Economic Theory in the Mathematical Mode

By GERARD DEBREU*

I

If a symbolic date were to be chosen for the birth of mathematical economics, our profession, in rare unanimous agreement, would select 1838, the year in which Augustin Cournot published his *Recherches sur les Principes Mathématiques de la Théorie des Richesses*. Students of the history of economic analysis could point out contributions made to mathematical economics as early as the beginning of the eighteenth century. They could also point out Johann Heinrich von Thünen's *Der Isolierte Staat*, 1826, a prototypical example of the use of mathematical reasoning in economic theory with little mathematical formalism. But Cournot stands out as the first great builder of mathematical models explaining economic phenomena. Among his successors in the nineteenth century and the early twentieth century, the highest prominence will be given in this lecture to Léon Walras (1834–1910), the founder of the mathematical theory of general economic equilibrium, to Francis Y. Edgeworth (1845–1926), and to Vilfredo Pareto (1848–1923). All three lived long enough into the twentieth century to have increased, for all Nobel Laureates, the value of the economics prize, had it, like the other prizes, been initiated in 1901.

If 1838 is the symbolic birthdate of mathematical economics, 1944 is the symbolic beginning of its contemporary period. In that year, John von Neumann and Oskar Morgenstern published the first edition of the *Theory of Games and Economic Behavior*, an event that announced a profound and extensive transformation of economic theory. In the following decade, powerful intellectual stimuli also came from many other directions. In addition to von Neumann and Morgenstern's book, Wassily Leontief's input-output analysis, Paul Samuelson's *Foundations of Economic Analysis*, Tjalling Koopmans' activity analysis of production, and George Dantzig's simplex algorithm were frequent topics of discussion, notably at the Cowles Commission when I joined it on June 1, 1950. To become associated at that time with a strongly interactive group which provided the optimal environment for the type of research that I wanted to do was an exceptional privilege.

One leading motivation for that research was the study of the theory of general economic equilibrium. Its goals were to make the theory rigorous, to generalize it, to simplify it, and to extend it in new directions. The execution of such a program required the solution of several problems in the theory of preferences, utility, and demand. It led to the introduction into economic theory of new analytical techniques borrowed from diverse fields of mathematics. Occasionally it made it necessary to find answers to purely mathematical questions. The number of research workers involved was, at first, small and slowly increasing, but in the early 1960's it began to grow more rapidly.

The most primitive of the concepts of the theory I will survey and discuss is that of the commodity space. One makes a list of all the commodities in the economy. Let l be their finite number. Having chosen a unit of measurement for each one of them, and a sign convention to distinguish inputs from outputs (for a consumer inputs are positive, outputs negative; for a producer inputs are

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negative, outputs positive), one can describe the action of an economic agent by a vector in the commodity space R^l. The fact that the commodity space has the structure of a real vector space is a basic reason for the success of the mathematization of economic theory. In particular convexity properties of sets in R^l, a recurring theme in the theory of general economic equilibrium, can be fully exploited. If, in addition, one chooses a unit of account, and if one specifies the price of each one of the l commodities, one defines a price-vector in R^l, a concept dual to that of a commodity-vector. The value of the commodity-vector z relative to the price-vector p is then the inner product $p \cdot z$.

One of the aims of the mathematical theory that Walras founded in 1874–77 is to explain the price-vector and the actions of the various agents observed in an economy in terms of an equilibrium resulting from the interaction of those agents through markets for commodities. In such an equilibrium, every producer maximizes his profit relative to the price-vector in his production set; every consumer satisfies his preferences in his consumption set under the budget constraint defined by the value of his endowment-vector and his share of the profits of the producers; and for every commodity, total demand equals total supply. Walras and his successors for six decades perceived that his theory would be vacuous without an argument in support of the existence of at least one equilibrium, and noted that in his model the number of equations equals the number of unknowns, an argument that cannot convince a mathematician. One must, however, immediately add that the mathematical tools that later made the solution of the existence problem possible did not exist when Walras wrote one of the greatest classics, if not the greatest, of our science. It was Abraham Wald, starting from Gustav Cassel's (1918) formulation of the Walrasian model, who eventually in Vienna in 1935–36 provided the first solution in a series of papers that attracted so little attention that the problem was not attacked again until the early 1950's.

Kenneth Arrow has told in his Nobel lecture (1974) about the path that he followed to the point where it joined mine. The route that led me to our collaboration was somewhat different. After having been influenced at the Ecole Normale Supérieure in the early 1940's by the axiomatic approach of N. Bourbaki to mathematics, I became interested in economics toward the end of World War II. The tradition of the School of Lausanne had been kept alive in France, notably by François Divisia and by Maurice Allais, and it was in Allais' formulation in *A la Recherche d'une Discipline Economique* (1943) that I first met, and was captivated by, the theory of general economic equilibrium. To somebody trained in the uncompromising rigor of Bourbaki, counting equations and unknowns in the Walrasian system could not be satisfactory, and the nagging question of existence was posed. But in the late 1940's several essential elements of the answer were still not readily available.

