The Apollonian Circles and Isodynamic Points

Tarik Adnan Moon

Abstract

This paper is on the Apollonian circles and isodynamic points of a triangle. Here we discuss some of the most intriguing properties of Apollonian circles and isodynamic points, along with several Olympiad problems, which can be solved using those properties.

Introduction

The idea of Apollonian circles of a triangle is derived from a problem that was first proposed by a geometer of ancient Greece. Isodynamic points are two common points of three Apollonian Circles of a triangle.

In this paper, we shall first explore several properties of Apollonian circles; then we shall discuss some of the most interesting results related to isodynamic points. After that we shall analyze several related problems, which will demonstrate how the knowledge of these properties can help a problem solver to solve some interesting problems.

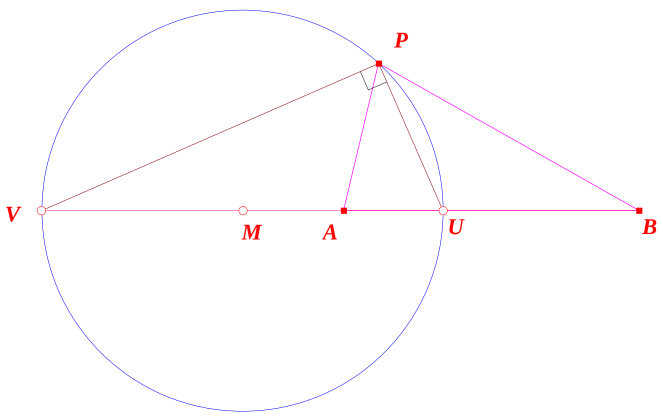
1 Apollonian Circle

Apollonius of Perga, a geometer of ancient Greece, proposed the following problem:

Problem 1. Find the locus of a point the ratio of whose from two fixed points is constant.

Solution We assume that we are given two fixed points A, B on a plane, and we need to find the locus of a point P such that $\frac{AP}{PB} = r$, where r is a given ratio. We assume that P is a point on the locus. Now we divide the line AB internally and externally in the given ratio. We have:

$$AU : UB = AV : VB = AP : PB = r$$



But from the angle bisector theorem we know that PU and PV are the internal and external angle bisectors of the $\angle APB$ respectively. As the internal and external bisectors of an angle are inclined at right angle, we have $\angle VPU = \frac{\pi}{2}$,. Let M be the midpoint of VU. Then the locus is indeed a circle with radius $r = MU$, and center M.

However, there is a special case. When $r = 1$, $V \to \infty$, and therefore the circle degenerates into a line.

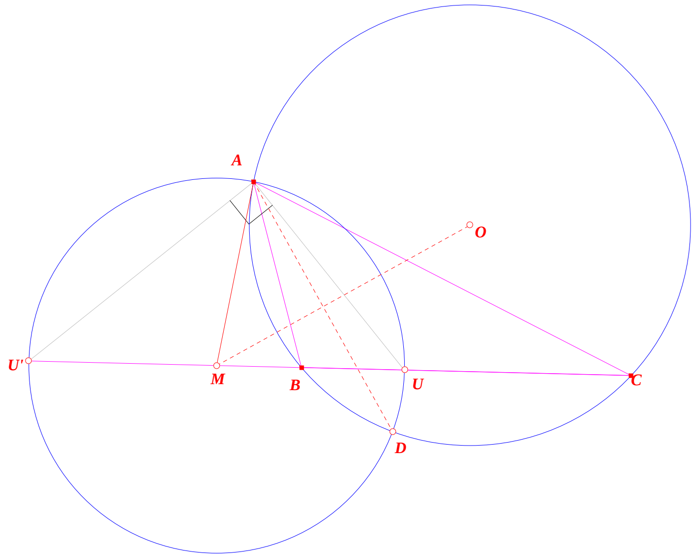
Now we define the **Apollonian circles** of a triangle. If the internal and external bisectors of the angles A, B, C of a triangle ABC meet the opposite sides BC, CA, AB in the points U, U'; V, V'; W, W', respectively, then the circles with UU', VV', WW' as diameters are called the $A-, B-, C-$ *Apollonian circles*(respectively) or *the circles of Apollonius* of a triangle ABC. (Section 3,Figure 1)
From Problem 1 we can infer that the Apollonian circles pass through the respective vertices of the triangle, and $BU : UC = BU' : U'C = BA : AC$ etc.
We continue our discussion with a classic problem related to the Apollonian circle. [1]

Problem 2. Let (M)[1], the A-Apollonian circle of $\triangle ABC$ meet (O), the circumcircle of this triangle, at A and D. Prove that $\angle ODM = \frac{\pi}{2}$.

First Solution We know that MO is the perpendicular bisector of the segment AD. So from symmetry, it is enough to prove that $\angle MAO = \frac{\pi}{2}$, i.e. MA is tangent to (O) at A.We have

$$\angle MAB + \frac{\angle A}{2} = \angle MAU = \angle MUA = \frac{\angle A}{2} + \angle C \iff \angle MAB = \angle C$$

Hence by the alternate segment theorem the result follows.



Before showing another solution to this problem, we would like to inform the reader that we shall frequently use the ideas of pole-polar, inversion, and harmonic conjugates in this paper. So, interested readers may refer to [1], [3], [4], [5].

Second Solution The problem actually asks to prove that these two circles are *orthogonal.* [2] From the definition of harmonic conjugate it follows that $(BCUU') = -1$. Now we prove a well known lemma for the convenience of the reader.

Lemma 1. If $(BCUU') = -1$, i.e. U, U' divide the segment BC harmonically, B and C are inverses w.r.t.[3] the circle with diameter UU'.

Proof. Let M be the midpoint of BC, and $MU = MU' = MA = R$. We have

$$\frac{BU}{UC} = \frac{BU'}{U'C} \iff \frac{BU}{BU'} = \frac{UC}{U'C} \iff \frac{(R - MB)}{(R + MB)} = \frac{(MC - R)}{(MC + R)} \iff R^2 = MB \times MC$$

□

So B and C are inverses with respect to the circle (M). Therefore we have $MB \cdot MC = MA^2$. Hence from the converse of the power of the point theorem MA is tangent to (O) from M. We have proved a very useful theorem:

[1]For brevity we shall denote a circle with diameter r and center O by (O, r), or simply by (O).
[2]Two circles (O, r) and (O', r') are called *orthogonal* iff $|OO'|^2 = r^2 + r'^2$.
[3]w.r.t.=with respect to.

