Twenty-Four Hours of Local Cohomology

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To our teachers

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Preface

This book is an outgrowth of the summer school *Local cohomology and its interactions with algebra, geometry, and analysis* that we organized in June 2005 in Snowbird, Utah. This was a joint program under the AMS-IMS-SIAM Summer Research Conference series and the MSRI Summer Graduate Workshop series. The school centered around local cohomology, and was intended for graduate students interested in various branches of mathematics. It consisted of twenty-four lectures by the authors of this book, followed by a three-day conference.

We thank our co-authors for their support at all stages of the workshop. In addition to preparing and delivering the lectures, their enthusiastic participation, and interaction with the students, was critical to the success of the event. We also extend our hearty thanks to Wayne Drady, the AMS conference coordinator, for cheerful and superb handling of various details.

We profited greatly from the support and guidance of David Eisenbud and Hugo Rossi at MSRI, and Jim Maxwell at AMS. We express our thanks to them, and to our Advisory Committee: Mel Hochster, Craig Huneke, Joe Lipman, and Paul Roberts. We are also indebted to the conference speakers: Markus Brodmann, Ragnar-Olaf Buchweitz, Phillippe Gimenez, Gennady Lyubeznik, Paul Roberts, Peter Schenzel, Rodney Sharp, Ngo Viet Trung, Kei-ichi Watanabe, and Santiago Zarzuela.

Finally, we thank the AMS and the MSRI for their generous support in hosting this summer school, and the AMS for publishing this revised version of the "Snowbird notes".

Anurag K. Singh and Uli Walther

Introduction

Local cohomology was invented by Grothendieck to prove Lefschetz-type theorems in algebraic geometry. This book seeks to provide an introduction to the subject which takes cognizance of the breadth of its interactions with other areas of mathematics. Connections are drawn to topological, geometric, combinatorial, and computational themes. The lectures start with basic notions in commutative algebra, leading up to local cohomology and its applications. They cover topics such as the number of defining equations of algebraic sets, connectedness properties of algebraic sets, connections to sheaf cohomology and to de Rham cohomology, Gröbner bases in the commutative setting as well as for D-modules, the Frobenius morphism and characteristic p methods, finiteness properties of local cohomology modules, semigroup rings and polyhedral geometry, and hypergeometric systems arising from semigroups.

The subject can be introduced from various perspectives. We start from an algebraic one, where the definition is elementary: given an ideal $\mathfrak{a}$ in a Noetherian commutative ring, for each module consider the submodule of elements annihilated by some power of $\mathfrak{a}$. This operation is not exact, in the sense of homological algebra, and local cohomology measures the failure of exactness. This is a simple-minded algebraic construction, yet it results in a theory rich with striking applications and unexpected interactions.

On the surface, the methods and results of local cohomology concern the algebra of ideals and modules. Viewing rings as functions on spaces, however, local cohomology lends itself to geometric and topological interpretations. From this perspective, local cohomology is sheaf cohomology with support on a closed set. The interplay between invariants of closed sets and the topology of their complements is realized as an interplay between local cohomology supported on a closed set and the de Rham cohomology of its complement. Grothendieck's local duality theorem, which is inspired by and extends Serre duality on projective varieties, is an outstanding example of this phenomenon.

Local cohomology is connected to differentials in another way: the only known algorithms for computing local cohomology in characteristic zero employ rings of differential operators. This connects the subject with the study of Weyl algebras and holonomic modules. On the other hand, the combinatorics of local cohomology in the context of semigroups turns out to be the key to understanding certain systems of differential equations.

Prerequisites. The lectures are designed to be accessible to students with a first course in commutative algebra or algebraic geometry, and in point-set topology. We take for granted familiarity with algebraic constructions such as localizations, tensor products, exterior algebras, and topological notions such as homology and fundamental groups. Some material is reviewed in the lectures, such as dimension theory for commutative rings and Čech cohomology from topology. The main body of the text assumes knowledge of the structure theory of injective modules and resolutions; these topics are often omitted from introductory courses, so they are treated in the Appendix.

