Tamagawa Numbers in the Function Field Case (Lecture 2)

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In the previous lecture, we defined the *Tamagawa number* of a connected semisimple algebraic group G over the field $\mathbf{Q}$, and formulated *Weil's conjecture*: if G is simply connected, then the Tamagawa number of G is equal to 1. In this lecture, we will discuss the analogous conjecture in the case of a function field.

Notation 1. Let $\mathbf{F}_q$ denote a finite field with q elements, and let X be an algebraic curve over $\mathbf{F}_q$ (which we assume to be smooth, proper, and geometrically connected). We let $\mathcal{K}$ denote the function field of the curve X (that is, the residue field of the generic point of X).

We will write $x \in X$ to mean that x is a *closed* point of the curve X. For each point $x \in X$, we let κ_x denote the residue field of X at the point x. Then κ_x is a finite extension of the finite field $\mathbf{F}_q$. We will denote the degree of this extension by $\deg(x)$ and refer to it as the *degree* of x. We let $\mathcal{O}_x$ denote the completion of the local ring of X at the point x: this is a complete discrete valuation ring with residue field κ_x, noncanonically isomorphic to a power series ring $\kappa_x[[t]]$. We let $\mathcal{K}_x$ denote the fraction field of $\mathcal{O}_x$. We let $\mathbf{A}$ denote the restricted product of the local fields $\mathcal{K}_x$: that is, the subset of the product $\prod_{x \in X} \mathcal{K}_x$ consisting of those elements $\{f_x\}_{x \in X}$ such that $f_x \in \mathcal{O}_x$ for all but finitely many values of x. We will refer to $\mathbf{A}$ as the *ring of adeles of* $\mathcal{K}$. It is a locally compact commutative ring, and the diagonal embedding $\mathcal{K} \rightarrow \mathbf{A}$ embeds $\mathcal{K}$ as a discrete subgroup of $\mathbf{A}$. We let $\mathbf{A}_0 = \prod_{x \in X} \mathcal{O}_x$ denote the ring of *integral adeles*: a compact open subring of $\mathbf{A}$.

Let G_0 be an affine algebraic group of dimension d defined over the field $\mathcal{K}$. It will often be convenient to assume that we are given an *integral model* of G_0: that is, that G_0 is given as the generic fiber of an affine group scheme G over the curve X. Later in this course, it will be useful to choose an integral model G with some nice properties. For the moment, we will assume the following:

(a) The map $G \rightarrow X$ is smooth.

(b) The fibers of the map $G \rightarrow X$ are connected.

If G satisfies (a) and the generic fiber G_0 is connected, then we can always arrange that G satisfies (b) by discarding any extraneous connected components of the remaining fibers.

For every commutative ring R equipped with a map $\operatorname{Spec} R \rightarrow X$, we let $G(R)$ denote the group of R-points of G. Then $G(\mathbf{A})$ is a locally compact group, containing $G(\mathcal{K})$ as a discrete subgroup. We can identify $G(\mathbf{A})$ with the restricted product of the locally compact groups $G(\mathcal{K}_x)$ with respect to the family of compact open subgroups $\{G(\mathcal{O}_x) \subseteq G(\mathcal{K}_x)\}$. Our first goal in this lecture is to describe a canonical Haar measure on $G(\mathbf{A})$, which we will refer to as *Tamagawa measure*.

Let $\Omega_{G/X}$ denote the relative cotangent bundle of the smooth morphism $\pi : G \rightarrow X$. Then $\Omega_{G/X}$ is a vector bundle on G of rank $d = \dim(G_0)$. We let $\Omega^d_{G/X}$ denote the top exterior power of $\Omega_{G/X}$, so that $\Omega^d_{G/X}$ is a line bundle on G. Let $\mathcal{L}$ denote the pullback of $\Omega^d_{G/X}$ along its zero section. Equivalently, we can identify $\mathcal{L}$ with the subbundle of $\pi_* \Omega^d_{G/X}$ consisting of left-invariant sections. Let $\mathcal{L}_0$ denote the generic fiber of $\mathcal{L}$, so that $\mathcal{L}_0$ is a 1-dimensional vector space over $\mathcal{K}$. Let us fix a nonzero element $\omega \in \mathcal{L}_0$, which we can identify with a left-invariant differential form of top degree on the algebraic group G_0.