In the meantime, an easier problem was solved, and its solution contributed significantly to that of the existence problem. At the turn of the century, Pareto had given a characterization of an optimal state of an economy in terms of a price system, using the differential calculus. A long phase of development of Pareto's ideas in the same mathematical framework came to a resting point with the independent contributions of Oscar Lange (1942) and of Allais (1943). In the summer of 1950, Arrow, at the Second Berkeley Symposium on Mathematical Statistics and Probability, and I, at a meeting of the Econometric Society at Harvard, separately treated the same problem by means of the theory of convex sets. Two theorems are at the center of that area of welfare economics. The first asserts that if all the agents of an economy are in equilibrium relative to a given price-vector, the state of the economy is Pareto optimal. Its proof is one of the simplest in mathematical economics. The second provides a deeper economic insight and rests on a property of convex sets. It asserts that associated with a Pareto optimal state s of an economy, there is a price-vector p relative to which all the agents are in equilibrium (under conditions that, here as elsewhere, I cannot fully specify). Its proof is based on the observation that in the commodity space R^l, the a priori given en-

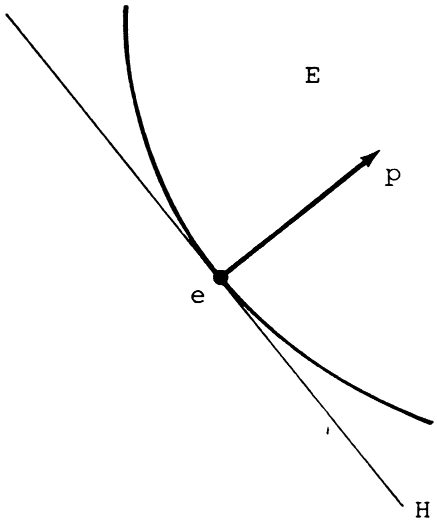

FIGURE 1

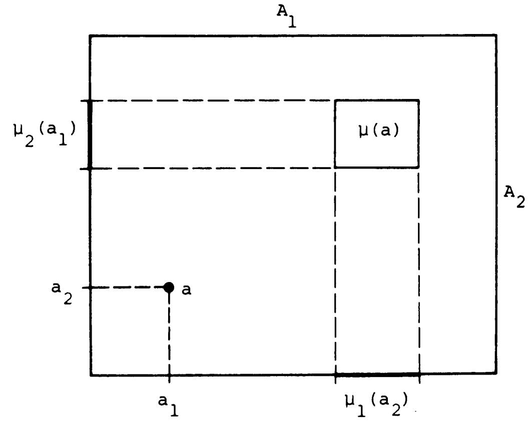

FIGURE 2

dowment-vector e of the economy is a boundary point of the set E of all the endowment-vectors with which it is possible to satisfy the preferences of all consumers at least as well as in the state s. Under conditions insuring that the set E is convex, there is a supporting hyperplane H for E through e. A vector p orthogonal to the hyperplane H, pointing towards E has all the required properties. (See Figure 1.) The treatment of the problem thus given by means of convexity theory was rigorous, more general and simpler than the treatment by means of the differential calculus that had been traditional since Pareto. The supporting hyperplane theorem (more generally the Hahn-Banach theorem, Debreu, 1954a) seemed to fit the economic problem perfectly. Especially relevant to my narrative is the fact that the restatement of welfare economics in set-theoretical terms forced a reexamination of several of the primitive concepts of the theory of general economic equilibrium. This was of great value for the solution of the existence problem.

In the year I joined the Cowles Commission, I learned about the Lemma in von Neumann's article of 1937 on growth theory that Shizuo Kakutani reformulated in 1941 as a fixed point theorem. I also learned about the applications of Kakutani's theorem made by John Nash in his one-page note of 1950 on "Equilibrium Points in N-Person Games" and by Morton Slater in his unpublished paper, also of 1950, on Lagrange multipliers. Again there was an ideal tool, this time Kakutani's theorem, for the proof that I gave in 1952 of the existence of a social equilibrium generalizing Nash's result. Since the transposition from the case of two agents to the case of n agents is immediate, we shall consider only the former which lends itself to a diagrammatic representation. Let the first agent choose an action a_1 in the a priori given set A_1, and the second agent choose an action a_2 in the a priori given set A_2. Knowing a_2, the first agent has a set $\mu_1(a_2)$ of equivalent reactions. Similarly, knowing a_1, the second agent has a set $\mu_2(a_1)$ of equivalent reactions. (See Figure 2.) $\mu_1(a_2)$ and $\mu_2(a_1)$ may be one-element sets, but in the important case of an economy with some producers operating under constant returns to scale, they will not be. The state $a = (a_1, a_2)$ is an equilibrium if and only if $a_1 \in \mu_1(a_2)$ and $a_2 \in \mu_2(a_1)$, that is, if and only if $a \in \mu(a) = \mu_1(a_2) \times \mu_2(a_1)$.

In other words, a is an equilibrium state if and only if it is a fixed point of the correspondence $a \mapsto \mu(a)$ from $A = A_1 \times A_2$ to A itself. Conditions insuring that Kakutani's theorem applies to A and μ guarantee the existence of an equilibrium state. In our article of 1954, Arrow and I cast a competitive economy in the form of a social system of the preceding type. The agents are the consumers, the producers, and a fictitious price setter. An appropriate definition of the set of reactions of the price setter to an excess demand vector makes the concept of equilibrium for that social system equivalent to the concept of competitive equilibrium for the original economy. In this manner a proof of existence, resting ultimately on Kakutani's theorem, was obtained for an equilibrium of an economy made up of interacting consumers and producers. In the early 1950's, the time had undoubtedly come for solutions of the existence problem. In addition to the work of Arrow and me, begun independently and completed jointly, Lionel McKenzie at Duke University proved the existence of an "Equilibrium in Graham's Model of World Trade and Other Competitive Systems" (1954), also using Kakutani's theorem. A different approach taken independently by David Gale (1955) in Copenhagen, Hukukane Nikaido (1956) in Tokyo, and Debreu (1956) in Chicago permitted the substantial simplification given in my *Theory of Value* (1959) of the complex proof of Arrow and Debreu.