Local cohomology is best understood with a mix of algebraic and geometric perspectives. However, while prior exposure to algebraic geometry and sheaf theory is helpful, it is not strictly necessary for reading this book. The same is true of homological algebra: although we assume some comfort with categories and functors, the rest can be picked up along the way either from references provided, or from the twenty-four lectures themselves. For example, concepts such as resolutions, limits, and derived functors are covered as part and parcel of local cohomology.

Suggested reading plan. This book could be used as a text for a graduate course; in fact, the exposition is directly based on twenty-four hours of lectures in a summer school at Snowbird (see the Preface). That being said, it is unlikely that a semester-long course would cover all of the topics; indeed, no single one of us would choose to cover all the material, were we to teach a course based on this book. For this reason, we outline possible choices of material to be covered in, say, a semester-long course on local cohomology.

Lectures 1, 2, 3, 6, 7, 8, and 11 are fundamental, covering the geometry, sheaf theory, and homological algebra leading to the definition and alternative characterizations of local cohomology. Many readers will have seen enough of direct and inverse limits to warrant skimming Lecture 4 on their first pass, and referring back to it when necessary.

A course focusing on commutative algebra could include also Lectures 9, 10, 12, and 13. An in-depth treatment in the same direction would follow up with Lectures 14, 15, 18, 21, and 22.

For those interested mainly in the algebraic geometry aspects, Lectures 12, 13, and 18 would be of interest, while Lectures 18 and 19 are intended to describe connections to topology.

For applications to combinatorics, we recommend that the core material be followed up with Lectures 5, 16, 20, and 24, although Lecture 24 also draws on Lectures 17 and 23. Much of the combinatorial material—particularly the polyhedral parts—needs little more than linear algebra and some simplicial topology.

From a computational perspective, Lectures 5, 17, and 23 give a quick treatment of Gröbner bases and related algorithms. These lectures can also serve as an introduction to the theory of Weyl algebras and D-modules.

A feature that should make the book more appealing as a text is that there are exercises peppered throughout. Some are routine verifications of facts used later, some are routine verifications of facts not used later, and others are not routine. None are open problems, as far as we know. To impart a more comprehensive feel for the depth and breadth of the subject, we occasionally include landmark theorems with references but no proof. Results whose proofs are omitted are identified by the end-of-proof symbol $\square$ at the conclusion of the statement.

There are a number of topics that we have not discussed: Grothendieck's parafactoriality theorem, which was at the origins of local cohomology; Castelnuovo-Mumford regularity; the contributions of Lipman and others to the theory of residues; vanishing theorems of Huneke and Lyubeznik, and their recent work on local cohomology of the absolute integral closure. Among the applications, a noteworthy absence is the use of local cohomology by Benson, Carlson, Dwyer, Greenlees, Rickard, and others in representation theory and algebraic topology. Moreover, local cohomology remains a topic of active research, with new applications and new points of view. There have been a number of spectacular developments in the two years that it has taken us to complete this book. In this sense, the book is already dated.

Acknowledgements. It is a pleasure to thank the participants of the Snowbird summer school who, individually and collectively, made for a lively and engaging event. We are grateful to them for their comments, criticisms, and suggestions for improving the notes. Special thanks are due to Manoj Kummini for enthusiastically reading several versions of these lectures.

We learned this material from our teachers and collaborators: Lucho Avramov, Ragnar-Olaf Buchweitz, Sankar Dutta, Bill Dwyer, David Eisenbud, Hans-Bjørn Foxby, John Greenlees, Phil Griffith, Robin Hartshorne, David Helm, Mel Hochster, Craig Huneke, Joe Lipman, Gennady Lyubeznik, Tom Marley, Laura Matusevich, Arthur Ogus, Paul Roberts, Rodney Sharp, Karen Smith, Bernd Sturmfels, Irena Swanson, Kei-ichi Watanabe, and Roger Wiegand. They will recognize their influence—points of view, examples, proofs—at various places in the text. We take this opportunity to express our deep gratitude to them.

Sergei Gelfand, at the AMS, encouraged us to develop the lecture notes into a graduate text. It has been a pleasure to work with him during this process, and we thank him for his support; it is a relief that we no longer have to hide from him at various AMS meetings. We also thank Natalya Pluzhnikov, production editor at AMS, for her expert assistance.

