The main result of McMullen on Teichmüller curves in $\mathcal{M}_2$ is the following.

Theorem 10. *(McMullen) Suppose that $SL(X, \omega)$ is a non-arithmetic lattice and X is of genus 2. Then the $SL(2, \mathbb{R})$-orbit of (X, ω) projects to $\mathcal{A}_2$ to be an algebraic curve contained in some symmetric Hilbert modular surface.*

In fact, Remark 3 can be invoked to show that in genus 2 if $\mathrm{SL}(X, \omega)$ is arithmetic, then $\mathrm{Jac}(X)$ admits real multiplication by $\mathbb{Q} \times \mathbb{Q}$, and the $\mathrm{SL}(2, \mathbb{R})$-orbit then projects to an appropriate symmetric Hilbert modular surface [Mc3].

The previous theorem easily leads to the following result, which can also be deduced from Calta's results.

Theorem 11. *(McMullen) Suppose that $(X, \omega) \in \mathcal{H}(2)$. If there is a hyperbolic element in $SL(X, \omega)$, then (X, ω) is a Veech surface.*

The situation is completely different for $\mathcal{H}(1, 1)$. Indeed, let $\mathcal{D}$ denote the translation surface given by identifying by translation opposite sides of the regular decagon. In [Mc4], McMullen conjectured, and in [Mc5] proves, the following.

Theorem 12. *(McMullen) The only non-arithmetic Veech surface of $\mathcal{H}(1, 1)$ is $\mathcal{D}$.*

McMullen [Mc4] gives an algorithm for determining those (X, ω) whose $\mathrm{SL}(2, \mathbb{R})$-orbit projects to a Hilbert modular surface for a given discriminant of order. In particular, he shows that Veech's original examples of a double pentagon and a double decagon account for all lattice groups giving rise to curves on the symmetric Hilbert modular surface of real multiplication by the order with discriminant $D = 5$.

Remark 4. For reasons of time and space, we have not discussed an important aspect of the projections of $SL(2, \mathbb{R})$-orbits in $\Omega\mathcal{M}_g$ to each of $\mathcal{M}_g$ and A_g: These projections are isometries for the appropriate metrics. This result is due to Kra [Kr]. This isometry is in some sense what allows one to use the structure of the homogenous space $\mathcal{A}_g$ to study Veech groups. As well, there are many curves in moduli space, but very few of them are isometrically embedded with respect to the Teichmüller metric.

Using the above, McMullen [Mc3] proves an analog of the celebrated Ratner Theorem, see [E].

Theorem 13. *(McMullen) The closure of the $SL(2, \mathbb{R})$-orbit of any $(X, \omega) \in \Omega\mathcal{M}_2$ projects to $\mathcal{M}_2$ as exactly one of the following: an algebraic curve; a Hilbert modular surface; all of $\mathcal{M}_2$.*

In recent work, M. Möller [Moe] has extended McMullen's result for lattice $SL(X, \omega)$. In particular, for $g > 2$, he shows that even though the action by the trace field identified by McMullen may not extend to the full Jacobian of X, it does identify special properties, which he studies in terms of variation of Hodge structures. (For an introduction to this study of splittings of bundles generalizing the study of the Hodge decomposition, see [Voi].) An *isogeny* of an abelian variety is a surjective morphism of algebraic varieties to some abelian variety, and this morphism is a group homomorphism, of finite kernel. (Isogenous abelian varieties are thus morally equivalent.)

Theorem 14. *(Möller) Suppose that $SL(X, \omega)$ is a lattice. Then the $SL(2, \mathbb{R})$-orbit of (X, ω) projects to $\mathcal{A}_g$ to be an algebraic curve contained in the locus parametrizing abelian varieties A splitting up to isogeny to a product $A_1 \times A_2$, where A_1 admits real multiplication by the trace field of $SL(X, \omega)$.*

2.4 Infinitely Generated Veech Groups

In [Vch3], Veech asked if a $SL(X, \omega)$ can ever be an infinitely generated Fuchsian group. This has recently been answered in the affirmative, [HS3], [Mc2].