Following that approach we consider a price-vector p different from 0 in R^l_+, the closed positive orthant of R^l. The reactions of the consumers and of the producers in the economy to p yield an excess demand vector z in R^l, whose coordinates represent for each commodity the (positive, zero, or negative) excess of demand over supply. Since the vector z may not be uniquely determined, one is led to introduce the set $Z(p)$ of the excess demand vectors associated with p, a set which is invariant if p is multiplied by a strictly positive real number. If every commodity in the economy can be freely disposed of, p^* is an equilibrium price-vector if and only if there is in $Z(p^*)$ a vector all of whose coordinates are negative or zero, that is, if and only if $Z(p^*)$ intersects R^l_-, the closed negative orthant of R^l. The fact that every consumer satisfies his budget constraint implies that all the points of $Z(p)$ are in or below the hyperplane through the origin of R^l orthogonal to p. (See Figure 3.) Additional conditions on Z suggested by Kakutani's theorem establish the existence of an equilibrium p^*.

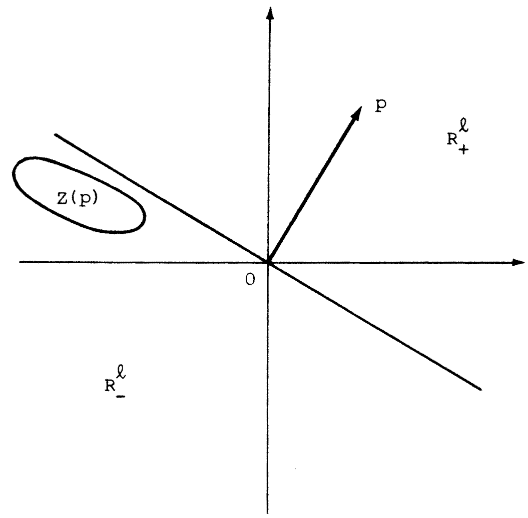

FIGURE 3

A proof of existence is now considered a necessary adjunct of a model proposing a concept of economic equilibrium, and in a recent survey (Debreu, 1982) more than 350 publications containing existence proofs of that type were listed. One of the most complex among these, because of the generality at which it aimed, was my article (1962).

During the past three decades, several other approaches to the problem of existence have been developed. Without attempting a systematic survey such as those prepared for Arrow and Intriligator (1981–84) by Stephen Smale (ch. 8), by Debreu (ch. 15), by E. Dierker (ch. 17), and by Herbert Scarf (ch. 21), one must explicitly mention two of them here.

Given an arbitrary strictly positive price-vector p, we now consider the case in which the reactions of the consumers and of the producers in the economy determine a unique excess demand vector $F(p)$. We also assume that the budget constraint of every consumer is exactly satisfied. Then one has

$$\text{Walras' Law} \qquad p \cdot F(p) = 0.$$

This equality suggests that the price-vector p be normalized by restricting it to the strictly positive part S of the unit sphere in R^l, for then the vector $F(p)$, being orthogonal to p, can be represented as being tangent to the

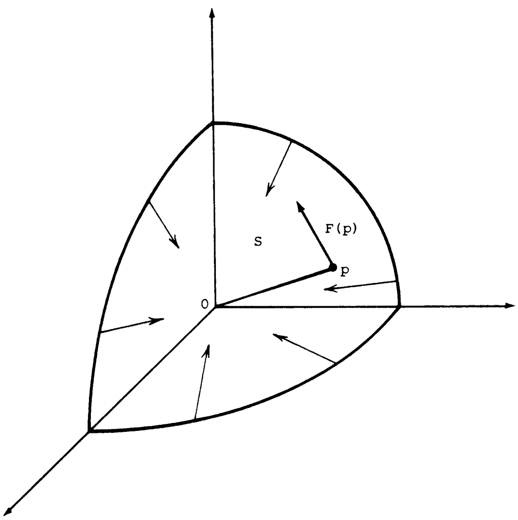

FIGURE 4

sphere S at p. (See Figure 4.) In mathematical terms, the excess demand function F defines a vector field on S. This representation turned out to be the key to the general characterization of excess demand functions that I will discuss later. It also provides an existence proof in the case of a boundary condition on F, meaning in economic terms that excess demand becomes large when some prices tend to zero, and in mathematical terms that the excess demand points inward near the boundary of S. For a continuous vector field, this property implies that there is at least one point p^* of S for which $F(p^*) = 0$. This equality of demand and supply for every commodity expresses that p^* is an equilibrium price-vector.

The second approach concerns the development of efficient algorithms for the computation of approximate equilibria, an area of research in which Scarf (1973) played the leading role. The search for algorithms of that class is a natural part of the program of study of general economic equilibrium. Yet the decisive stimulus came unexpectedly from the solution of a problem in game theory, when C. E. Lemke and J. T. Howson (1964) provided an algorithm for the solution of two-person non-zero-sum games. The computation of equilibria has found its way into a large number of applications and has added an important new aspect to the theory of general economic equilibrium.