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Lecture 1

Basic Notions

This lecture provides a summary of basic notions concerning *algebraic sets*, i.e., solution sets of polynomial equations. We will discuss the notion of dimension of an algebraic set and review the required results from commutative algebra along the way.

Throughout this lecture, $\mathbb{K}$ will denote a field. The rings considered will be commutative and with an identity element.

1. Algebraic sets

Definition 1.1. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring in n variables over a field $\mathbb{K}$, and consider polynomials $f_1, \ldots, f_m \in R$. Their zero set

$$\{(\alpha_1, \ldots, \alpha_n) \in \mathbb{K}^n \mid f_i(\alpha_1, \ldots, \alpha_n) = 0 \text{ for } 1 \leqslant i \leqslant m\}$$

is an *algebraic set* in $\mathbb{K}^n$, denoted $\operatorname{Var}(f_1, \ldots, f_m)$. These are our basic objects of study, and they include many familiar examples.

Example 1.2. If $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$ are homogeneous linear polynomials, their zero set is a vector subspace of $\mathbb{K}^n$. If V and W are vector subspaces of $\mathbb{K}^n$, then we have the following inequality:

$$\operatorname{rank}_{\mathbb{K}}(V \cap W) \geqslant \operatorname{rank}_{\mathbb{K}} V + \operatorname{rank}_{\mathbb{K}} W - n\,.$$

One way to prove this inequality is by using the exact sequence

$$0 \longrightarrow V \cap W \xrightarrow{\ \alpha\ } V \oplus W \xrightarrow{\ \beta\ } V + W \longrightarrow 0$$

where $\alpha(u) = (u, u)$ and $\beta(v, w) = v - w$. Then

$$\begin{aligned}\operatorname{rank}_{\mathbb{K}}(V \cap W) &= \operatorname{rank}_{\mathbb{K}}(V \oplus W) - \operatorname{rank}_{\mathbb{K}}(V + W)\\ &\geqslant \operatorname{rank}_{\mathbb{K}} V + \operatorname{rank}_{\mathbb{K}} W - n\,.\end{aligned}$$

Example 1.3. A *hypersurface* is the zero set of one equation. The unit circle in $\mathbb{R}^2$ is a hypersurface—it is the zero set of the polynomial $x^2 + y^2 - 1$.

Example 1.4. If $f \in \mathbb{K}[x_1, \dots, x_n]$ is a homogeneous polynomial of degree d, then $f(\alpha_1, \dots, \alpha_n) = 0$ implies that

$$f(c\alpha_1, \dots, c\alpha_n) = c^d f(\alpha_1, \dots, \alpha_n) = 0 \qquad \text{for all } c \in \mathbb{K}\,.$$

Hence if an algebraic set $V \subseteq \mathbb{K}^n$ is the zero set of homogeneous polynomials, then, for all $(\alpha_1, \dots, \alpha_n) \in V$ and $c \in \mathbb{K}$, we have $(c\alpha_1, \dots, c\alpha_n) \in V$. In this case, the algebraic set V is said to be a *cone*.

Example 1.5. Fix integers $m, n \geqslant 2$ and an integer $t \leqslant \min\{m, n\}$. Let V be the set of all $m \times n$ matrices over $\mathbb{K}$ which have rank less than t. A matrix has rank less than t if and only if its size t minors (i.e., the determinants of $t \times t$ submatrices) all equal zero. Take

$$R = \mathbb{K}[x_{ij} \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n]\,,$$

which is a polynomial ring in mn variables arranged as an $m \times n$ matrix. Then V is the zero set of the $\binom{m}{t}\binom{n}{t}$ polynomials arising as the $t \times t$ minors of the matrix (x_{ij}). Hence V is an algebraic set in $\mathbb{K}^{mn}$.

Exercise 1.6. This is taken from [**95**]. Let $\mathbb{K}$ be a finite field.

