ANCIENT NUMERALS AND ARITHMETIC

By Paul J. Renne

Overview

This unit encourages the practice of using the numerals and arithmetic of Ancient Egyptians, Ethiopians and Babylonians. A general history is given of the development of these systems, and during the history the activities in the unit are introduced.

The unit focuses on the Ancient Egyptian and Babylonian because it is believed that they were the major influence on the ancient Greeks, which in turn were the precursor to Europe's mathematical development after the dark ages.

The ancient Egyptian methods were very effective, and the Babylonian methods were even more so. They are different enough than ours. Studying them deepens the understanding of mathematics, as well as expands the repertoire of mathematical tools available to oneself.

The mathematical ability necessary to practice the activities of the unit spans a wide range. Some of the activities are quite suitable for the youngest of students. Ethiopian and Egyptian multiplication could quite easily be performed with kindergarteners or first graders. It is possible that a student that has not quite learned to add could perform the techniques, using Ethiopian and Egyptian numerals. Perhaps the activity could be used to reinforce the learning of addition. The process is easy, perhaps easier than our own. Understanding and explaining why the processes produce the product in a multiplication of two numbers, for example, is a venture, however, more suitable to a college mathematics course.

The Babylonian numerals and arithmetic may fall into the category of being suitable for kindergarteners, however knowledge of high school mathematics will be necessary to understand the Babylonian algorithms.

In addition to enhancing our mathematical development, studying these ancient systems broadens our understanding of computers. Interestingly, many of these ancient methods are similar to the ones used by computers.

This unit is written as a history, filled with an experimentation of doing simple processes. Inside these processes are numerous questions and activities that are suitable to any level of mathematical knowledge. Ideas, suggestions and examples are shown of the arithmetic. The reader is to use their imagination as to what they want to use it for. The possibilities are endless.

To get the full benefit of the unit, it is highly suggested that the reader get out pencil and paper and practice the activities after or during reading the Rational next.

Rational

We can now solve many problems in the physical universe, but do we really understand the numerals and arithmetic we are using to calculate them? It is the belief of the author that one never completely understands anything, but that understanding is a process that is deepened as one experiments and conjectures. The arithmetic you are about to see is different enough than our own, that it causes one to question and experiment.

Throughout the brief history next, the numerals and arithmetic are introduced. It is suggested that the reader briefly familiarize themselves with the different numerals found in the Appendix before reading the description of the activities.

All history is speculation. It is not dead, but alive and constantly changes as we conjecture, reproduce pieces of the past and find more evidence. The author has taken liberties in reproducing the numerals and the arithmetic. How each person draws numerals in the present is inherently unique to that moment and that person. The numerals, especially The Babylonian, evolved as the author practiced drawing them. A reader with background knowledge will note that the numerals do not look like ones they had seen. The Babylonian time period in which they were using these numerals spans over 1,000 years. The numerals evolved over this time of course and did not look exactly the same throughout. If one looks at Babylonian numerals from several different sources, one will see that they are not consistent. The author himself was startled to find, after nearly completing this unit, while leafing through his favorites of books that he had read as research, *Zero, The Biography of a Dangerous Idea,* Charles Seife, that his sign for the empty space was going the wrong way. After leafing through other research, such as ironically the “other” book about zero, *The Nothing that Is,* by Robert Kaplan, it was found that several had made the empty space symbol, tilted to the left as does the author.

The ancient symbols and processes evolved during their use and are still evolving. This is to say that the symbols are not engraved in stone. No, instead they are preserved on tablets that were engraved in clay, written on papyrus and carved in bone. What these symbols mean is pure speculation. There are differing opinions among the experts.

We should also be careful not to assume that what we can clearly see today using modern mathematics was in the mind of the ancient mathematician. Contrarily we should not assume that the meaning behind what they were doing is deeper than we realize. (Idea from *Pythagoras, Theorem in Babylonian Mathematics,* by J.J. O'Connor and E.F. Robertson)

There is a lot we take for granted in our modern math. As the number zero is to us, for example. Imagine not having it. Zero was not natural to the earliest civilizations known to have number systems. Using numerals to count physical objects, such as the number of animals in a clearing, does not require it.

We also assume that numbers are base ten as our own. To "feel" that we understand, we will want to convert a different base to the numbers we are comfortable with. The ancient Egyptians used a base ten as we do, but the Babylonians used a base sixty system. Many people, including the author, have problems with different bases. Many, including those non mathematical have suggested our time units as a simple example of base sixty. Five hours, twenty-three minutes and twenty-five seconds is base sixty. ($5 + \frac{23}{60} + \frac{25}{3600} = 5.39027$ hours)

The earliest example of human beings using numerals is found as tick marks on bone said to be 35,000 years old. Many mathematical historians assume that these "numbers" were used in counting the number of a particular kind of object, such as the number of sheep. Obviously this method is quite cumbersome with large numbers. Using 1000 tick marks to represent the number one thousand, obviously, would require a lot of time and space making the representation perhaps more trouble than it is worth.

Regardless of the simplicity, cultures, which used such systems, developed methods of multiplication and division, such as the Ethiopian Method

The method involves placing each number being multiplied (multiplicand) into one of two columns. It doesn't matter which. The left hand column is thought of as pebbles and the right side is thought of as houses. The algorithm is to double the pebbles and take half of the houses, neglecting fractions by taking them away. The rows with an even number of houses are considered evil and are crossed off. The sum of the remaining pebbles is then summed. This is the product. Below is an example using modern numerals. The problem is done twice to show that order does not matter.

