The Probability Plot Correlation Coefficient Test for the Normal, Lognormal, and Gumbel Distributional Hypotheses

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Filliben (1975) and Looney and Gulledge (1985) developed powerful tests of normality which are conceptually simple and computationally convenient and may be readily extended to testing nonnormal distributional hypotheses. The probability plot correlation coefficient test consists of two widely used tools in water resource engineering: the probability plot and the product moment correlation coefficient. Many water resource applications require powerful hypothesis tests for nonnormal distributions. This technical note develops a new probability plot correlation coefficient test for the Gumbel distribution. Critical points of the test statistic are provided for samples of length 10 to 10,000. Filliben's and Looney and Gulledge's tests were originally developed for testing the normal hypothesis for sample sizes less than 100. Since many water resource research applications require a test of normality for samples of length greater than 100, Filliben's test is extended here for samples of length 100 to 10,000.

INTRODUCTION

Many water resource research investigations require tests for both normal and nonnormal hypotheses. *Filliben* [1975] and *Looney and Gulledge* [1985] developed powerful probability plot correlation coefficient (PPCC) tests for normality which have the following attractive features:

1. The test statistic is conceptually easy to understand because it combines two fundamentally simple concepts: the probability plot and the correlation coefficient.

2. The test is computationally simple since it only requires computation of a simple correlation coefficient.

3. The test statistic is readily extendible for testing some nonnormal distributional hypotheses, as is shown in this technical note.

4. The test compares favorably with seven other tests of normality on the basis of empirical power studies performed by *Filliben* [1975] and *Looney and Gulledge* [1985].

5. The test is invariant to the parameter estimation procedure employed to fit the probability distribution.

6. The test allows a comparison of the results in both a graphical (probability plot) and a numerical (correlation coefficient) form.

Given these attractive features and the fact that water resource applications often require tests of normal and nonnormal hypotheses, this study was undertaken to extend Filliben's original PPCC test for normality to samples of length 100 to 10,000 and to provide a new PPCC test for the Gumbel distribution.

A significant portion of the existing water resource literature has sought to determine which theoretical probability distribution best describes sequences of observed annual peak streamflows. *Beard*'s [1974] study, summarized by the Water Resource Council's Bulletin 17 [*Interagency Advisory Committee on Water Data*, 1982], represents perhaps the most comprehensive study. Other studies, too numerous to mention here, have compared the precision of quantile estimates derived from various combinations of probability distributions and parameter estimation procedures. *Wallis and Wood* [1985] provide a recent example of this type of study, and *Thomas* [1985] reviews the general problem of fitting flood-

flow frequency distributions. In general, these studies rarely include hypothesis tests to determine the probability of type I errors associated with the choice of an assumed probability distribution. This paper provides a new hypothesis test for the Gumbel distribution which may be employed in future flood frequency studies to examine the "goodness of fit." For example, this test could have provided a valuable contribution if it had been incorporated into the study by *Rossi et al.* [1984] which sought to approximate the distribution of 39 annual floodflow records in Italy using a generalized Gumbel distribution.

Kottegoda [1985] recommended the use of Filliben's PPCC test of normality as a preliminary outlier-detection procedure for sequences of annual peak floodflows. *Kottegoda* [1984] also found Filliben's PPCC test of normality useful for testing the normal hypothesis when fitting autoregressive moving average (ARMA) models to annual streamflow sequences.

THE PROBABILITY PLOT

Probability plots are used widely in the statistics literature. For example, *Johnson and Wichern* [1982, pp. 152–156], *Snedecor and Cochran* [1980, pp. 59–63], and *Mage* [1982] recommend use of probability plots for assessing the goodness of fit of a hypothesized distribution. A number of investigators have proposed goodness-of-fit tests which are based upon information contained in probability plots such as the tests proposed by *Filliben* [1975], *LaBrecque* [1977] and *Looney and Gulledge* [1985].

Probability plots have been used widely in water resource investigations. While analytic approaches for fitting probability distributions to observed data are, in theory, more efficient statistical procedures than graphical curve fitting procedures, many hydrologists would not make engineering decisions without the use of a graphical display (probability plot). Probability plots were recently recommended by the *National Research Council* [1985, Appendixes D and E] as a basis for extrapolation of flood frequency curves in dam safety evaluations. Similarly, the *Federal Emergency Management Agency* [1982, Appendix 3] recommends the use of probability plots in the determination of the probability distribution of annual maximum flood elevations which arises from the combined effects of ice jam and storm-induced flooding.