(1) For every point $p \in \mathbb{K}^n$, construct a polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$ such that $f(p) = 1$ and $f(q) = 0$ for all points $q \in \mathbb{K}^n \smallsetminus \{p\}$.
(2) Given a function $g \colon \mathbb{K}^n \longrightarrow \mathbb{K}$, show that there is a polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$ with $f(p) = g(p)$ for all $p \in \mathbb{K}^n$.
(3) Prove that any subset of $\mathbb{K}^n$ is the zero set of a single polynomial.

Remark 1.7. One may ask: is the zero set of an infinite family of polynomials also the zero set of a finite family? To answer this, recall that a ring is Noetherian if its ideals are finitely generated, and that a polynomial ring R is Noetherian by Hilbert's basis theorem; see [**6**, Theorem 7.5]. Let $\mathfrak{a} \subseteq R$ be the ideal generated by a possibly infinite family of polynomials $\{g_\lambda\}$. The zero set of $\{g_\lambda\}$ is the same as the zero set of all elements in the ideal $\mathfrak{a}$. The ideal $\mathfrak{a}$ is finitely generated, say $\mathfrak{a} = (f_1, \dots, f_m)$, so the zero set of $\{g_\lambda\}$ is precisely the zero set of $f_1, \dots, f_m$.

Given a set of polynomials $f_1, \dots, f_m$ generating an ideal $\mathfrak{a} \subseteq R$, their zero set $\mathrm{Var}(f_1, \dots, f_m)$ equals $\mathrm{Var}(\mathfrak{a})$. Note that $\mathrm{Var}(f)$ equals $\mathrm{Var}(f^k)$ for any integer $k \geqslant 1$. Hence if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals with the same radical, then $\mathrm{Var}(\mathfrak{a}) = \mathrm{Var}(\mathfrak{b})$. A theorem of Hilbert states that over an algebraically closed field, the converse is true as well.

Theorem 1.8 (Hilbert's Nullstellensatz)**.** *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over an algebraically closed field $\mathbb{K}$. If $\operatorname{Var}(\mathfrak{a}) = \operatorname{Var}(\mathfrak{b})$ for ideals $\mathfrak{a}, \mathfrak{b} \subseteq R$, then $\operatorname{rad}\mathfrak{a} = \operatorname{rad}\mathfrak{b}$.*

Consequently the map $\mathfrak{a} \longmapsto \operatorname{Var}(\mathfrak{a})$ is a containment-reversing bijection between radical ideals of $\mathbb{K}[x_1, \ldots, x_n]$ and algebraic sets in $\mathbb{K}^n$. □

For a proof, solve [**6**, Problem 7.14]. The following corollary, also referred to as the Nullstellensatz, tells us when polynomial equations have a solution.

Corollary 1.9. *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over an algebraically closed field $\mathbb{K}$. Then polynomials $f_1, \ldots, f_m \in R$ have a common zero if and only if $(f_1, \ldots, f_m) \neq R$.*

Proof. Since $\operatorname{Var}(R) = \varnothing$, an ideal $\mathfrak{a}$ of R satisfies $\operatorname{Var}(\mathfrak{a}) = \varnothing$ if and only if $\operatorname{rad}\mathfrak{a} = R$, which happens if and only if $\mathfrak{a} = R$. □

Corollary 1.10. *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over an algebraically closed field $\mathbb{K}$. Then the maximal ideals of R are the ideals*

$$(x_1 - \alpha_1, \ldots, x_n - \alpha_n), \qquad \text{where } \alpha_i \in \mathbb{K}.$$

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. Then $\mathfrak{m} \neq R$, so Corollary 1.9 implies that $\operatorname{Var}(\mathfrak{m})$ contains a point $(\alpha_1, \ldots, \alpha_n)$ of $\mathbb{K}^n$. But then

$$\operatorname{Var}(x_1 - \alpha_1, \ldots, x_n - \alpha_n) \subseteq \operatorname{Var}(\mathfrak{m}),$$

so $\operatorname{rad}\mathfrak{m} \subseteq \operatorname{rad}(x_1 - \alpha_1, \ldots, x_n - \alpha_n)$. Since $\mathfrak{m}$ and $(x_1 - \alpha_1, \ldots, x_n - \alpha_n)$ are maximal ideals, it follows that they must be equal. □

Exercise 1.11. This is also taken from [**95**]. Prove that if $\mathbb{K}$ is not algebraically closed, any algebraic set in $\mathbb{K}^n$ is the zero set of a single polynomial.

