COMPLEX AND UNPREDICTABLE CARDANO

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At a purely instrumental level, quantum theory is all about multiplication, addition and taking mod squares of complex numbers called probability amplitudes. The rules for combining amplitudes are deceptively simple. When two or more events are independent you multiply their respective probability amplitudes and when they are mutually exclusive you add them. Whenever you want to calculate probabilities you take mod squares of respective amplitudes. That's it. If you are prepared to ignore the explanatory power of the theory (which you should not) the rest is just a set of convenient mathematical tools developed for the purpose of book-keeping of amplitudes. Thus we tabulate amplitudes into state vectors and unitary matrices, and place them in Hilbert spaces. We introduce tensor products, partial traces, density operators and completely positive maps, and often get lost in between. Make the formalism as complicated as you wish, still, at the bottom of it you will find only two basic ingredients - complex numbers and probabilities - both of them discovered, amazingly enough, by the same person...

1. Gambling scholar

Girolamo Cardano, a 25 year old medical student from Padua, knew the tricks of the trade and yet he was losing money at an alarming rate. When he finally discovered that the cards were marked he did not hesitate to draw his dagger. He stabbed the cheat in the face and forced his way out of the gambling den into the narrow streets of Venice, recovering his money on the way. Running for his life in complete darkness he slipped and plunged into muddy waters of a canal. Not the best place to be, especially if you cannot swim. It was sheer luck that he managed, somehow, to grab the side of a passing boat and get himself lifted to safety by a helpful hand. However, once he regained his posture on the board, Cardano found himself facing the man with a bandaged face, the same who had cheated him at the gambling table few hours previously. Perhaps it was the chill of the night that cooled the tempers or perhaps neither of the two wanted trouble with the notoriously strict Venetian authorities, the fact is, there was no brawl. Instead, Cardano was given clothing and travelled back to Padua in an agreeable company of his fellow gambler.

Girolamo Cardano

Yes, Cardano could have popped out of a cloak and dagger novel, but he did exist. He was born in 1501 in Pavia, rose to become one of the finest minds of his time, and died in squalor and solitude in 1576 in Rome. The gambling incident in Venice took place in September 1526 and was carefully recorded fifty years later in Cardano's autobiography, *De vita propria liber* (The book of my life). From the same autobiography we learn that Cardano was "...hot tempered, single minded, and given to woman,... cunning, crafty, sarcastic, diligent, impertinent, sad and treacherous, miserable, hateful, lascivious, obscene, lying, obsequious,.." and "...fond of the prattle of old men." In the chapter dedicated to "stature and appearance" we learn that he was a man of medium height with narrow chest, long neck and exceedingly thin arms. His eyes were very small and half-closed, and his hair blond. He had high-pitched and piercing voice, and suffered from insomnia. He was afraid of heights and "...places where there is any report of a mad dog having been seen". The narrative veers from his conduct, appearance, diet, and sex life to meetings with supernatural beings and academic intrigues. A patchy but surprisingly readable account of the mind-set of a Renaissance man.

Cardano grew up in and around Milan. His father, a proud Milanese and a friend of Leonardo da Vinci, was a lawyer by training but his interests were mostly in mathematics and occult. His mother, Cardano wrote in his autobiography, "...was easily provoked...quick of memory and wit". In 1520 he enrolled at the University of Pavia, about twenty miles south of Milan, to study medicine. At the time Milan, which few decades earlier had enjoyed wealth and prosperity under the Sforza family, had become a battleground for French and Spanish armies, repeatedly struck by plaque and other disasters. Amid uncertainties and lack of funds the University of Pavia closed temporarily in 1523 and Cardano transferred to Padua, which was effectively the university of the Republic of Venice. This is when he became known for excesses at the gaming tables. Card games, dice and chess, which, by his own admission, occupied him almost every day, were the methods he used to make a living.

2. Probability

We do not know for sure, but it could have been the Venetian incident that prompted Cardano to write notes on probability. He knew that cheating at cards and dice was a risky endeavor, so he learned to win "honestly" by applying his discoveries concerning probabilities. His *Liber de ludo aleæ* (The book on games of chance), is a compilation of his scattered writings on the subject, some of them written as early as 1525, some of them later, around 1565 or so. What survived is a treatise of 32 short chapters that were rescued from the pile of posthumous manuscripts, collated and included in the magnificent edition of Cardano's extant works, ten large folio volumes, published in 1663. It contains the first study of the principles of probability, the first attempt to quantify chance.

Of course, games of chance and the drawing of lots were discussed in a number of ancient texts and a number of mystics, loonies and mathematicians enumerated the possible outcomes of various games. The snag was, most of these enumerations were not enumerations of equally likely cases, so they could hardly be used to calculate odds in a systematic way. Cardano was more careful. He started with the notion of fairness or, as he put it, "equal conditions":

> The most fundamental principle of all in gambling is simply equal conditions, e.g. of opponents, of bystanders, of money, of situation, of the dice box and of the die itself. To the extent to which you depart from that equity, if it is in your opponent's favour, you are a fool, and if in your own, you are unjust.