Although the U.S. Water Resources Council [*Interagency Advisory Committee on Water Data*, 1982] advocates the use of method of moments to fit the Log-Pearson type III distri-

bution to observed floodflow data, their recommendations also include the use of probability plots. Clearly, probability plots play an important role in statistical hydrology.

Since the introduction of probability plots in hydrology by *Hazen* [1914], the choice of which plotting position to employ in a given application has been a subject of debate for decades; *Cunnane* [1978] provides a review of the problem. Although the debate regarding which plotting position to employ still continues, most studies have failed to acknowledge how imprecise all such estimates must be. *Loucks et al.* [1981, p. 109] document the sampling properties of plotting positions and raise the question whether differences in the bias among many competing plotting positions are very important considering their large variances [see *Loucks et al.*, 1981, pp. 179–180]. This technical note need not address that issue, since a probability plot is used here as a basis for the construction of hypothesis tests, rather than for selecting a quantile of the cumulative distribution function as the design event.

A probability plot is defined as a graphical representation of the ith order statistic $y_{(i)}$ versus a plotting position which is simply a measure of the location of the ith order statistic from the standardized distribution. One is often tempted to choose the expected value of the ith order statistic, $E(y_{(i)})$, as a measure of the location parameter. However, Filliben argued that computational inconveniences associated with selection of the order statistic mean can, in general, be avoided by choosing to measure the location of the ith order statistic by its median, M_i, instead of its mean, $E(y_{(i)})$. Filliben chose to define a probability plot for the normal distribution as a plot of the ith order statistic versus an approximation to the median value of the ith order statistic. Approximations to the expected value of the ith order statistic, $E(y_{(i)})$, are now available for a wide variety of probability distributions (see, for example, *Cunnane* [1978]). *Looney and Gulledge* [1985] and *Ryan et al.* [1982] define a probability plot for the normal distribution as a plot of the ith order statistic versus an approximation to the mean value of the ith order statistic. There appears to be no particularly convincing reason why one should use the order statistic's mean or median as a measure of the location parameter when constructing a probability plot for the purpose of hypothesis testing.

The Probability Plot Correlation Coefficient Test

If the sample to be tested is actually distributed as hypothesized, one would expect the plot of the ordered observations $y_{(i)}$ versus the order statistic means or medians to be approximately linear. Thus the product moment correlation coefficient which measures the degree of linear association between two random variables is an appropriate test statistic. Filliben's PPCC test is simply a formalization of a technique used by statistical hydrologists for many decades; that is, it determines the linearity of a probability plot. Prior to the introduction of Filliben's PPCC test of normality into the water resources literature by *Loucks et al.* [1981, p. 181], determination of the linearity of a probability plot was largely a graphical and subjective procedure.

Filliben's PPCC test statistic is defined as the product moment correlation coefficient between the ordered observations $y_{(i)}$ and the order statistic medians M_i for a standardized normal distribution. His test statistic becomes

$$\hat{r} = \frac{\sum_{i=1}^{n} (y_{(i)} - \bar{y})(M_i - \bar{M})}{\sqrt{\sum_{i=1}^{n} (y_{(i)} - \bar{y})^2 \sum_{j=1}^{n} (M_j - \bar{M})^2}} \tag{1}$$

Here the M_i correspond to the median (or mean) values of the ith largest observation in a sample of n standardized random variables from the hypothesized distribution.

Filliben's PPCC test was developed for a two-parameter normal (or log normal) distribution. Generalized PPCC tests may be developed for any one- or two-parameter distribution which exhibits a fixed shape. However, distributions which do not exhibit a fixed shape such as the gamma family or distributions with more than two independent parameters are not suited to the construction of a general and exact PPCC test. For example, the PPCC test for normality presented here could be employed to test the two-parameter lognormal hypothesis, however, the test would not be suited to testing the three-parameter lognormal hypothesis. Use of the critical points of the test statistic $\hat{r}$ provided here or in the work by *Filliben* [1975] for testing the three-parameter lognormal hypothesis will lead to fewer rejections of the null hypothesis than one would anticipate. This is because only two parameters are estimated in the construction of the PPCC tests developed here, yet three parameters are required to fit a three-parameter log normal distribution.

