Chapter II

Complex Differentiation

Having introduced the complex number system, we proceed to the development of the theory of functions of a complex variable, beginning with the notion of derivative. Although the definition of the derivative of a complex-valued function of a complex variable is formally the same as that of the derivative of a real-valued function of a real variable, the concept holds surprises, as we shall see.

Generally, we shall let z denote a variable point in the complex plane; its real and imaginary parts will be denoted by x and y, respectively.

II.1. Definition of the Derivative

Let the complex-valued function f be defined in an open subset G of $\mathbf{C}$. Then f is said to be differentiable (in the complex sense) at the point z_0 of G if the difference quotient $\dfrac{f(z)-f(z_0)}{z-z_0}$ has a finite limit as z approaches z_0. That limit is then called the derivative of f at z_0 and denoted by $f'(z_0)$:

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}.$$

What does the last equality mean? In ϵ–δ language, it is the statement that for every positive number ϵ there is a positive number δ such that

$$\left|\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)\right|<\epsilon$$

whenever $0<|z-z_0|<\delta$.

As in calculus, the "d-notation" is also used for the derivative: if f is differentiable at z_0, then $f'(z_0)$ is according to convenience denoted alternatively by $\dfrac{df(z_0)}{dz}$.

NB. According to our definition, $f'(z_0)$ cannot be defined unless z_0 belongs to an open set in which f is defined.

II.2. Restatement in Terms of Linear Approximation

Let the complex-valued function f be defined in an open subset of $\mathbf{C}$ *containing the point z_0. Then f is differentiable in the complex sense at z_0 if and only if there is a complex number c such that the function $R(z) = f(z) - f(z_0) - c(z - z_0)$ satisfies $\displaystyle\lim_{z \to z_0} \frac{R(z)}{z - z_0} = 0$, in which case $f'(z_0) = c$.*

The statement is obvious in view of the equality

$$\frac{R(z)}{z - z_0} = \frac{f(z) - f(z_0)}{z - z_0} - c.$$

The statement says that f is differentiable at z_0, with $f'(z_0) = c$, if and only if f is well approximated near z_0 by the linear function $f(z_0) + c(z - z_0)$, in the sense that the remainder $R(z)$ in the approximation is small compared to the distance from z_0.

II.3. Immediate Consequences

The following properties of complex differentiation are proved from the basic definition in exactly the same way as the corresponding properties in the theory of functions of a real variable.

(i) *If f is differentiable at z_0 then f is continuous at z_0.*

(ii) *If f and g are differentiable at z_0, then $f + g$ and fg also are, and*

$$(f + g)'(z_0) = f'(z_0) + g'(z_0) \qquad \textit{(sum rule)};$$

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0) \qquad \textit{(product rule)}.$$

If in addition $g(z_0) \neq 0$, then f/g is differentiable at z_0, and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2} \qquad \textit{(quotient rule)}.$$

(iii) *If f is differentiable at z_0 and g is differentiable at $f(z_0)$, then the composite function $g \circ f$ is differentiable at z_0 and*

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0) \qquad \textit{(chain rule)}.$$

The proofs are left to the reader.

Exercise II.3.1. Prove statements (i)–(iii) in detail.

II.4. Polynomials and Rational Functions

From the definition of derivative it is immediate that a constant function is differentiable everywhere, with derivative 0, and that the identity function (the function $f(z) = z$) is differentiable everywhere, with derivative 1. Just as in elementary calculus one can show from the last statement, by repeated applications of the product rule, that, for any positive integer n, the function $f(z) = z^n$ is differentiable everywhere, with derivative nz^{n-1}. This, in conjunction with the sum and product rules, implies that every polynomial is everywhere differentiable: If $f(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$, where $c_0, \ldots, c_n$ are complex constants, then $f'(z) = nc_n z^{n-1} + (n-1)c_{n-1} z^{n-2} + \cdots + c_1$.