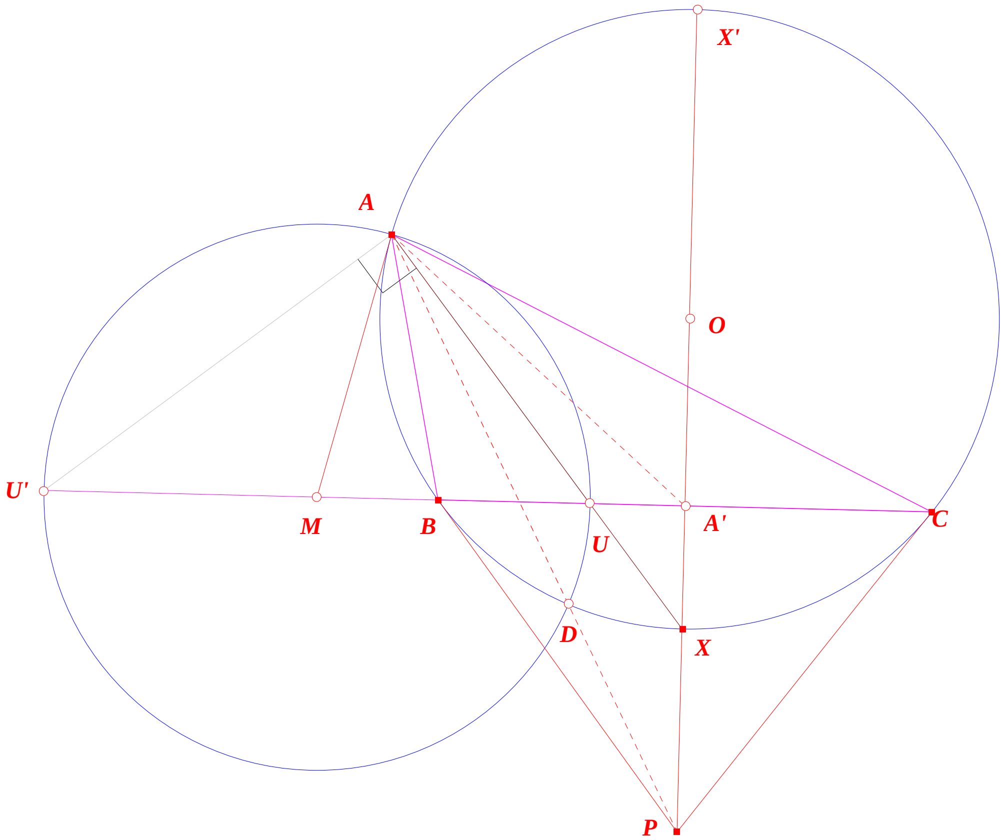
Theorem 1.1. *The Apollonian circle and the circumcircle of a triangle are orthogonal.*

Problem 3. If A' is the midpoint of segment BC, with the same conditions of the previous problem prove that $\angle A'AC = \angle BAD$.

Solution Let the tangents to (O) at B and C meet at P. We draw a diameter XX' through P. Now we prove a lemma.

Lemma 2. P lies on the extension of AD.

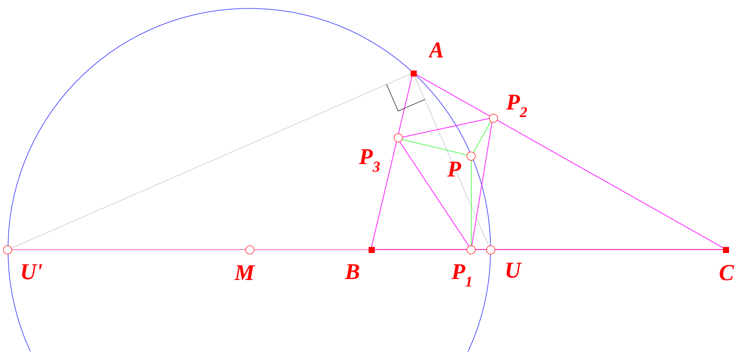
Proof. From symmetry X, the midpoint of the arc BC; A', the midpoint of the segment BC, lies on PX'. From Problem 2 MA, MD are tangents to (O). So M is the pole of the polar AD, and M lies on BC, which is the polar of of the pole P. So from *La Hire's Theorem*, we deduce that P lies on the extension of AD. □



Here P is the inverse of A' w.r.t (O). Thus $(PA'XX') = -1$. As XX' is a diameter, $\angle XAX' = \frac{\pi}{2}$. So we deduce that XA and $X'A$ are the internal and external angle bisectors of $\angle PAA'$, respectively. So $\angle XAA' = \angle PAX$. Therefore the conclusion follows.
Actually this is a very interesting property of the symmedians - the reader may have already noticed that AD is the A-symmedian of $\triangle ABC$.

Theorem 1.2. *The common chord of the circumcircle and the Apollonian circle is a symmedian of the triangle.*

Problem 4. If P_1, P_2, P_3 are feet of perpendiculars from a point P to the sides BC, CA, AB respectively, find the locus of the point P such that $P_1P_2 = P_1P_3$. [2]



Solution We shall prove that the locus is the Apollonian Circle. We can prove by the Sine Law that

$$P_1P_3 = BP \sin B, \quad \text{and} \quad P_1P_2 = CP \sin C$$

So

$$P_1P_2 = P_1P_3 \iff \frac{BP}{CP} = \frac{\sin C}{\sin B} = \frac{AB}{AC}$$

So the locus is the A-Apollonian circle of $\triangle ABC$.

2 Isodynamic Points

Now we are ready to prove the main result, the existence of the isodynamic points.

