Portfolio Optimization in Secondary Spectrum Markets

Praveen Kumar Muthuswamy*, Koushik Kar*, Aparna Gupta*, Saswati Sarkar† and Gaurav Kasbekar†
*Rensselaer Polytechnic Institute
Troy, NY 12180, USA
Email: {muthup, kark, guptaa}@rpi.edu
†University of Pennsylvania
Philadelphia, PA 19104, USA
Email: {kgaurav, swati}@seas.upenn.edu

***Abstract*—In this paper, we address the spectrum portfolio optimization (SPO) question in the context of secondary spectrum markets, where bandwidth (spectrum access rights) can be bought in the form of *primary* and *secondary* contracts. While a primary contract on a channel provides guaranteed access to the channel bandwidth (possibly at a higher per-unit price), the bandwidth available to use from a secondary contract (possibly at a discounted price) is typically uncertain/stochastic. The key problem for the buyer (service provider) in this market is to determine the amount of primary and secondary contract units needed to satisfy uncertain user demand.**

We initially consider a single-region problem in which the spectrum contracts are valid only in the single-region in which the buyer wishes to provide service. We formulate the problem as one of minimizing the cost of the spectrum portfolio subject to constraints on bandwidth shortage. Two different forms of bandwidth shortage constraints are considered, namely, the demand satisfaction rate constraint, and the demand satisfaction probability constraint. While the SPO problem under demand satisfaction rate constraint is shown to be convex for all density functions, the SPO problem under demand satisfaction probability constraint is not convex in general. We derive some sufficient conditions for convexity for this case. The SPO problems can therefore be solved efficiently using standard convex optimization techniques. Later, we extend the problem formulation and the convexity results to the multiple-region setting, where the buyer's portfolio is intended to serve a set of disjoint geographical locations, each having its own customer demand.

Finally, we perform a thorough simulation-based study of the single-region and the multiple-region problems for different choices of the problem parameters, and provide key insights regarding the portfolio composition and demonstrate the convexity of the efficient frontier. We provide several insights about the scaling behavior of the unit prices of the secondary contracts, as the stochastic characterization of the bandwidth available from secondary contracts change.

I. Introduction and Background

The number of users of the wireless spectrum, as well as the demand for bandwidth per user, has been growing at an enormous pace in recent years. Since spectrum is limited, its effective management is vitally important to meet this growing demand. The spectrum available for public use can be broadly categorized into the unlicensed and licensed zones. In the unlicensed part of the spectrum, any wireless device is allowed to transmit. To use the licensed part, however, license must be obtained from appropriate government authority – the Federal Communications Commission (FCC) in the United States, for example - for the exclusive right to transmit in a certain block of the spectrum over the license time period, typically for a fee. While spectrum management in licensed bands has mostly been controlled by responsible government bodies, the need for bringing market based reform in spectrum trading is being increasingly recognized [1], [2], [3]. In order to achieve spectrum-usage efficiency, spectrum markets should allow dynamic trading of spectral resources and derived contracts of different risk-return characteristics. Providers can then choose to buy/sell one or more of these spectrum contracts depending on the level of service they wish to provide to their customers.

We consider a spectrum market in which a wireless service provider (buyer) can purchase spectrum access rights from another provider (seller) in the form of two types of spectrum contracts: *primary contract* and *secondary contract*. Typically, the buyer will be a smaller or local provider, buying access rights over its operational area from a larger regional or national provider which acts as the seller, although the framework and results that we present in this paper does not make any such assumption. Primary contract offers unrestricted access rights on a channel – a specific channel or one of a set of channels "owned" by the seller. On the other hand, secondary contract offers restricted access rights on a channel or a set of channels – it provides access to the "leftover" bandwidths on the channel(s) that the primary members on the channel(s) do not need at that specific time. At their core, primary and secondary contracts differ in the risk-return tradeoff. A primary contract represents a risk-free contract in terms of its bandwidth return characteristics, while the secondary contract is inherently risky in terms of the bandwidth it can provide. Primary contracts would generally be more expensive (in terms of cost per unit bandwidth provided/used), since they provide full access rights. Secondary contracts would typically be cheaper due to their riskiness. These two contracts represent two fundamental forms of spectrum access contracts – analogous to bonds and stocks in terms of the risk characteristics. In financial markets, it is well known that bonds and stocks help investors achieve their expected risk-return tradeoff on investment. Similarly, we envisage that the wireless service providers can efficiently tradeoff the level of service they wish to provide against their cost by using these two types of contracts.

A key challenge for a provider in this market is to determine a spectrum portfolio of primary and secondary contracts that can provide the desired level of service to its users at a low cost. We formulate and study this *Spectrum Portfolio*

Optimization (SPO) problem from the perspective of a buyer. In standard financial portfolio optimization, the objective is to maximize the expected portfolio return while satisfying some constraint on the variance of return. However, in our case, the constraint can be specified meaningfully in two ways – either in terms of the expected bandwidth shortage, or in terms of the probability of bandwidth shortage. We refer to these constraints as the *demand satisfaction rate constraint* and the *demand satisfaction probability constraint*, respectively. We study the SPO problem under the above two constraints separately.

The specific technical contributions of this paper are as follows. Firstly, we show that the SPO problem under demand satisfaction rate constraint is convex under any assumptions on the user demand and the bandwidth return distributions. Secondly, we show that the SPO problem under demand satisfaction probability constraint is not convex in general and also derive sufficient conditions on the density functions for convexity. The motivation behind showing convexity of the optimization problems is that convex problems can be solved efficiently using standard techniques such as gradient descent and Newton's methods, whereas there are no general techniques for solving non-convex problems efficiently. We outline how the optimal portfolios can be computed in these cases. We also show that the efficient frontier of both the SPO problems is convex. In the next step, we extend the SPO problem formulation and the convexity results to a multiple-region scenario, where the buyer's portfolio is intended to serve a set of disjoint geographical locations, each having its own user demand, using available primary and secondary contracts that provide access rights only over subsets of all locations of interest. Finally, we perform a detailed simulation-based study of the single-region and the multiple-region SPO problems and provide several insights about the portfolio composition and the price characteristics of the secondary contracts.

Economics of spectrum allocation and auction mechanisms have been discussed widely in the literature [4], [5], [6], [7]. Spectrum sharing games and/or pricing issues have been considered in [8], [9], [10], [11]. Discussions and recommendations for transition to spectrum markets and secondary markets for spectrum trading have emerged [12], [13], [14]. In [14], the authors consider a spectrum secondary market analogous to the stock market for dynamically trading their channel holdings. The proposed auction-based market mechanism is shown to improve user performance and spectrum utilization. However, a clear design of the contract types and trade off analysis using portfolio theory has been not been considered before.