The explanation of equilibrium given by a model of the economy would be complete if the equilibrium were unique, and the search for satisfactory conditions guaranteeing uniqueness has been actively pursued (an excellent survey is found in Arrow and Hahn, 1971, ch. 9). However, the strength of the conditions that were proposed made it clear by the late 1960's that global uniqueness was too demanding a requirement and that one would have to be satisfied with local uniqueness. Actually, that property of an economy could not be guaranteed even under strong assumptions about the characteristics of the economic agents. But one can prove, as I did in 1970, that, under suitable conditions, in the set of all economies, the set of economies that do *not* have a set of locally unique equilibria is negligible. The exact meaning of the terms I have just used and the basic mathematical result on which the proof of the preceding assertion rests can be found in Sard's theorem to which Stephen Smale introduced me in conversations in the summer of 1968. The different parts of the solution fell into place at Milford Sound on the South Island of New Zealand. On the afternoon of July 9, 1969, when my wife Françoise and I arrived, intermittent rain and overcast weather that dulled the view tempted me to work once more on what had become a long tantalizing problem, and, this time, ideas quickly crystallized. The next morning a cloudless sky revealed the Sound in its mid-winter splendor.

The "suitable conditions" to which I alluded are differentiability conditions which, in the present situation, are essentially unavoidable. As for the term "negligible," it means, in the case of a finite-dimensional set of economies, "contained in a closed set of Lebesgue measure zero." The main ideas of the proof can be conveyed intuitively in the simple case of an exchange economy with m consumers. The demand function f_i of the ith consumer associates with every pair (p, w_i) of a strictly positive price-vector p and a positive wealth (or income) w_i the demand $f_i(p, w_i)$ in the closed positive orthant R^l_+ of the commodity space. The ith consumer is characterized by his demand function f_i and by his endowment-vector e_i in the strictly positive orthant P of R^l. The functions f_i are kept fixed and assumed to be continuously differentiable. Therefore, the economy is described by the list $e = (e_1, \ldots, e_m)$ of the m endowment-vectors in P^m. The price-vector p being restricted to belong to S, the strictly positive part of the unit sphere, the excess demand vector associated with a pair (p, e) in $S \times P^m$ is

$$F(p,e) = \sum_{i=1}^{m} [f_i(p, p \cdot e_i) - e_i].$$

The equilibrium manifold M (Smale, 1974; Balasko, 1975) is the subset of $S \times P^m$ defined by $F(p,e) = 0$, an equality which, because of Walras' Law, imposes only $l-1$ constraints. Under the assumptions made, M is a differentiable manifold and its dimension is $\dim M = \dim P^m + \dim S - (l-1) = lm = \dim P^m$. Now let T be the projection from M into P^m, and define a critical economy e as an economy such that it is the projection of a point (e, p) of M where the Jacobian of T is singular, geometrically where the tangent linear manifold of dimension lm does not project onto P^m. (See Figure 5.) By Sard's theorem the set of critical economies is closed and of Lebesgue measure zero. A regular economy, outside the negligible critical set, not only has a discrete set of equilibria; it also has a neighborhood in which the set of equilibria varies continuously as a function of the parameters defining the economy. The study of regular economies thus forms a basis for the analysis of the determinateness of equilibrium and of the stability of economic systems. Moreover, the continuity of the set of equilibria in a neighborhood of a regular economy insures that the explanation of equilibrium provided by the model is robust with respect to unavoidable errors in the measurement of the parameters. Once again, a mathematical result, Sard's theorem, was found to fit exactly the needs of economic theory. The study of regular economies has been an active research area in the last decade, and Smale, Balasko, and Andreu Mas-Colell (1984) are among its main contributors.

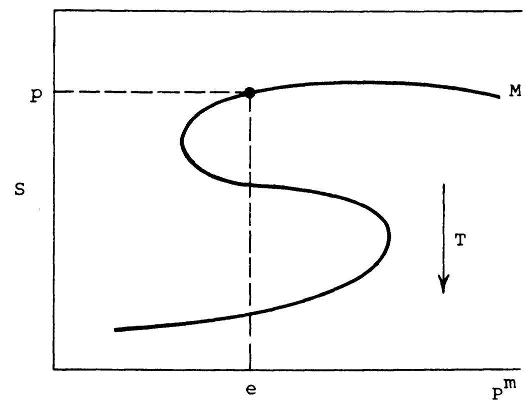

FIGURE 5

Departing from chronological order, I now return to the late 1950's and to the early 1960's, and to the beginning of the theory of the core of an economy. Edgeworth (1881) had given a persuasive argument in support of the common imprecise belief that markets become more competitive as the number of their agents increases in such a way that each one of them tends to become negligible. He had specifically shown that his "contract-curve" tends to the set of competitive equilibria in a two-commodity economy with equal numbers of consumers of each one of two types. His brilliant contribution stimulated no further work until Martin Shubik (1959) linked Edgeworth's contract curve with the game theoretical concept of the core (D. B. Gillies, 1953). The first extension of Edgeworth's result was obtained by Scarf (1962), and the complete generalization to the case of an arbitrary number of commodities and of types of consumers was given by Debreu and Scarf (1963). Associated with our joint paper is one of my most vivid memories of the instant when a problem is solved. Scarf, then at Stanford, had met me at the San Francisco airport in December 1961, and as he was driving to Palo Alto on the freeway, one of us, in one sentence, provided a key to the solution; the other, also in one sentence, immediately provided the other key; and the lock clicked open. Once again, the basic mathematical result was the supporting hyperplane theorem for convex sets. The theorem that we had proved remained special, because it applied only to economies with a given number of types of consumers and an equal, increasing number of consumers of each type. Generalizations were soon forthcoming. Robert Aumann (1964) introduced the concept of an atomless measure space of economic agents, a natural mathematical formulation of the concept of an economy with a large number of agents, all of them negligible. Under notably weak conditions, Aumann proved that for such an economy the core coincides with the set of competitive equilibria. Karl Vind (1964) then pointed out that the proper mathematical tool for the proof of that striking result was Lyapunov's (1940) theorem on the convexity, and compactness, of the range of an atomless finite-dimensional vector measure. Out of these contributions grew an extensive literature that included among its high points Yakar Kannai's (1970) and Truman Bewley's (1973) articles, and that culminated in Werner Hildenbrand's book (1974). This was surveyed recently in Arrow and Intriligator (1982) by Hildenbrand (ch. 18).