In the simplest case of two players with equal stakes, the game is fair if the number of favorable and unfavourable outcomes is the same for each player. More generally, Cardano argued, fairness requires that the stakes in an equitable wager should be in proportion to the number of ways in which each player can win. He then went on to find fair odds for wagering with dice. He correctly enumerated the various possible throws, i.e. 6 for one die, 6×6 for two dice, and $6 \times 6 \times 6$ for three dice. For example, when discussing the case of rolling two symmetric dice he wrote

> ...there are six throws with like faces, and fifteen combinations with unlike faces, which when doubled gives thirty, so that there are thirty-six throws in all,...

Trivial? Perhaps, but, for the time, Cardano showed remarkable understanding that the outcomes for two rolls should be taken to be the 36 ordered pairs rather than the 21 unordered pairs. In contrast, as late as the 18th century the famous French mathematician Jean le Rond d'Alembert (1717 – 1783), author of several works on probability, made a silly mistake claiming that when

FIGURE 1. Caravaggio *The Cardsharps* c. 1594, oil on canvas, The Kimbell Art Museum, Fort Worth, Texas, USA. Two cheats and one dupe, beautifully painted by the young Caravaggio. One cheat, who concealed extra cards behind his back, plays with the unworldly boy while his accomplice peeps at the victim's hand and signals with his fingers. This was Cardano's world.

a coin is tossed twice the number of heads that turn up would be 0, 1 or 2, which he viewed as three equiprobable outcomes.[1] Cardano chose the correct sample space for his dice problems and effectively defined probability, or the odds, if you wish, as an appropriate ratio of favorable and unfavorable cases. For example,

> If therefore, someone should say, I want an ace, a deuce, or a trey, you know that there are 27 favourable throws, and since the circuit is 36, the rest of the throws in which these points will not turn up will be 9; the odds will therefore be 3 to 1.

Here the "circuit" is the number of possible elementary outcomes, that is, the size of the sample space, and the favorable outcomes are all throws which result in at least one face showing one, two or three points. In other parts of the text he also quantifies odds as a ratio of favorable to all possible cases.

Cardano's careful enumerations provided, at the very least, good explanations why certain numbers of points were more advantageous than others. This was something many dice-players had known from their experience, and even though they could relate it to the number of ways the throws can come out their counting was very problematic. For example, it was known, and regarded as puzzling[2], that in a throw of three dice the sum of points is more likely to be 10 than 9, even though there are six ways in which the sum can be nine

$$1+2+6, \quad 1+3+5, \quad 1+4+4, \quad 2+2+5, \quad 2+3+4, \quad 3+3+3, \tag{1}$$

[1]J. d'Alembert "Croix ou Pile" in *L'Encyclopédie*, ed. Diderot, vol. 4 (Paris, 1754).

[2]Galileo Galilei was explicitly asked, by one of the gambling noblemen at the court in Florence, to explain this puzzle, and so he did in his brief *Considerazione sopra il Giuoco dei Dadi*, written around 1620.

and there are also six ways for the sum to be ten,

$$1+4+5, \quad 1+3+6, \quad 2+4+4, \quad 2+2+6, \quad 2+3+5, \quad 3+3+4. \tag{2}$$

The fact that the outcomes should be taken to be ordered triples (27 of which sum up to ten but only 25 to nine) was not well understood. Thus even if Cardano's discussion had been limited to calculating the correct chances on dice, astragals and cards, it could have been regarded as a great achievement, but he went further than that. He made several insightful general statements about the nature of probability. For example, he realized that when the probability of an event is p, then by a large number n of repetitions the number of times the event will occur is not far from np. Although claims that he anticipated the laws of large numbers are difficult to justify, it is clear that his intuition was leading him in the right direction. The most remarkable part of *Liber de ludo aleæ* is Cardano's discussion of the probabilities for repeated throws of dice. It led him, after few unsuccessful attempts, to the correct power formula; given the probability p of a success in a single trial the probability of n successes in n independent trials is p^n. We can follow the process of his discovery in the text as it goes by trial and errors and he did not hide the errors; on the contrary, they are brought to reader's attention by chapter headings such as "On an Error Which I Made About This".

All this was written more than a century before a certain Chevalier de Méré, an expert gambler, consulted Blaise Pascal (1623–1662) on some "curious problems" in games of chance. Pascal wrote to his older colleague Pierre de Fermat (1601–1665), and it was through their correspondence, as we are often told, the rules of probability were derived. The thing is, *Liber de ludo aleæ* appeared in print over eighty years after Cardano's death and about nine years after Pascal's first letter. Thus, it is reasonable to assume that it had no impact on the subsequent development of the subject. However, in all fairness, one should recognize the fact that Cardano was the first to calculate probabilities correctly and the first to attempt to write down the laws of chance. According to Øystein Ore, a Norwegian mathematician who elucidated many obscure parts of Cardano's gambling studies, it would be more just to date the beginning of probability theory from *Liber de ludo aleæ* rather than the correspondence between Pascal and Fermat.[3] I certainly agree with that.