A function of the form f/g, where f and g are polynomials, is called a rational function. Such a function is defined wherever its denominator, g, does not vanish, hence everywhere except on a finite set. The quotient rule and the differentiability of polynomials imply that a rational function is differentiable at every point where it is defined and that its derivative is a rational function.

II.5. Comparison Between Differentiability in the Real and Complex Senses

Recall that a real-valued function u defined in an open subset G of $\mathbf{R}^2$ is said to be differentiable (in the real sense) at the point (x_0, y_0) of G if there are real numbers a and b such that the function $R(x, y) = u(x, y) - u(x_0, y_0) - a(x - x_0) - b(y - y_0)$ satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0.$$

In that case u has first partial derivatives at (x_0, y_0) given by

$$\frac{\partial u(x_0,y_0)}{\partial x} = a, \ \frac{\partial u(x_0,y_0)}{\partial y} = b.$$

The reader will find this notion discussed in any multivariable calculus book.

To facilitate a comparison with complex differentiation, we restate the preceding definition in complex notation: the real-valued function u in the open subset G of $\mathbf{C}$ is by definition differentiable at the point $z_0 = x_0 + iy_0$ of G if there are real numbers a and b such that the function $R(z) = u(z) - u(z_0) - a(x - x_0) - b(y - y_0)$ satisfies

$$\lim_{z\to z_0} \frac{R(z)}{z - z_0} = 0.$$

Now suppose that f is a complex-valued function defined in the open subset G of $\mathbf{C}$, and let u and v denote its real and imaginary parts: $f = u+iv$. Given a point $z_0 = x_0 + iy_0$ of G and a complex number $c = a + ib$, we can write

$$\begin{aligned}R(z) = f(z) - f(z_0) - c(z - z_0) &= [u(z) - u(z_0) - a(x - x_0) + b(y - y_0)] \\ &\quad + i\,[v(z) - v(z_0) - b(x - x_0) - a(y - y_0)] \\ &= R_1(z) + iR_2(z).\end{aligned}$$

Clearly, $\lim_{z\to z_0} \frac{R(z)}{z - z_0} = 0$ if and only if $\lim_{z\to z_0} \frac{R_1(z)}{z - z_0} = 0$ and $\lim_{z\to z_0} \frac{R_2(z)}{z - z_0} = 0$.

Referring to II.2, we can draw the following conclusion:

The function f is differentiable (in the complex sense) at z_0 if and only if u and v are differentiable (in the real sense) at z_0 and their first partial derivatives satisfy the relations $\frac{\partial u(z_0)}{\partial x} = \frac{\partial v(z_0)}{\partial y}$, $\frac{\partial u(z_0)}{\partial y} = -\frac{\partial v(z_0)}{\partial x}$. In that case,

$$f'(z_0) = \frac{\partial u(z_0)}{\partial x} + i\frac{\partial v(z_0)}{\partial x} = \frac{\partial v(z_0)}{\partial y} - i\frac{\partial u(z_0)}{\partial y}\,.$$

II.6. Cauchy-Riemann Equations

The two partial differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are called the Cauchy-Riemann equations for the pair of functions u, v. As seen above, the equations are satisfied by the real and imaginary parts of a complex-valued function at each point where that function is differentiable.

Exercise II.6.1. At which points are the following functions f differentiable?

$$\text{(a) } f(z) = x, \qquad \text{(b) } f(z) = \overline{z}, \qquad \text{(c) } f(z) = \overline{z}^2.$$

Exercise II.6.2. Prove that the function $f(z) = \sqrt{|xy|}$ is not differentiable at the origin, even though it satisfies the Cauchy-Riemann equations there.