Portfolio optimization problem has been studied extensively in finance since the development of the mean-variance optimization framework in [15]. Several attempts have been made to improve the model and the risk measure [16], [17], [18], [19]. In [19], the authors propose a new measure of risk, namely, the expected shortfall and show that the problem of minimizing expected shortfall subject to a linear equality constraint is convex. The expected shortfall function considered in [19] measures the shortfall of return with respect to the α-quantile of the return distribution. But the demand satisfaction rate constraint that we consider measures the shortfall of the bandwidth return relative to a stochastic quantity and is therefore different from the shortfall function in [19]. However, we are still able to make use of some of their analysis techniques to our problem. Probabilistic constraints have not been studied much, until recently in [20] and [21]. In [20], the authors study probabilistically constrained linear programs and present conditions for convexity of the constraint. While we apply some their results in our context, we also provide additional conditions for convexity on the SPO problem.

The novelty of our contribution stems from the following aspects. Though the notion of primary and secondary users and their spectrum access rights have been extensively discussed recently, our modeling of these access rights as bond-like riskless and stock-like risky contracts, and the rigorous formulation of the spectrum portfolio optimization problem are novel. Convexity of various versions of the portfolio optimization question have been studied in the finance and optimization literature; however, very limited results exist on the specific demand satisfaction constraints that appear meaningful in the spectrum access context. We provide several interesting results for the SPO problem with such constraints in this paper. The formulation and analysis of the multi-region SPO problem, and the insights obtained from our numerical studies, also constitute novel contributions of this work.

The rest of the paper is organized as follows. In Section II, we formally define the SPO problems under demand satisfaction rate and probability constraints. In Sections III and IV, we study the convexity properties of the two SPO problems. In Section V, we study the multiple-region SPO problem. Finally, in Section VI, we present the results of our simulation study.

II. Spectrum Portfolio Optimization Problem Formulation

In this section, we formally define the spectrum portfolio optimization (SPO) problem for a single region. The formulation and discussion of the multi-region SPO problem is deferred to Section V. Although not necessary for the mathematical formulation or subsequent analytical treatment of the SPO problem, it is easy to motivate the development of the framework by considering a (secondary) spectrum market in which N "higher level" spectrum providers are selling access contracts in the form of primary and secondary contracts to other "lower level" providers. These seller spectrum providers will typically be large providers (like VerizonWireless, AT&T, and Sprint in the US for example) who have directly leased spectrum from the governing body (like FCC), and might want to offer their excess bandwidth in the form of primary and secondary contracts. The buyers of the contracts can be smaller, possibly local, wireless spectrum service providers who are trying to obtain bandwidth at the cheapest price to serve their user (customer) demand. We assume that primary and secondary contracts can be obtained in multiple units.

Without loss of generality, we can assume that each unit of primary contract provides exclusive access to 1 unit of bandwidth in some channel that the seller provider operates on. On the other hand, each unit of secondary contract provides exclusive access to bandwidth that is a random variable varying between 0 and 1 unit. While this assumption is for the ease of exposition, it can be easily generalized. A simple way to view this setting would be to consider a seller provider having C units of free bandwidth, offering C units of primary and C units of secondary contracts. If at any time slot, the primary members in totality use $\alpha < C$ units of bandwidth, each unit of secondary contract has access to $0 < (C-\alpha)/C < 1$ units of bandwidth. A buyer holding x units of secondary contracts with this seller provider will then have access to $x(C-\alpha)/C$ units of bandwidth in that time slot.

Note that we are associating contracts – primary or secondary – with the seller providers, not specific channels. All primary contracts (no matter which seller provider provides it) can be considered equivalent, since they offer the same bandwidth return (one unit, guaranteed). This also argues for the fact that they must be priced the same; without loss of generality, we assume that the cost of one unit of any primary contract is unity. Secondary contracts offered by different seller providers will differ from one another, depending on the access pattern of the primary members of the seller provider, and their price per unit with also differ. However, since each unit of secondary contract offers an average return of less than one unit bandwidth, and have some risk associated with the return, the price per unit for each secondary contract must be less than unity (the price of an unit of primary contract).

With this abstraction, the SPO problem can be viewed in the context of a market where a single type of primary contract, and N different types of secondary contracts, are being offered.[1] Each unit of primary contract sold in the secondary spectrum market offers guaranteed access to 1 unit of bandwidth at a cost of 1. The secondary contract offered by the provider i can be described by the pair (p_i, B_i), where, p_i is the unit price of the secondary contracts offered by the i^{th} seller provider and B_i is the random variable (varying between 0 and 1) characterizing the bandwidth return from one unit of secondary contract of the i^{th} provider. From the above discussion, $p_i < 1, \forall i$.

In the following, we assume that each seller provider has a large pool of available bandwidth, and so any amount of primary or secondary contract units can be bought from the providers. This is for ease of exposition, and can be easily generalized by incorporating into the SPO problem additional upper bounds on the number of primary and secondary contract units available from a seller provider.

Now we are ready to formally define the SPO problem from the perspective of a single buyer provider. The buyer's objective is to create a spectrum portfolio consisting of primary and secondary contract units from the N seller providers in order to provide service to its customer base. Let $x_i, 1 \leq i \leq N$ denote the amount of secondary contract units purchased from i^{th} seller provider. Since the primary contracts offered by all the N providers are identical, we denote the total amount of primary contract units bought by x_0. We assume a relaxation that $x_0, x_1, x_2, ..., x_N$ are non-negative real numbers, not necessarily integers. Let the vector $\overline{x} = (x_0, x_1, x_2, ..., x_N)$, denote the buyer's spectrum portfolio. The buyer wishes to satisfy its customers' demand for bandwidth using the spectrum portfolio, $\overline{x}$. The customer demand is often unknown in advance. We model it as a random variable Q.

[1]Note that the basic portfolio optimization question in financial markets, while considering multiple risky (stock) assets, assumes only a single risk-free (bond) asset, for similar reasons.

The bandwidth return or the actual units of bandwidth available from a spectrum portfolio $\overline{x}$, is uncertain, due to presence of the secondary contracts. The bandwidth return of the portfolio $\overline{x}$, denoted by $B(\overline{x})$, is defined as

$$B(\overline{x}) = x_0 + \sum_{i=1}^{N} x_i \times B_i. \tag{1}$$

Since the bandwidth return and the demand are stochastic, it is impossible or highly expensive to construct a portfolio that always offers enough bandwidth to satisfy the customer demand. But, it is desirable to construct portfolios with low levels of bandwidth shortage. Let us denote the bandwidth shortage of a portfolio by $S(\overline{x})^+$. $S(\overline{x})^+$ is defined as,

$$S(\overline{x})^+ = \max(Q - B(\overline{x}), 0) \tag{2}$$

Note that the shortage, $S(\overline{x})^+$, is also a stochastic quantity.