In a different direction, a formalization of an economy with a large number of negligible agents was proposed by Donald Brown and Abraham Robinson (1972), who introduced the sophisticated techniques of Nonstandard Analysis in economic theory. Remarkably, this approach eventually led to the elementary inequalities of Robert Anderson (1978) on the extent of competitiveness of allocations in the core in an economy with a finite number of agents.

In the mid-1970's, the theory of the core and the theory of regular economies were joined in the study of the rate of convergence of the core to the set of competitive equilibria. Lloyd Shapley (1975) had shown that convergence could be arbitrarily slow. Debreu (1975) then proved that in the case of increasing equal numbers of agents of each of a finite number of types, the rate of convergence to the set of competitive equilibria of a *regular* economy is of the same order as the reciprocal of the number of agents. The extension of this result from replicated economies to more general sequences of economies was provided by Birgit Grodal (1975).

Intimately linked with the contemporary development of the theory of general economic equilibrium was that of the theory of preferences, utility, and demand. New results in the latter were in some cases required, in others motivated by the former. The primitive concepts in the theory of preferences of a consumer are his consumption set X, a subset of R^l, and his preference relation $\lesssim$, a complete preorder on X. We shall say that a real-valued function u on X is a utility function if it represents the preference relation $\lesssim$ in the sense that

$$[x \lesssim y] \Leftrightarrow [u(x) \leqq u(y)].$$

A necessary and sufficient condition for the existence of a continuous utility function is that the set $G = \{(x, y) \in X \times X | x \lesssim y\}$ be closed relative to $X \times X$ (Debreu, 1954b; 1964). Although more abstract than the familiar concept of an infinite family of indifference sets in R^l, the concept of a single set G in $R^l \times R^l$ is far simpler as two more instances illustrate.

To say that an agent has preferences similar to that of another means for a mathematical economist that a topology has been introduced on the set of preferences. This was done by Kannai (1970), in an article whose publication was long delayed. The prospect of comparing two preference relations $\lesssim$ and $\lesssim'$ on the two consumption sets X and X' (now assumed to be closed) is daunting if one thinks of each preference relation as an infinite family of indifference sets in R^l. It becomes appealing if one thinks of each preference relation as a closed subset of $R^l \times R^l$ (Debreu, 1969). The topology on the set of preferences was at the basis of the theory of the core in Hildenbrand (1974). It was indispensable for the work that Kannai (1974) and Mas-Colell (1974) did on the approximation of a convex preference relation by convex preference relations representable by concave utility functions.

The other instance pertains to preference relations representable by differentiable utility functions. The traditional approach, by focusing on the consumption set X in R^l, raised delicate integrability questions (extensively surveyed by Leonid Hurwicz, 1971, ch.

9). In contrast, a differentiable preference relation $\lesssim$ can simply be defined by the condition that the boundary of the associated set G is a differentiable manifold in $R^l \times R^l$ (Debreu, 1972).

In all these developments, the theory of preferences was stimulated and helped by questions asked about the utility function u such as "When is u continuous?," "When is u concave?," "When is u differentiable?" Yet another instance is provided by the study of a preference relation $\lesssim$ defined on the product X of n sets $X_1, \ldots, X_n$. The question now is whether the preference relation $\lesssim$ can be represented by a utility function of the form

$$u(x) = \sum_{i=1}^{n} u_i(x_i),$$

where x is the n-list $(x_1, \ldots, x_n)$ and for every i, $x_i \in X_i$. This problem was studied by Leontief (1947a, b) and by Samuelson (1947, ch. 7), by means of the differential calculus. It can be studied by topological methods (Debreu, 1960) which bring out more clearly the essential independence property on which the solution is based.

The last example from the theory of preferences, utility, and demand will be the problem of the characterization of the excess demand function of an economy. We consider an exchange economy $\mathscr{E}$ with m consumers. As before, the demand function f_i of the ith consumer associates with a pair (p, w_i) of a price-vector p in the strictly positive part S of the unit sphere in R^l and of a wealth (or income) w_i in the set R_+ of nonnegative real numbers, a consumption vector $f_i(p, w_i)$ in the closed positive orthant R^l_+ of R^l. If the ith consumer has a preference relation $\lesssim_i$ on R^l_+, then $f_i(p, w_i)$ is a commodity-vector that satisfies $\lesssim_i$ under the budget constraint $p \cdot z \leqq w_i$. The economy $\mathscr{E}$ is defined by specifying for the ith consumer $(i = 1, \ldots, m)$ the demand function f_i and the endowment-vector e_i in R^l_+. The aggregate excess demand function of the economy is the function F defined by