Exercise* II.6.3. Prove that the Cauchy-Riemann equations in polar coordinates are

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}\,, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}\,.$$

II.7. Sufficient Condition for Differentiability

A theorem from the theory of functions of a real variable states that if a real-valued function of several variables has first partial derivatives, then it is differentiable at every point where those partial derivatives are continuous. (This can be found in any multivariable calculus book. For the convenience of readers who have not seen a proof, one is given in Appendix 1.) In combination with the necessary and sufficient condition from II.5, this gives the following useful sufficient condition for complex differentiability: *Let the complex-valued function $f = u + iv$ be defined in the open subset G of $\mathbf{C}$, and assume that u and v have first partial derivatives in G. Then f is differentiable at each point where those partial derivatives are continuous and satisfy the Cauchy-Riemann equations.*

II.8. Holomorphic Functions

A complex-valued function that is defined in an open subset G of $\mathbf{C}$ and differentiable at every point of G is said to be holomorphic (or analytic) in G. The simplest examples are polynomials, which are holomorphic in $\mathbf{C}$, and rational functions, which are holomorphic in the regions where they are defined. Later we shall see that the elementary functions of calculus—the exponential function, the logarithm function, trigonometric and inverse trigonometric functions, and power functions—all have complex versions that are holomorphic functions.

By II.5 we know that the real and imaginary parts of a holomorphic function have partial derivatives of first order obeying the Cauchy-Riemann equations. In the other direction, by II.7, if the real and imaginary parts of a complex-valued function have continuous first partial derivatives obeying the Cauchy-Riemann equations, then the function is holomorphic.

The asymmetry in the two preceding statements—the inclusion of a continuity condition in the second but not in the first—relates to an interesting and subtle theoretical point. The derivative of a holomorphic function, as will be shown later (in Section VII.8), is also holomorphic, so that in fact a holomorphic function is differentiable to all orders, and its real and imaginary parts have continuous partial derivatives to all orders. We shall only be able to prove this, however, after developing a fair amount of machinery. Meanwhile, we shall have to skirt around it occasionally.

Although, as we have seen above, some of the basic properties of real and complex differentiability are formally identical, the repeated differentiability of holomorphic functions points to a glaring dissimilarity. There are well-known examples of continuous real-valued functions on $\mathbf{R}$ that are nowhere differentiable. An indefinite integral of such a function is differentiable everywhere while its derivative is differentiable nowhere. By taking an n-fold indefinite integral, one can produce a function that is differentiable to order n yet whose n-th derivative is nowhere differentiable. Such "pathology" does not occur in the realm of complex differentiation.

From the basic rules of differentiation noted in Section II.3 one sees that if f and g are holomorphic functions defined in the same open set G, then $f+g$ and fg are also holomorphic in G, and f/g is holomorphic in $G\backslash g^{-1}(0)$. If f is holomorphic in G and g is holomorphic in an open set containing $f(G)$, then the composite function $g\circ f$ is holomorphic in G.

Exercise* II.8.1. Let the function f be holomorphic in the open disk D. Prove that each of the following conditions forces f to be constant: (a) $f'=0$ throughout D; (b) f is real-valued in D; (c) $|f|$ is constant in D; (d) $\arg f$ is constant in D.

Exercise* II.8.2. Let the function f be holomorphic in the open set G. Prove that the function $g(z)=\overline{f(\overline{z})}$ is holomorphic in the set $G^*=\{\overline{z}:\, z\in G\}$.

II.9. Complex Partial Differential Operators

The partial differential operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are applied to a complex-valued function $f=u+iv$ in the natural way:

$$\frac{\partial f}{\partial x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x},\quad \frac{\partial f}{\partial y}=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}.$$

We define the complex partial differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ by

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\quad \frac{\partial}{\partial \overline{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

Thus, $\dfrac{\partial}{\partial x}=\dfrac{\partial}{\partial z}+\dfrac{\partial}{\partial \overline{z}},\ \ \dfrac{\partial}{\partial y}=i\left(\dfrac{\partial}{\partial z}-\dfrac{\partial}{\partial \overline{z}}\right)$.