Theorem 2.1. *The three Apollonian circles of a triangle have two points in common.*

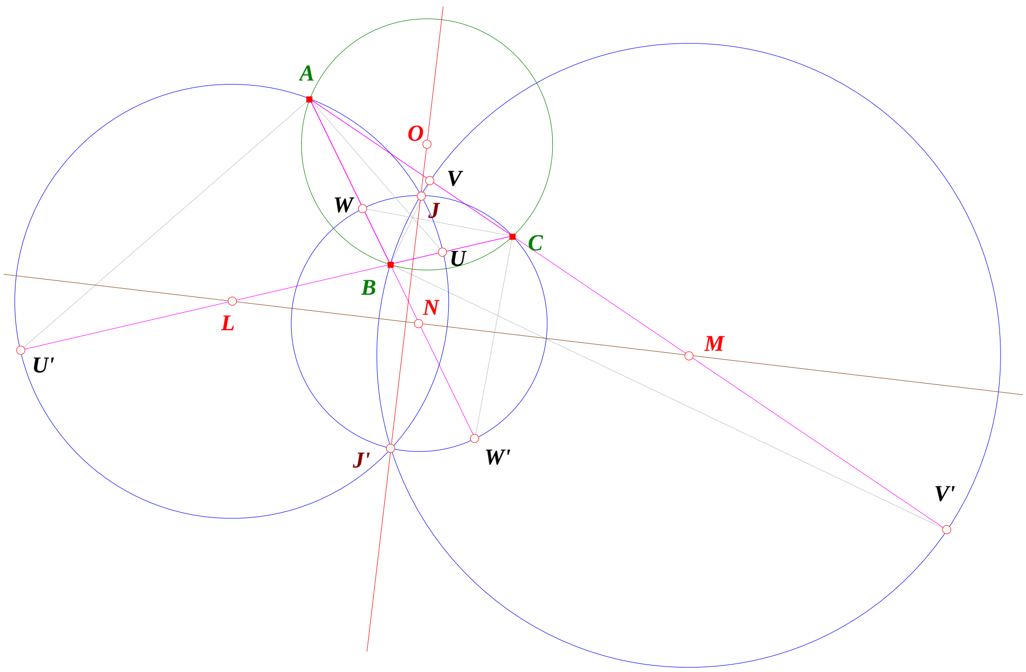


Figure 1: *Three Apollonian circles* $(L), (M), (N)$*; and the isodynamic points* J, J'.

Before proving the theorem, we examine the figure carefully. Here J, the intersection point inside the triangle, is called the first isodynamic point, while J' is called the second isodynamic point. We can also see from the figure that J, J' are two points of intersection of the Apollonian circles. So JJ' is the radical axis of these three circles. Several other interesting properties that are evident from the diagram will be proved in this section.

Proof. From the definition of Apollonian circles we have:

$$JB : JC = BA : CA, \quad JC : JA = BC : BA \iff JB : JA = BC : CA$$

Therefore J lies on the circle (N). □

Theorem 2.2. *L, M, N are collinear.*[4]

Proof. We shall at first prove that $BL : LC = c^2 : b^2$.
From Theorem 1.1 we know that AM is tangent to (O) at A. So $\triangle LAB \sim \triangle LCA$. Therefore

$$AL : LC = c : b \quad \iff AL^2 : LC^2 = c^2 : b^2$$

But from power of the point L we have $LA^2 = LB \cdot LC$. So $BL : LC = c^2 : b^2$.
Now multiplying three similar expressions, from the converse of the Menelaus's theorem we conclude that L, M, N are collinear. □

Theorem 2.3. *$(L), (M), (N)$ are coaxal.*

Proof. L, M, N are collinear, and they share the same radical axis JJ'. So they are coaxal. □

Problem 5. Let O and K be the circumcenter and the Lemoine point (the point of concurrency of the symmedians). Prove that J, J', K, O are collinear. Also prove that J is the inverse of J' w.r.t. (O).

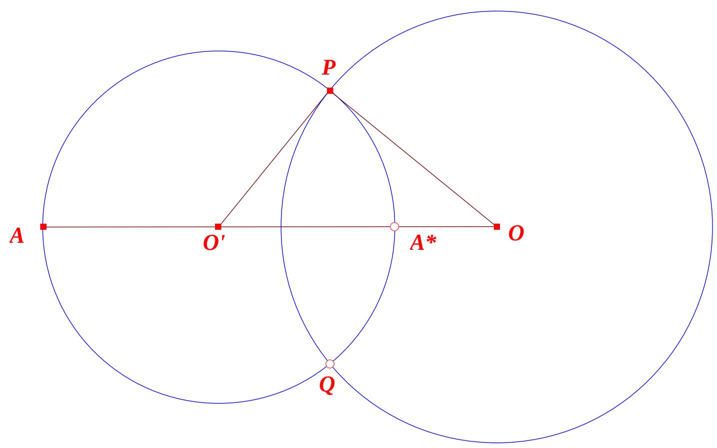
Solution We prove a lemma before we start.

Lemma 3. If a circle is orthogonal to two given circles, its center lies on the radical axis of those two circles.

Proof. If a circle (O, r) is orthogonal to two given circles (A, p) and (B, q), the power of O w.r.t (A), $P_{(A)}(O) = OA^2 - p^2 = r^2$. Similarly the power of O w.r.t. (B) is equal to r^2. So O lies on the radical axis of those two circles. □

From Lemma 3, apparently O lies on JJ'. From Problem 3 (or Theorem 1.2) we can deduce that L is the pole of symmedian AD (We use the notations of Figure 1). Applying the same logic we can say that N, M are the pole of the other two symmedians. We know that the symmedians are concurrent at Lemonie point, K. So K is the pole of the polar LNM. But the pole of the polar LNM must be on the perpendicular line from O to LMN. As OJJ' is the radical axis of the circles $(L), (M), (N)$; K lies on the line OJJ'.
We need to prove another lemma to complete the second part.

Lemma 4. If two orthogonal circles are given, one remains invariant under inversion w.r.t. the other.



[4] *Throughout this paper we shall often refer to* **Figure 1** *and its notations.*

Proof. Let (O, r), (O', R) be two orthogonal circles, and OO' intersect (O') at A and A^*. It is enough to prove that A and A^* are inverses w.r.t. (O). We have

$$OA \times OA^* = (OO' + R)(OO' - R) = |OO'|^2 - R^2 = r^2$$

Hence the conclusion follows.
The most important implication of this lemma is that if we take any line passing through O (or O'), and if the line intersects (O') (or (O)) at A and A^*; A, A^* are inverses. This is because the center of the circle and A, A^* are collinear. □

As J, J' are points collinear with O; and $(O), (L)$ are orthogonal, from Lemma 4 we deduce that J and J' are inverses of each other w.r.t. (O).