The spectrum portfolio optimization (SPO) problem for the buyer is to find the least costly portfolio with low levels of bandwidth shortage. The SPO objective is

$$\text{minimize } C(\overline{x}) = x_0 + \sum_{i=1}^{N} x_i \times p_i \tag{3}$$

The constraint on bandwidth shortage can be specified either in terms of expected shortage or probability of shortage. Therefore, we consider two versions of constraints for the SPO problem – the Demand Satisfaction Rate (DSR) constraint, and the Demand Satisfaction Probability (DSP) constraint, as expressed below:

$$\textit{DSR Constraint:} \quad E[S(\overline{x})^+] \; < \; \delta; \tag{4}$$

$$\textit{DSP Constraint:} \quad Pr(S(\overline{x})^+ > 0) \; < \; \epsilon. \tag{5}$$

Here $C(\overline{x}) = x_0 + \sum_{i=1}^{N} x_i \times p_i$ is the cost of the spectrum portfolio $\overline{x}$. The DSR constraint ensures that the expected amount of bandwidth shortage is below certain acceptable level δ. On the other hand, the DSP constraint bounds the probability of shortage to a low value ϵ. We devote the following sections to the study of the SPO problem under these two types of constraints.

III. SPO under Demand Satisfaction Rate (DSR) Constraint

In this section, we study the properties of the SPO problem under demand satisfaction rate constraint, and provide the expressions for certain useful quantities that can be utilized to compute the optimal portfolio solution efficiently.

A. Convexity Analysis

The objective function of the SPO problem (Equation 3) is linear and therefore convex. The demand satisfaction rate function (i.e. $E[S(\overline{x})^+]$), however, is non-linear in $\overline{x}$. We show below that $E[S(\overline{x})^+]$ is also convex in $\overline{x}$. This implies that the feasibility set represented by the DSR constraint (Equation 4) is also convex, and therefore the SPO problem under DSR constraint is a convex optimization question.

Theorem 1: $E[S(\overline{x})^+]$ is convex in $\overline{x}$.

Proof: We show that the Hessian of the function $E[S(\overline{x})^+]$ is positive semi-definite. We obtain the gradient and Hessian of $E[S(\overline{x})^+]$ as follows.

Let $g(\overline{x}) = E[S(\overline{x})^+] = E[S(\overline{x}) \times I(S(\overline{x}) > 0)]$, where $I(\cdot)$ is an indicator function and $S(\overline{x}) = Q - (x_0 + \sum_{i=1}^{N} x_i \times B_i)$. Also let random vector $\overline{B} = [B_1\ B_2\ ...\ B_N]$. We first obtain $\frac{\partial g(\overline{x})}{\partial x_i}$, for $i = 1$ to N. Given i, define $u = Q - x_0 - \sum_{j \neq i} x_j \times B_j$ and $v = B_i$. Note that $S(\overline{x}) = u - x_i v$. Now,

$$g(\overline{x}) = \int_{-\infty}^{\infty} \int_{x_i v}^{\infty} (u - x_i v) f_{U,V}(u, v) du dv$$

$$\begin{aligned}
\frac{\partial g(\overline{x})}{\partial x_i} &= \frac{\partial}{\partial x_i} \int_{-\infty}^{\infty} \int_{x_i v}^{\infty} (u - x_i v) f_{U,V}(u, v) du dv \\
&= \int_{-\infty}^{\infty} \int_{x_i v}^{\infty} (-v) f_{U,V}(u, v) du dv \\
&= -E[B_i \times I(S(\overline{x}) > 0)] \qquad (6)
\end{aligned}$$

We apply Leibniz Integral rule in the second step above to obtain the derivative of the integral. $\frac{\partial g(\overline{x})}{\partial x_0}$ can be obtained similarly by defining $u = Q - \sum_j x_j \times B_j$.

$$\begin{aligned}
\frac{\partial g(\overline{x})}{\partial x_0} &= \frac{\partial}{\partial x_0} \int_{u=x_0}^{\infty} (u - x_0) f_U(u) du \\
&= \int_{u=x_0}^{\infty} (-1) f_U(u) du \\
&= -E[I(S(\overline{x}) > 0)]
\end{aligned}$$

We next obtain the Hessian of the shortfall constraint, i.e. $\nabla^2 g(\overline{x})$, using a similar approach. First, we find $\frac{\partial^2 g(\overline{x})}{\partial x_k \partial x_i}$, where $k \neq i$ and $k, i \geq 1$. Define $u = Q - x_0 - \sum_{j \neq i,k} x_j \times B_j$, $v = B_k$, $w = B_i$, which have a joint density $f_{U,V,W}(.,.,.)$. Now, $S(\overline{x}) = u - x_k v - x_i w$ and

$$\frac{\partial g(\overline{x})}{\partial x_i} = -\int_{R^3} w I(S(\overline{x}) > 0) f_{U,V,W}(u, v, w) du dv dw$$

$\frac{\partial^2 g(\overline{x})}{\partial x_k \partial x_i}$ is given by,

$$\begin{aligned}
&= -\frac{\partial}{\partial x_k} \int_{-\infty}^{\infty} w \int_{-\infty}^{\infty} \int_{x_k v + x_i w}^{\infty} f_{U,V,W} du dv dw \\
&= -\int_{-\infty}^{\infty} w \int_{-\infty}^{\infty} (-v) f_{U,V,W}(x_k v + x_i w, v, w) dv dw \\
&= -\int b_i (-b_k) f_{\overline{B},S(\overline{x})}(\overline{b}, 0) d\overline{b} \\
&= f_{S(\overline{x})}(0) E[B_i B_k | S(\overline{x}) = 0]
\end{aligned}$$

$\frac{\partial^2 g(\overline{x})}{\partial x_0^2}$ can be obtained by defining $u = Q - \sum_{j \neq k} x_j \times B_j$, $v = 1$, $w = B_k$, for some k.

$$\begin{aligned}
\frac{\partial^2 g(\overline{x})}{\partial x_0^2} &= -\frac{\partial}{\partial x_0} \int_{-\infty}^{\infty} \int_{x_0 + x_k w}^{\infty} f_{U,W}(u, w) du dw \\
&= -\int_{-\infty}^{\infty} (-1) f_{U,W}(x_0 + x_k w, w) dw \\
&= \int f_{\overline{B},S(\overline{x})}(\overline{b}, 0) d\overline{b} \\
&= f_{S(\overline{x})}(0) \int f_{\overline{B}|S(\overline{x})}(\overline{b}|0) d\overline{b} = f_{S(\overline{x})}(0)
\end{aligned}$$

$f_{\overline{B}}$ is the joint density function of the bandwidth return vector $\overline{B}$. Similarly, $\frac{\partial^2 g(\overline{x})}{\partial x_0 \partial x_k} = f_{S(\overline{x})}(0) E[B_k | S(\overline{x}) = 0]$. Thus, the Hessian of the constraint can be written as

$$\nabla^2 g(\overline{x}) = f_{S(\overline{x})}(0) \times E[\overline{A}\overline{A}^T | S(\overline{x}) = 0] \qquad (7)$$

where $\overline{A} = [1\ B_1\ B_2\ ...\ B_N]^T$.