2. Krull dimension of a ring

We would like a notion of dimension for algebraic sets which agrees with the vector space dimension if the algebraic set is a vector space and gives a suitable generalization of the inequality in Example 1.2. The situation is certainly more complicated than with vector spaces. For example, not all points of an algebraic set have similar neighborhoods—the algebraic set defined by $xy = 0$ and $xz = 0$ is the union of a line and a plane.

A good theory of dimension requires some notions from commutative algebra, which we now proceed to recall.

Definition 1.12. Let R be a ring. The *spectrum* of R, denoted $\operatorname{Spec} R$, is the set of prime ideals of R with the *Zariski topology*, which is the topology where the closed sets are

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subseteq \mathfrak{p}\} \qquad \text{for ideals } \mathfrak{a} \subseteq R.$$

It is easily verified that this is indeed a topology: the empty set is both open and closed, an intersection of closed sets is closed since

$$\bigcap_{\lambda} V(\mathfrak{a}_\lambda) = V(\bigcup_{\lambda} \mathfrak{a}_\lambda)\,,$$

and the union of two closed sets is closed since

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})\,.$$

The *height* of a prime ideal $\mathfrak{p}$, denoted $\operatorname{height}\mathfrak{p}$, is the supremum of integers t such that there exists a chain of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_2 \supsetneq \cdots \supsetneq \mathfrak{p}_t\,, \qquad \text{where } \mathfrak{p}_i \in \operatorname{Spec} R\,.$$

The height of an arbitrary ideal $\mathfrak{a} \subseteq R$ is

$$\operatorname{height}\mathfrak{a} = \inf\{\operatorname{height}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R, \mathfrak{a} \subseteq \mathfrak{p}\}\,.$$

The *Krull dimension* of R is

$$\dim R = \sup\{\operatorname{height}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}\,.$$

Note that for every prime ideal $\mathfrak{p}$ of R we have $\operatorname{height}\mathfrak{p} = \dim R_\mathfrak{p}$, where $R_\mathfrak{p}$ denotes the localization of R at the multiplicative set $R \smallsetminus \mathfrak{p}$.

Example 1.13. The prime ideals of $\mathbb{Z}$ are (0) and (p) for prime integers p. Consequently the longest chains of prime ideals in $\operatorname{Spec}\mathbb{Z}$ are those of the form $(p) \supsetneq (0)$, and so $\dim \mathbb{Z} = 1$.

Exercise 1.14. If a principal ideal domain is not a field, prove that it has dimension one.

Exercise 1.15. What is the dimension of $\mathbb{Z}[x]$?

The result below is contained in [**6**, Corollary 11.16].

Theorem 1.16 (Krull's height theorem)**.** *Let R be a Noetherian ring. If an ideal $\mathfrak{a} \subsetneq R$ is generated by n elements, then each minimal prime of $\mathfrak{a}$ has height at most n. In particular, every ideal $\mathfrak{a} \subsetneq R$ has finite height.* □

A special case is Krull's *principal ideal theorem*: every proper principal ideal of a Noetherian ring has height at most one.

While it is true that every prime ideal in a Noetherian ring has finite height, the Krull dimension of a ring is the *supremum* of the heights of its prime ideals, and this supremum may be infinite; see [**6**, Problem 11.4] for an example due to Nagata. Over *local* rings, that is to say Noetherian rings with a unique maximal ideal, this problem does not arise.

The following theorem contains various characterizations of dimension for local rings; see [**6**, Theorem 11.14] for a proof.

Theorem 1.17. *Let $(R, \mathfrak{m})$ be a local ring, and d a nonnegative integer. The following conditions are equivalent:*

(1) $\dim R \leqslant d$;

(2) *there exists an $\mathfrak{m}$-primary ideal generated by d elements;*

(3) *for $t \gg 0$, the length function $\ell(R/\mathfrak{m}^t)$ agrees with a polynomial in t of degree at most d.* □

Definition 1.18. Let $(R, \mathfrak{m})$ be a local ring of dimension d. Elements $x_1, \ldots, x_d$ are a *system of parameters* for R if $\operatorname{rad}(x_1, \ldots, x_d) = \mathfrak{m}$.