Filliben's Test for Normality Extended

Filliben employed an estimate of the order statistic median

$$M_i = \Phi^{-1}(F_Y(y_{(i)})) \tag{2}$$

in (1); here $\Phi(x)$ is the cumulative distribution function of the standard normal distribution, and $F_Y(y_{(i)})$ is equal to its median value, which Filliben approximated as

$$\hat{F}_Y(y_{(i)}) = 1 - (0.5)^{1/n} \qquad i = 1$$
$$\hat{F}_Y(y_{(i)}) = (i - 0.3175)/(n + 0.365) \qquad i = 2, \cdots, n - 1 \tag{3}$$
$$\hat{F}_Y(y_{(i)}) = (0.5)^{1/n} \qquad i = n$$

Filliben's approximation to the median of the ith order statistic in (3) is employed in this study. The Minitab computer program [*Ryan et al.*, 1982] and *Looney and Gulledge* [1985] implement the PPCC test by employing *Blom*'s [1953] approximation to the order statistic means for a normal population. Hence the tables of critical points which *Ryan et al.* [1982] and *Looney and Gulledge* [1985] provide differ slightly from Filliben's results.

Filliben tabulated critical values of $\hat{r}$ for samples of length 100 or less. In Monte-Carlo experiments one is often confronted with the need for tests of normality with samples of greater length. Thus critical points (or significance levels) for Filliben's test statistic were computed for samples of length $n = 100$, 200, 300, 500, 1000, 2000, 3000, 5000, and 10,000. This was accomplished by generating 10,000 sequences of standard normal random variables each of length n and applying (1), (2), and (3) to obtain 10,000 corresponding estimates of $\hat{r}$, denoted $\hat{r}_i$, $i = 1, \cdots, 10,000$. Critical points of the distribution of $\hat{r}$ were obtained by using the empirical sampling procedure

$$\hat{r}_p = \hat{r}_{(10,000p)} \tag{4}$$

where $\hat{r}_p$ denotes the pth quantile of the distribution of $\hat{r}$ and $\hat{r}_{(10,000p)}$ denotes the 10,000p largest observation in the sequence of 10,000 generated values of $\hat{r}$. As the sample size, n, becomes very large, the percentage points of the distribution of $\hat{r}$ approach unity and, in fact, become indistinguishable from that value. Therefore it is more convenient to tabulate the percentage points of the distribution of $(1 - \hat{r})$. The results of these experiments are summarized in Table 1, which also provides a comparison with Filliben's results for the case when

TABLE 1. Critical Points of $1000(1 - \hat{r})$ Where $\hat{r}$ is the Normal Probability Plot Correlation Coefficient

	Significance Level					
n	0.005	0.01	0.05	0.10	0.25	0.50
100*	21.	19.	13.	11.	8.	6.
100	21.3	18.8	13.0	10.7	7.84	5.54
200	11.2	9.85	6.96	5.83	4.26	3.07
300	7.61	6.46	4.75	3.98	2.98	2.18
500	4.62	4.18	3.04	2.52	1.89	1.38
1,000	2.66	2.45	1.76	1.46	1.09	0.746
2,000	1.23	1.09	0.816	0.698	0.533	0.400
3,000	0.846	0.752	0.546	0.468	0.363	0.276
5,000	0.493	0.450	0.343	0.293	0.228	0.171
10,000	0.252	0.226	0.174	0.150	0.117	0.0890

This table is based upon 10,000 replicate experiments. The first row, marked *, gives Filliben's [1975] results. An example documents the use of this table. The 10th percentile of $\hat{r}$'s distribution when $n = 500$ is determined from

$$\hat{r}_{10} = 1 - 2.52 \times 10^{-3} = 0.99748$$

Interpolation of the critical points may be accomplished by noting that $\ln (n)$ and $\ln (1000(1 - \hat{r}))$ are linearly related for each significance level.

$n = 100$ in the first two rows of the table. The agreement is generally very good; discrepancies seem to be due to Filliben's rounding off of the values he reported.