Since $f_{S(\overline{x})}(0) \geq 0$ and $E[\overline{A}\overline{A}^T | S(\overline{x}) = 0]$ is positive semi-definite, $\nabla^2 E[S(\overline{x})^+]$ is also positive semi-definite. Therefore, $E[S(\overline{x})^+]$ is convex. ∎

B. Computational Methods

Convex optimization problems can be solved efficiently using techniques such as Newton's method and gradient descent if the gradients of the objective function and the constraints are available. Therefore, we briefly discuss how the gradients of the SPO problem under the DSR constraint can be computed. The gradients of the objective function (linear) can be computed trivially; the constraint gradient can be computed numerically as outlined below. We know,

$$\begin{aligned}
\frac{\partial E[S(\overline{x})^+]}{\partial x_0} &= -E[I[S(\overline{x}) > 0]], \\
\frac{\partial E[S(\overline{x})^+]}{\partial x_k} &= -E[B_k \times I[S(\overline{x}) > 0]]. \qquad (8)
\end{aligned}$$

$$E[B_i \times I[S(\overline{x}) > 0]] = \int_{\overline{b}} \int_q b_i \times I[S(\overline{x}) > 0] \times f_{\overline{B},Q}(.) d\overline{b} dq.$$

Once the joint density function of $\overline{B}$ and Q, $f_{\overline{B},Q}(.)$, is known, the $N+1$-dimensional integration can be computed numerically by approximating the integral using a summation. For our simulations, we assume that $B_i s$ are independent of each other and also independent of Q, therefore, $f_{\overline{B},Q}(.) = \prod_{i=1}^{N} f_{B_i} \times f_Q$. We consider the truncated gaussian density function (for both the B_is and Q) defined below:

$$f_Y(y) = \frac{1}{\sigma_Y} \times \frac{\phi(\frac{y - \mu_Y}{\sigma_Y})}{\Phi(\frac{b - \mu_Y}{\sigma_Y}) - \Phi(\frac{a - \mu_Y}{\sigma_Y})}, a \leq y \leq b, \qquad (9)$$

where ϕ and Φ are the standard normal density and distribution functions.

IV. SPO UNDER DEMAND SATISFACTION PROBABILITY (DSP) CONSTRAINT

Next, we study the convexity properties of the SPO problem under the demand satisfaction probability constraint. We first show that the DSP constraint is non-convex, without any assumptions on the distribution of the demand Q and the bandwidth return variable B_is. Later, we present the conditions under which the constraint and therefore the SPO problem becomes convex.

A. Convexity Analysis

1) Non-convexity of SPO: We present an example where the feasible set of the SPO problem under the DSP constraint (Equation 5) is non-convex. Consider a simple case, when there are two secondary contracts, i.e $N = 2$. Let the B_1 and B_2 be uniformly distributed between 0 and 1. Let Q have a triangular density function given by, $f_Q(q) = 2 \times q, 0 \leq q \leq 1$. Note that $Pr(S(\overline{x})^+ > 0) = Pr(B(\overline{x}) < Q)$, where $S(\overline{x}) = Q - B(\overline{x})$ and $B(\overline{x}) = x_0 + \sum_{i=1}^{N} x_i \times B_i$. Consider the portfolio vectors $\overline{x}_1 = (0, 1, 0)$, $\overline{x}_2 = (0, 0, 1)$. We have

$$Pr(S(\overline{x}_1)^+ > 0) = Pr(B_1 < Q) = \frac{2}{3} = Pr(S(\overline{x}_2)^+ > 0).$$

Choose $\epsilon = 0.67$, and denote the feasibility set by,

$$\mathcal{X}_{0.67} = \{\overline{x} : Pr(S(\overline{x})^+ > 0) < 0.67\}. \tag{10}$$

We see that $\overline{x}_1, \overline{x}_2 \in \mathcal{X}_{0.67}$. However, consider the convex combination, $\overline{x}_3 = \frac{1}{2} \times \overline{x}_1 + \frac{1}{2} \times \overline{x}_2 = (0, 1/2, 1/2)$. It can be shown that

$$Pr(S(\overline{x}_3)^+ > 0) = Pr(\frac{1}{2} \times B_1 + \frac{1}{2} \times B_2 < Q) = \frac{17}{24} > 0.67.$$

i.e. $\overline{x}_3 \notin \mathcal{X}_{0.67}$. So, the feasibility set is not convex in general.

2) Conditions for convexity: For a given ϵ, denote the feasibility set (from (5)) by

$$\mathcal{X}_\epsilon = \{\overline{x} : Pr(S(\overline{x})^+ > 0) < \epsilon\} \tag{11}$$

Theorem 2: $\mathcal{X}_\epsilon$ is convex if the random vector $\overline{B} = [B_1 \;\; B_2 \;\; ...B_N]^T$ and the demand Q have log-concave and symmetric density functions and $0 \leq \epsilon \leq 0.5$.

We invoke the results from [20] to show this. From [20], we know that $\{\overline{x} : Pr(\overline{x}^T a < b) \geq 1 - \epsilon\}$ is convex, if the density function of the random vector a and the random variable b is log-concave and symmetric. This result readily applies to our case, by rewriting the constraint (Equation 5) as $Pr(-B(\overline{x}) < -Q) \geq 1 - \epsilon$.

In addition, we derive another condition for convexity, which only requires a non-increasing assumption on the probability density function of Q, and none on the B_is.

Theorem 3: $\mathcal{X}_\epsilon$ is convex if $f_Q^{'} \leq 0$ everywhere.