For every point $x \in X$, ω determines a Haar measure $\mu_{x,\omega}$ on the locally compact topological group $G(\mathcal{K}_x)$. Concretely, we can describe this measure as follows. Let t denote a uniformizing parameter for $\mathcal{O}_x$ (so that $\mathcal{O}_x \simeq \kappa_x[[t]]$), and let G^x denote the fiber product $\operatorname{Spec} \mathcal{O}_x \times_X G$. Choose a local coordinates $y_1, \ldots, y_d$ for the group G^x near the identity: that is, coordinates which induce a map $u : G^x \to \mathbb{A}^d$ which is étale at the origin of $G(x)$. Let $v_x(\omega)$ denote the order of vanishing of ω at the point x. Then, in a neighborhood of the origin in $G(x)$, we can write $\omega = t^{v_x(\omega)} \lambda dy_1 \wedge \cdots \wedge dy_d$, where λ is an invertible regular function. Let $\mathfrak{m}_x$ denote the maximal ideal of $\mathcal{O}_x$, and let $G(\mathfrak{m}_x)$ denote the kernel of the reduction map $G(\mathcal{O}_x) \to G(\kappa_x)$. Since $y_1, \ldots, y_d$ are local coordinates near the origin, the map u induces a bijection $G(\mathfrak{m}_x) \to \mathfrak{m}_x^d$. The measure defined by the differential form $dy_1 \wedge \cdots \wedge dy_d$ on $G(\mathfrak{m}_x)$ is obtained by pulling back the "standard" measure on $\mathcal{K}_x^d$ along the map u, where this standard measure is normalized so that $\mathcal{O}_x^d$ has measure 1. It follows that the measure of $G(\mathfrak{m}_x)$ (with respect to the differential form $dy_1 \wedge \cdots \wedge dy_d$) is given by $\frac{1}{|\kappa_x|^d}$. We therefore have

$$\mu_{\omega,x}(G(\mathfrak{m}_x)) = q^{-\deg(x) v_x(\omega)} \frac{1}{|\kappa_x|^d}.$$

The smoothness of G implies that the map $G(\mathcal{O}_x) \to G(\kappa_x)$ is surjective, so that we have

$$\mu_{\omega,x}(G(\mathcal{O}_x)) = q^{-\deg(x) v_x(\omega)} \frac{|G(\kappa_x)|}{|\kappa_x|^d}.$$

Remark 2. If you prefer, you can take the above formula as the *definition* of the measure $\mu_{x,\omega}$. One should then show that this measure depends only on the underlying algebraic group G_0 and the choice of differential form ω, and not on the choice of integral model G.

A key fact is the following:

Proposition 3. *Suppose that G_0 is connected and semisimple, and let ω be a nonzero element of $\mathcal{L}_0$. Then the product of the measures $\mu_{x,\omega}$ on the groups $G(\mathcal{K}_x)$ determines a well-defined measure on the restricted product $G(\mathbf{A}) = \prod_{x \in X}^{\text{res}} G(\mathcal{K}_x)$. Moreover, this product measure is independent of ω.*

To check that the product measure is well-defined, it suffices to show that it is well-defined when evaluated on a compact open subgroup of $G(\mathbf{A})$, such as $G(\mathbf{A}_0)$. This is equivalent to the absolute convergence of the infinite product

$$\prod_{x \in X} \mu_{x,\omega}(G(\mathcal{O}_x)) = \prod_{x \in X} q^{-\deg(x) v_x(\omega)} \frac{|G(\kappa_x)|}{|\kappa_x|^d}.$$

Let us assume this for the moment. The fact that the product measure is independent of the choice of ω follows from the fact that the infinite sum

$$\sum_{x \in X} \deg(x) v_x(\omega) = \deg(\mathcal{L})$$

is independent of ω.

Definition 4. Let G_0 be a connected semisimple algebraic group over $\mathcal{K}$. Let d denote the dimension of G_0, and let g denote the genus of the curve X. The *Tamagawa measure* on $G(\mathbf{A})$ is the Haar measure given informally by the product

$$\mu_{\text{Tam}} = q^{d(1-g)} \prod_{x \in X} \mu_{x,\omega}$$

Remark 5. More precisely, we can say that Tamagawa measure μ_{Tam} is the Haar measure on $G(\mathbf{A})$ which is normalized by the requirement

$$\mu_{\text{Tam}}(G(\mathbf{A}_0)) = q^{d(1-g) - \deg(\mathcal{L})} \prod_{x \in X} \frac{|G(\kappa_x)|}{|\kappa_x|^d}.$$

Remark 6. In order for Tamagawa measure to be well-defined, it is important that the quotients $\frac{|G(\kappa_x)|}{|\kappa_x|^d}$ be close to 1, so that the infinite product $\prod_{x \in X} \frac{|G(\kappa_x)|}{|\kappa_x|^d}$ is absolutely convergent. This can fail dramatically if G is not connected (in this case, we expect each factor to be approximately equal to the number of connected components of G). However, it is satisfied for many groups which are not semisimple: for example, for the additive group $\mathbf{G}_a$.