Theorem 15. *([HS3]) For each genus $g \geq 4$, there exist $(Y, \alpha) \in \Omega\mathcal{M}_g$ with $SL(Y, \alpha)$ infinitely generated. In particular, the genus four translation surface arising from the triangle of angles $(3\pi/10, 3\pi/10, 2\pi/5)$ has infinitely generated Veech group.*

Theorem 16. (McMullen) *Suppose that $(X, \omega) \in \Omega\mathcal{M}_2$ is such that $SL(X, \omega)$ admits a hyperbolic element. Then the limit set of $SL(X, \omega)$ is the full boundary $\partial\mathbb{H}$. Furthermore, there exist infinitely many distinct $(X, \omega) \in \Omega\mathcal{M}_2$ with $SL(X, \omega)$ infinitely generated.*

2.4.1   Commonalities of Proofs

Other than the specifics of the examples, the proofs of these two results have common logic, both beginning with the fact that a non-lattice Fuchsian group whose limit set is all of $\partial\mathbb{H}$ must be infinitely generated. Now, it is often quite easy to show that the Veech group of a given translation surface is not a lattice: simply exhibit a saddle connection in whose direction the surface does not admit a decomposition into cylinders of commensurable moduli.

To show that the limit set of the Veech groups under consideration in the two theorems have all of $\partial\mathbb{H}$ as limit sets, both proofs show that the *parabolic directions* of the corresponding translation surfaces — that is, the directions for which there is a cylinder decomposition with commensurable moduli, and thus a corresponding parabolic element in the group — form a dense set in the unit circle of all directions. In both cases, one exhibits some point $p \in X$ such that every direction in which there is a separatrix passing through p is in fact a parabolic direction. This is the difficult step in each proof.

2.4.2   Sketch: Proof of Theorem 16

Suppose that X is of genus two and $SL(X, \omega)$ admits a hyperbolic element, of trace say t. Let $K = \mathbb{Q}(t)$ be the trace field. By results of the appendix of [KS], one can assume that the relative (to the singularities of ω) periods of ω on X lie in $K(i)$. Let ϕ be an affine diffeomorphism corresponding to the hyperbolic element. As in the previous section, $T^* := \phi^* + (\phi^*)^{-1}$ acts as multiplication by t on V, the real subspace spanned in $H^1(X, \mathbb{R})$ by the real and imaginary parts of ω. Once again, we let σ denote the non-trivial Galois group element. One finds that T^* thus acts as multiplication by $\sigma(t)$ on the subspace V^σ spanned by the real and imaginary parts of $\sigma(\omega)$. Since T^* is appropriately self-adjoint, V and V^σ are orthogonal. One thus has that the integral over X of each of $\omega \wedge \sigma(\omega)$ and $\omega \wedge \sigma(\overline{\omega})$ is zero, where the bar here denotes complex conjugation. From this, $\int_X \rho \wedge \sigma(\rho) = 0$ when ρ is the closed real form associated to any directional flow of slope in $\mathbb{P}^1(K) = K \cup \{\infty\}$.

However, if f is the interval exchange transformation on a transversal of the measured foliation associated to ρ, then $\int_X \rho \wedge \sigma(\rho) = \text{flux}(f)$, where flux$(f)$ is a version of the SAF-invariant introduced by McMullen, the *Galois flux*: Suppose that all the translations for some interval exchange transformation T are contained in some quadratic number field K, then one defines

$$\text{flux}(T) = \sum_{j=1}^{n} \lambda_j \sigma(t_j) \in \mathbb{R} \ .$$

Now, if this flux vanishes, then the directional flow for ρ cannot be uniquely ergodic. But, Masur's criterion now tells us that $g_t \text{SL}(X, \omega)$ leaves every compact set. This implies in turn that there are very short saddle connections on the corresponding translation surfaces $g_t \circ (X, \omega)$ for large t. Using the quadratic nature of K, elementary Diophantine approximation considerations (to wit: quadratic numbers cannot be well-approximated by rationals) then allow McMullen to conclude that for t sufficiently large, such a short saddle connection must in fact lie in the direction of the foliation. Restricting to genus 2, he then can give a complete analysis of such loops, to conclude that either the foliation is periodic, or else surgery along a leaf presents (X, ω) as a connected sum of irrationally foliated tori. In particular, it turns out that if there is a Weierstrass point lying on a saddle connection in the direction of flow for ρ, then this a parabolic direction.