Proof:

$$\begin{aligned} Pr(S(\overline{x})^+ > 0) &= Pr(Q > B(\overline{x})) \\ &= \int_{\overline{b}} f_{\overline{B}}(\overline{b}) P(Q > x_0 + \sum_{i=1}^{N} x_i \times b_i) d\overline{b} \\ &= \int_{\overline{b}} f_{\overline{B}}(\overline{b}) (1 - F_Q(x_0 + \sum_{i=1}^{N} x_i \times b_i)) d\overline{b}. \end{aligned}$$

Here F_Q is the distribution function of the demand Q. Since $f_{\overline{B}}(\overline{b}) \geq 0$ and independent of $\overline{x}$, we see that $Pr(S(\overline{x})^+ > 0)$ is convex when $F_Q(x_0 + \sum_{i=1}^{N} x_i \times b_i)$ is concave in $\overline{x}$ for all $\overline{b}$. The second order derivatives of $F_Q(x_0 + \sum_{i=1}^{N} x_i \times b_i)$ are given by,

$$\frac{\partial^2 F_Q(.)}{\partial x_0^2} = f_Q^{'}, \frac{\partial^2 F_Q}{\partial x_i^2} = f_Q^{'} \times b_i^2, \frac{\partial^2 F_Q}{\partial x_i x_j} = f_Q^{'} \times b_i \times b_j.$$

For any $\overline{z} \in R^{N+1}$, $\overline{z}^T \nabla^2 F_Q(.) \overline{z} = f_Q^{'}(.) \times (z_0 + \sum_{i=1}^{N} z_i \times b_i)^2$. Therefore, $Pr(S(\overline{x})^+ > 0)$ and hence $\mathcal{X}_\epsilon$ is convex, if $f_Q^{'}(.) \leq 0$. That is, the density function of the demand is non-increasing everywhere. ■

In the proof above, Q and $\overline{B}$ are assumed to be independent of each other. If not, the sufficient condition for convexity is, $f_{Q|\overline{B}}^{'}(.|\overline{b}) \leq 0$ everywhere, $\forall \overline{b}$.

Theorem 2 covers important distributions such as the gaussian and the uniform density function (both $\overline{B}$ and Q must follow some symmetric, log-concave distribution, although they need not be the same distribution). Theorem 3 covers exponential and other asymmetric decreasing density functions for Q that are not included in Theorem 2 (the distribution of $\overline{B}$ can be arbitrary).

Remark 1: Let $N = 1$. If Q is deterministic, then the DSP constraint reduces to a linear constraint. In this case, the optimal portfolio consists of entirely primary or entirely secondary contracts. The optimal portfolio is $(Q, 0)$, if $\epsilon < F_{B_1}(p_1)$ and $(0, \frac{Q}{F_{B_1}^{-1}(\epsilon)})$, if $\epsilon >= F_{B_1}(p_1)$, where F_{B_1} is the cumulative distribution function of B_1.

Remark 2: It can be shown that the efficient frontier (curve showing optimal spectrum portfolio cost vs ϵ, δ) for the SPO problem under DSP as well as DSR constraint is convex. The proof of this can be found in the technical report [22].

B. Computational Methods

We obtain the gradient of the DSP constraint in order to solve the SPO problem efficiently. From (8), we find that

$$\frac{\partial E[S(\overline{x})^+]}{\partial x_0} = -E[I[S(\overline{x}) > 0]] = -Pr(S(\overline{x})^+ > 0) \tag{12}$$

Therefore, the gradient of the DSP constraint is readily obtained from the second derivatives of the DSR constraint (7).

$$\begin{aligned} \frac{\partial Pr(S(\overline{x})^+ > 0)}{\partial x_0} &= -f_{S(\overline{x})}(0) \\ \frac{\partial Pr(S(\overline{x})^+ > 0)}{\partial x_k} &= -f_{S(\overline{x})}(0) \times E[B_k | S(\overline{x}) = 0] \\ &= -\int w \times f_{U,W}(x_0 + x_k w, w) dw, \end{aligned}$$

where $U = Q - \sum_{i \neq k} x_i \times B_i$ and $W = B_k$. Under independence assumptions, $f_{U,W} = f_U \times f_W$ and f_U can be written in terms of the density functions of B_i $(i \geq 1, i \neq k)$ and Q, as $f_U = \{\bigodot_{i \neq k, x_i \neq 0} \frac{1}{|x_i|} f_{B_i}(\frac{-b_i}{x_i})\} * f_Q(q)$, where, f_{B_i} is the density function of B_i and $\bigodot$ denotes repeated convolution.

V. SPO over multiple regions

The spectrum contracts typically come with clauses that restricts the use of the spectrum to certain geographical regions. This could be due to licensing or coverage limitations of the seller provider. For example, a seller provider may only have the license to use a part of the spectrum in certain regions (say certain counties or states in the United States), and not others. Alternatively, the base stations of the seller provider may only cover certain sub-areas of the overall area of interest to the buyer, which can span multiple regions. This adds additional complexity to the SPO problem, since the spectrum portfolio should satisfy the buyer provider's requirements for each of these regions. In this section, we formulate the SPO problem over multiple regions and show that the results for the single region problem extend to multi-region case as well.

Let us assume that the buyer of spectrum contracts operate over a set of K disjoint geographical regions. The buyer's objective is to construct a portfolio of spectrum contracts in order to satisfy the user demand in each of the K regions. Denote the set of regions by $\mathcal{R}$, i.e, $\mathcal{R} = \{1, 2, .., K\}$. Let there be M primary and N secondary contracts in the market. Let z_i, p_j denote the unit price of i^{th} primary contract and j^{th} secondary contract, respectively. Let $\mathcal{R}^p_i \subset \mathcal{R}, 1 \leq i \leq M$ denote the set of regions in which the i^{th} primary contract is valid. Similarly, let $\mathcal{R}^s_j \subset \mathcal{R}, 1 \leq j \leq N$ denote the set of regions in which the j^{th} secondary contract is valid. The user demand for each region is uncertain, denoted by the random variable Q_k, $1 \leq k \leq K$.

The multi-region SPO problem under DSR constraint can be stated as follows:

$$\text{Minimize} \quad C(\overline{x}) = \sum_{i=1}^{M} y_i \times z_i + \sum_{j=1}^{N} x_j \times p_j, \tag{13}$$

$$\text{s.t.} \quad E[\{Q_k - \textstyle\sum_{i\in\mathcal{C}^p_k} y_i - \sum_{j\in\mathcal{C}^s_k} x_j \times B_{jk}\}^+] < \delta_k \quad \forall k, \tag{14}$$

$$E[\textstyle\sum_{k=1}^{K}\{Q_k - \sum_{i\in\mathcal{C}^p_k} y_i - \sum_{j\in\mathcal{C}^s_k} x_j \times B_{jk}\}^+] < \delta. \tag{15}$$

Here $\{y_1, ..., y_M, x_1, ..., x_N\}$ denotes the spectrum portfolio. $\mathcal{C}^p_k$ and $\mathcal{C}^s_k$ denote the set of primary and secondary contracts that are valid in the k^{th} region ($1 \leq k \leq K$), respectively. $\mathcal{C}^p_k$ and $\mathcal{C}^s_k$ can be obtained from $\mathcal{R}^p_i, 1 \leq i \leq M$ and $\mathcal{R}^s_i, 1 \leq j \leq N$. Note that $\mathcal{C}^p_k \subset \{1, 2, .., M\}$ and $\mathcal{C}^s_k \subset \{1, 2, .., N\}$. The random variable B_{jk} represents the bandwidth return of the j^{th} secondary contract in the k^{th} region. For the multiple region problem, there are totally $K + 1$ inequality constraints; one DSR constraint for each of the K regions and one overall DSR constraint for all the regions. The LHS of the $(K+1)^{th}$ constraint is simply the summation of the LHS of the first K constraints. However, note that $\sum_{k=1}^{K} \delta_k > \delta$, else the last constraint would be redundant; typically, the buyer provider may want have $\delta_k > \delta/K$, for each k. The motivation of both types of constraints (per-region as well as overall) is as follows. While the buyer provider would be interested in the ensuring a certain DSR over its overall customer base, it may also want to ensure a certain DSR (possibly a smaller normalized DSR than the overall DSR) is ensured in each of its regions of operation, to limit excessive customer dissatisfaction in each individual region.