3. Three obvious axioms

Cardano's "definition" of probability as a ratio of favorable to all possible outcomes is perfectly acceptable as long as you know (somehow) that all elementary outcomes are equiprobable. But how would you know? In many physical experiments the assumption of equiprobability can be supported by underlying symmetry or homogeneity. If we toss coins or roll dice we often assume they are symmetrical in shape and therefore unbiased. However, Cardano himself pointed out that "every die, even if it is acceptable, has its favoured side". No matter how close a real object resembles a perfect Platonic die, for mathematicians this approach is far from satisfactory for it is circular - the concept of probability depends on the concept of equiprobability.

You may be surprised to learn that the search for a widely acceptable definition of probability took nearly three centuries and was marked by much controversy. In fact the meaning of randomness and probability is still debated today. Are there genuinely random, or stochastic, phenomena in nature or is randomness just a consequence of incomplete descriptions? What does it really mean to say that the probability of a particular event is, say, 0.75? Is this a relative frequency with which this event happens? Or is it the degree to which we should believe the event will happen or has happened? Is probability objective or subjective?

Most physicists would *probably* (and here I express my degree of belief) vote for objective probability. Indeed, physicists even *define* probability as a relative frequency in a long series of independent repetitions. But how long is long enough? Suppose you toss a coin 1000 times and wish to calculate

[3]Øystein Ore, *Cardano, the Gambling Scholar* (Princeton, 1953).

the relative frequency of heads. Is 1000 enough for convergence to "probability" to happen? The best you can say is that the relative frequency will be close to the probability of heads with at least such and such probability. Once again, a circular argument. Is there a way out of this vicious circle?

If you are prepared to forget about the meaning of probabilities and focus on the form rather than substance then the issue was resolved in the 1930's, when Andrey Nikolaevich Kolmogorov (1903–1987) put probability on an axiomatic basis in his monograph with the impressive German title *Grundbegriffe der Wahrscheinlichkeitsrechnung* (Foundations of Probability Theory). The Kolmogorov axioms are simple and intuitive. Once you identify all elementary outcomes, or events, you may then assign probabilities to them. Probability is a number between 0 and 1, and an event which is certain has probability 1. These are the first two axioms. There is one more. The probability of any event can be calculated using a deceptively simple rule - the additivity axiom:

> Whenever an event can occur in several mutually exclusive ways, the probability for the event is the sum of the probabilities for each way considered separately.

Obvious, isn't it? So obvious, in fact, that probability theory was accepted as a mathematical framework theory, a language that can be used to describe actual physical phenomena. Physics should be able to identify elementary events and assign numerical probabilities to them. Once this is done you may revert to mathematical formalism of probability theory. The Kolmogorov axioms will take care of the mathematical consistency and will guide you whenever there is a need to calculate probabilities of more complex events. This is a very sensible approach apart from the fact that it does not work! Today, we know that probability theory, as ubiquitous as it is, fails to describe many common quantum phenomena. The main culprit, as we shall see soon, is our innocuous and "obvious" additivity axiom. In order to fix the problem we need another mathematical tool, namely, complex numbers. They were discovered as a by-product of a fascinating search for an algebraic solution to the cubic equation, and this brings us back to Cardano.

4. The cubic equation

In 1526 Cardano was awarded his doctorate in medicine. The first few years of his medical practice, in a small village near Padua, were difficult, but eventually fortune smiled on him and he was appointed a public lecturer in mathematics in Milan, a position sponsored by the Piatti foundation. His interesting and entertaining lectures attracted large audiences. Married, with three children, he settled to a relatively comfortable life. He gradually established a new medical practice and acquired influential patrons and patients. In 1536, a fifteen year-old Lodovico Ferrari entered Cardano's service as an errand boy. Cardano soon realized that he had acquired an exceptional servant. For all his irascible temper, Lodovico was a mathematical prodigy and before long he became Cardano's most loyal friend and disciple. The two became very close collaborators and went on to work out a general algebraic solution to the cubic and quartic equations, probably the most important mathematical achievement of the 16th century.

Today, with the benefit of modern mathematical notation, we write the cubic equation as

$$ax^3 + bx^2 + cx + d = 0 \tag{3}$$

with a, b, c and d being given real numbers. The Renaissance mathematicians knew that one can get rid off the square term and reduce this equation to the "depressed form",

$$x^3 = px + q. \tag{4}$$

The trick involved replacing x by $x - b/3a$. Of course, not a single Renaissance mathematician would write these equations in this way. They were usually described in words, for example, expression

$$x^3 = 8x + 3. \tag{5}$$

would have been written by Cardano as

(6) cubus æqualis 8. rebus ṕ. 3. ,

where the Latin *rebus* ("things") refers to unknown quantities. In Italian texts the unknown "thing" was *cosa* and for a time the early algebraists were known as "cossists". I should also mention here that in Europe negative numbers were not considered seriously until the 17th century, so in Cardano's time different versions of the cubic equation must have been written down depending on the signs of the coefficients. Although solutions to some particular cubics had been known for some time it was a general algebraic solution that still eluded the best minds of the Renaissance. Then, sometime around 1515, Scipione del Ferro (1465–1526), a professor of mathematics in Bologna, found a general rule for solving a specific cubic equation of the form $x^3 + cx = d$, with c and d positive. It was a real breakthrough, but del Ferro kept the solution secret. Why?