Theorem 2.4. *If OK intersects (O) at Q and R,*[5] $(QRJJ') = -1$.

Proof. By the previous problem J and J' are inverses w.r.t. (O). So by Lemma 1 $(QRJJ') = -1$. □

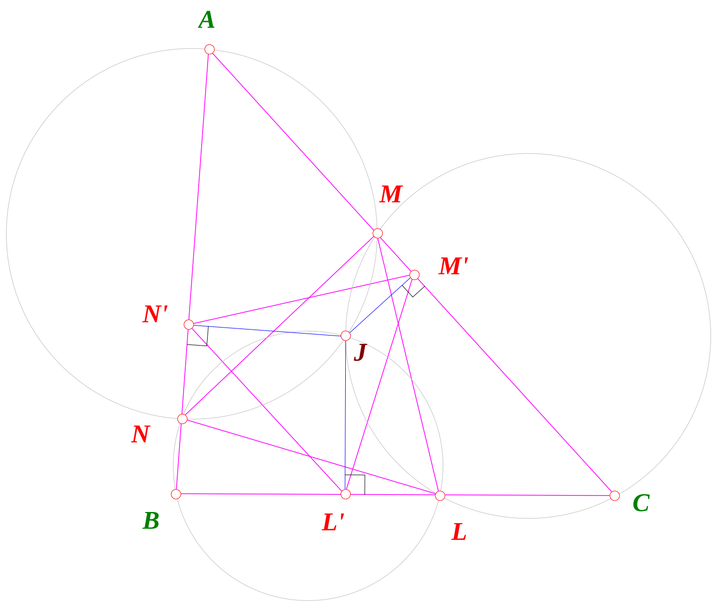
Now here is a problem that appeared in the *Tournament of the Towns 1995.* [7]

Problem 6. Show that there are exactly two points for a triangle such that the feet of the perpendiculars to the three sides form an equilateral triangle.

Solution From problem 4 we know that the Apollonian circle is the locus of the point P, which has isosceles pedal triangle. So the points for which we get an equilateral pedal triangle are their intersections, i.e. the isodynamic points of a triangle.

The pedal triangle of the isodynamic points has many other marvelous features.

Theorem 2.5. *Among all equilateral triangles having vertices on the sides of a triangle, the pedal triangle of J, the first isodynamic point, has the minimum area.*



Proof. Let LMN be an equilateral triangle which has vertices on the sides of $\triangle ABC$. If we draw the circumcircles of the triangles LCM, MAN, NBL, they will concur in a point J, by Miquel's Theorem (we can prove this easily by angle chasing). Now we draw the pedal triangle $L'M'N'$ of the point J. From the cyclic quadrilaterals we have

$$\angle JLM = \angle JCM = \angle JL'M'$$
$$\angle JLN = \angle JBN = \angle JL'N'.$$

[5] QR is called the *Brocard diameter* of a triangle.

Adding these two we get, $\angle MLN = \angle M'L'N' = 60°$. So a spiral similarity with center J, ratio $r = \frac{JL'}{JL} \leq 1$, and angle $\alpha = \angle LJL'$ maps $\triangle LMN \to \triangle L'M'N'$. From Problem 6, we deduce that J is the first isodynamic point of $\triangle ABC$. Hence the conclusion follows. □

Several interesting problems can be solved using this property. For example:

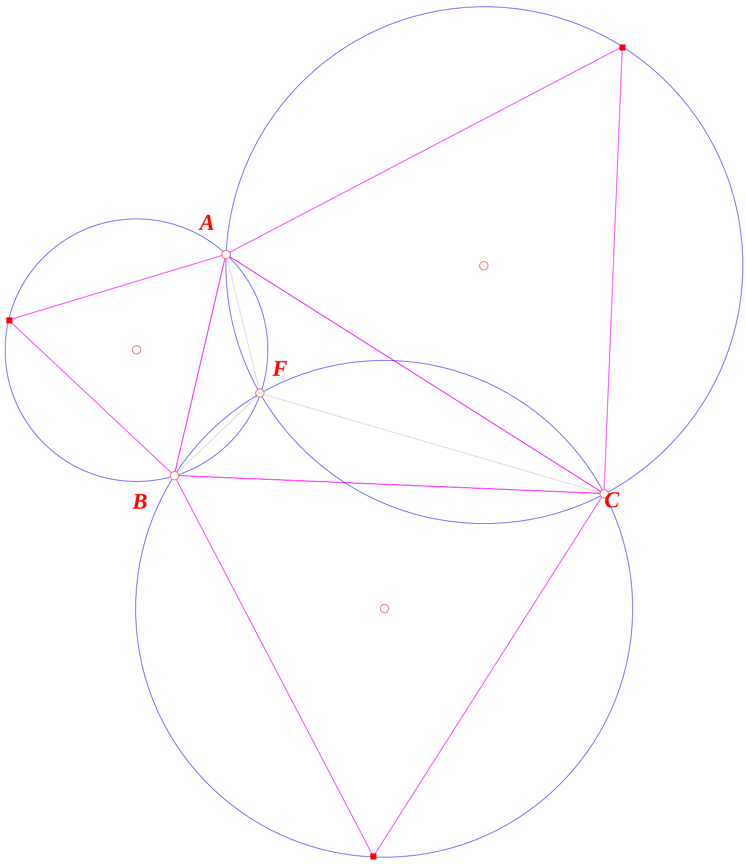
Let P, Q, and R be the points on sides BC, CA, and AB of an acute triangle ABC such that triangle PQR is equilateral and has minimal area among all such equilateral triangles. Prove that the perpendiculars from A to line QR, from B to line RP, and from C to line PQ are concurrent.

We end this section with a real gem: the relation between the famous Fermat point and isodynamic point.

Theorem 2.6. *The isodynamic point and the Fermat point are isogonal conjugates.*

Proof. At first we prove this for the first Fermat point. From the construction of the first Fermat point, (i.e. by erecting equilateral triangles externally on the sides of the triangle, and drawing their circumcircles) we can easily see that it is the only point satisfying

$$\angle AFB = \angle BFC = \angle CFA = 120°.$$



So it will be enough to prove that the isogonal conjugate F (suppose) of J satisfies the property. We prove the following lemma at first[6].