The SPO problem under DSP constraint can be defined similarly as above, but by replacing the expectation constraints with the corresponding probability constraints, and δ_k and δ by ϵ_k and ϵ, respectively.

For both the SPO problems, we see that the k^{th} constraint ($1 \leq k \leq K$) is similar to the constraint for the single region problem ((4) and (5)) except for the presence or absence of few variables inside the two summations. First, consider the SPO problem under DSR constraint (13-(15)). Let the k^{th} rate constraint be denoted by g_k. g_k involves only some of the $y_i^{'}$s and $x_j^{'}$s. It can be rewritten as,

$$E[\{Q_k - \sum_{1\leq i\leq M} y_i \times I(i \in C^p_k) - \sum_{1\leq j\leq N} x_j \times B^{'}_j\}^+] < \delta_k, \tag{16}$$

where $B^{'}_j = B_{jk}$, if $j \in \mathcal{C}^s_k$, else $B^{'}_j = 0$. $I(i \in C^p_k)$ is the indicator function for the set C^p_k. Now, the proof technique for the single-region problem can be readily extended to show that g_k is convex in $y_i^{'}s, x_j^{'}s$. The final constraint (g_{K+1}) is also convex, since it is the sum of several convex functions. Therefore, the feasible set for this problem is convex, since the intersection of several convex sets is convex. Similarly, the feasible set for the multiple-region SPO problem under DSP constraint is also convex, if the density functions of all the random parameters involved are log-concave and symmetric.

Since the problems are convex, the constraint gradients can be used to solve the optimization problems efficiently. The gradient of the k^{th} DSR constraint is given by,

$$\frac{\partial g_k}{\partial y_i} = -E[I(S_k > 0)], \; i \in C^p_k; \quad 0, \text{ otherwise},$$

where $S_k = Q_k - \sum_{i\in\mathcal{C}^p_k} y_i - \sum_{j\in\mathcal{C}^s_k} x_j \times B_{jk}$.

$$\frac{\partial g_k}{\partial x_j} = -E[B_{jk} \times I(S_k > 0)], \; j \in C^s_k; \quad 0, \text{ otherwise}.$$

Similarly, the gradient of k^{th} demand satisfaction probability constraint (denoted by h_k) can be computed using the results for the single region problem, as given below:

$$\frac{\partial h_k}{\partial y_i} = -f_{S_k}(0), \; i \in C^p_k; \quad 0, \text{ otherwise},$$

where f_{S_k} is the density function of S_k.

$$\frac{\partial h_k}{\partial x_j} = -f_{S_k}(0) \times E[B_{jk}|S_k = 0], \; j \in C^s_k; \quad 0, \text{ otherwise}.$$

For both the problems, the gradient of the $(K+1)^{th}$ constraint is the summation of the gradients of the first K constraints. The gradients can be evaluated numerically as explained in the previous sections.

VI. Simulation-based evaluation

We solve the SPO problems using Matlab to study the characteristics of the spectrum portfolio. Our goal is examine how the parameters of the problem, namely, the price of the secondary contracts, the bandwidth return distributions, and the constraints (ϵ, δ) influence the portfolio composition. The results for the single-region SPO problems are presented in sections VI-A and VI-B, while the results for the multiple-region problem are presented in section VI-C.

A. Single Primary and Single Secondary contract

We first consider the simplest case of there being a single secondary contract seller in the market. The bandwidth return B_1 and the demand Q are assumed to have truncated normal distributions (Refer to (9)). B_1 has a mean of 0.5, while the demand Q has a mean of 1.5. The distribution of Q is restricted to the interval $[0, 3]$. We obtain optimal portfolio when the key parameters of the problem (ϵ, δ, p_1) are changed.

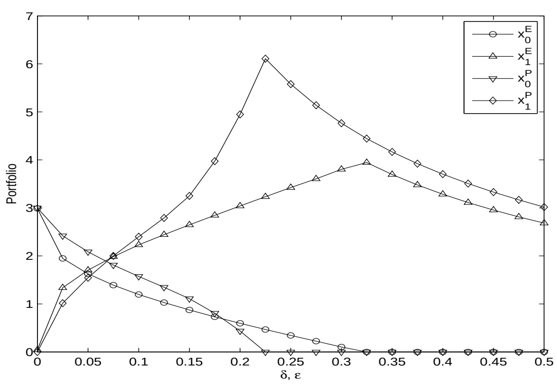

Fig. 1. Number of primary (x_0^E, x_0^P) and secondary (x_1^E,x_1^P) contract units in the optimal portfolio for the SPO problem under DSR and DSP constraint.

Figure 1 shows the spectrum portfolio composition for different choices of the DSR constraint (δ) and DSP constraint (ϵ), respectively. The unit price of the secondary contract, $p_1 = 0.25$. In the figure, $\overline{x}^E = \{x_0^E, x_1^E\}$ and $\overline{x}^P = \{x_0^P, x_1^P\}$ denote the portfolios for SPO problems with DSR and DSP constraints, respectively. As expected, when $\delta = \epsilon = 0$, we observe that the portfolio consists of primary contract units only. This is due to the fact that the secondary contracts due to their stochastic returns introduce bandwidth shortage (or demand violation) even if they are bought in large quantities. Moreover, the number of primary contract units in both the cases is equal to the maximum possible demand (i.e 3). As the constraint (ϵ, δ) is relaxed, we find that the number of primary contract units reduces sharply until it becomes zero. On the other hand the number of secondary contract units (x_1^E, x_1^P) increases initially, but starts decreasing as soon as the number of primary contract units becomes zero. This can be explained as follows: As the constraint (ϵ, δ) is increased from zero, it becomes unnecessary to meet the demand with probability one. Therefore, total cost of the portfolio can be reduced, by reducing the number of primary contract units, while adding the requisite amount of secondary contract units to keep the demand violation below the desired value. This happens until the number of primary contract units becomes zero. Beyond this point, the only way to reduce the cost is to reduce the number of secondary contract units, which can be reduced as ϵ, δ increases.