$$\text{(a)} \qquad F(p) = \sum_{i=1}^{m} [f_i(p, p \cdot e_i) - e_i].$$

Under weak standard assumptions, the function F (1) is continuous and (2) satisfies Walras' Law. Hugo Sonnenschein (1972, 1973) asked whether these two properties characterize F. Specifically, given F satisfying (1) and (2), can one find m consumers with demand functions f_i and endowment-vectors e_i satisfying (a)? Sonnenschein conjectured that the answer was affirmative and made the first attack on this problem. Rolf Mantel (1974) proved Sonnenschein's conjecture in the case of continuously differentiable demand functions, and Debreu (1974) in the general case. The proof appearing in this last article was inspired by, and rests on, the representation of the excess demand function F as a vector-field on the strictly positive part of the unit sphere. The characterization of aggregate excess demand functions so obtained has several applications. It shows that the hypothesis of preference satisfaction (or equivalently of utility maximization) puts essentially no restriction on F, that a theorem on the existence of a general economic equilibrium is equivalent to a fixed point theorem (via an observation of Hirofumi Uzawa, 1962), and that any dynamic behavior can be observed for an economy operating under a tâtonnement process (as the examples of global instability of Scarf, 1960, presaged). One impact of that characterization has been the redirection of research on aggregate demand functions toward a specification of the distribution of the characteristics of the economic agents. The first theoretical result explaining the "Law of Demand" (Hildenbrand, 1983) was a product of that redirected research.

II

Having surveyed in some detail, as tradition requires, the work cited by the Royal Swedish Academy of Sciences, I turn to issues of methodology in economic theory.

Contemporary developments in the theory of general economic equilibrium took Walras' work as their point of departure, but some of Walras' ideas had a long lineage that included Adam Smith's (1776) profound insight. Smith's idea that the many agents of an economy, making independent decisions, do not create utter chaos but actually contribute to producing a social optimum, raises indeed a scientific question of central importance. Attempts to answer it have stimulated the study of several of the problems that every economic system must solve, such as the efficiency of resource allocation, the decentralization of decisions, the incentives of decision makers, the treatment of information.

In the past few decades, that wide range of problems has been the subject of an axiomatic analysis in which primitive concepts are chosen, assumptions concerning them are formulated, and conclusions are derived from those assumptions by means of mathematical reasoning disconnected from any intended interpretation of the primitive concepts. The benefits of the axiomatization of economic theory have been numerous. Making the assumptions of a theory entirely explicit permits a sounder judgment about the extent to which it applies to a particular situation. Axiomatization may also give ready answers to new questions when a novel interpretation of primitive concepts is discovered. As an illustration, consider the concept of a commodity, which had meant traditionally a good or a service whose physical properties and whose delivery date and location are specified. In the case of an uncertain environment, Arrow (1953) added to those characteristics of a commodity the event in which delivery will take place. In this manner one obtains, without any change in the form of the model, a theory of uncertainty in which all the results of the theory of certainty are available (Debreu, 1959, ch. 7). Axiomatization, by insisting on mathematical rigor, has repeatedly led economists to a deeper understanding of the problems they were studying, and to the use of mathematical techniques that fitted those problems better. It has established secure bases from which exploration could start in new directions. It has freed researchers from the necessity of questioning the work of their predecessors in every detail. Rigor undoubtedly fulfills an intellectual need of many contemporary economic theorists, who therefore seek it for its own sake, but it is also an attribute of a theory that is an effective thinking tool. Two other major attributes of an effective theory are simplicity and generality. Again, their aesthetic appeal suffices to make them desirable ends in themselves for the designer of a theory. But their value to the scientific community goes far beyond aesthetics. Simplicity makes a theory usable by a great number of research workers. Generality makes it applicable to a broad class of problems.

In yet another manner, the axiomatization of economic theory has helped its practitioners by making available to them the superbly efficient language of mathematics. It has permitted them to communicate with each other, and to think, with a great economy of means. At the same time, the dialogue between economists and mathematicians has become more intense. The example of a mathematician of the first magnitude like John von Neumann devoting a significant fraction of his research to economic problems has not been unique. Simultaneously, economic theory has begun to influence mathematics. Among the clearest instances are Kakutani's theorem, the theory of integration of correspondences (Hildenbrand, 1974), algorithms for the computation of approximate fixed points (Scarf's ch. 21 in Arrow and Intriligator, 1981–84), and of approximate solutions of systems of equations (Smale's ch. 8 in Arrow and Intriligator, 1981).

III

In narratives of their careers, scientists try to acknowledge the main influences to which they responded, and the support they received from other scientists and from different institutions, even though such attempts are unlikely to be entirely successful. To all the persons and organizations I have named, I want to add the outstanding education system I have known in France, and the Centre National de la Recherche Scientifique which made my conversion from mathematics to economics possible. After my move to the United States in 1950, I was associated with three great universities (Chicago, Yale, and Berkeley) where scientific research is a natural way of life; and during the last two

decades the Economics Program of the National Science Foundation has given me, more than anything else, time for that research. All those institutions have provided a superb environment for the task that had to be performed.

REFERENCES

Allais, M., *A la Recherche d'une Discipline Economique*, Paris: Imprimerie Nationale, 1943.

Anderson, R. M., "An Elementary Core Equivalence Theorem," *Econometrica*, November 1978, *46*, 1483–87.

Arrow, K. J., "An Extension of the Basic Theorems of Classical Welfare Economics," in J. Neyman, ed., *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley: University of California Press, 1951, 507–32.

______, "Le Rôle des Valeurs Boursières pour la Répartition la Meilleure des Risques," *Econométrie*, Paris: Centre National de la Recherche Scientifique, 1953, 41–48.

______, "General Economic Equilibrium: Purpose, Analytic Techniques, Collective Choice," *American Economic Review*, June 1974, *64*, 253–72.

______ **and Debreu, G.,** "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, July 1954, *22*, 265–90.

______ **and Hahn, F. H.,** *General Competitive Analysis*, San Francisco: Holden Day, 1971.

______ **and Intriligator, M. D.,** *Handbook of Mathematical Economics*, Vols. I, II, III, Amsterdam: North-Holland, 1981–84.