Intuitively one can think of a holomorphic function as a complex-valued function in an open subset of $\mathbf{C}$ that depends only on z, i.e., is independent of $\overline{z}$. We can make this notion precise as follows. Suppose the function $f=u+iv$ is defined and differentiable in an open set. One then has

$$\frac{\partial f}{\partial z}=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right),$$

$$\frac{\partial f}{\partial \overline{z}}=\frac{1}{2}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right).$$

The Cauchy-Riemann equations thus can be written $\frac{\partial f}{\partial \overline{z}} = 0$. As this is the condition for f to be holomorphic, it provides a precise meaning for the statement: "A holomorphic function is one that is independent of $\overline{z}$." If f is holomorphic, then (not surprisingly) $f' = \frac{\partial f}{\partial z}$, as the following calculation shows:

$$f' = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial z}\,.$$

II.10. Picturing a Holomorphic Function

One can visualize a real-valued function of a real variable by means of its graph, which is a curve in $\mathbf{R}^2$. A complex-valued function of a complex variable also has a graph, but its graph is a two-dimensional object in the four-dimensional space $\mathbf{C}\times\mathbf{C}$, something ordinary mortals cannot easily visualize. A more sensible approach, if one wants to obtain a geometric picture of a holomorphic function, is to think of the function as a map from the complex plane to itself, and to try to understand how the map deforms the plane; for example, how does it transform lines and circles?

The simplest case is that of a linear function, a function f of the form $f(z) = az + b$, where a and b are complex numbers, and $a \neq 0$ (to exclude the trivial case of a constant function). The map $z \mapsto az + b$ can be written as the composite of three easily understood transformations:

$$z \;\mapsto\; |a|z \;\mapsto\; az \;\mapsto\; az + b.$$

The first transformation in the chain is a scaling with respect to the origin by the factor $|a|$, a so-called homothetic map about the origin. The second transformation is multiplication by the number $a/|a|$, which is just rotation about the origin by the angle $\arg a$. The last transformation is translation by the vector b. We see in particular that the linear function $f(z) = az + b$ maps straight lines onto straight lines and preserves the angles between intersecting lines.

Linear functions are very special, but remember that a holomorphic function is a function that is well approximated locally by linear functions. If the function f is holomorphic in a neighborhood of the point z_0, one would expect it to behave near z_0 approximately like the linear function $z \mapsto f'(z_0)(z - z_0) + f(z_0)$. If $f'(z_0) = 0$ this will tell us little, but if $f'(z_0) \neq 0$ it should say something about the "infinitesimal" deformation produced by f near z_0. As we shall see, this is indeed the case: if $f'(z_0) \neq 0$, the holomorphic function f preserves the angles between curves intersecting at z_0. To make this precise we need some preliminaries about curves in the complex plane.

II.11. Curves in C

By a curve in **C** we shall mean a continuous function γ that maps an interval I of **R** into **C**. Thus, curves for us will always be parametrized curves. However, we shall often speak of curves as if they were subsets of **C**. For example, we shall say that the curve γ is contained in a given region of **C** if the range of γ is contained in that region.

Here are a few simple examples.

1. $\gamma(t) = (1-t)z_1 + tz_2$ $(-\infty < t < \infty)$.

Here, z_1 and z_2 are distinct points of **C**. This curve is a parametrization of the straight line determined by z_1 and z_2, the direction of the parametrization being from z_1 to z_2.

2. $\gamma(t) = \cos t + i \sin t$ $(0 \le t \le 2\pi)$.

This is a parametrization of the unit circle, the circle being traversed once in the counterclockwise direction as t moves from the initial to the terminal point of the parameter interval $[0, 2\pi]$.

3. $\gamma(t) = \cos t - i \sin t$ $(-2\pi \le t \le 2\pi)$.

This also is a parametrization of the unit circle, but this time the circle is traversed twice in the clockwise direction.