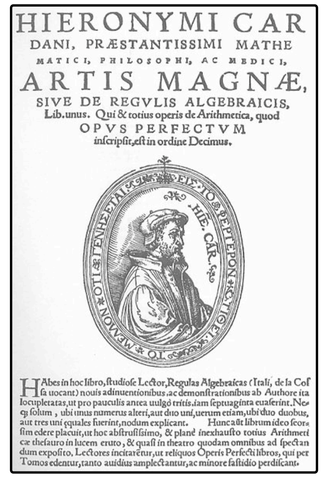
It may sound strange in our publish-or-perish age, but keeping some mathematical discoveries secret was quite common at the time. Many mathematical tools were treated like trade secrets. After all, del Ferro and his colleagues made their living by offering their services to whomever offered the best pay. Patronage was hard to come by, there was no tenure, university positions were few and held by virtue of eminence and reputation, and challenges could come at any time. Scholars were often involved in animated public debates that attracted large crowds. These were very reminiscent of the knight's tournaments in the Middle Ages with all the rituals - challenges, often publicly distributed in a form of printed pamphlets (*cartelli*), responses (*risposte*), witnesses, judges etc. Town and gown, students and merchants, and all kinds of spectators would gather in public squares or churches to watch the spectacle. The basic rule of combat was that no one should propose a problem that he himself could not solve. Reputations, jobs and salaries were at stake.

We do not know whether del Ferro ever used his result in a mathematical contest, but he was aware of its value, and shortly before his death in 1526, he passed the secret to his son-in-law, Annibale della Nave, and to one of his students, Antonio Maria Fiore. Although neither man published the solution, rumors began to spread that the cubic equations had been solved. In particular Niccolò Fontana of Brescia (1499 or 1500–1557), better known under his nickname Tartaglia (meaning the stammerer, a reference to his speech impediment) boasted that he had discovered the solution to cubics of the form $x^3 + bx^2 = d$ (again b and d positive). At the time Tartaglia was a teacher of mathematics in Venice and Fiore, a native of Venice, was very keen on a good teaching job in his hometown. Confident in his mathematical abilities, Fiore challenged Tartaglia to a public contest. It was a bad idea. Tartaglia was a much better mathematician, and, as it happened, the night before the contest on the 13th of February 1535, he had figured out del Ferro's solution. The contest, held in Venice, was a humiliating defeat for Fiore and a great victory for Tartaglia. Overnight, an unknown teacher of mathematics from Venice became a nationwide celebrity. His star was rising. However, it ain't over 'til the fat lady sings. Enter Cardano.

5. Ars Magna

When he heard of Tartaglia's triumph, Cardano asked him to reveal the secret and to give him permission to include the solution to cubic equations in *Practica arithmeticae generalis*, a book which he was preparing for publication. He promised to give full credit to Tartaglia, but Tartaglia categorically refused, stating that in due time he himself would write a book on the subject. When? He would not say, as he was preoccupied with his work on ballistics and translating Euclid's *Elements* into Italian. Cardano did not give up. After several exchanges of letters Tartaglia accepted an invitation to visit Milan, possibly in the hope that through Cardano's connections he could secure a lucrative job with the Spanish governor in Milan. The visit took place on the 25th of March 1539. This much we know for sure, but what exactly happened during the visit is not clear. We have two contradicting stories, one by Tartaglia and one by Ferrari. Tartaglia claimed he divulged the secret to Cardano in the form of an enigmatic poem but only after Cardano had taken a solemn oath to keep the solution secret. Ferrari, who was present at the meeting, swore that Cardano took no oath of secrecy. The word of one man stands against that of the other.

One way or another, in May 1539 *Practica arithmeticae generalis* appeared without Tartaglia's solution. However, Ferrari and Cardano were working hard and managed to extend Tartaglia's method to cover the most general case of the cubic equation. Ferrari went even further and worked out solutions to quartic equations. Meanwhile, Tartaglia still had not published anything on the cubics. In 1543, following rumors about the original discovery by del Ferro, Cardano and Ferrari travelled to Bologna to meet Annibale della Nave. After examining del Ferro's papers they found a clear evidence that twenty years earlier he indeed discovered the same solution as Tartaglia. Thus, even if Cardano had been sworn to secrecy the oath was no longer valid.


HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib.unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

HIE. CAR.

HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cossa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neq; solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aũt librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarẽtur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

Frontispiece of *Ars Magna*

In 1545 Cardano's *Artis magnæ sive de regulis algebraicis* (in short *Ars Magna*) was published. It was a breakthrough in mathematics, a masterpiece comparable in its impact only to *De revolutionibus orbium coelestium* by Copernicus and *De humani corporis fabrica* by Vesalius, both published two years earlier. In the book Cardano explores in detail the cubic and quartic equations and their solutions. He demonstrates for the first time that solutions can be negative, irrational, and in some cases may involve square roots of negative numbers. He acknowledges that he originally received the solution to the special cubic equation from Tartaglia. Chapter 11, titled *De Cubo & rebus æqualibus Numero* opens with the following line: "Scipio Ferro of Bologna well-nigh thirty years ago discovered this rule and handed it on to Antonio Maria Fior of Venice, whose contest with Niccolò Tartaglia of Brescia gave Niccolò occasion to discover it. He [Tartaglia] gave it to me in response to my entreaties, though withholding the demonstration". Cardano also credits Ferrari with the solution to quartic equations.

After the publication of *Ars Magna* Tartaglia flew into a wild rage and started a campaign of public abuse directed at Cardano and Ferrari. He published his own work *Quesiti et Inventioni Diverse* (New Problems and Inventions) which included correspondence with Cardano and what he maintained were word-by-word accounts of their meetings. They can hardly be regarded as objective, in fact, they read like a public rebuke. However, Cardano, who was now regarded as the world's leading mathematician, couldn't care less. He did not pick up the fight, letting his loyal secretary, Ferrari, deal with it. And so Ferrari did. He wrote a *cartello* to Tartaglia, with copies to fifty Italian mathematicians, challenging him to a public contest. Tartaglia however did not consider Ferrari as worthy of debate - he was after Cardano. Ferrari and Tartaglia wrote *cartelli* and *risposte*, trading insults for over a year until 1548 when Tartaglia received an offer of a good teaching job in his native town of Brescia. Most likely, in order to establish his credentials for the post, he was asked to take part in the debate with Ferrari. Tartaglia was an experienced debater and expected to win. The contest took place in the Church of Santa Maria del Giardino in Milan, on the 10th of August 1548. The place was packed with curious Milanesi, and the governor himself was presiding. Ferrari arrived with a large entourage of supporters, Tartaglia only with his own brother; Cardano was, conveniently, out of town. There are no accounts of the debate but we know that Tartaglia decided to flee Milan that night. Ferrari was declared an undisputed winner.

6. Real roots, complex numbers

So, how does it all lead us to complex numbers? Recall that back in the Renaissance negative numbers were treated with a bit of suspicion, so taking roots of the suspicious numbers must have been almost heretical. After all solving equations meant solving specific mercantile or geometric problems. Thus "things" were measurable entities and whenever solving the quadratic equations, such as $x^2 + 1 = 0$, led to the square root of a negative number it was assumed that the problem was meaningless with no solutions. Cubic equations were different. Some of them had perfectly respectable solutions, which could be easily guessed, and yet the square roots of negative numbers popped up halfway through, in the derivations of these solutions, and there was no way to avoid or to ignore them. This, to say the least, was puzzling.

The general solution to the depressed cubic reads

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{q}{2} - \sqrt{\Delta}}, \tag{7}$$

where

$$\Delta = \left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3. \tag{8}$$

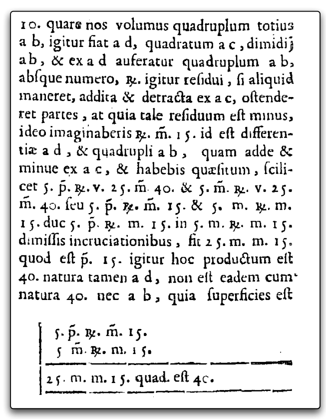
With their confusing notation and their reluctance to accept negative numbers the Renaissance mathematicians initially failed to grasp that this is indeed the general formula, which solves all cubics not just some specific cases. The most intriguing case, known as *casus irreducibilis*, occurs when $\Delta < 0$, for it involves square roots of negative numbers and always leads to three real solutions. Cardano, after learning the "general" solution from Tartaglia, tried to make sense out of *casus irreducibilis*. In 1539 he raised the matter with Tartaglia only to learn that he had not "mastered the true way of solving problems of this kind". Tartaglia, it seems, had little understanding of his own solution. Cardano, to be sure, did not elaborate on this case in *Ars Magna* but he did not avoid it either. Take, for example, equation

$$x^3 = 8x + 3, \tag{9}$$

which appears in Chapter 13 of *Ars Magna* with the comment: "Solving $x^3 = 8x + 3$ according to the preceding rule, I obtain 3." That must have baffled any careful reader who tried to work it out "according to the rule" since it is a clear *casus irreducibilis* with $\Delta = -1805/108$ and the solution which could only be expressed as

$$x = \sqrt[3]{\frac{3}{2} + \sqrt{-\frac{1805}{108}}} + \sqrt[3]{\frac{3}{2} - \sqrt{-\frac{1805}{108}}}. \tag{10}$$