6 x 9

6	9
~~12~~	~~4~~
~~24~~	~~2~~
48	1
54	

9 x 6

~~9~~	~~6~~
18	3
36	1
54	

It has been stated in *Ancient Puzzles* by Dominic Olivastro that although the method may seem bizarre to us, it actually is a very logical way to multiply if one does not have a complete number system. It is also noted that computers represent numbers in binary and that many entry level computer books exemplify using a trick to convert decimal numbers to binary using a similar method as that of Ethiopian Multiplication.

The process begins with a decimal number for example, 100. The algorithm involves halving the number as the Ethiopians did their houses, including the discarding of fractions. This process is repeated until a one is arrived. To the right of each number is to be written either a 0 if even or a 1 if odd. Reading this list from bottom to top leads to the conversion of the decimal to binary. The example of 100 from decimal to binary is below:

100	0
50	0
25	1
12	0
6	0
3	1
1	1

The decimal number one hundred is therefore 1100100 in binary, which indeed it is for:

1 1 0 0 1 0 0

$$\mathbf{2^{6}\ \ 2^{5}\ \ 2^{4}\ \ 2^{3}\ \ 2^{2}\ \ 2^{1}\ \ 2^{0}}$$

2^6 + 2^5 + 2^2
64 + 32 + 4 = 100

The Ancient Egyptians developed similar methods of multiplication ,perhaps derived from the Ethiopians (note the proximity), however, their numbering system was more advanced. The ancient Egyptians used a base 10 system as does our modern one. Instead of writing 10 as 10 little slashes they derived a new symbol for 10 as well as for each successive power of 10; 100, 1000, 10000 etc.

This enabled them to represent larger numbers with fewer symbols. According to Olivastro and other sources the Egyptians had no rules for what position each symbol should take. For example the number 21 could be represented by the symbol for 10, followed by the symbol for 1 and ending with another symbol for 10. It did not matter what order the symbols took place. So the question that could be asked is how many different ways could an Egyptian write the number 21? (ways to write 73 is much more difficult). Below is the number of ways to arrange the Egyptian numerals representing 21.

Writing Egyptian numerals creates this classic question from combinatorics. How many different ways can we arrange n things with n_1 of one kind, n_2 of another, …, n ith of an ith kind.

Once one begins to write numerals in Egyptian it is easy to hypothesize that an Egyptian addition problem might look like a putting together of two groups of characters and simplifying by converting 10 of one type of symbol into that of the next highest power of 10. The process of Egyptian addition is not covered in the unit, but it is easy to reproduce and is required in order to multiply in ancient Egyptian.

Ancient Egyptian multiplication and division is very interesting. It has the doubling and halving of Ethiopian style, but is more sophisticated in that many large numbers and fractions can be dealt with easily and efficiently. The author is ashamed to admit that he does not “feel” a deep understanding of exactly why Egyptian and Ethiopian multiplication and division make sense. He uses the excuse that he has just not sat down and experimented and done it. An explanation can be recited, which mimics the partial explanation given by Olivastro in *Ancient Puzzles*, but the author has decided to leave this blank and up to the reader to conjecture and prove.

Ancient Egyptian multiplication is done by creating two columns. An Egyptian numeral one always begins in the left column and either multiplicand in the right. It doesn't matter which one. Both columns are doubled until the point where any combination of numbers in the left column can be summed to the unchosen multiplicand. These are checked and their corresponding numbers in the right column are summed. This is the product. To get the "real" Egyptian thought process one ought to experiment with the process using Egyptian numerals as is suggested in the activity. Although the Egyptian flavor is missing below is an example of Egyptian multiplication, using our modern numerals. The problem is 6 x 8 and is done twice to verify that it does not matter which multiplicand is chosen.

1	6
2	12
4	24
8	☑48
	48

1	8
2	☑ 16
4	☑ 32
	48

Next are 14 x 9, using 14 as the multiplicand

1	☑ 14
2	28
4	56
8	☑ 112
	126

Note that 1 and 8 sum to 9 "the unchosen multiplicand" and that their corresponding numbers to the right sum to the product, 126. It is the reader's job to practice the process in Egyptian and then explain why it works.

Before going onto division we will take a step back to Olivastro's statement about the ordering of the characters of Egyptian numerals. Although he makes this claim, copies of actual calculations done by a scribe preserved on papyrus that are represented in his book, show the numerals written in an orderly arrangement read from **right** to **left**, with the highest powers of 10 down to the lowest. One of the calculations represents a scribe's work in attempting to perform the operation $3 + \frac{1}{3} + \frac{1}{5}$. “A trivial problem by today’s standards”, (Olivastro), but it was quite difficult using The Ancient Egyptian System. It could be conjectured that an Egyptian left it as such:

The Egyptian’s had fractions. Because of the representation for fractions they used, they would only use unit fractions, meaning fractions where the numerator is one (i.e. 1/2, 1/3, 1/4, 1/10).

Can an Egyptian represent any fraction? According to Olivastro, it was proven in 1880 that every fraction is a sum of unit fractions. Try to come up with a counterexample.

Another question to ask is why the Egyptians represent fractions as a sum of unit fraction (none repeating). For example: 3/4 = 1/2 + 1/4. But the Egyptians would not write it as 1/4 + 1/4 +1/4. The reason why this is so is inherent in the Egyptian representation of fractions and their method of division. Three more points in reference to answer this question are:

The Egyptian method of division leaves sums of unit fractions as a by-product. If two of the same unit fractions did happen to exist side-by-side the Egyptians had a clever way to reduce, and the representation of fractions used by the Egyptians only allowed for unit fractions.

The first to be covered here is Egyptian division. Of course the richness cannot be found, nor can the Egyptian unit fractions using our numerals but below is an example of how the Egyptians would perform the operation: $35 \div 8$.