Lemma 5. For any two isogonal conjugate points F and J we have:

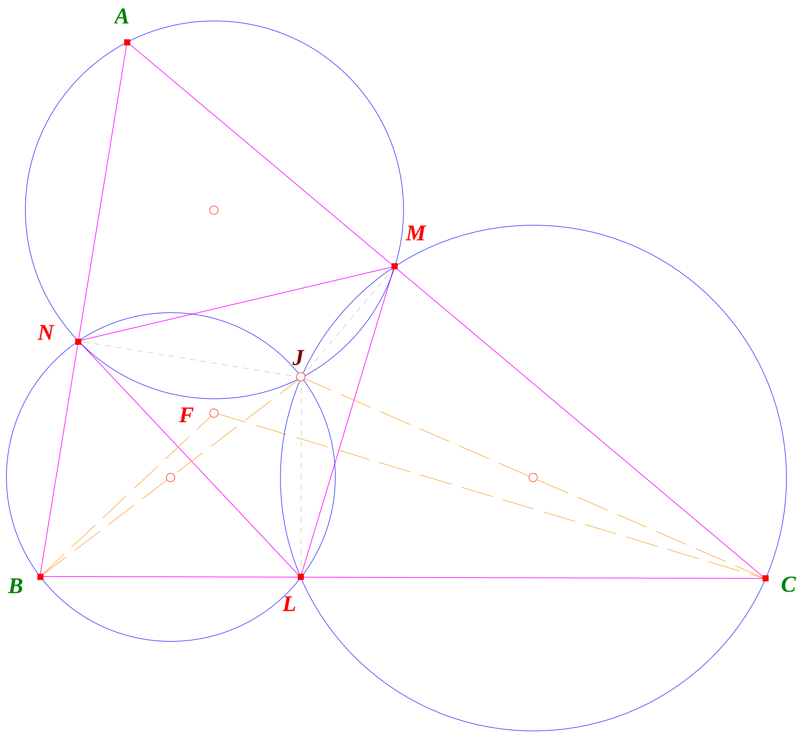
$$\angle BFC + \angle BJC = 180° + \angle A.$$

[6] The proof would be more rigorous if we used directed angles modulo π, but we compromise rigor for the sake of simplicity.

Proof. As J and F are isogonal conjugates. We have $\angle FBC = \angle JBA$, and $\angle FCB = \angle JCA$. Also

$$\begin{aligned}\angle BFC + \angle BJC =& (180^\circ - \angle FBC - \angle FCB) + (180^\circ - \angle JBC - \angle JCB)\\ =& 360^\circ - (\angle B + \angle C)\\ =& 180^\circ + \angle A\end{aligned}$$

□



Let LMN be the pedal triangle of J. Then from the cyclic quadrilaterals $JMAN, JNBL$, and $JLCM$ we have

$$\begin{aligned}\angle BJC =& \angle JBA + \angle A + \angle JCB = \angle JLN + \angle A + \angle MLJ\\ =& 60^\circ + \angle A\\ \iff \angle BFC =& 180^\circ + \angle A - (60^\circ + \angle A) = 120^\circ\end{aligned}$$

Similarly we can show that

$$\angle AFB = \angle CFA = 120^\circ$$

So F is the isogonal conjugate of J. In the same way we can prove the result for the second Fermat point. We leave this as an exercise for the reader. □

3 Olympiad Problems and More Applications

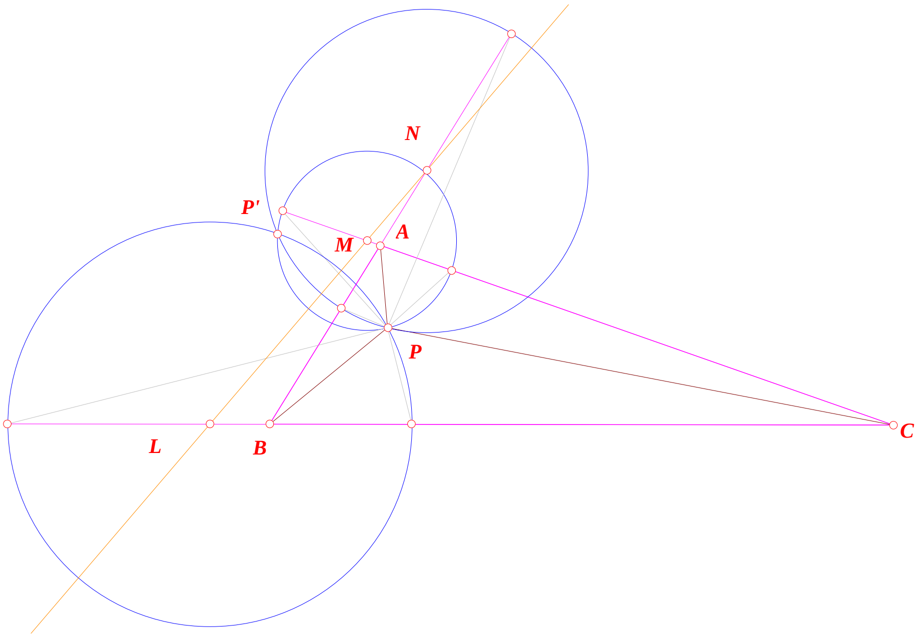
In this section we discuss some Olympiad caliber problems, several of which appeared in different Olympiads. We shall also prove more properties of the Apollonian circles and the isodynamic points.

Problem 7. Show that the intersections of the perpendicular bisectors of the internal angle bisectors meet the respective sides of the triangle in three collinear points.

Solution It is easy to notice that the perpendicular bisector of the AU, intersect AB at L. We have proved the required collinearity as Theorem 2.2 .
A similar problem asks to show that U', J', W' are collinear. We have $\frac{BU'}{U'C} = \frac{BA}{AC}$ etc. Multiplying the similar expressions, again we can easily prove the result by Menelaus's Theorem.

Problem 8. $\triangle ABC$ and a point P is given. Draw Apollonius circles of $\angle APB, \angle APC$, and $\angle CPB$. Prove that these three circles pass through a common point other than P.