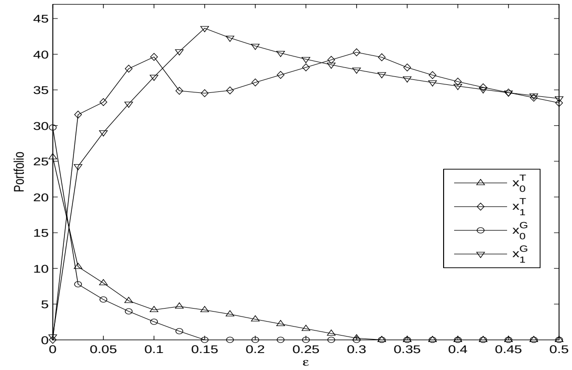

Fig. 2. Number of primary and secondary contract units in the portfolio for the SPO problem under DSR constraint. $\{x_0^T, x_1^T\}$, $\{x_0^G, x_1^G\}$ denote the spectrum portfolio for empirical and gaussian fitted distributions, respectively.

Next, we study the sensitivity of the portfolio composition to changes in the distribution of the demand (Q) and the bandwidth return (B_1). We obtain the empirical distribution of the total daily traffic of a Verizon Wi-Fi HotSpot network from [23] (Refer to Figure 12 of [23]) and consider this distribution for the user demand Q. From this, we compute the distribution of B_1 as $f_{B_1}(b) = f_Q(\beta(1-b))$, $0 \le b \le 1$, since bandwidth availability is related negatively to the user demand (the scaling factor β is used for normalization). The results for SPO problem under DSR constraint are shown in Figure 2, where, $\{x_0^T, x_1^T\}$ represents the spectrum portfolio when Q and B_1 have the empirical distributions discussed above. We also solve the SPO problem with Q and B_1 modeled as gaussian distributions that approximate the above empirical distributions (See $\{x_0^G, x_1^G\}$). Some small differences notwithstanding, we see that the general trend in the optimal portfolio composition for the empirical and fitted gaussian distributions is the same. Additionally, we ran simulations with uniform distribution and observed similar results. Therefore, in the following we only present the results for gaussian distributions.

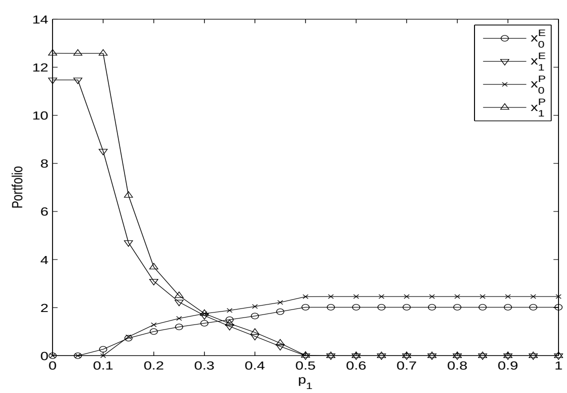

Fig. 3. Optimal spectrum portfolio composition for different choices of the unit price of secondary contract.

Figure 3 shows the effect of unit price of the secondary contract on the portfolio. The simulation parameters for this figure are same as the ones chosen for Figure 1. But, we now fix ϵ and δ at 0.1, and increase the price of the secondary contract p_1. As p_1 is increased from 0 to 1, the spectrum portfolio composition gradually changes from those with entirely secondary contract units to those with entirely primary contract units. The transition in this case happens when $p_1 = 0.5$. For prices between 0.1 and 0.5, the portfolio has a mix of primary as well as secondary contract units. Since the mean bandwidth return of the secondary contract is 0.5, we find that buying secondary contract units make sense only when their price is roughly half the price of the primary contract (i.e. 1×0.5).

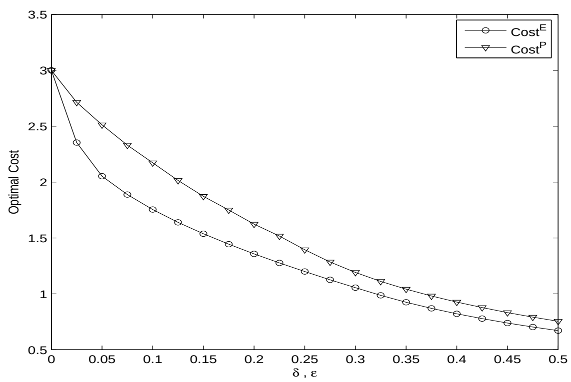

Fig. 4. Efficient Frontier (convex): Optimal portfolio cost vs demand satisfaction constraint (ϵ,δ). $Cost^E$ and $Cost^P$ denote the cost under the DSR and DSP constraints, respectively.

Figure 4 shows the minimum cost that can be achieved for the SPO problems as the constraint (ϵ, δ) is increased. The cost decreases as expected. Moreover, this curve, popularly known as the Efficient Frontier in the context of financial portfolio optimization, is observed to be convex. A formal proof of convexity of the efficient frontier is presented in [22].

B. Single Primary and Two Secondary contracts

We next consider two types of secondary contracts and study how the price and bandwidth return characteristics of a contract affects the choice of the secondary contract. Due to space limitations, we only present results on the SPO problem under the DSR constraint. As before, the demand has normal distribution between 0 and 3. The price of the single primary contract is 1. The bandwidth returns of the two secondaries, B_1 and B_2, have normal distribution between 0 and 1, but with different mean and variance.

We obtain the optimal portfolio $\overline{x} = \{x_0, x_1, x_2\}$ as the ratio of the unit prices of the two secondaries, i.e. $\frac{p_1}{p_2}$, is increased. The results are shown in Figures 5 and 6. For the results shown in Figure 5, B_1 and B_2 have same mean (of 0.5) but different variances. Figure 5 shows $x_1 - x_2$ as $\frac{p_1}{p_2}$ is increased from 0.5 to 2. Each of the three curves corresponds to a fixed choice of the variance (σ_1, σ_2) of the bandwidth returns. Consider the curve corresponding to the variance choice $\sigma_1 = 0.2\sigma_2$. We find that $x_1 - x_2 > 0$, until $\frac{p_1}{p_2} \leq 1.4$. This implies that the contribution of the first secondary contract units to the overall portfolio is higher than that of the second contract even if the unit price of the first contract is higher than the unit price of the second contract. This is clearly due to the fact that B_1 has lesser variance than B_2. However, if $\frac{p_1}{p_2} > 1.4$, the second contract units are more, since it is much lesser priced. On the other hand, when $\sigma_1 = 2\sigma_2$, the first secondary contract is preferred over second contract, only if it costs lesser than the second contract. These results suggest that secondary contracts that have lower variance of bandwidth return can be priced higher than those with higher variances, provided they have the same mean bandwidth return. Moreover, it was observed that the portfolio consisted of non-zero units of both the secondary contracts for price ratios shown, i.e. $x_1 \neq 0, x_2 \neq 0$, for $0.5 \leq \frac{p_1}{p_2} \leq 2$. This suggests that it is cost efficient to buy a mix of secondary contract units from multiple sellers, instead of just one, provided their prices are not very different. x_1 or x_2 became zero only when $\frac{p_1}{p_2}$ is either too high or too low, respectively.