And how do you get 3 out of that? Cardano does not say. This is surprising. After all Cardano is hardly afraid of square roots of negative numbers. On the contrary, a few chapters later he constructs explicit examples to show how to deal with them. The examples must have been motivated by the *casus irreducibilis*. He discusses a problem of finding two numbers which sum to 10 and such that their product is 40. The solution is, of course, $5 \pm \sqrt{-15}$, or rather 5. p̃. R_x. m̃. 15 and 5. m̃. R_x. m̃. 15, as it was written by Cardano. Here R_x stands for the Latin word "radix" which means "root". Finding it difficult to make sense out of such "numbers" Cardano took a purely instrumental approach. He noticed that if you are prepared to ignore the question of what the square root of minus fifteen meant, and just pretend it worked like any other square root, then you could check that these mathematical entities actually fit the equation. He wrote: "Putting aside the mental tortures involved, multiply $5+\sqrt{-15}$ by $5-\sqrt{-15}$, making $25-(-15)$ which is $+15$. Hence this product is 40." Richard Witmer, in his translation of *Ars Magna*, points out that the Latin phrase used by Cardano, namely *dimissis incruciationibus*, can also be translated "the cross-multiples having cancelled out". The sentence would then read "Multiply $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$ and, the cross-multiples having cancelled out, the result is $25 - (-15)$, which is $+15$. Hence this product is 40". No "mental tortures"


10. quare nos volumus quadruplum totius a b, igitur fiat a d, quadratum a c, dimidij a b, & ex a d auferatur quadruplum a b, absque numero, ℞. igitur residui, si aliquid maneret, addita & detracta ex a c, ostenderet partes, at quia tale residuum est minus, ideo imaginaberis ℞. m̃. 15. id est differentiæ a d, & quadrupli a b, quam adde & minue ex a c, & habebis quæsitum, scilicet 5. p̃. ℞. v. 25. m̃. 40. & 5. m̃. ℞. v. 25. m̃. 40. seu 5. p̃. ℞. m̃. 15. & 5. m. ℞. m. 15. duc 5. p̃. ℞. m. 15. in 5. m. ℞. m. 15. dimissis incruciationibus, fit 25. m. m. 15. quod est p̃. 15. igitur hoc productum est 40. natura tamen a d, non est eadem cum natura 40. nec a b, quia superficies est

5. p̃. ℞. m̃. 15.
5. m̃. ℞. m. 15.
25. m. m. 15. quad. est 40.

Complex numbers in *Ars Magna*

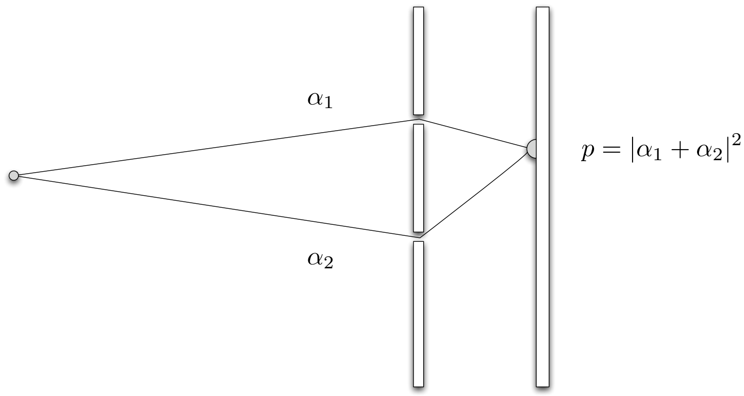


FIGURE 2. Addition of probability amplitudes in the double-slit experiment. Single particle quantum interference experiments of this type have been performed with photons, electrons, neutrons, atoms and even large molecules.

in this version. In another book, *Ars Magna Arithmeticæ*, Cardano remarks that $\sqrt{-9}$ is neither $+3$ nor -3 but some "obscure third sort of thing" (*quaedam tertia natura abscondita*). This is how complex numbers were announced to the world.

It took another thirty years or so, for one of those baffled readers of *Ars Magna*, Rafael Bombelli (1526–1572), the son of a Bolognese wool merchant, to provide a clear discussion of *casus irreducibilis* and to set up formal rules that allowed to perform consistent calculations with complex numbers. Still, it was Cardano, slightly ahead of his time, who wrote them down and, suppressing his uneasiness, performed simple operations on them. Unfortunately he never took them seriously and commented that they were as refined as they were useless!

7. Amplitudes

Today, these "useless" discoveries are indispensable to all practicing physicists. Indeed, complex numbers and probabilities underpin the best framework theory we have today - a superb description of the inner working of the whole physical world - the quantum theory. The theory asserts that probabilities are less fundamental than probability amplitudes, which are complex numbers α such that $|\alpha|^2$ are interpreted as probabilities.

I have already alluded to the fact that Nature does not conform to the Kolmogorov axioms. Indeed, there is no a priori reason it should. We have now an overwhelming experimental evidence that by manipulating probabilities alone we cannot describe our physical world. Example? The simplest and the most convincing is still the good, old double-slit experiment.