(MathLinks) [10]



Solution Let the centers of the Apollonian circles of those angles be M, N, L respectively. By Theorem 2.2 we have $BL : LC = BP^2 : PC^2$, etc. So

$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = \frac{BP^2}{PC^2} \cdot \frac{CP^2}{PA^2} \cdot \frac{AP^2}{PB^2} = 1$$

Thus L, M, N are collinear by the converse of Menelaus's Theorem. As these circles have one point, P, in common they must have another point in common, which will be on the common radical axis of these three circles.

The following problem is from 9^{th} Iberoamerican Olympiad 1994. [13]

Problem 9. Let A, B and C be given points on a circle K such that the triangle $\triangle ABC$ is acute. Let P be a point in the interior of K. Let X, Y, and Z be the other intersection of AP, BP and CP with the circle. Determine the position of P to obtain $\triangle XYZ$ equilateral.

First Solution We shall prove the point is the first isodynamic point of $\triangle ABC$. We invert $\triangle ABC$ w.r.t a circle (P), which has an arbitrary radius r. Now we have

$$A'B' = AB \cdot \frac{r^2}{PA \cdot PB}, B'C' = BC \cdot \frac{r^2}{PB \cdot PC}, C'A' = CA \cdot \frac{r^2}{PC \cdot PA}$$

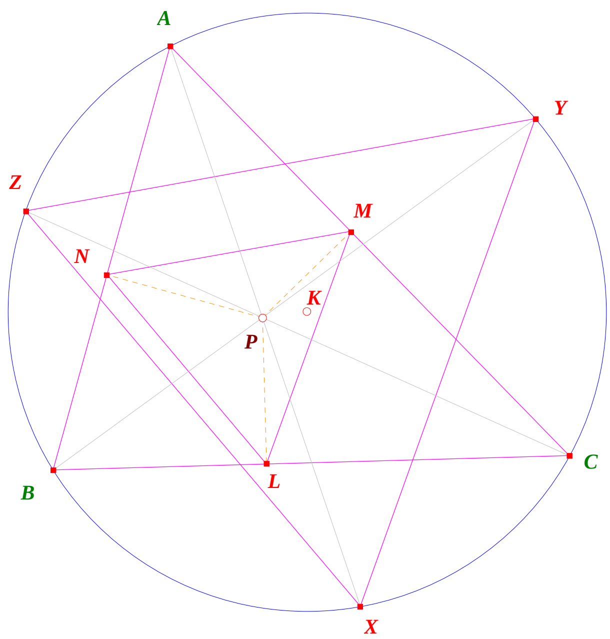
As P is on the Appolonian circles, we have

$$\frac{AB}{BC} = \frac{PA}{PC} \quad \text{and} \quad \frac{A'B'}{B'C'} = \frac{AB}{BC} \cdot \frac{PB \cdot PC}{PA \cdot PB} = 1$$

Similarly $B'C' = C'A'$. Thus the inverted triangle is equilateral. Now we are going to prove two very useful lemmas to finish this problem. These two lemmas are true for any point P which is not on the circumcircle of $\triangle ABC$.

Lemma 6. Let P be any point inside a triangle ABC, and let X, Y, Z be the intersection of AP, BP, CP with the circumcircle of $\triangle ABC$. Then $\triangle LMN$, the pedal triangle of P, is similar to $\triangle XYZ$.

Proof. Here $\angle AXY = \angle ABP = \angle NLP$ and $\angle AXZ = \angle ACP = \angle MLP$. Adding these two, we get $\angle ZXY = \angle NLM$.
Similarly we get the relations for the other angles. □



Lemma 7. With the same configuration, if $\triangle A'B'C'$ is obtained from $\triangle ABC$ by an inversion w.r.t a circle with center P and arbitrary radius $(= r)$, $\triangle LMN \sim \triangle A'B'C' \sim \triangle XYZ$.

Proof. From the power of the point P, we have,

$$AP \cdot XP = BP \cdot YP = CP \cdot ZP.$$

From the definition of inversion

$$AP \cdot A'P = BP \cdot B'P = CP \cdot C'P = r^2.$$

Therefore

$$\frac{XP}{A'P} = \frac{YP}{B'P} = \frac{ZP}{C'P}$$

Hence $\triangle LMN \sim \triangle A'B'C' \sim \triangle XYZ$. □

From these lemmas we get the conclusion.
In this problem, we have proved a terrific property of isodynamic points. *The isodynamic points of a triangle are the only points, w.r.t which we can invert the triangle into an equilateral triangle.*
However there is a shorter solution which does not use inversion, but rather uses the idea of Theorem 2.6.

Second Solution Let F be the isogonal conjugate of P. From the proof of Theorem 2.6 we know that $\angle APC + \angle AFC = 180° + A$. But

$$\begin{aligned}\angle APC =& 180° - (\angle PAC + \angle ACP) = 180° - (\angle XYC + \angle AYZ)\\ =& 180° - (\angle AYC - \angle ZYX) = 180° - (180° - \angle A - 60°)\\ =& 60 + \angle A\end{aligned}$$

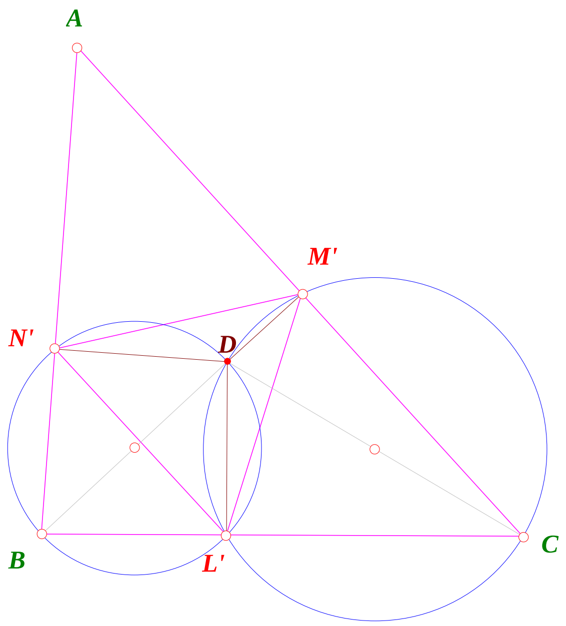
So $\angle AFC = 120°$. We know that the Fermat point is the only point satisfying the condition. So P is the isogonal conjugate of F, i.e., the isodynamic point.