1	8
2	16
4	☑ 32

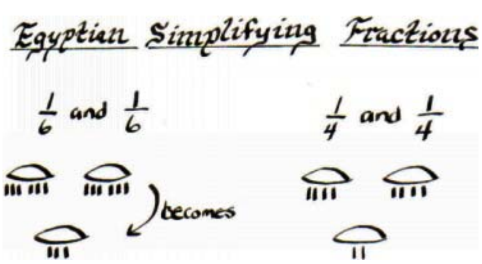
$\frac{1}{2}$	4
$\frac{1}{4}$	☑ 2
$\frac{1}{8}$	☑ 1

$$4 + \frac{1}{4} + \frac{1}{8}$$

The process begins with two columns, a one to the left and the divisor, in this case 8, to the right. Each are doubled to the point that if they were doubled again the divisor column (right) would pass the dividend, in this case 35. After this point the numbers are halved, beginning with the original numbers (row), until the divisor column reaches 1. The combinations of numbers in the right column that sum to the dividend are checked and their corresponding numbers to the left are summed. This is the quotient.

After one experiments with some Egyptian division problems it is quite clear that we run into problems when the divisor is not a power of two. The column to the right does not reach one. Perhaps the Egyptians threw out fractions as the Ethiopians. Using this conjecture it was found that the Egyptian division algorithm works fairly well in that quite good approximations result. By modifying exact solutions could be obtained.

Another reason why an Egyptian scribe would not leave two of the same unit fractions in a numeral is found in the ingenious method the Egyptians used to simplify. Below are a few examples in Egyptian numerals:



Note that two bug like symbols with six legs each becomes one with three and two with four legs each become one with two. The analogy to our own is of course $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$ and $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. How the Egyptians dealt with odd denominators is unknown by this author and is left as conjecture.

There is a famous problem from preserved papyrus that shows an example of a situation where the Egyptians method does a “better” job at solving a problem.

It is in answering the question how do you divide 3 loaves of bread among 5 people? (Evenly and Equitably)

For us the question is easy. We should give each person 3/5 of a loaf. But what would each person really be getting? Three men would literally get 3/5 of a loaf and the other two would get two pieces of bread, one worth 1/5 the other worth 2/5 (Olivastro)

The Ancient Egyptians would give each of the five people exactly the three same pieces, 1/3 of a loaf, 1/5 of a loaf, and 1/15 of a loaf. So perhaps the Egyptian method is more fair and equitable (neglecting the end pieces, which could be a good math twist on the problem).

The Egyptian system was quite advanced for its time, but it lacked much. One of which is the number 0. The Egyptians had no symbol for Zero. According to Charles Seife, they simply represented nothing with nothing. If an Egyptian where to write the numeral 302 they would simply not draw any symbols which represented 10.

Most historians believe that the first cultures to use 0 were the Mayans and the Babylonians.

The Mayans appear to have been quite sophisticated in their use of 0. For them it was not only a placeholder, but was used in counting. According to Seife, it may be the only culture known that begins the day of each of their months with day 0. It may seem bizarre, but let’s say you were just born yesterday. How many days old are you right now? 24 hours or one day, so the first day of your life was day 0.

Seife makes the claim that the Ancient Mayan Calendar is superior to ours of modern day. It is perhaps the best calendar ever created. The Mayans were very advanced, before the time period of the Babylonians. It is safe to say, though, that there was absolutely no collaboration between the Mayan and Babylonian mathematicians. The Mayan development was a separate time and place and is not the subject of this paper.

The early Greeks took the Egyptian system and slightly improved it using similar notation for the numerals as well as fractions. It is questionable whether later Greeks improved the system or impeded it. For instead of writing 43 with 4 ten symbols and 3 one symbols, the new and improved Greek system used one symbol for 40 and one for 3. (Seife) These symbols came from the alphabet.

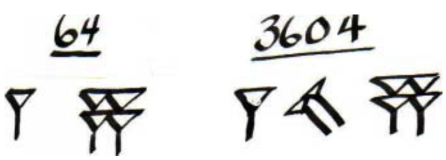
It seems better in that there are less symbols to represent numbers but in doing calculations, having different symbols for each place value of say 10 for example (10, 20, 30, etc.) causes problems. This is especially so if one is using Egyptian type methods.

The arithmetic was so difficult, Seife notes that Greek astronomers converted to the Babylonian system, did their calculations, and then converted them back to Greek.

What ever the Babylonians did, it was effective. It was the best of the day. This is why it is the focus of study now.

The Babylonians, once again had a base 60 numeral system. They kept track of which power of 60 a place designated with a physical space. There is an author of whom the websites referenced in this unit referred to as someone whom had studied and written about Babylonian mathematics. His name is Neugebauer and his apparently most readable book on the subject is *Exact Sciences in Antiquity*. He uses a notation that will be referred to as Neugebauer notation.

So the Babylonians are considered the first to use zero. In actuality the Babylonians did not use zero as a number, but as an empty space. It was a positional system. The empty space symbol allowed them to distinguish between 64 and 3604.



Despite this characteristic, the Babylonians never wrote their empty space symbol by itself or even at the end of a numeral. Note they read their numerals from left to right as we do. As a consequence of not placing the empty space symbol at the end of a numeral, the character for 1,60 and 3600 as well as all other powers of 60 is the same symbol. Historians speculate that Babylonians noticed the difference by the context of the problem. In *A History of Zero*, by J.J. O'Connor and E.F.Robertson, it is noted that we still use context to interpret numbers today, with an example referring to bus fares. If one is told that the bus fare from their house to downtown Pittsburgh is one fifty, it is clear that $1.50 is meant. (This person of course resides in Pittsburgh as does the author) If one is told that the bus fair from Pittsburgh to New York is one fifty, it is clear that $150.00 is the price.

The Babylonians dealt with fractions in much the same way as we use a decimal point in decimals, however they had no point or symbol indicating where we would have our decimal point. This was written as a space the same as the space between every other power of 60.