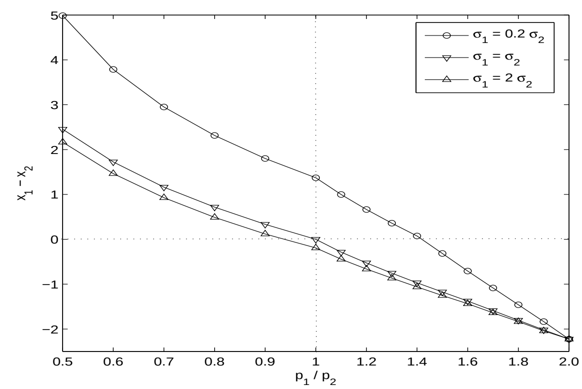

Fig. 5. Relative contribution of the two secondary contract units as the ratio of the unit prices of the two secondary contracts is increased. Each curve corresponds to a fixed choice of the variance of B_1 and B_2.

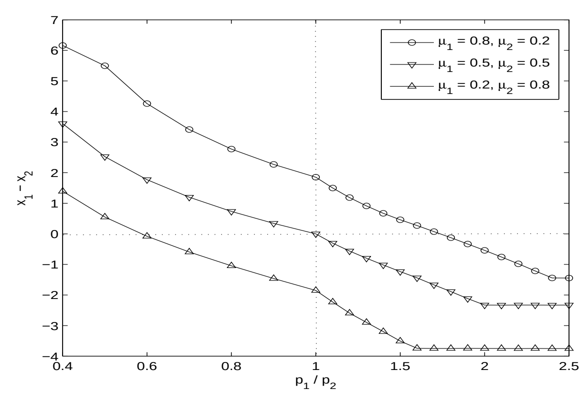

Fig. 6. Relative contribution of the two secondaries. Each curve corresponds to a fixed choice of the mean of B_1 and B_2.

Figure 6 shows $x_1 - x_2$ vs $\frac{p_1}{p_2}$, for different choices of means of B_1 and B_2, keeping the variance fixed at 0.1. When $\mu_1 = 0.8$ and $\mu_2 = 0.2$, we find that $x_1 - x_2 > 0$ as long as $\frac{p_1}{p_2} \leq 1.75$. That is, the secondary contract with 4 times higher mean bandwidth return is preferred even at 75% higher price. We also observe that the secondary contract with lesser mean is preferred only if it has lower price (For the curve with $\mu_1 = 0.2, \mu_2 = 0.8$, $x_1 - x_2 > 0$ only for $\frac{p_1}{p_2} \leq 0.6$). Figures 5 and 6 suggest that the mean as well as the variance of the bandwidth return of a secondary contract play equally important roles in determining the unit price of the secondary contract.

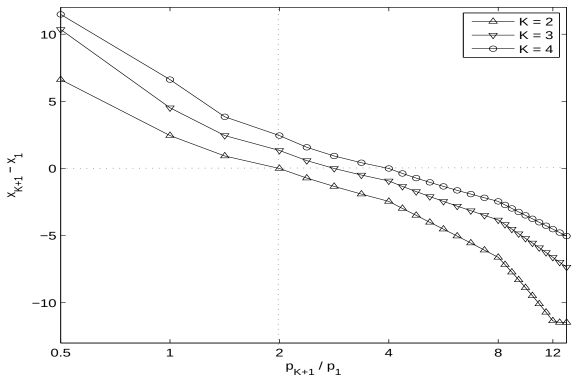

Fig. 7. Figure showing the effect of the ratio of unit price of the $K+1^{th}$ secondary contract to that of the first secondary contract, for different K.

C. Multiple Region

For the multiple-region problem, we consider the following simulation setup. There are totally K regions, $K+1$ primary contracts, and $K+1$ secondary contracts. The i^{th} primary and secondary contract, where $1 \leq i \leq K$, is valid in the i^{th} region only. In other words, the first K primary and secondary contracts are single-region contracts each valid in one of the K regions. However, the $K+1^{th}$ primary and secondary contract is valid over all the K regions. We examine the composition of secondary contract units in the optimal portfolio, when the price of the K-region secondary contract, i.e. p_{K+1}, changes.

The price of all the primary contracts, i.e. $z_1, z_2..., z_{K+1}$, is set to a large value such that the portfolio consists of only secondary contract units. The first K secondary contracts are identical in terms of their bandwidth return distributions and unit prices. The prices of all the single-region secondary contracts, $p_1, p_2.., p_K$, are set to 1. The bandwidth return variables (B_i, $1 \leq i \leq K+1$) follow truncated normal distribution with mean 0.5 and variance 0.25. Due to space limitations, we only consider the multiple-region SPO problem under DSR constraints. The constraint δ_i is set to 0.1 for all the K regions, while δ is set to $K \times 0.1$. We increase the price of the $(K+1)^{th}$ secondary contract, i.e. p_{K+1}, from 0.5 and observe the portfolio composition.

Figure 7 shows the simulation results for $K = 2, 3, 4$. It was observed that the total number of primary contract units is zero as expected, i.e. $y_1 = y_2 = .. = y_{K+1} = 0$. Moreover, all the single-region secondary contracts contributed equal units to the portfolio, i.e. $x_1 = x_2 = .. = x_K$. Therefore, we plot $x_{K+1} - x_1$ as a function of the ratio $\frac{p_{K+1}}{p_1}$ for $K = 2, 3, 4$. For each K, when the price ratio $\frac{p_{K+1}}{p_1} < K$, we find the portfolio consists of higher quantity of $(K+1)^{th}$ secondary contract units compared to the single-region secondary contract, i.e. $x_{K+1} - x_1 > 0$. However, when $\frac{p_{K+1}}{p_1} \geq K$, the single-region secondary contracts are preferred over the K-region contract ($x_{K+1} - x_1 < 0$, if $\frac{p_{K+1}}{p_1} \geq K$). Therefore, we find that the provider (seller) of K-region secondary contract can scale up its price upto a factor of K and still enjoy preference over the single-region contracts offered by smaller providers. This happens due to the fact that the provider can either buy one unit of the K-region secondary contract or one unit from each of the K single-region secondary contracts to provide the same service over the K regions at the same cost. The portfolio shifts completely in favor of single-region contracts only when the price, p_{K+1}, is too high. For the above choice of parameters, x_{K+1} became zero when $\frac{p_{K+1}}{p_1} > 12, 18$, and 24, respectively, for $K = 2, 3$, and 4. That is, when the price of K-region secondary contract is roughly $6K$ times (or higher) the price of the single-region secondary contract, the portfolio no longer consists of K-region secondary contract units.

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