Remark 7. If $G = \mathbf{G}_a$, then we have $d = 1$, $\deg(\mathcal{L}) = 0$, and $|G(\kappa_x)| = |\kappa_x|$ for each $x \in X$. Consequently, the Tamagawa measure μ_{Tam} is characterized by the formula $\mu_{\mathrm{Tam}}(G(\mathbf{A}_0)) = q^{1-g}$. Note that we have an exact sequence

$$0 \to \mathrm{H}^0(X; \mathcal{O}_X) \to G(\mathbf{A}_0) \to G(\mathcal{K})\backslash G(\mathbf{A}) \to \mathrm{H}^1(X; \mathcal{O}_X) \to 0,$$

so that the Tamagawa measure of the quotient $G(\mathcal{K})\backslash G(\mathbf{A})$ is given by

$$\frac{|\,\mathrm{H}^1(X; \mathcal{O}_X)|}{|\,\mathrm{H}^0(X; \mathcal{O}_X)|} \mu_{\mathrm{Tam}}(G(\mathbf{A}_0)) = \frac{q^g}{q} q^{1-g} = 1.$$

Remark 8. One might ask the motivation for the auxiliary factor $q^{d(1-g)}$ appearing in the definition of the Tamagawa measure. Remark 7 provides one answer: the auxiliary factor is exactly what we need in order to guarantee that Weil's conjecture holds for the additive group $\mathbf{G}_a$.

Another answer is that the auxiliary factor is necessary to obtain invariance under *Weil restriction*. Suppose that $\pi : X \to Y$ is a separable map of algebraic curves over $\mathbf{F}_q$. Let $\mathcal{K}_Y$ be the fraction field of Y, so that $\mathcal{K}$ is a finite extension of $\mathcal{K}_Y$, let $\mathbf{A}_Y$ denote the ring of adeles of $\mathcal{K}_Y$, and let H_0 denote the algebraic group over $\mathcal{K}_Y$ obtained from G_0 by Weil restriction along the inclusion of fields $\mathcal{K}_Y \hookrightarrow \mathcal{K}$. Then we have a canonical isomorphism of locally compact groups $G_0(\mathbf{A}) \simeq H_0(\mathbf{A}_Y)$. This isomorphism is compatible with the Tamagawa measures on each side, but only if we include the auxiliary factor $q^{d(1-g)}$ indicated in Definition 4.

The goal of this course is to prove the following:

Conjecture 9 (Weil)**.** Suppose that G_0 is semisimple and simply connected. Then $\mu_{\mathrm{Tam}}(G(\mathcal{K})\backslash G(\mathbf{A})) = 1$.

Note that the quotient $G(\mathcal{K})\backslash G(\mathbf{A})$ carries a right action of the compact group $G(\mathbf{A}_0)$. We may therefore write $G(\mathcal{K})\backslash G(\mathbf{A})$ as a union of orbits, indexed by the collection of double cosets

$$G(\mathcal{K})\backslash G(\mathbf{A})/G(\mathbf{A}_0).$$

Moreover, if $Z \subseteq G(\mathcal{K})\backslash G(\mathbf{A})$ is the orbit corresponding to the double coset of an element $\gamma \in G(\mathbf{A})$, then Z can be identified with the quotient of $G(\mathbf{A}_0)$ by the intersection $G(\mathbf{A}_0) \cap \gamma^{-1} G(\mathcal{K}) \gamma$. We therefore have

$$\begin{aligned} \mu_{\mathrm{Tam}}(G(\mathcal{K})\backslash G(\mathbf{A})) &= \mu_{\mathrm{Tam}}(G(\mathbf{A}_0)) \sum_{\gamma} \frac{1}{|G(\mathbf{A}_0) \cap \gamma^{-1} G(\mathcal{K}) \gamma|} \\ &= q^{d(1-g)-\deg(\mathcal{L})} \Big(\prod_{x \in X} \frac{|G(\kappa_x)|}{|\kappa_x|^d} \Big) \sum_{\gamma} \frac{1}{|G(\mathbf{A}_0) \cap \gamma^{-1} G(\mathcal{K}) \gamma|}. \end{aligned}$$

Our next goal is to give an algebro-geometric interpretation to many of the expressions appearing on the right hand side of this equation.