Problem 10. Let D be a point in the interior of an acute angled ABC such that $AB = a \cdot b$, $AC = a \cdot c$, $AD = a \cdot d$, $BC = b \cdot c$, $BD = b \cdot d$ and $CD = c \cdot d$. Prove that $\angle ABD + \angle ACD = \frac{\pi}{3}$.

(Singapore TST 2004) [11]

Solution From the relations we get

$$\frac{AB}{AC} = \frac{a \cdot b}{a \cdot c} = \frac{b \cdot d}{c \cdot d} = \frac{BD}{CD}$$
$$\frac{AC}{BC} = \frac{a \cdot c}{b \cdot c} = \frac{a \cdot d}{b \cdot d} = \frac{AD}{BD}$$
$$\frac{BC}{AB} = \frac{b \cdot c}{a \cdot b} = \frac{c \cdot d}{a \cdot d} = \frac{CD}{AD}.$$



So we conclude that D is the first isodynamic point of $\triangle ABC$. Let $L'M'N'$ be the pedal triangle of D. From Problem 7 we know $L'M'N'$ is equilateral. Finally, from the cyclic quadrilaterals $CL'DM'$ and $BL'DN'$,

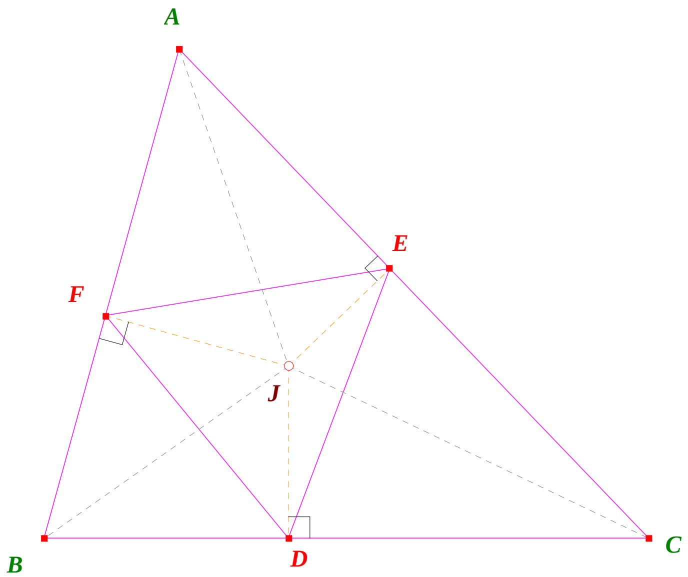
$$\angle ABD + \angle ACD = \angle N'L'D + \angle M'N'D = \angle N'L'M' = 60^\circ$$

We end our discussion with a geometric inequality that appeared as *G 8* in the IMO shortlist 1993. Indeed, this problem would be a quite hard one if we did not know the properties of the Apollonian cirlces and isodynamic points (or Fermat point). This solution is due to Vladimir Zajic [15].

Problem 11. The vertices D, E, F of an equilateral triangle lie on the sides BC, CA, AB respectively of a triangle ABC. If a, b, c are the respective lengths of these sides, and S the area of ABC, prove that

$$DE \geq \frac{2 \cdot \sqrt{2} \cdot S}{\sqrt{a^2 + b^2 + c^2 + 4 \cdot \sqrt{3} \cdot S}}.$$

Solution We shall prove that the given length, in the right hand side, is the side length of the pedal triangle of the first isodynamic point J. By Problem 6 $\triangle DEF$, the pedal triangle of J, is equilateral. From the second solution of Problem 9 we have $\angle AJB = \angle C + 60^\circ$ and also, $\angle BJC = \angle A + 60^\circ$, $\angle CJA = \angle B + 60^\circ$. Let $e = DE = EF = FD$ be the side length of the equilateral pedal triangle $\triangle DEF$.



The area S of the triangle $\triangle ABC$ with circumradius R is

$$\begin{aligned}
S =& \frac{1}{2}\left[AJ \cdot BJ \sin(C+60^\circ) + BJ \cdot CJ \sin(A+60^\circ) + CJ \cdot AJ \sin(B+60^\circ)\right] \\
=& \frac{e^2}{2}\left[\frac{\sin(C+60^\circ)}{\sin A \sin B} + \frac{\sin(A+60^\circ)}{\sin B \sin C} + \frac{\sin(B+60^\circ)}{\sin C \sin A}\right] \\
=& \frac{4R^3e^2}{abc}\left[\sin A \sin(A+60^\circ) + \sin B \sin(B+60^\circ) + \sin C \sin(C+60^\circ)\right] \\
=& \frac{R^2e^2}{S}\left[\frac{1}{2}(\sin^2 A + \sin^2 B + \sin^2 C) + \frac{\sqrt{3}}{2}(\sin A \cos A + \sin B \cos B + \sin C \cos C)\right] \\
=& \frac{e^2}{8S}\left[a^2 + b^2 + c^2 + \frac{4\sqrt{3}R^2}{2}\ (\sin 2A + \sin 2B + \sin 2C)\right] \\
=& \frac{e^2}{8S}(a^2 + b^2 + c^2 + 4S\sqrt{3}) \\
\iff\ e =& \frac{2S\sqrt{2}}{\sqrt{a^2 + b^2 + c^2 + 4S\sqrt{3}}}.
\end{aligned}$$

Here we have used the identity

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$$

Thus the expression on the right side of the inequality in question is precisely the side length of the equilateral pedal triangle $\triangle DEF$ of the 1st isodynamic point J. Any other equilateral triangle $\triangle D'E'F'$ inscribed in the triangle $\triangle ABC$, so that $D' \in BC,\ E' \in CA,\ F' \in AB$, is obviously obtained from the equilateral pedal triangle $\triangle DEF$ by a spiral similarity with the center J and similarity coefficient greater than 1, hence its side $e' = D'E'$ is greater than the side $e = DE$. (This part was discussed as Theorem 2.5)
So the inequality follows.