Aumann, R. J., "Markets with a Continuum of Traders," *Econometrica*, January-April, 1964, *32*, 39–50.

Balasko, Y., "On the Graph of the Walras Correspondence," *Econometrica*, September-November 1975, *43*, 907–12.

Bewley, T. F., "Edgeworth's Conjecture," *Econometrica*, September-November 1973, *41*, 425–54.

Brown, D. J. and Robinson, A., "A Limit Theorem on the Cores of Large Standard Exchange Economies," *Proceedings of the National Academy of Sciences of the U.S.A.*, 1972, *69*, 1258–60.

Cassel, K. G., *Theoretische Sozialökonomie*, Leipzig: C. F. Winter, 1918.

Cournot, A., *Recherches sur les Principes Mathématiques de la Théorie des Richesses*, Paris: L. Hachette, 1838.

Dantzig, G. B., "Maximization of a Linear Function of Variables Subject to Linear Inequalities," in T. C. Koopmans, ed., *Activity Analysis of Production and Allocation*, New York: Wiley & Sons, 1951, 339–47.

Debreu, G., "The Coefficient of Resource Utilization," *Econometrica*, July 1951, *19*, 273–92.

______, "A Social Equilibrium Existence Theorem," *Proceedings of the National Academy of Sciences*, 1952, *38*, 886–93.

______, (1954a) "Valuation Equilibrium and Pareto Optimum," *Proceedings of the National Academy of Sciences*, 1954, *40*, 588–92.

______, (1954b) "Representation of a Preference Ordering by a Numerical Function," in R. M. Thrall et al., eds., *Decision Processes*, New York: Wiley & Sons, 1954, 159–65.

______, "Market Equilibrium," *Proceedings of the National Academy of Sciences*, 1956, *42*, 876–78.

______, *Theory of Value, An Axiomatic Analysis of Economic Equilibrium*, New York: Wiley & Sons, 1959.

______, "Topological Methods in Cardinal Utility Theory," in: K. J. Arrow et al., eds., *Mathematical Methods in the Social Sciences, 1959*, Stanford: Stanford University Press, 1960, 16–26.

______, "New Concepts and Techniques for Equilibrium Analysis," *International Economic Review*, September 1962, *3*, 257–73.

______, "Continuity Properties of Paretian Utility," *International Economic Review*, September 1964, *5*, 285–93.

______, "Neighboring Economic Agents," in *La Décision*, Colloques Internationaux du Centre National de la Recherche Scientifique No. 171, Paris, 1969, 85–90.

______, "Economies with a Finite Set of Equilibria," *Econometrica*, May 1970, *38*, 387–92.

______, "Smooth Preferences," *Econo-*

metrica, July 1972, *40*, 603–15; "Smooth Preferences: A Corrigendum," *Econometrica*, July 1976, *44*, 831–32.

______, "Excess Demand Functions," *Journal of Mathematical Economics*, 1974, *1*, 15–21.

______, "The Rate of Convergence of the Core of an Economy," *Journal of Mathematical Economics*, 1975, *2*, 1–7.

______, "Regular Differentiable Economies," *American Economic Review Proceedings*, May 1976, *66*, 280–87.

______, "Existence of Competitive Equilibrium," in K. J. Arrow and M. D. Intriligator, eds., *Handbook of Mathematical Economics*, Vol. II, Amsterdam: North-Holland, 1982, ch. 15.

______ **and Scarf, H.**, "A Limit Theorem on the Core of an Economy," *International Economic Review*, September 1963, *4*, 235–46.

______ **and Koopmans, T. C.**, "Additively Decomposed Quasiconvex Functions," *Mathematical Programming*, 1982, *24*, 1–38.

Dierker, E., "Regular Economies," in K. J. Arrow and M. D. Intriligator, eds., *Handbook of Mathematical Economics*, Vol. II, Amsterdam: North-Holland, 1982, ch. 17.

Divisia, F., *Economique Rationnelle*, Paris: Doin, 1928.

Edgeworth, F. Y., *Mathematical Psychics*, London: Kegan Paul, 1881.

Gale, D., "The Law of Supply and Demand," *Mathematica Scandinavica*, 1955, *3*, 155–69.

Gillies, D. B., "Some Theorems on *n*-Person Games," unpublished doctoral dissertation, Princeton University, 1953.

Grodal, B., "The Rate of Convergence of the Core for a Purely Competitive Sequence of Economies," *Journal of Mathematical Economics*, 1975, *2*, 171–86.

Hildenbrand, W., *Core and Equilibria of a Large Economy*, Princeton: Princeton University Press, 1974.

______, "Core of an Economy," in K. J. Arrow and M. D. Intriligator, eds., *Handbook of Mathematical Economics*, Vol. II, Amsterdam: North-Holland, 1982, ch. 18.

______, "On the 'Law of Demand'," *Econometrica*, July 1983, *51*, 997–1019.

Hurwicz, L., "On the Problem of Integrability of Demand Functions," in J. S. Chipman et al., eds., *Preferences, Utility, and Demand*, New York: Harcourt Brace Jovanovich, 1971, 174–214.

Kakutani, S., "A Generalization of Brouwer's Fixed Point Theorem," *Duke Mathematical Journal*, 1941, *8*, 457–59.

Kannai, Y., "Continuity Properties of the Core of a Market," *Econometrica*, November 1970, *38*, 791–815.