Construction 10 (Regluing)**.** Let γ be an element of the group $G(\mathbf{A})$. We can think of γ as given by a collection of elements $\gamma_x \in G(\mathcal{K}_x)$, having the property that there exists a finite set S such that $\gamma_x \in G(\mathcal{O}_x)$ whenever $x \notin S$.

We define a G-bundle $\mathcal{P}_\gamma$ on X as follows:

(*a*) The bundle $\mathcal{P}_\gamma$ is trivial on the open set $U = X - S$.

(b) The bundle $\mathcal{P}_\gamma$ is trivial on a formal neighborhood $\operatorname{Spec} \mathcal{O}_x$ of each point $x \in S$.

(c) For each $s \in S$, the trivializations of $\mathcal{P}_\gamma$ on U and $\operatorname{Spec} \mathcal{O}_x$ given by (a) and (b) differ by the element $\gamma_x \in G(\mathcal{K}_x)$ on the overlap $\operatorname{Spec} \mathcal{K}_x \simeq \operatorname{Spec} \mathcal{O}_x \times_X U$.

Note that the G-bundle $\mathcal{P}_\gamma$ is canonically independent of the choice of S, so long as S contains all points x such that $\gamma_x \notin G(\mathcal{O}_x)$.

Remark 11. Let $\gamma, \gamma' \in G(\mathbf{A})$. The G-bundles $\mathcal{P}_\gamma$ and $\mathcal{P}_{\gamma'}$ come equipped with trivializations at the generic point of X. Consequently, giving an isomorphism between the restrictions $\mathcal{P}_\gamma\,|_{\operatorname{Spec} \mathcal{K}}$ and $\mathcal{P}_{\gamma'}\,|_{\operatorname{Spec} \mathcal{K}}$ is equivalent to giving an element $\beta \in G(\mathcal{K})$. Unwinding the definitions, we see that this isomorphism admits an (automatically unique) extension to an isomorphism of $\mathcal{P}_\gamma$ with $\mathcal{P}_{\gamma'}$ if and only if $\gamma'^{-1}\beta\gamma$ belongs to $G(\mathbf{A}_0)$. This has two consequences:

(a) The G-bundles $\mathcal{P}_\gamma$ and $\mathcal{P}_{\gamma'}$ are isomorphic if and only if γ and γ' determine the same element of $G(\mathcal{K})\backslash G(\mathbf{A})/G(\mathbf{A}_0)$.

(b) The automorphism group of the G-torsor $\mathcal{P}_\gamma$ is the intersection $G(\mathbf{A}_0) \cap \gamma^{-1} G(\mathcal{K})\gamma$.

Remark 12. Let $\mathcal{P}$ be a G-bundle on X. Then $\mathcal{P}$ can be obtained from Construction 10 if and only if the following two conditions are satisfied:

(i) There exists an open set $U \subseteq X$ such that $\mathcal{P}\,|_U$ is trivial.

(ii) For each point $x \in X - U$, the restriction of $\mathcal{P}$ to $\operatorname{Spec} \mathcal{O}_x$ is trivial.

By a direct limit argument, condition (i) is equivalent to the requirement that $\mathcal{P}\,|_{\operatorname{Spec} \mathcal{K}}$ be trivial: that is, that $\mathcal{P}$ is classified by a trivial element of $\mathrm{H}^1(\operatorname{Spec} \mathcal{K}; G_0)$. If G_0 is semisimple and simply connected, then $\mathrm{H}^1(\operatorname{Spec} \mathcal{K}; G_0)$ vanishes (this is a theorem of Harder: a strong version of the Hasse principle from the previous lecture) so that condition (i) is automatic. If the map $G \to X$ is smooth and geometrically connected, then condition (ii) is automatic (the restriction $\mathcal{P}\,|_{\operatorname{Spec} \kappa_x}$ can be trivialized by Lang's theorem, and any trivialization of $\mathcal{P}\,|_{\operatorname{Spec} \kappa_x}$ can be extended to a trivialization of $\mathcal{P}\,|_{\operatorname{Spec} \mathcal{O}_x}$ by virtue of the smoothness of G).