A Babylonian would write the decimal number 70.25 as the following: A unit's symbol, representing 60 in decimal. Following a space would be the symbol for 10. The Babylonians were base 60, yet they also had a symbol for 10. Despite it's representation what it means in this case are 10, 60^0s, or ones in the number. After the next space would be a ten character and a five character, representing fifteen. Upon first reading or experimenting, the careful reader and mathematician might suppose that the author has made a mistake or there is a typo. Why does the symbol for fifteen in Babylonian represent .25 in decimal? The decimal .25 is fifteen in Babylonian if it is the sixtieth place as opposed to the tenth and hundredth. In Neugebauer notation the number 70.25 in decimal notation is 1,10;15 in Babylonian. The semicolon denotes where our decimal point would occur. Notice it is $1 \text{ x } 60^1 + 10 \text{ x } 60^0 + 15 \text{ x } 60^{-1} = 60 + 10 + \frac{15}{60} = 70.25$.

Below is the Babylonian version:

Notice that without signifying where the "decimal point" occurs, the numeral is ambiguous. It could just as easily represent $1 \text{ x } 60^2 + 10 \text{ x } 60^1 + 15 \text{ x } 60^0 = 3600 + 600 + 15 = 4215$. As was stated before scholars believe the Babylonian mathematicians knew the difference in its context.

Many of the recovered Babylonian clay tablets are actually multiplication tables which give clues as to how they did their math. The Babylonians can be imagined as kings of the multiplication table. Many of their computations could be done in reference to or in memorizing these tables. Perhaps Babylonian scribes knew their multiplication tables. It is possible. There are fair numbers of people of today that know their multiplication tables.

Babylonian tables were not like ours. Rather than squares they were lists of single multiples. For example a Babylonian would represent the three tables vertically as:

1 a-ra′	3	3	
2 “	3	6	
3 “	3	9	
4 “	3	12	
5 “	3	15	...etc.

“A-ra′ “ of course is the anglicized version of multiply, used by the Babylonians. Babylonian versions of these tables are found in the Activities and Appendix of the unit.

The Babylonians could also divide. They did this by multiplying by the reciprocal of the divisor, much in the spirit of elementary school division of fractions. Consequently they also had reciprocal tables, of which are also presented in Activities and Appendix section of unit.

In addition to multiplication and reciprocal tables Babylonians also had square tables. Besides being able to multiply by using the tables or memorizing them, the Babylonians had another method which looked like they were using the following relationship:

$$ab = \frac{(a+b)^2 - a^2 - b^2}{2}$$

Note: $(a + b)^2 = (a + b)^2$

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$2ab = (a + b) - a^2 - b^2$$

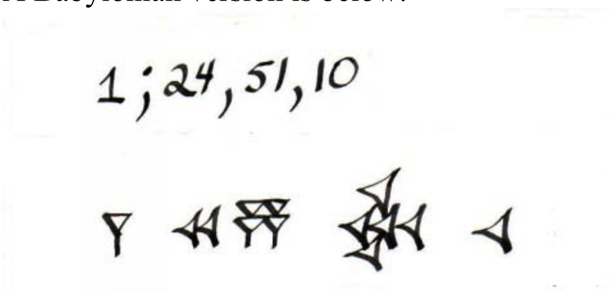
$$ab = \frac{(a+b)^2 - a^2 - b^2}{2}$$

The Babylonians did not write variables but did their calculation in a step-by-step manner which alluded to this logic. When observing this method it can be seen that the Babylonians had good use for their square tables as well as the multiplication ones. Substituting 3 for *a* and 4 for *b* leads to:

$$(3)(4) = \frac{(3+4)^2 - 3^2 - 4^2}{2} = (49 - 9 - 16)/2 = 24/2 = 12$$

Every product is in terms of square numbers.

It has been found that the Babylonians had written a numeral on several of the clay tablets and it was found that it was the value of the square root of two correct to nine decimal places. In Neugebauer notation the number is 1;24,51,10. A Babylonian version is below:



There are several hypothesizes as to how the Babylonians did this. One from *Pythagoras's Theorem in Babylonian Mathematics* by J.J O'Connor and E.F Robertson suggests that the Babylonians followed the following algorithm:

Make two guesses as to a number's square root- one high and one low. Take the average of the two and square the result. If the result is greater than the original number we are computing the square root of replace the high guess with this better bound ((High + Low) $\div$ 2). If the result is less than the original number then replaces the low guess with this better bound.
Repeat the process as many times as desired.

Following is an example choosing 2 as the high guess and 1 as the low:

$(\frac{2+1}{2})^2 = 2.25$ so replace high guess with $\frac{3}{2}$ or 1.5

$(\frac{1.5+1}{2})^2 = 1.5625$ ……. $\frac{1.5+1}{2} = 1.25$

$(\frac{1.5+1.25}{2})^2 = 1.890625$ …. $\frac{1.5+1.25}{2} = 1.375$

$(\frac{1.5+1.375}{2})^2$ etc.

The method takes several iterations before a good approximation is obtained, but it is possible and can be done to any level of desired accuracy. The Babylonians had numerous algorithms, which appeared as what we think of as an algebraic process, without the variables. The process has also been stated as being more similar to what a computer would do as opposed to those of our algebraic methods.

Since it has been hypothesized in this unit that the Egyptians and Babylonians were the major influence on the ancient Greeks which in turn were the influence on Western Europe after the dark ages, it is implied that they were in turn the major influence on our own modern one.

A major omission, when referring to the influence on our modern one and Europe is that of ancient India. As history is itself, our beliefs about what came out of the mathematics of India is shrouded in mystery. Despite the controversy, it seems likely that our present day numerals, including 0, began in India. For those that see the Babylonian mathematics as silly let it be stated that many believe that having a positional system as the Babylonians was the greatest achievement in mathematics ever. According to Seife, the Indians got the idea of zero as a placeholder from the Babylonians and developed it into a number itself, including the negative numbers.