A Probability Plot Correlation Coefficient Test for the Gumbel Distribution

As discussed earlier, an important and distinguishing property of Filliben's PPCC test statistic in (1) is that it is extendible to some nonnormal distributional hypotheses. In this section a probability plot correlation test for the extreme value type I distribution is presented. The extreme value type I distribution is often called the Gumbel distribution, since *Gumbel* [1941] first applied it to flood frequency analysis. Its CDF may be written as

$$F_Y(y) = \exp\left(-\exp\left(-(a + by)\right)\right) \tag{5}$$

Method of moments estimators of the parameters a and b are given by

$$\hat{a} = \gamma - \frac{\bar{y}\pi}{s_y 6^{1/2}} \tag{6a}$$

$$\hat{b} = \frac{\pi}{s_y 6^{1/2}} \tag{6b}$$

where γ is Euler's constant ($\gamma = 0.57721$) and $\bar{y}$ and s_y are the sample mean and standard deviation. Although maximum likelihood estimators of a distribution's parameters are usually preferred over method of moments estimators, [see *Lettenmaier and Burges*, 1982], in this case, the method of moments estimators are much simpler than the corresponding maximum likelihood estimators, which require a numerical algorithm to solve the resulting system of nonlinear equations. Method of moments estimators are employed here, since they are computationally convenient and they have no impact upon the hypothesis tests (see (11) which follows). This CDF is unique because sequences of Gumbel random variables may be conveniently generated by noting that (5) can be written in its inverse form as

$$y_i = F_Y^{-1}(y_i) = \frac{-a - \ln\left(-\ln\left(F_Y(y_i)\right)\right)}{b} \tag{7}$$

where the cumulative probabilities $F_y(y_i)$ are generated from a uniform distribution over the interval $(0, 1)$.

In this case the test statistic is defined as the product moment correlation coefficient between the ordered observations y_i and M_i using

$$M_i = F_Y^{-1}(\hat{F}_Y(y_{(i)})) \tag{8}$$

where $\hat{F}_y(y_{(i)})$ is *Gringorten*'s [1963] plotting position for the Gumbel distribution:

$$\hat{F}_Y(y_{(i)}) = \frac{i - 0.44}{n + 0.12} \tag{9}$$

Gringorten's plotting position was derived with the objective of setting

$$F_Y(y_{(n)}) = F_Y(E(y_{(n)})) \tag{10}$$

where $E(y_{(n)})$ is the expected value of the largest observation of a Gumbel distribution. Thus Gringorten's plotting position is only unbiased for the largest observation. *Cunnane* [1978] recommends the use of Gringorten's plotting position over several competing alternatives for use with the Gumbel distribution.

For testing the Gumbel hypothesis the test statistic is given by (1) with M_i obtained from (6), (7), (8), and (9). Since critical points of this test statistic are unavailable in the literature even for small samples, critical points (or significance levels) were computed for sample sizes in the range $n = 10$ to 10,000. This was accomplished by generating 10,000 sequences of Gumbel random variables (using (7)) each of length n and applying equations (1), (6), (7), (8), and (9) to obtain 10,000 corresponding estimates of $\hat{r}$ denoted $\hat{r}_i$, $i = 1, \cdots, 10,000$. Critical points of $\hat{r}$ were obtained by using the empirical sampling procedure given in (4). The results of these experiments

TABLE 2. Critical Points of $1000(1 - \hat{r})$ Where $\hat{r}$ is the Gumbel Probability Plot Correlation Coefficient

	Significance Level					
n	0.005	0.01	0.05	0.10	0.25	0.50
10	156.	137.	91.6	74.0	49.6	32.0
20	114.	294.	61.0	48.3	33.3	21.7
30	99.2	80.9	47.4	37.8	25.4	16.9
40	85.9	71.4	40.6	31.1	21.4	13.8
50	73.7	61.1	35.4	27.1	18.2	12.1
60	66.6	53.3	31.5	24.0	16.1	10.6
70	57.2	49.4	28.0	21.3	14.4	9.43
80	59.7	47.5	25.3	19.6	13.1	8.61
90	53.0	44.6	23.6	18.1	11.9	7.97
100	48.3	40.4	22.1	16.9	11.2	7.39
200	30.1	23.7	13.4	10.2	6.73	4.45
300	22.5	18.1	9.79	7.49	4.91	3.23
500	15.3	12.2	6.67	5.01	3.33	2.20
1,000	8.23	6.66	3.78	2.92	1.93	1.28
2,000	4.77	3.82	2.09	1.61	1.09	0.736
3,000	3.23	2.61	1.50	1.16	0.779	0.528
5,000	1.95	1.59	0.975	0.756	0.507	0.344
10,000	1.12	0.858	0.525	0.414	0.277	0.190

This table is based upon 10,000 replicate experiments. An example documents the use of this table. The 10th percentage point of $\hat{r}$ when $n = 1000$ is determined from

$$\hat{r}_{10} = 1 - 2.92 \times 10^{-3} = 0.99708$$

Interpolation of the critical points may be accomplished by noting that $\ln (n)$ and $\ln (1000(1 - \hat{r}))$ are linearly related for each significance level.

are summarized in Table 2, where again, as in Table 1, the percentage points of $(1 - \hat{r})$ are more convenient to tabulate.