Theorem 1.17 implies that every local ring has a system of parameters.

Example 1.19. Let $\mathbb{K}$ be a field, and take

$$R = \mathbb{K}[x, y, z]_{(x,y,z)}/(xy, xz)\,.$$

Then R has a chain of prime ideals $(x) \subsetneq (x, y) \subsetneq (x, y, z)$, so $\dim R \geqslant 2$. On the other hand, the maximal ideal (x, y, z) is the radical of the 2-generated ideal $(y, x - z)$, implying that $\dim R \leqslant 2$. It follows that $\dim R = 2$ and that $y, x - z$ is a system of parameters for R.

Exercise 1.20. Let $\mathbb{K}$ be a field. For the following local rings $(R, \mathfrak{m})$, compute $\dim R$ by examining $\ell(R/\mathfrak{m}^t)$ for $t \gg 0$. In each case, find a system of parameters for R and a chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d\,, \qquad \text{where } d = \dim R\,.$$

(1) $R = \mathbb{K}[x^2, x^3]_{(x^2,x^3)}$.

(2) $R = \mathbb{K}[x^2, xy, y^2]_{(x^2,xy,y^2)}$.

(3) $R = \mathbb{K}[w, x, y, z]_{(w,x,y,z)}/(wx - yz)$.

(4) $R = \mathbb{Z}_{(p)}$ where p is a prime integer.

If a finitely generated algebra over a field is a domain, then its dimension may be computed as the transcendence degree of a field extension:

Theorem 1.21. *If a finitely generated $\mathbb{K}$-algebra R is a domain, then*

$$\dim R = \operatorname{tr.deg}_{\mathbb{K}} \operatorname{Frac}(R)\,,$$

where $\operatorname{Frac}(R)$ *is the fraction field of R. Moreover, any chain of primes in* $\operatorname{Spec} R$ *can be extended to a chain of length* $\dim R$ *which has no repeated terms. Hence* $\dim R_{\mathfrak{m}} = \dim R$ *for every maximal ideal $\mathfrak{m}$ of R, and*

$$\operatorname{height} \mathfrak{p} + \dim R/\mathfrak{p} = \dim R \qquad \textit{for all } \mathfrak{p} \in \operatorname{Spec} R\,.$$ □

When $\mathbb{K}$ is algebraically closed, this is [**6**, Corollary 11.27]; for the general case see [**115**, Theorem 5.6, Exercise 5.1].

Example 1.22. A polynomial ring $R = \mathbb{K}[x_1, \dots, x_n]$ over a field $\mathbb{K}$ has dimension n since $\operatorname{tr.deg}_{\mathbb{K}} \mathbb{K}(x_1, \dots, x_n) = n$. Now let $f \in R$ be a nonzero polynomial with irreducible factors $f_1, \dots, f_k$. The minimal primes of the ideal (f) are the ideals $(f_1), \dots, (f_k)$. Each of these has height 1 by Theorem 1.16. It follows that $\dim R/(f) = n - 1$.

Remark 1.23. We say that a ring R is $\mathbb{N}$-*graded* if $R = \bigoplus_{i \in \mathbb{N}} R_i$ as an Abelian group, and

$$R_i R_j \subseteq R_{i+j} \qquad \text{for all } i, j \in \mathbb{N}\,.$$

When R_0 is a field, the ideal $\mathfrak{m} = \bigoplus_{i \geqslant 1} R_i$ is the unique *homogeneous maximal ideal* of R.