Combining Remarks 11 and 12, we obtain the formula

$$\mu_{\mathrm{Tam}}(G(\mathcal{K})\backslash G(\mathbf{A})) \simeq q^{d(1-g)-\deg(\mathcal{L})} \Big(\prod_{x \in X} \frac{|G(\kappa_x)|}{|\kappa_x|^d}\Big) \sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|}.$$

Here the sum is taken over all isomorphism classes of G-bundles on X. We may therefore reformulate Conjecture 9 as follows:

Conjecture 13 (Weil)**.** Suppose that G_0 is semisimple and simply connected. Then

$$q^{d(1-g)-\deg(\mathcal{L})} \sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|} = \prod_{x \in X} \frac{|\kappa_x|^d}{|G(\kappa_x)|}.$$

Remark 14. Note that neither side of the equation of Conjecture 13 is *a priori* well defined. The absolute convergence of the product on the right hand side is equivalent to the well-definedness of Tamagawa measure. The left hand side is usually an infinite sum (unless the algebraic group G_0 is anisotropic), but the conjecture asserts that this infinite sum converges to the right hand side.

The assertion of Conjecture 13 can be regarded as a function field analogue of the Siegel mass formula (in its original formulation). However, there are tools available for attacking Conjecture 13 that have no analogue in the case of a number field. More specifically, we would like to take advantage of the fact that the collection of all G-bundles on X admits an algebro-geometric parametrization.

Definition 15. If Y is a scheme equipped with a map $Y \to X$, we define a *G-bundle* on Y to be a principal homogeneous space for the group scheme $G_Y = Y \times_X G$ over Y. The collection of G-bundles on Y forms a category (in which all morphisms are isomorphisms).

For every $\mathbf{F}_q$-algebra R, we let $\mathrm{Bun}_G(R)$ denote the category of G-bundles on the relative curve

$$\operatorname{Spec} R \times_{\operatorname{Spec} \mathbf{F}_q} X.$$

The construction $R \mapsto \mathrm{Bun}_G(R)$ is an example of an *algebraic stack*, which we will denote by Bun_G. We will refer to Bun_G as the *moduli stack of G-bundles on X*.

Remark 16. The algebraic stack Bun_G is smooth over $\mathbf{F}_q$, and its dimension is given by $d(1-g) - \deg(\mathcal{L})$. By definition, the category $\mathrm{Bun}_G(\mathbf{F}_q)$ is the category of G-bundles on X. We will denote the sum $\sum_{\mathcal{P}} \frac{1}{|\mathrm{Aut}(\mathcal{P})|}$ by $|\mathrm{Bun}_G(\mathbf{F}_q)|$: we can think of it as a (weighted) count of the objects of $\mathrm{Bun}_G(\mathbf{F}_q)$, which properly takes into account the fact that $\mathrm{Bun}_G(\mathbf{F}_q)$ is a category rather than a set. With this notation, we can rephrase Conjecture 13 as an equality

$$\frac{|\mathrm{Bun}_G(\mathbf{F}_q)|}{q^{\dim(\mathrm{Bun}_G)}} = \prod_{x \in X} \frac{|\kappa_x|^d}{|G(\kappa_x)|}$$

Remark 17. For every point $x \in X$, let G_x denote the fiber product $\operatorname{Spec} \kappa_x \times_X G$, so that G_x is a connected algebraic group over κ_x. Let BG_x denote the classifying stack of G_x: this is a smooth algebraic stack of dimension $-d$ over $\operatorname{Spec} \kappa_x$. Then $BG_x(\mathbf{F}_q)$ is the category of G_x-bundles on $\operatorname{Spec} \kappa_x$. It follows from Lang's theorem that every G_x-bundle on $\operatorname{Spec} \kappa_x$ is trivial. Moreover, the automorphism group of the trivial G_x-bundle is given by $G_x(\kappa_x) = G(\kappa_x)$. We may therefore rewrite Weil's conjecture in the suggestive form

$$\frac{|\mathrm{Bun}_G(\mathbf{F}_q)|}{q^{\dim(\mathrm{Bun}_G)}} = \prod_{x \in X} \frac{|BG_x(\kappa_x)|}{|\kappa_x|^{\dim(BG_x)}}.$$

Heuristically, this should reflect the idea that Bun_G is a product of the classifying stacks BG_x as x varies over the curve X.

References

[1] Harder, G. *Über die Galoiskohomologie halbeinfacher algebraischer Gruppen. III.* Journal für die reine und angewandte Mathematik 274, 1975, 125-138.