Interestingly, the PPCC test is invariant to the fitting procedure employed to estimate a and b in (7). This result is evident for the Gumbel PPCC test when one rewrites the test statistic as

$$r = \frac{\text{cov}\,(y_i, M_i)}{[\text{Var}\,(y_i)\,\text{Var}\,(M_i)]^{1/2}}$$

$$= \frac{\text{cov}\left(\ln\,[-\ln\,[U_i]],\,\ln\left[-\ln\left[\dfrac{i - 0.44}{n + 0.12}\right]\right]\right)}{[\text{Var}\,(\ln\,[-\ln\,[U_i]])\,\text{Var}\,(\ln\,[-\ln\,[U_i]])]^{1/2}} \quad (11)$$

here the U_i are uniform random variables generated to equal $F_Y(y_i)$. An attractive property of this test is that the test statistic in (11) does not depend on either of the distribution parameters. This result is general in that it applies to any PPCC test for a one- or two-parameter distribution which exhibits a fixed shape.

SUMMARY

The probability plot correlation coefficient test is an attractive and useful tool for testing the normal, lognormal, and Gumbel hypotheses. The advantages of the PPCC hypothesis tests developed in this technical note include:

1. The PPCC test consists of two widely used tools in water resource engineering: the probability plot and the product moment correlation coefficient. Since hydrologists are well acquainted with both these tools, the PPCC test provides a conceptually simple, attractive, and powerful alternative to other possible hypothesis tests.

2. The PPCC test is flexible because it is not limited to any sample size. In addition, the test is readily extended to nonnormal hypotheses, as was accomplished here for the Gumbel hypothesis. Critical points for the test statistic r in (1) could readily be developed for other one- or two-parameter probability distributions which exhibit a fixed shape.

3. *Filliben* [1975] and *Looney and Gulledge* [1985] found that the PPCC test for normality compares favorably, in terms of power, with seven other normal test statistics.

4. The PPCC test statistic in (1) does not depend upon the procedure employed to estimate the parameters of the probability distribution.

5. While this technical note has developed the PPCC test statistic for the purposes of constructing composite hypothesis tests, the PPCC test statistic in (1) can readily be employed to compare the goodness of fit of a family of admissible distributions. That is, a sample could be fit to a number of reasonable distribution functions, and corresponding estimates of the PPCC could be used to compare the goodness of fit of each distribution. *Filliben* [1972] has found the PPCC to be a promising criterion for selection of a reasonable distribution function among several competing alternatives.

6. Filliben's PPCC test of normality has recently been incorporated into the Minitab computer program [*Ryan et al.*, 1982]. Although *Ryan et al.* [1982] recommend the use of *Blom*'s [1953] plotting position rather than Filliben's approximation given in (3), the Minitab computer program could readily be employed to implement the tests reported here.

REFERENCES

Beard, L. R., Flood flow frequency techniques, *Tech. Rep. CRWR-119*, Cent. for Res. in Water Resour., The Univ. of Tex. at Austin, Austin, 1974.

Blom, G., *Statistical Estimates and Transformed Beta Variables*, pp. 68–75, 143–146, John Wiley, New York, 1953.

Cunnane, C., Unbiased plotting positions—A review, *J. Hydrol., 37*, 205–222, 1978.

Federal Emergency Management Agency, Flood insurance study—Guidelines and specifications for study contractors, Washington, D. C., September 1982.

Filliben, J. J., Techniques for tail length analysis, *Proc. Conf. Des. Exp. Army Res. Dev. Test., 18th*, 1972.

Filliben, J. J., The probability plot correlation coefficient test for normality, *Technometrics, 17*(1), 1975.

Gringorten, I. I., A plotting rule for extreme probability paper, *J. Geophys. Res., 68*(3), 813–814, 1963.