The Greeks were geometry based and Seife believes feared 0. This is exemplified in Zeno's Paradox, but is most so because built into the Greek logic system it denoted the existence of god.

This belief may have seriously impeded western advancement until quite recently. This argument taken from the Greeks was referenced to and taken very seriously up until the 1600's in Europe.

Despite Fibonacci's*, Liber Abaci*, (1200 A.D.), Indian numerals did not begin to be experimented with in Europe until the 1500's, and zero went from being out right banned to being unpopular. Up until this time Europe was using Roman Numerals, which according to Seife and others, was a step in the negative towards advancement.

Since this is a brief history of ancient arithmetic another culture to be mentioned is the ancient Chinese. It is arguable that the ancient Chinese appear to have done everything that was done in western culture (some of which not until the 1800's) thousands of years before anyone else. In addition it can be assumed that there was no contact between the Chinese and Babylonians and Egyptians.

Objectives

The intent of the unit is to introduce the reader to two different numeral systems and processes of arithmetic. The reader is to experiment, hypothesize and conjecture about these characters and methods. In these are numerous ideas of activities that could be performed with students of any mathematical ability. Some are presented as activities, others suggested, but it is also left to the reader to use their imagination.

The unit is to inspire and be a stepping-stone to the study of different mathematical systems. It is also to conjecture that studying these systems will aid us in the present, such as understanding how computers operate.

Strategies

The author provides written descriptions of the processes of ancient arithmetic through a history. Examples, further explanations and representations of the numerals are provided in the Activities and Appendix sections of the unit.

The real goal is to have the reader actually experiment with the ancient arithmetic. From this the reader will have expanded their mathematical horizons. New ideas and insights are naturally the result. The reader will be likely to conjecture and continue the work of discovering how these ancients did their math.

Activities

Ethiopian Multiplication

The method involves placing each number being multiplied (multiplicand) into one of two columns. It doesn't matter which. The left hand column is thought of as pebbles and the right side is thought of as houses. The algorithm is to double the pebbles and take half of the houses, neglecting fractions by taking them away. The rows with an even number of houses are considered evil and are crossed off. The sum of the remaining pebbles is then summed. This is the product. Below is an example using modern numerals. The problem is done twice to show that order does not matter. Below is an example of 24 x 36, using modern numerals:

~~24~~ ~~36~~ Note that using Ethiopian

numerals

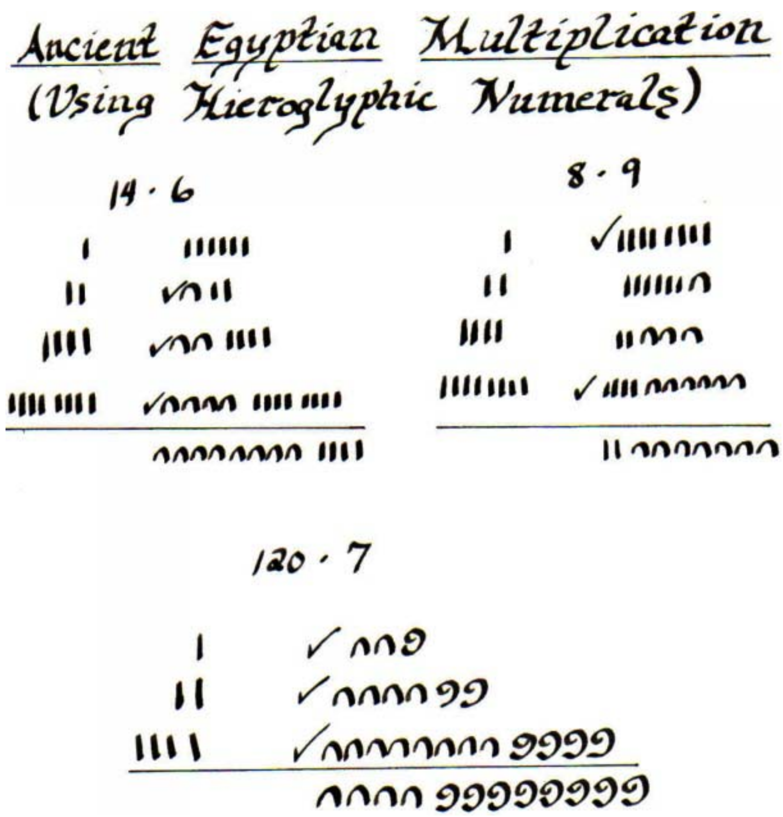
48	18
96	9
192	4
384	2
768	1
864	

could be a good activity to use with younger students. (With smaller numbers of course)

Egyptian Multiplication

Once again, ancient Egyptian multiplication is done by creating two columns. An Egyptian numeral one always begins in the left column and either multiplicand in the right. It doesn't matter which one. Both columns are doubled until the point where any combination of numbers in the left column can be summed to the unchosen multiplicand. These are checked and their corresponding numbers in the right column are summed. This is the product.