However, (upon developing (X, ω) such that a singularity lies at the origin, every developed image of) each non-singular Weierstrass point has co-ordinates in K. Thus, any separatrix passing through a non-singular Weierstrass point lies in a direction whose slope is in $\mathbb{P}^1(K)$. From the above, this direction is hence a parabolic direction. But, for any given point of a translation surface, the directions of separatrices passing through this point are dense, see say Lemma 1 of [HS3]. The density of parabolic limit points then follows.

Remark 5. A side-product of the above is that a Veech surface of genus two defined over $\mathbb{Q}(\sqrt{d})$ allows a normalization such that the set of slopes of its periodic directions equals $\mathbb{Q}(\sqrt{d}) \cup \{\infty\}$, see also [Ca]. This is specific to genus two, see [AS].

McMullen [Mc3] gives an infinite family of genus two translation surfaces of infinitely generated Veech group by explicit construction, see Figure 1 there. Indeed, given 3 squares, of side length 1, a and $a + 1$ respectively,

one can place these squares so as to construct a genus two surface. If a is irrational of the form $b - 1 + \sqrt{b^2 - b + 1}$ for non-zero $b \in \mathbb{Q}$, then the Veech group of the translation surface is infinitely generated.

2.4.3   Sketch: Proof of Theorem 15

On the other hand, the proof of Theorem 15 constructs examples by use of ramified covers of Riemann surfaces $f : Y \rightarrow X$: the pull-back $\alpha = f^*(\omega)$ can have an infinitely generated group even if $\mathrm{SL}(X, \omega)$ is a lattice. (Some background for this can be found in [HS].) Indeed, suppose that the ramification is at the singularities of ω and at a point p — called a *connection point* — such that every separatrix of (X, ω) passing through p extends to a saddle connection. Again by Lemma 1 of [HS3], this is a dense set of directions. Since $\mathrm{SL}(X, \omega)$ is a lattice, the direction of any saddle connections is a parabolic direction; one easily shows that each of our dense set of parabolic directions for (X, ω) is a lattice, the direction of any saddle connections is a parabolic direction; one easily shows that each of our dense of parabolic directions for (X, ω) is also a parabolic direction for (Y, α). It follows that the parabolic limit points of $\mathrm{SL}(Y, \alpha)$ are dense.

The main part of the proof of Theorem 15 consists of showing that there are (X, ω) with connection points p such that the corresponding $\mathrm{SL}(Y, \alpha)$ is not a lattice. For this, it suffices to show that one can find points that are at the same time connection points and have infinite orbit under the group of oriented affine diffeomorphisms. Amusingly enough, the genus two example of Figure **??** admits such points. After an innocuous normalization, these are the points of coordinates in $\mathbb{Q}(\sqrt{5})$ (other than the regular Weierstrass points, which are given by the middle of the sides). This results from the fact that the parabolic (limit) points of Γ_5 (recall that this is the Veech group here, up to a normalization) is $\mathbb{Q}(\sqrt{5})$, [L]. This latter fact can be recovered by direct use of Remark 5. By way of [HS], one then finds that the translation surface to which the triangle angles $(3\pi/10, 3\pi/10, 2\pi/5)$ unfolds is a ramified cover of the genus two example, with ramification above singularities and connection points.

In [HS4], it is shown that the geometry of the projection to $\mathcal{M}_g$ of the $\mathrm{SL}(2, \mathbb{R})$-orbit of such (Y, α) is very complicated: $\mathrm{SL}(Y, \alpha)$ has infinitely many non-equivalent parabolic points and infinitely many "infinite ends".

2.5 Classification

The fundamental classification problem of determining when two given translation surfaces are in the same $SL(2, \mathbb{R})$-orbit seems far from being resolved. Indeed, this remains open even for square-tiled surfaces, with the exception of the stratum $\mathcal{H}(2)$.

In the setting of square-tiled surfaces, it suffices to classify the *primitive* square-tiled surfaces: those such that the lattice generated by their relative periods is $\mathbb{Z}^2$. One easily shows that in this setting $SL(X, \omega) \subset SL(2, \mathbb{Z})$. There is an action of $SL(2, \mathbb{Z})$ on the set of primitive square-tiled surfaces of fixed number of squares, n; two such surfaces are in the same $SL(2, \mathbb{R})$-orbit if and only if they are in the same $SL(2, \mathbb{Z})$-orbit.