Assume that $R_0 = \mathbb{K}$ and that R is a finitely generated $\mathbb{K}$-algebra. Let M be a finitely generated $\mathbb{Z}$-graded R-module, i.e., $M = \bigoplus_{i \in \mathbb{Z}} M_i$ as an Abelian group, and $R_i M_j \subseteq M_{i+j}$ for all $i \geqslant 0$ and $j \in \mathbb{Z}$. The *Hilbert-Poincaré series* of M is the generating function for $\operatorname{rank}_{\mathbb{K}} M_i$, i.e., the series

$$P(M, t) = \sum_{i \in \mathbb{Z}} (\operatorname{rank}_{\mathbb{K}} M_i) t^i \in \mathbb{Z}[[t]][t^{-1}]\,.$$

It turns out that $P(M, t)$ is a rational function of the form

$$\frac{f(t)}{\prod_j (1 - t^{d_j})} \qquad \text{where } f(t) \in \mathbb{Z}[t]\,,$$

[**6**, Theorem 11.1], and that the dimension of R is precisely the order of the pole of $P(R, t)$ at $t = 1$. We also discuss $P(M, t)$ in Lecture 5.

Example 1.24. Let R be the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ where $\mathbb{K}$ is a field. The vector space dimension of R_i is the number of monomials of degree i, which is the binomial coefficient $\binom{i+n-1}{i}$. Hence

$$P(R, t) = \sum_{i \geqslant 0} \binom{i + n - 1}{i} t^i = \frac{1}{(1 - t)^n}\,.$$

Exercise 1.25. Compute $P(R, t)$ in the following cases:

(1) $R = \mathbb{K}[wx, wy, zx, zy]$ where each of wx, wy, zx, zy has degree 1.

(2) $R = \mathbb{K}[x^2, x^3]$ where the grading is induced by $\deg x = 1$.

(3) $R = \mathbb{K}[x^4, x^3y, xy^3, y^4]$ where $\deg x^4 = 1 = \deg y^4$.

(4) $R = \mathbb{K}[x, y, z]/(x^3 + y^3 + z^3)$ where each of x, y, z has degree 1.

3. Dimension of an algebraic set

For simplicity, in this section we assume that $\mathbb{K}$ is algebraically closed.

Definition 1.26. An algebraic set V is *irreducible* if it is not the union of two algebraic sets which are proper subsets of V; equivalently if each pair of nonempty open sets intersect.

Exercise 1.27. Prove that an algebraic set $V \subseteq \mathbb{K}^n$ is irreducible if and only if $V = \operatorname{Var}(\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ of $\mathbb{K}[x_1, \dots, x_n]$.

Remark 1.28. Every algebraic set in $\mathbb{K}^n$ can be written uniquely as a finite union of irreducible algebraic sets, where there are no redundant terms in the union. Let $V = \operatorname{Var}(\mathfrak{a})$ where $\mathfrak{a}$ is a radical ideal. Then we can write

$$\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t \qquad \text{for } \mathfrak{p}_i \in \operatorname{Spec} \mathbb{K}[x_1, \dots, x_n]\,,$$

where this intersection is irredundant. This gives us

$$V = \operatorname{Var}(\mathfrak{p}_1) \cup \cdots \cup \operatorname{Var}(\mathfrak{p}_t),$$

and $\operatorname{Var}(\mathfrak{p}_i)$ are precisely the irreducible components of V. Note that the map $\mathfrak{a} \longmapsto \operatorname{Var}(\mathfrak{a})$ gives us the following bijections:

$$\begin{array}{lcl} \text{radical ideals of } \mathbb{K}[x_1, \dots, x_n] & \longleftrightarrow & \text{algebraic sets in } \mathbb{K}^n, \\ \text{prime ideals of } \mathbb{K}[x_1, \dots, x_n] & \longleftrightarrow & \text{irreducible algebraic sets in } \mathbb{K}^n, \\ \text{maximal ideals of } \mathbb{K}[x_1, \dots, x_n] & \longleftrightarrow & \text{points of } \mathbb{K}^n. \end{array}$$

Definition 1.29. Let $V = \operatorname{Var}(\mathfrak{a})$ be an algebraic set defined by a radical ideal $\mathfrak{a} \subseteq \mathbb{K}[x_1, \dots, x_n]$. The *coordinate ring* of V, denoted $\mathbb{K}[V]$, is the ring $\mathbb{K}[x_1, \dots, x_n]/\mathfrak{a}$.