Below are several examples using Egyptian numerals:

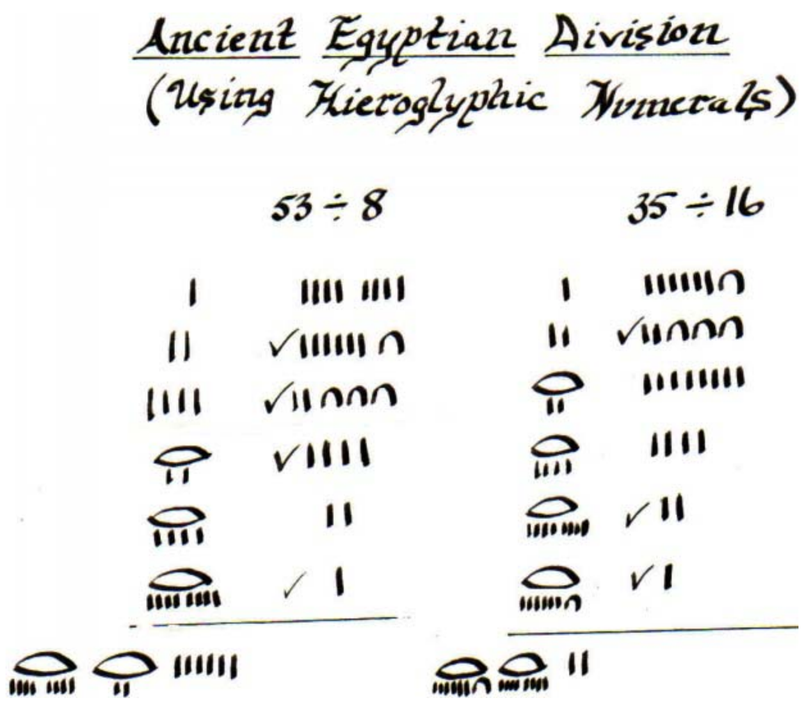


Note that the first example does not write the characters from largest power of ten to the smallest from right to left as the scribe had done in his calculations on copies of papyrus in *Ancient Puzzles* by Dominic Olivastro. But this is acceptable since it has been assumed that an Egyptian numeral can be written with the characters in any order.

Create your own conjecture as to why the Egyptian multiplication process works. Explain your stance using modern numerals.

Egyptian Division

The process begins with two columns, a one to the left and the divisor to the right. Each are doubled to the point that if they were doubled again the divisor column (right) would pass the dividend. After this point the numbers are halved, beginning with the original numbers (row), until the divisor column reaches 1. The combinations of numbers in the right column that sum to the dividend are checked and their corresponding numbers to the left are summed. This is the quotient.



Note the process has worked, for $53 \div 8 = 6\,{}^{5}\!/\!{}_{8} = 6.625 = 6 + \frac{1}{2} + \frac{1}{8}$

While the Egyptian multiplication will work with any integers, there are flaws in the division process. As was noted it is only completely effective if the divisor is a power of two. The author conjectured that perhaps the Egyptians used a process similar to the discarding of fractions used by Ethiopians. The method results in quite a good approximation of the quotient, but not the exact result.

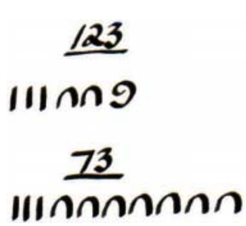
Create your own conjecture about how the Egyptians dealt with the problem of divisors of powers besides two.

Combinatorics with Egyptian Numerals

Recall that an Egyptian numeral can be written with the characters in any order. How many different ways can an Egyptian write the numeral 32?

The answer is that there are 10 ways. Below are all ten in Egyptian characters:

How many different ways can an Egyptian write the number 73 or 123? Listing all of the possibilities for 123 would take a while and even longer for 73.



Statisticians use the following relationship. One twenty three uses 6 characters, 1 of one kind, 2 of another kind and three of a third. The number of different ways to arrange these characters is $\frac{6!}{1!2!3!} = \frac{720}{12} = 60$ ways.

The number of ways to arrange the characters of the Egyptian numeral 73 is $\frac{10!}{7!3!}$, $\frac{3628800}{(5040)(6)} = \frac{3628800}{30240}$ = 120 ways.

The Egyptian numeral 32 is once again arranged in $\frac{5!}{3!2!} = \frac{120}{12}$ = 10 ways.

Converting between Decimal, Old Greek and Babylonian

First let's note an algorithm created by the author to convert a decimal number to Babylonian. This will be shown with the example of the decimal number 400,023.6.

Divide the number by the highest power of 60 that does not exceed it. In this case it is 60^3 or 216,000.

$$\frac{400{,}023.6}{216{,}000} = 1.85196$$

The integer part of the quotient represents the number of 60^3 units, in this case it is 1. Next subtract 1 x 60^3 units from the original number.

$400{,}023.6 - 1 \times 60^3$
$400{,}023.6 - 216{,}000$

184,023.6

Divide this by the next smallest power of sixty and subtract the integer multiple of that power of 60.

$$\frac{184{,}023.6}{60^2} = \frac{184{,}023.6}{3600} = 51.117\mathbf{6}$$

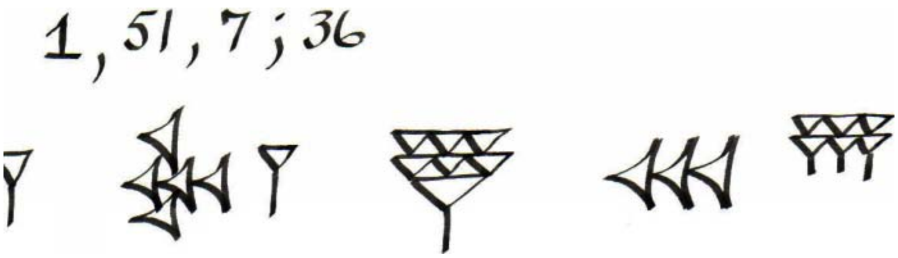
$184{,}023.6 - 51 \times 60^2 = 184{,}023.6 - 51 \times 3600 = 423.6$

Continuing this algorithm leads to:

$$\frac{423.6}{60^1} = 7.06$$

$$423.6 - 7 \times 60^1 = 3.6$$

There are three, 60^0 parts and $\frac{6}{10}(60) = 36$, 60^{-1} or 60^{th} parts. In Neugebauer notation the numeral is 1,51,7;36. Below is the Babylonian version:



Incidentally note that there is a one in ten chances of a decimal place being zero and a one in sixty chance of a Babylonian place being empty.