4. In this example, $\gamma(t)$ is defined piecewise:

$$\gamma(t) = \begin{cases} t, & 0 \le t \le 1, \\ 1 + (t-1)i, & 1 \le t \le 2, \\ i + 3 - t, & 2 \le t \le 3, \\ (4-t)i, & 3 \le t \le 4. \end{cases}$$

This is a parametrization of the square with vertices $0,\ 1,\ 1+i,\ i$. The square is traversed once in the counterclockwise direction.

The curve $\gamma : I \longrightarrow \mathbf{C}$ is said to be differentiable at the point t_0 of I if its real and imaginary parts are differentiable at t_0, or, what is equivalent, if the difference quotient $\dfrac{\gamma(t) - \gamma(t_0)}{t - t_0}$ approaches a finite limit as t tends to t_0. That limit is then denoted by $\gamma'(t_0)$. The curve γ is called differentiable if it is differentiable at each of its points; it is said to be of class C^1 if it is differentiable and its derivative, γ', is continuous.

The curve γ is said to be regular at the point t_0 if it is differentiable at t_0 and $\gamma'(t_0) \ne 0$. If γ is of class C^1 and regular at each point of its interval of definition, we call it a regular curve. The curves in the first three examples above are regular. The one in the fourth example is regular except at the points $1, 2, 3$ of the parameter interval $[0, 4]$.

A curve γ has a well-defined direction at each point t_0 where it is regular, namely, the direction determined by the derivative $\gamma'(t_0)$, referred to as the tangent direction. We can describe that direction, for example, by specifying the argument of $\gamma'(t_0)$, or by specifying the unit tangent vector, $\dfrac{\gamma'(t_0)}{|\gamma'(t_0)|}$.

Suppose γ_1 and γ_2 are two curves in $\mathbf{C}$, and suppose they have a point of intersection, say $\gamma_1(t_1) = \gamma_2(t_2)$. Suppose further that γ_j is regular at t_j, $j = 1,\ 2$. Then by the angle between γ_1 and γ_2 we shall mean the angle $\arg\ \gamma_2'(t_2) - \arg\ \gamma_1'(t_1)$ $(= \arg\ \gamma_2'(t_2)\overline{\gamma_1'(t_1)})$. In geometric terms, this is the angle through which one must rotate the unit tangent vector to γ_1 at t_1 to make it coincide with the unit tangent vector to γ_2 at t_2. Note that the angle depends on the order in which we take γ_1 and γ_2; reversal of the order leaves the magnitude of the angle the same but changes its sign. (To be completely precise, perhaps we should speak of the "angle between γ_1 and γ_2 corresponding to the parameter values t_1 and t_2" because the two curves might intersect for other pairs of parameter values. This degree of precision would not be worth the awkwardness of expression it would entail.)

Suppose that f is a holomorphic function in an open set G and that γ is a curve in G. Then we can apply f to γ to obtain the curve $f \circ \gamma$. Suppose γ is differentiable at t_0, and let $z_0 = \gamma(t_0)$. Then the standard argument justifying the chain rule applies to show that $f \circ \gamma$ is differentiable at t_0 and that $(f \circ \gamma)'(t_0) = f'(z_0)\gamma'(t_0)$. (Details are in Appendix 2.) Thus, if γ is regular at t_0 and if $f'(z_0) \neq 0$, then $f \circ \gamma$ is regular at t_0, and one obtains the direction of $f \circ \gamma$ at t_0 from that of γ at t_0 by adding $\arg\ f'(z_0)$.

II.12. Conformality

Let f be a holomorphic function defined in the open subset G of $\mathbf{C}$, and let z_0 be a point of G such that $f'(z_0) \neq 0$. Let γ_1 and γ_2 be curves such that $\gamma_1(t_1) = \gamma_2(t_2) = z_0$, and such that γ_j is regular at t_j, $j = 1,\ 2$. Then the angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ equals the angle between γ_1 and γ_2.