4 More Problems!

Here are a few problems that are related to the discussion of this paper. Using the properties we have discussed will often be the crux move for solving these problems. However, some problems may have solutions that do not use the ideas we have discussed, and obviously they will often need other ideas that we have not discussed.

Problem 1. An Apollonian circle of a triangle make an angle of 120° with the remaining two circles.

Problem 2. Let $\triangle ABC$ be right and AH be the altitude to the hypotenuse. Prove that Apollonius circles of $\angle AHB$ and $\angle AHC$ intersect at the center of Apollonius circle of $\angle BAC$.

Problem 3. Consider a triangle ABC and its internal angle bisector BD ($D \in BC$). The line BD intersects the circumcircle Ω of triangle ABC at B and E. Circle ω with diameter DE cuts Ω again at F. Prove that BF is the symmedian line of triangle ABC.

Problem 4. Let F be the Fermat's point of a triangle ABC. Let X, Y, Z be the feet of the perpendiculars from this Fermat point F to the sides BC, CA, AB of triangle ABC. The circumcircle of triangle XYZ intersects the sides BC, CA, AB at the points X', Y', Z' (apart from X, Y, Z). Show that the triangle $X'Y'Z'$ is equilateral.
(Hint: F, J are isogonal conjugates.)

Problem 5 (Romanian Olympiad). Given four points A_1, A_2, A_3, A_4 in the plane, no three collinear, such that

$$A_1A_2 \cdot A_3A_4 = A_1A_3 \cdot A_2A_4 = A_1A_4 \cdot A_2A_3,$$

denote by O_i the circumcenter of $\triangle A_jA_kA_l$ with $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Assuming $\forall i, A_i \neq O_i$, prove that the four lines A_iO_i are concurrent or parallel.

Problem 6. An equilateral triangle XYZ is inscribed in the circle (O). Let P be an arbitrary point inside the triangle which is not on the sides, so PX, PY, PZ cut (O) at A, B, C, respectively. Let D, E, F be the centers of the inscribed circles of the triangle PBC, PCA, PAB respectively. Prove that AD, BE, CF are concurrent.

Problem 7. A circle with chord BC is given. A is an arbitrary point on the circle. Prove that

1. When A varies, the loci of isodynamic points are a pair of circles.
2. Let R be the radius of the given circle, R_1 and R_2 be the radii of the locus circles. Then

$$\left| \frac{1}{R_1} \pm \frac{1}{R_2} \right| = \frac{1}{R}.$$

Problem 8. Let ABC be a triangle inscribed in circumcircle (O). Denote A_1, B_1, C_1 respectively to be the projections of A, B, C onto BC, CA, AB. Let A_2, B_2, C_2 respectively be the intersections of AO, BO, CO with BC, CA, and AB. A circle Ω_a passes through A_1, A_2 and lies tangent to the arc of BC that does not contain A of (O) at T_a. The same definition holds for T_b, T_c. Prove that AT_a, BT_b and CT_c are concurrent.

Problem 9. Prove that $FF' \parallel OH$ where F is the Fermat point, F' the isogonal conjugate of the Fermat point, and O and H are the circumcenter and orthocenter of a triangle.

Problem 10 (USA MOSP 1996). Let $AB_1C_1, AB_2C_2, AB_3C_3$ be directly congruent equilateral triangles. Prove that the pairwise intersections of the circumcircles of triangles AB_1C_2, AB_2C_3, AB_3C_1 form an equilateral triangle congruent to the first three.

Acknowledgments

The author would like to thank Son Hong Ta, Pranon Rahman Khan, and Kazi Hasan Zubaer for their helpful comments and encouragement. The author would also like to thank Vladimir Zajic for providing an excellent solution to Problem 11, and motivation for several other problems. Most of these problems have been taken from MathLinks forum. This document was prepared using LaTeX, and the figures were drawn using Cabri Geometry® II Plus.

References

[1] Nathan Altshiller-Court, *College Geometry: An Introduction to the Modern Geometry of the Triangle and the Circle*, Dover Books on Mathematics.

[2] Roger A. Johnson, *Advanced Euclidean Geometry* , Dover Books on Mathematics.

[3] Kiran Kedlaya, *Geometry Unbound*, version of January 18, 2006.
http://math.mit.edu/~kedlaya/geometryunbound/

[4] Cosmin Pohoata, *Harmonic Division and its Applications*,
Mathematical Reflection 4, 2007.
http://reflections.awesomemath.org/2007_4/harmonic_division.pdf

[5] Kin Y. Li, *Pole and Polar*, Mathematical Excalibur, Volume 11, Number 4.
http://www.math.ust.hk/excalibur/v11_n4.pdf

[6] Tarik Adnan Moon, *Pole-Polar:* Key Facts.
http://sites.google.com/site/kmckbd/Home/documents/pole_polar.pdf

[7] Alexander Bogomolny, *Apollonian Circles Theorem.*
http://www.cut-the-knot.org/Curriculum/Geometry/ApollonianCircle.shtml

[8] MathLinks topic, *17th Junior Tournament of the Towns 1995 Autumn problems.*
http://www.mathlinks.ro/Forum/viewtopic.php?p=1365322

[9] Jean-Louis AYME, *La fascinante figure de Cundy.*
http://pagesperso-orange.fr/jl.ayme/Docs/La%20fascinante%20figure%20de%20Cundy.pdf

[10] MathLinks topic, *3 Apollonius circles pass through one point.*
http://www.mathlinks.ro/Forum/viewtopic.php?p=1602320

[11] MathLinks topic, *Singapore TST 2004.*
http://www.mathlinks.ro/Forum/viewtopic.php?p=18403

[12] MathLinks topic, *Smallest equilateral triangle.*
http://www.mathlinks.ro/Forum/viewtopic.php?p=272246

[13] MathLinks topic, *9th ibmo - brazil 1994/q4.*
http://www.mathlinks.ro/Forum/viewtopic.php?p=506218

[14] MathLinks topic, *Two fermat points - variety of hard results.*
http://www.mathlinks.ro/Forum/viewtopic.php?p=497143

[15] MathLinks topic, *Highly recommended by the Problem Committee.*
http://www.mathlinks.ro/Forum/viewtopic.php?p=463149

Tarik Adnan Moon
Student, Class 12, Kushtia Government College,
Kushtia, Bangladesh
Email: moonmathpi469@gmail.com