______, "Approximation of Convex Preferences," *Journal of Mathematical Economics*, 1974, *1*, 101–06.

Koopmans, T. C., *Activity Analysis of Production and Allocation*, New York: Wiley & Sons, 1951.

Lange, O., "The Foundations of Welfare Economics," *Econometrica*, July–October 1942, *10*, 215–28.

Lemke, C. E. and Howson, J. T., Jr., "Equilibrium Points of Bimatrix Games," *Journal of the Society of Industrial and Applied Mathematics*, 1964, *12*, 413–23.

Leontief, W. W., *The Structure of the American Economy, 1919–1929*, Cambridge: Harvard University Press, 1941.

______, (1947a) "A Note on the Interrelation of Subsets of Independent Variables of a Continuous Function with Continuous First Derivatives," *Bulletin of the American Mathematical Society*, 1947, *53*, 343–50.

______, (1947b) "Introduction to a Theory of the Internal Structure of Functional Relationships," *Econometrica*, October 1974, *15*, 361–73.

Lyapunov, A. A., "On Completely Additive Vector-Functions," *Izvestija Akademii Nauk SSSR*, 1940, *4*, 465–78.

McKenzie, L. W., "On Equilibrium in Graham's Model of World Trade and Other Competitive Systems," *Econometrica*, April 1954, *22*, 147–61.

Mantel, R., "On the Characterization of Aggregate Excess Demand," *Journal of Economic Theory*, March 1974, *7*, 348–53.

Mas-Colell, A., "Continuous and Smooth Consumers: Approximation Theorems," *Journal of Economic Theory*, July 1974, *8*, 305–36.

______, *The Theory of General Economic Equilibrium; a Differentiable Approach*, Cambridge: Cambridge University Press, 1984.

Nash, J. F., "Equilibrium Points in *N*-Person Games," *Proceedings of the National Academy of Sciences of the U.S.A.*, 1950, *36*, 48–49.

Neumann, J. von, "Über ein Ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes," *Ergebnisse eines Mathematischen Kolloquiums*, 1937, *8*, 73–83.

______ **and Morgenstern, O.**, *Theory of Games and Economic Behavior*, Princeton: Princeton University Press, 1944.

Nikaido, H., "On the Classical Multilateral Exchange Problem," *Metroeconomica*, August 1956, *8*, 135–45.

Pareto, V., *Manuel d'Economie Politique*, Paris: Giard, 1909.

Samuelson, P. A., *Foundations of Economic Analysis*, Cambridge: Harvard University Press, 1947.

Sard, A., "The Measure of the Critical Points of Differentiable Maps," *Bulletin of the American Mathematical Society*, 1942, *48*, 883–90.

Scarf, H., "Some Examples of Global Instability of Competitive Equilibrium," *International Economic Review*, September 1960, *1*, 157–72.

______, "An Analysis of Markets with a Large Number of Participants," *Recent Advances in Game Theory*, The Princeton University Conference, 1962.

______, (with the collaboration of T. Hansen), *The Computation of Economic Equilibria*, New Haven: Yale University Press, 1973.

______, "The Computation of Equilibrium Prices: An Exposition," in K. J. Arrow and M. D. Intriligator, eds., *Handbook of Mathematical Economics*, Vol. II, Amsterdam: North-Holland, 1982, ch. 21.

Shapley, L. S., "An Example of a Slow-converging Core," *International Economic Review*, June 1975, *16*, 345–51.

Shubik, M., "Edgeworth Market Games," *Contributions to the Theory of Games*, Vol. IV, *Annals of Mathematical Studies, 40*, Princeton University Press, 1959.

Slater, M., "Lagrange Multipliers Revisited," Cowles Commission Discussion Paper, Mathematics 403, 1950.

Smale, S., "Global Analysis and Economics," IIA, *Journal of Mathematical Economics*, 1974, *1*, 1–14.

______, "Global Analysis and Economics," in K. J. Arrow and M. D. Intriligator, eds., *Handbook of Mathematical Economics*, Vol. I, Amsterdam: North-Holland, 1981, ch. 8.

Smith, A., *An Inquiry into the Nature and Causes of the Wealth of Nations*, 2 vols., London: W. P. Strahan and T. Cadell, 1776.

Sonnenschein, H., "Market Excess Demand Functions," *Econometrica*, May 1972, *40*, 549–63.

______, "Do Walras' Identity and Continuity Characterize the Class of Community Excess Demand Functions?," *Journal of Economic Theory*, August 1973, *6*, 345–54.

Thünen, J. H. von, *Der Isolierte Staat*, Hamburg: F. Perthes, 1826.

Uzawa, H., "Walras' Existence Theorem and Brouwer's Fixed Point Theorem," *Economic Studies Quarterly*, 1962, *8*, 59–62.

Vind, K., "Edgeworth-allocations in an Exchange Economy with Many Traders," *International Economic Review*, May 1964, *5*, 165–77.

Wald, A., "Über die Eindeutige Positive Lösbarkeit der Neuen Produktionsgleichungen," *Ergebnisse eines Mathematischen Kolloquiums*, 1935, *6*, 12–20.

______, (1936a) "Über die Produktionsgleichungen der Ökonomischen Wertlehre," *Ergebnisse eines Mathematischen Kolloquiums*, 1936, *7*, 1–6.

______, (1936b) "Über einige Gleichungs-Systeme der Mathematischen Ökonomie," *Zeitschrift für Nationalökonomie*, 1936, *7*, 637–70.

Walras, L., *Éléments d'Economie Politique Pure*, Lausanne: L. Corbaz and Company, 1874–77.