This statement follows immediately from the discussion preceding it, from which one sees that

$$\arg\,(f \circ \gamma_j)'(t_j) = \arg f'(z_0) + \arg \gamma_j'(t_j), \qquad j = 1,\ 2.$$

The function $f(z) = z^2$ shows what can happen if the hypothesis $f'(z_0) \neq 0$ is dropped. This function, whose derivative vanishes at the origin, transforms two lines through the origin making an angle α into two lines making an angle 2α. On the other hand, as we shall see in a later chapter, the derivative of a nonconstant holomorphic function can vanish only on an isolated set of points, so the angle-preservation property given by the theorem above is the rule rather than the exception.

A map from the plane to the plane is called conformal at the point z_0 if it preserves the angles between pairs of regular curves intersecting at z_0. Thus, we can restate II.11 by saying that a holomorphic function is conformal at each point where its derivative does not vanish.

II.13. Conformal Implies Holomorphic

We shall now show that conformal maps are necessarily holomorphic. We begin with the simplest case, that of a linear transformation of the plane. Linear here means linear as a transformation of the real vector space $\mathbf{R}^2$ to itself. If f is such a map then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are constants, and f is uniquely determined by those constants plus the condition $f(0) = 0$. Letting $a = \frac{\partial f}{\partial z}$ and $b = \frac{\partial f}{\partial \overline{z}}$ (also constants), we see that $f(z) = az + b\overline{z}$. Suppose this map preserves the angles between pairs of directed lines intersecting at the origin. We shall then prove that $b = 0$. We may assume that $a + b \neq 0$ (since otherwise the transformation would send the whole real axis to the origin) and, that done, that $a \neq 0$ (since otherwise the map would be anticonformal—it would reverse the angles between pairs of directed lines). Let λ be a complex number of absolute value 1. Our map sends the real line to the directed line through the origin determined by $a+b$, and it sends the directed line through the origin determined by λ to the one determined by $a\lambda + b\overline{\lambda}$. Our assumption about angle preservation thus implies that

$$\arg(a\lambda + b\overline{\lambda}) \;-\; \arg(a+b) \;=\; \arg\lambda.$$

Since the left side in this equality equals

$$\arg\lambda + \arg\left(a + \frac{b\overline{\lambda}}{\lambda}\right) - \arg(a+b),$$

the equality reduces to

$$\arg\left(a + \frac{b\overline{\lambda}}{\lambda}\right) \;=\; \arg(a+b)\,.$$

Now, if $b \neq 0$ then, as λ traverses the unit circle, the point $a + \dfrac{b\overline{\lambda}}{\lambda}$ (twice) traverses the circle with center a and radius $|b|$, in violation of the preceding equality, which says that $a + \frac{b\overline{\lambda}}{\lambda}$ lies on the ray through the origin determined by $a+b$. We can conclude that $b = 0$; in other words, our map is given by $z \mapsto az$, a holomorphic function, as desired.

The preceding result will serve as a lemma for its generalization to C^1 maps. We consider a complex-valued function $f = u + iv$ defined on an open subset G of $\mathbf{C}$. We assume that u and v have continuous first partial derivatives. Then, if γ is a differentiable curve in G, the curve $f \circ \gamma$ is also differentiable. In fact, suppose z_0 is a point on γ, say $\gamma(t_0) = z_0$. Let

$$a = \frac{\partial f(z_0)}{\partial z}, \quad b = \frac{\partial f(z_0)}{\partial \overline{z}}.$$

An application of the chain rule, the details of which are in Appendix 2, then shows that

$$(f \circ \gamma)'(t_0) = a\gamma'(t_0) + b\overline{\gamma'(t_0)}.$$

Hence, if f preserves the angles between pairs of regular curves intersecting at z_0, then the linear map $z \mapsto az + b\overline{z}$ preserves the angles between pairs of directed lines through the origin. By what is established above, that means $b = 0$ and $a \neq 0$. The equality $b = 0$ just says that the functions u and v satisfy the Cauchy-Riemann equations at z_0, which, as noted in Section II.7, implies that f is differentiable (in the complex sense) at z_0. Moreover, $f'(z_0) = \frac{\partial f(z_0)}{\partial z} = a$.