Below is a table showing some conversions:

Some Conversions between Decimal, Old Greek, and Babylonian

Decimal	Greek	Babylonian
36	ΔΔΔ IIIIII	
100	H	
324	HHHΔΔIIII	
0.4	? IIII	
12	ΔII	
$10\frac{1}{3}$	?	

Incidentally the old Greek version of 36 is not quite correct in that the old Greeks had a symbol for five. Can you write any decimal number in Babylonian numerals?

Babylonian Multiplication Tables

Babylonian multiplication tables were not as ours. Rather than squares, they were lists of single multiples. For example a Babylonian would represent the three tables vertically as:

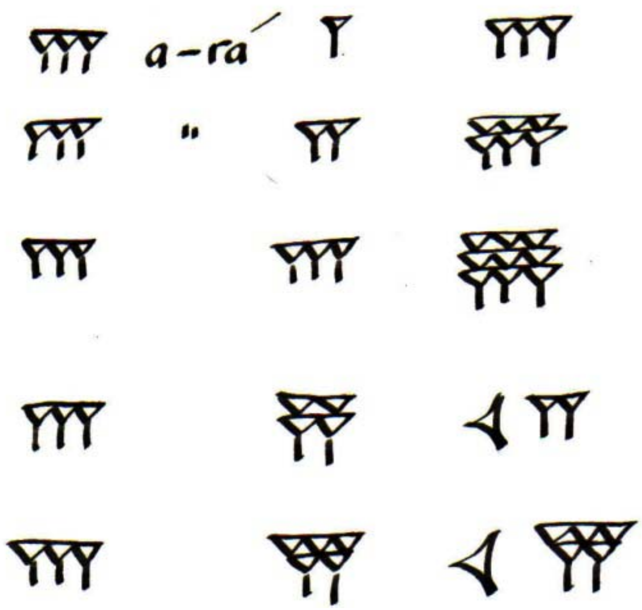
1	a-ra′	3	3
2	“	3	6

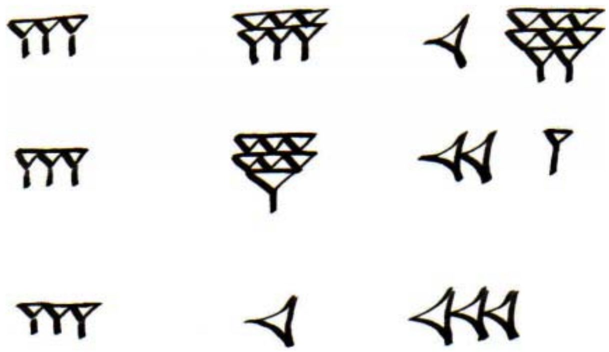
3	“	3	9	
4	“	3	12	
5	“	3	15	...etc.

In Babylonian the table would look something like:

Babylonian Multiplication Table

The following is a table of multiples of 3.





Babylonian Reciprocal Tables

Babylonian Reciprocal Table

Between each number are the words, "igi-n-gál-bi" in their letters

 the reciprocal of 2 is $\frac{1}{2}$ and $\frac{1}{2} = \frac{30}{60}$

 $\frac{1}{4} = \frac{15}{60}$

 $\frac{1}{5} = \frac{12}{60}$

 $\frac{1}{6} = \frac{10}{60}$

 $\frac{1}{8} = \frac{7\frac{1}{2}}{60}$

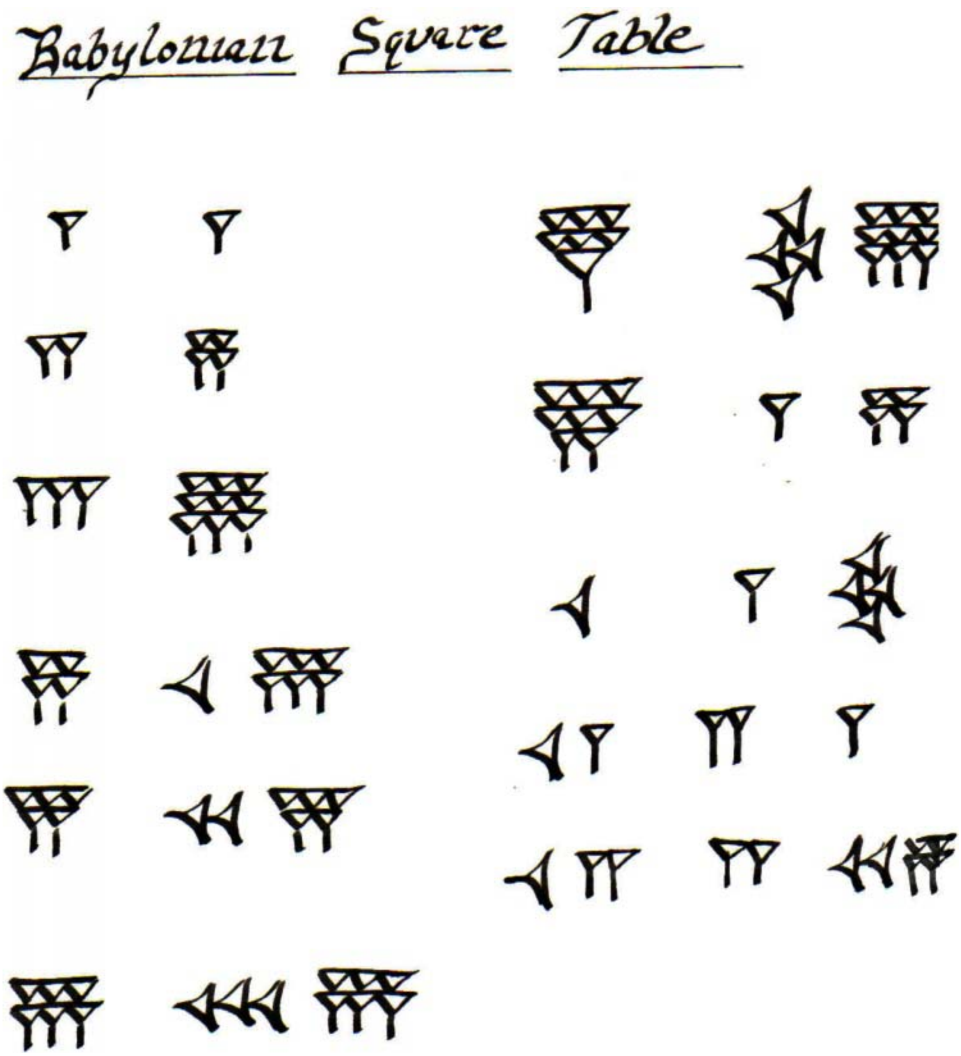
 $\frac{1}{9} = \frac{6\frac{2}{3}}{60}$