In $\mathcal{H}(2)$, the position of the Weierstrass points give an invariant for the $SL(2, \mathbb{Z})$-action. Informally: given a surface of our type, we develop in such a manner that singularity lies at the origin, the six Weierstrass points then each has coordinates that are integers are half-integers. To be more precise, one explicitly parametrizes the square-tiled surfaces of $\mathcal{H}(2)$, as in [EMS], [Z].

Proposition 5. ([HL]) *The number of integer coordinate Weierstrass points of a square-tiled surface of $\mathcal{H}(2)$ is invariant under the action of $SL(2, \mathbb{Z})$.*

If the number n of square tiles is even, there are two such Weierstrass points; if n is odd, there are either three or one such point. The invariant completely classifies the orbits.

Theorem 17. ([HL], McMullen) *Given an integer $n \geq 3$, the square-tiled surfaces of $\mathcal{H}(2)$ form two $SL(2, \mathbb{Z})$-orbits if n is odd and $n \geq 5$; they form a single orbit if either n is even or $n = 3$.*

The theorem was first proved in [HL] for prime n. McMullen generalized this to not only square-tiled surfaces, but also so as to give an analogous result for all Veech surfaces of $\mathcal{H}(2)$.

Combining Theorem 17 with a counting formula given by [EMS] shows that the genus of Teichmüller curves defined by primitive square-tiled surfaces tends to infinity with the number of tiles. This can be compared with the fact that there are no explicitly known Teichmüller curves of positive genus arising from non-arithmetic surfaces of $\mathcal{H}(2)$. (One expects that in fact almost all of these are of positive genus.)
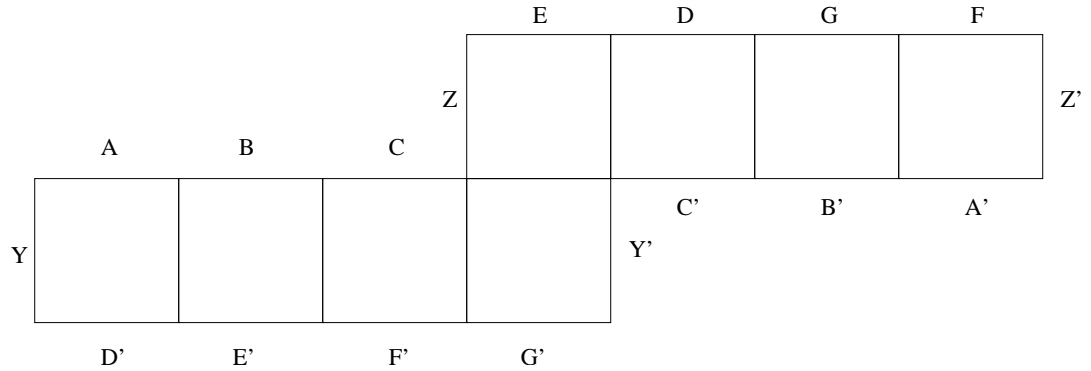
Figure 2.1: A square-tiled surface with $\mathrm{SL}(X, \omega) = \mathrm{SL}(2, \mathbb{Z})$.

One can also show the group $\mathrm{SL}(X, \omega)$ for a primitive square-tiled surface is a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$ only in the case of surfaces of three square tiles. See [S] for an example of a non-congruence subgroup, and [HL2] for the general case. Nevertheless, there are non-trivial examples of square-tiled surfaces whose group is exactly the full group $\mathrm{SL}(2, \mathbb{Z})$, see [S]. There has been work on this phenomenon by Herrlich, Schmoll, as well as by Möller. We thank M. Möller for kindly providing Figure 2.1, which represents one such surface.

2.6   Questions

We conclude with some more open questions.

1. Is there a general converse to the Veech Dichotomy (as found by Mc-Mullen for genus $g = 2$)?

2. Which Fuchsian groups are realized as Veech groups?

3. Is there an algorithm for determining the Veech group of a general translation surface?

4. Do there exist non-trivial Veech groups without parabolic elements?

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