The following theorem has been proved: *Let f be a complex-valued function, defined in an open subset G of $\mathbf{C}$, whose real and imaginary parts have continuous first partial derivatives. If f preserves the angles between regular curves intersecting in G, then f is holomorphic, and f' is never* 0.

II.14. Harmonic Functions

The complex-valued function f, defined in an open subset of $\mathbf{C}$, is said to be harmonic if it is of class C^2 and satisfies Laplace's equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

This equation and its higher-dimensional versions play central roles in many branches of mathematics and physics. Of course, the complex-valued function f is harmonic if and only if its real and imaginary parts are.

II.15. Holomorphic Implies Harmonic

A holomorphic function is harmonic, provided it is of class C^2.

As noted in Section II.8, we shall prove later that a holomorphic function is of class C^k for all k, at which point we can drop the proviso in the preceding statement. To establish the proposition, let the function $f = u + iv$ be holomorphic and of class C^2. By the Cauchy-Riemann equations, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\Big(\frac{\partial v}{\partial y}\Big) = \frac{\partial}{\partial y}\Big(\frac{\partial v}{\partial x}\Big) = -\frac{\partial^2 u}{\partial y^2}\,,$$

which proves that u is harmonic. Similar reasoning proves the same result for v, and thus f is harmonic.

II.16. Harmonic Conjugates

The reasoning in the preceding section shows that a pair of real-valued C^2 functions u and v, defined in the same open subset of $\mathbf{C}$, will be harmonic if they satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,.$$

In this situation one says that v is a harmonic conjugate of u. Phrased differently, if u and v are real valued and of class C^2, then v is a harmonic conjugate of u if and only if $u + iv$ is holomorphic.

Note that harmonic conjugates are not unique: if v is a harmonic conjugate of u then so are the functions that differ from v by constants. That is essentially the extent of nonuniqueness, as we shall see later (and, in a special case, in Exercise II.16.4 below). A natural question is whether every harmonic function has a harmonic conjugate. We shall eventually develop enough machinery to deal with this question.

Friendly Advice. When beginning the study of complex analysis and faced with a problem in the subject, the initial response of many students is to reduce the problem to one in real variables, a subject they have previously studied. Such a reduction can sometimes be helpful, but at other times it can make things overly complicated. (Some of the exercises below illustrate this point.) Try to get into the habit of "thinking complex."

Exercise II.16.1. For which values of the real constants $a,\ b,\ c,\ d$ is the function $u(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ harmonic? Determine a harmonic conjugate of u in the cases where it is harmonic.

Exercise* II.16.2. Prove that Laplace's equation can be written in polar coordinates as

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0\,.$$

Exercise II.16.3. Find all real-valued functions h, defined and of class C^2 on the positive real line, such that the function $u(x, y) = h(x^2 + y^2)$ is harmonic.

Exercise II.16.4. Prove that, if u is a real-valued harmonic function in an open disk D, then any two harmonic conjugates of u in D differ by a constant.

Exercise II.16.5. Suppose that u is a real-valued harmonic function in an open disk D, and suppose that u^2 is also harmonic. Prove that u is constant.

Exercise II.16.6. Prove that if the harmonic function v is a harmonic conjugate of the harmonic function u, then the functions uv and $u^2 - v^2$ are both harmonic.

Exercise II.16.7. Prove (assuming equality of second-order mixed partial derivatives) that

$$\frac{\partial^2}{\partial\overline{z}\partial z} \; = \; \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Thus, Laplace's equation can be written as $\dfrac{\partial^2 f}{\partial\overline{z}\partial z} \; = \; 0$.

Exercise II.16.8. Prove that if u is a real-valued harmonic function then the function $\dfrac{\partial u}{\partial z}$ is holomorphic.