 $\frac{1}{10} = \frac{6}{60}$

Note that the Babylonians skipped one seventh. While it was possible for them to write this fraction it was a difficult and laborious process. Why?

Many ancient Babylonian mathematicians were to approximate the fraction and have written that " seven does not divide." In addition to this nor does eleven, thirteen, seventeen, etc. Why not?

Babylonian Square Tables



After filling one column, Babylonians would continue to make table in the next column to the right, much as we do. The table is read as a number and then it's square. Recall that the Babylonian method of multiplication shown in the unit requires square numbers.

Appendix

Ancient Numerals

Ancient Egyptian

Greek (Old)

1	2	3	4	10	20
I	II	III	IIII	Δ	ΔΔ

100	200	344
H	HH	HHHΔΔΔΔIII

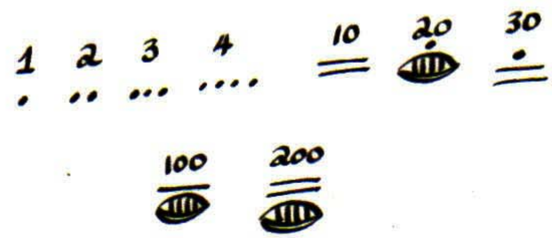
Greek (New)

1	2	3	4	10	20	30	100	200
α	β	γ	δ	ι	κ	λ	ρ	σ

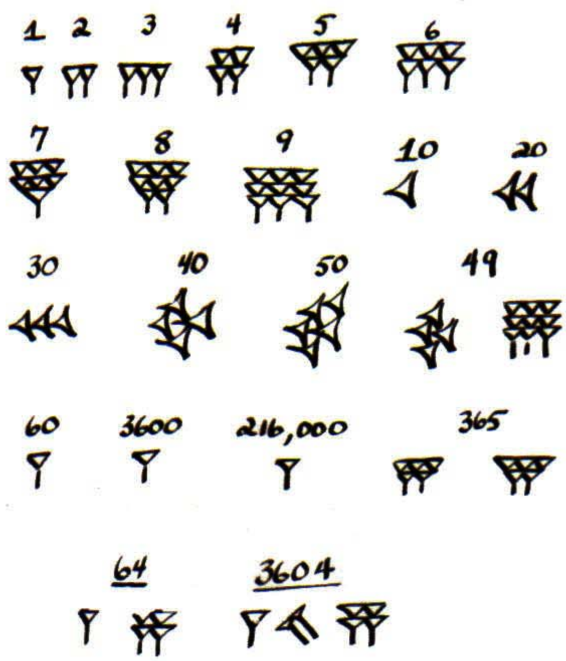
Ethiopian

1	2	3	4
I	II	III	IIII

Mayan



Babylonian



√2 (Correct to 9 decimal places)

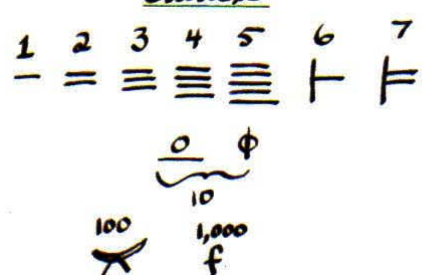
1; 24, 51, 10 (Neugebauer Not

π approximation

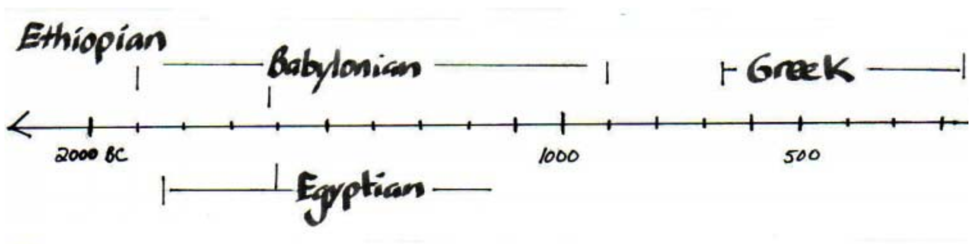
3 1/8

3; 7, 30

Chinese



Below is a very rough time line of the Babylonian, Egyptian and Greek time periods. There is much overlap, especially in the case of Babylonians. Apparently it reemerged around 900 B.C. and was definitely kept alive by being used again by the Greeks.



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- *Time and Tide, Babylonian Mathematics and Sexagesimal Notation* by John Harris

http://www-groups.dcs.st-and.ac.uk/~history/Indexes/Babylonians.html

-Another article by J.J. O'Connor and E.F. Robertson