Imagine a source of particles, say electrons, which are fired in the direction of a screen in which there are two small holes. Beyond the screen is a wall with a detector placed on it. If the lower hole is closed the electrons can arrive at the detector only through the upper hole. Of course, not all electrons will reach the detector, many of them will end up somewhere else on the wall, but given a location of the detector there is a probability p_1 that an electron emitted by the source reaches the detector through the upper hole. If we close the upper hole then there is a probability p_2 that an electron emitted by the source reaches the detector through the lower hole. If both holes are open it makes perfect sense to assume that each electron reaching the detector must have travelled either through the upper or the lower hole. The two events are mutually exclusive thus the total probability should be the sum $p = p_1 + p_2$. However, it is well established experimentally that this is not the case.

Quantum theory asserts that the probability of an event is given by the square of the modulus of a complex number α called the probability amplitude. Thus we associate amplitudes α_1 and α_2 with the two alternative events , namely "electron emitted by the source reaches the detector through the upper hole" and "electron emitted by the source reaches the detector through the lower hole", respectively. For consistency we must require that $|\alpha_1|^2 = p_1$ and $|\alpha_2|^2 = p_2$. However, and this makes quantum theory different, when an event can occur in several alternative ways, the amplitude for the event is the sum of the amplitudes for each way considered separately. In our case the amplitude that an electron reaches the detector when the two holes are open is

$$\alpha = \alpha_1 + \alpha_2 \tag{11}$$

and the associated probability

$$\begin{aligned} p = |\alpha|^2 = |\alpha_1 + \alpha_2|^2 &= |\alpha_1|^2 + |\alpha_2|^2 + \alpha_1^\star \alpha_2 + \alpha_1 \alpha_2^\star \qquad (12) \\ &= p_1 + p_2 + |\alpha_1||\alpha_2|(e^{i(\theta_1-\theta_2)} + e^{-i(\theta_1-\theta_2)}) \\ &= p_1 + p_2 + 2\sqrt{p_1 p_2}\ \cos(\theta_1 - \theta_2). \qquad (13) \end{aligned}$$

where we have expressed the amplitudes in their polar form $\alpha_1 = |\alpha_1|e^{i\theta_1}$ and $\alpha_2 = |\alpha_2|e^{i\theta_2}$. The last term on the r.h.s. marks the departure from the classical theory of probability. It blatantly ignores the additivity axiom. The probability of any two mutually exclusive events is the sum of the probabilities of the individual events, $p_1 + p_2$, modified by what is called the interference term, $2\sqrt{p_1 p_2}\ \cos(\phi_1 - \phi_2)$. Depending on the relative phase $\phi_1 - \phi_2$, the interference term can be either negative (destructive interference) or positive (constructive interference), leading to either suppression or enhancement of the total probability p. Note that the important quantity here is the relative phase $\phi_1 - \phi_2$ rather than the absolute values ϕ_1 and ϕ_2. This observation is not trivial at all. In simplistic terms - if an electron reacts only to the difference of the two phases, each pertaining to a separate path, then it must have, somehow, experienced the two paths. Thus we cannot say that the electron has travelled *either* through the upper *or* the lower hole, it has travelled through both. However weird it sounds, this is how it is.

Phases of probability amplitudes tend to be very fragile and may fluctuate rapidly due to spurious interactions with the environment. In this case, the interference term may average to zero and we recover the classical addition of probabilities. This phenomenon is known as *decoherence*. It is very conspicuous in physical systems made out of many interacting components and is chiefly responsible for our classical description of the world – without interference terms we may as well add probabilities instead of amplitudes.

Can we formulate quantum theory without complex numbers? Not really. One puzzling feature of quantum theory is that it can hardly be different from what it is. I mean, if you try to construct a framework theory with few reasonable assumptions, you end up with something very similar to quantum theory. In particular, once we request *continuity* of admissible time evolutions we will end up with quantum theory, and if this requirement is dropped we obtain classical probability theory! [4] We cannot avoid Cardano's "useless" discoveries.

8. Epilogue

Let us get back to 1548. What happened after the contest in the Church of Santa Maria del Giardino in Milan?

Tartaglia's appointment in Brescia was not renewed. He taught there for about a year but his stipend was not paid. After many lawsuits, he returned, seriously out of pocket, to his previous job in Venice. He spent much of the rest of his life plotting and collecting a dossier against Cardano. He died in poverty in Venice on the 13th of December 1557.

[4] See, for example, Lucien Hardy, *Quantum Theory From Five Reasonable Axioms* (quant-ph 0101012).

Ferrari became famous. He was appointed a tax assessor to the governor of Milan, and soon after retired as a young and rich man. He moved back to his hometown - Bologna - where he lived with his widowed sister Maddalena. In 1565 he was offered a professorship in mathematics at the university but, unfortunately, the same year he died of arsenic poisoning, most likely administered by his sister. Maddalena did not grieve much at his funeral and having inherited his fortune, remarried two weeks later. Her new husband promptly left her, taking with him all her dowry. She died in poverty.