Gumbel, E. J., The return period of flood flows, *Ann. Math Stat., 12*(2), 163–190, 1941.

Hazen, A., Storage to be provided in impounding reservoirs for municipal water supply, *Trans. Am. Soc. Civ. Eng., 77*, 1547–1550, 1914.

Interagency Advisory Committee on Water Data, Guidelines for determining flood flow frequency, Washington, D. C., 1982.

Johnson, R. A., and D. W. Wichern, *Applied Multivariate Statistical Analysis*, Prentice-Hall, Englewood Cliffs, N. J., 1982.

Kottegoda, N. T., Investigation of outliers in annual maximum flow series, *J. Hydrol., 72*, 105–137, 1984.

Kottegoda, N. T., Assessment of non-stationarity in annual series through evolutionary spectra, *J. Hydrol., 76*, 381–402, 1985.

LaBrecque, J., Goodness-of-fit tests based on nonlinearity in probability plots, *Technometrics, 19*(3), 293–306, 1977.

Lettenmaier, D. P., and S. J. Burges, Gumbel's extreme value I distribution: A new look, *J. Hydraul. Div. Am. Soc. Civ. Eng., 108*(HY4), 502–514, 1982.

Looney, S. W., and T. R. Gulledge, Jr., Use of the correlation coefficient with normal probability plots, *Am. Stat., 39*(1), 75–79, 1985.

Loucks, D. P., J. R. Stedinger, and D. A. Haith, *Water Resources Systems Planning and Analysis*, Prentice-Hall, Englewood Cliffs, N. J., 1981.

Mage, D. T., An objective graphical method for testing normal distributional assumptions using probability plots, *Am. Stat., 36*(2), 116–120, 1982.

National Research Council, *Safety of Dams—Flood and Earthquake Criteria*, National Academy Press, Washington, D. C., 1985.

Rossi, F., M. Fiorentino, and P. Versace, Two-component extreme value distribution for flood frequency analysis, *Water Resour. Res., 20*(7), 847–856, 1984.

Ryan, T. A. Jr., B. L. Joiner, and B. F. Ryan, *Minitab Reference Manual*, Minitab Project, Penn. State Univ., University Park, November 1982.

Snedecor, G. W., and Cochran, W. G., *Statistical Methods*, 7th ed., Iowa State University Press, Ames, 1980.

Thomas, W. O., Jr., A uniform technique for flood frequency analysis, *J. Water Resour. Plann. Manage. Div. Am. Soc. Civ. Eng., 111*(3), 321–337, 1985.

Wallis, J. R., and E. F. Wood, Relative accuracy of log Pearson III procedures, *J. Hydraul. Eng. Am. Soc. Civ. Eng., 111*(7), 1043–1056, 1985.

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Correction to "The Probability Plot Correlation Coefficient Test for the Normal, Lognormal, and Gumbel Distributional Hypotheses" by Richard M. Vogel

In the paper "The Probability Plot Correlation Coefficient Test for the Normal, Lognormal, and Gumbel Distributional Hypotheses" by R. M. Vogel (*Water Resources Research*, *22*(4), 587–590, 1986), the following corrections should be made.

Equation (1) should read

$$\hat{r} = \frac{\sum\limits_{i=1}^{n} (y_{(i)} - \bar{y})(M_i - \bar{M})}{\left[\sum\limits_{i=1}^{n} (y_{(i)} - \bar{y})^2 \sum\limits_{j=1}^{n} (M_j - \bar{M})^2 \right]^{1/2}} \tag{1}$$

Table 2 should be revised as follows: The critical point of $1000(1 - \hat{r})$ for $n = 20$ and a significance level of 0.01 should read 94.0 instead of 294.

Equation (11) should read

$$\hat{r} = \frac{\mathrm{cov}\,(y_i, M_i)}{[\mathrm{Var}\,(y_i)\,\mathrm{Var}\,(M_i)]^{1/2}}$$

$$= \frac{\mathrm{cov}\left\{ \ln\,[-\ln\,(U_i)],\, \ln\left[-\ln\left(\dfrac{i-0.44}{n+0.12} \right) \right] \right\}}{\left\{ \mathrm{Var}\,[\ln\,(-\ln\,(U_i))]\,\mathrm{Var}\left[\ln\left(-\ln\left(\dfrac{i-0.44}{n+0.12} \right) \right) \right] \right\}^{1/2}} \tag{11}$$

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