Cardano outlived both Tartaglia and Ferrari. In 1546, fifteen years into his marriage, his wife died leaving Cardano the sole caretaker of his three children. This did not seem to affect him that much. He remained in Milan, lecturing in geometry at the University of Pavia, making money both as a physician and as a writer. From the day he published *Ars Magna* till about 1560 he had everything he could possibly wish for - fame, position, money, respect. These were his golden years, but then, as it often happens, came difficult times. His eldest and most beloved son, Giambatista, married a woman who, by all accounts, was a despicable character. Publicly mocked and taunted about the paternity of his children he reached his limits of sanity and poisoned her. He was arrested. Cardano did everything he could for his son. He hired the best lawyers and paid all the expenses. Five doctors were brought in who stated that Giambatista's wife had not been poisoned, or at least, had not received a fatal dose. It seemed that a not-guilty verdict would be announced when, for some inexplicable reason, Giambatista confessed. He was sentenced to death, tortured, and beheaded in 1560. It was a real turning point in Cardano's life; he never recovered from his grief. He gave up his lucrative medical practice in Milan and moved to Pavia, where he became pathologically obsessed about his own safety. He reported a number of intrigues, attempts on his life and malicious accusations of professional incompetence and sexual perversion. In 1562, forced to resign his position in Pavia, Cardano, with some help from the influential Borromeo family, secured transfer to a professorship of medicine in Bologna.

For a while Cardano's life resumed its old order; he was happy to touch base with his old friend Ferrari. However, problems with his children continued. Cardano's daughter, Chiara, died of syphilis, contracted as a result of her prostitution. His second son, Aldo, a perpetual thief, moved with him to Bologna but did nothing but drink and gamble. On a number of occasions frustrated Cardano had to bail him out of the staggering debts. Finally, when in 1569 Aldo gambled away all of his personal possessions and was caught stealing a large amount of cash and jewelry from his father, Cardano had him banished from Bologna. In 1570, Cardano himself was imprisoned for a few months by the Inquisition in Bologna. The charges against Cardano are not known. Some point to Cardano's connections with Andreas Osiander, one of the leaders of the German Reformation, as a possible cause of the involvement of the Inquisition. By the time one of his most vicious enemies, Tartaglia, was already dead and it is very unlikely he had anything to do with it.

On his release from prison, Cardano was sworn to secrecy about the whole proceedings and forbidden to hold a university post. Helped by another of his loyal pupils, Rodolfo Silvestri, he moved to Rome and appealed to the Pope, Gregory XIII, who granted him a small pension and allowed him a limited practice in medicine. It was in Rome he started writing his autobiography *De vita propia liber*. He died on the 21st of September 1576. Some say that Cardano predicted the exact date of his own death, some say that he starved himself to death to make this prediction true, we will never know.

Further Reading

While I would not want to guarantee that all the statements about Cardano that appear in this article are true I made an effort to consult the best sources I had access to. The following books were particularly useful to me and are recommended to anyone interested in Cardano's colourful life and his diverse interests.

- Girolamo Cardano, *The Book of My Life (De vita propia liber).* Translated by Jean Stoner. New York Review Books, 2002.
- Girolamo Cardano, *The Great Art or The Rules of Algebra.* Translated and edited by T. Richard Witmer. MIT Press, 1968.
- Girolamo Cardano, *Opera Omnia Hieronymi Cardani, Mediolanesis*, 10 vols, Spoon, Lyons, 1663. All 10 volumes can be downloaded from the website of the Philosophy Department, University of Milan (http://www. filosofia.unimi.it/cardano/).
- Øystein Ore, *Cardano, the Gambling Scholar.* Princeton, 1953. Contains English translation of *Liber de ludo aleæ* by Sydney H. Gould.
- Florence N. David, *Games, Gods and Gambling. A History of Probability and Statistical Ideas.* Charles Griffin & Co, London, 1962.
- Anthony Grafton, *Cardano's Cosmos: The Worlds and Works of a Renaissance Astrologer.* Harvard University Press, 2001.
- Nancy G. Siraisi, *The Clock and the Mirror: Girolamo Cardano and Renaissance Medicine.* Princeton University Press, 1997.

Final Remarks

This is not intended to be a scholarly work on the history of probability and complex numbers. The main purpose of this recreational article, if there has to be a purpose, is to bring somewhat neglected and under-appreciated Cardano to the attention of my colleagues, especially those who teach quantum mechanics and who want to make their lectures more entertaining. Few years ago I was asked to explain the rudiments of quantum theory to the first year Cambridge undergraduates. In order to do that I had to cover both probabilities and complex numbers. I decided to start with a historical perspective on the two topics, and my first lecture was all about Cardano. Students liked it, and so did I.

This is a revised version of my contribution to the Festschrift in honor of my Italian friend and colleague Giuseppe Castagnoli. I welcome suggestions, corrections, additions.