The points of V correspond to maximal ideals of $\mathbb{K}[x_1, \dots, x_n]$ containing $\mathfrak{a}$, and hence to maximal ideals of $\mathbb{K}[V]$. Let $p \in V$ be a point corresponding to a maximal ideal $\mathfrak{m} \subset \mathbb{K}[V]$. The *local ring of V at p* is the ring $\mathbb{K}[V]_{\mathfrak{m}}$.

Definition 1.30. The *dimension* of an irreducible algebraic set V is the Krull dimension of its coordinate ring $\mathbb{K}[V]$. For a (possibly reducible) algebraic set V, we define

$$\dim V = \sup\{\dim V_i \mid V_i \text{ is an irreducible component of } V\}\,.$$

Example 1.31. The irreducible components of the algebraic set

$$V = \operatorname{Var}(xy, xz) = \operatorname{Var}(x) \cup \operatorname{Var}(y, z) \qquad \text{in } \mathbb{K}^3$$

are the plane $x = 0$ and the line $y = 0 = z$. The dimension of the plane is $\dim \mathbb{K}[x, y, z]/(x) = 2$, and that of the line is $\dim \mathbb{K}[x, y, z]/(y, z) = 1$, so the algebraic set V has dimension two.

Example 1.32. Let V be a vector subspace of $\mathbb{K}^n$, with $\operatorname{rank}_{\mathbb{K}} V = d$. After a linear change of variables, we may assume that the $n - d$ homogeneous linear polynomials defining the algebraic set V are a subset of the variables of the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$, say $V = \operatorname{Var}(x_1, \dots, x_{n-d})$. Hence

$$\dim V = \dim \mathbb{K}[x_1, \dots, x_n]/(x_1, \dots, x_{n-d}) = d = \operatorname{rank}_{\mathbb{K}} V\,.$$

The dimension of an algebraic set has several desirable properties:

Theorem 1.33. *Let $\mathbb{K}$ be an algebraically closed field, and let V and W be algebraic sets in $\mathbb{K}^n$.*

(1) *If V is a vector space over $\mathbb{K}$, then* $\dim V$ *equals the vector space dimension,* $\operatorname{rank}_{\mathbb{K}} V$.

(2) *Assume that W is irreducible of dimension d, and that V is defined by m polynomials. Then every nonempty irreducible component of $V \cap W$ has dimension at least $d - m$.*

(3) *The dimension of $V \cap W$ is at least* $\dim V + \dim W - n$.

(4) *If $\mathbb{K}$ is the field of complex numbers, then* $\dim V$ *is half the dimension of V as a real topological space.*

Note that (3) generalizes the rank inequality of Example 1.2.

Sketch of proof. (1) was observed in Example 1.32.

(2) Let $\mathfrak{p}$ be the prime of $R = \mathbb{K}[x_1, \dots, x_n]$ such that $W = \operatorname{Var}(\mathfrak{p})$, and let $V = \operatorname{Var}(f_1, \dots, f_m)$. An irreducible component of $V \cap W$ corresponds to a minimal prime $\mathfrak{q}$ of $\mathfrak{p} + (f_1, \dots, f_m)$. But then $\mathfrak{q}/\mathfrak{p}$ is a minimal prime of $(f_1, \dots, f_m)R/\mathfrak{p}$, so height $\mathfrak{q}/\mathfrak{p} \leqslant m$ by Theorem 1.16. Hence

$$\dim R/\mathfrak{q} = \dim R/\mathfrak{p} - \operatorname{height} \mathfrak{q}/\mathfrak{p} \geqslant d - m\,.$$

(3) Replacing V and W by irreducible components, we may assume that $V = \operatorname{Var}(\mathfrak{p})$ and $W = \operatorname{Var}(\mathfrak{q})$ for prime ideals $\mathfrak{p}, \mathfrak{q} \subseteq \mathbb{K}[x_1, \dots, x_n]$. Let $\mathfrak{q}' \subseteq \mathbb{K}[x'_1, \dots, x'_n]$ be the ideal obtained from $\mathfrak{q}$ by replacing each x_i with a new variable x'_i. We may regard $V \times W$ as an algebraic set in $\mathbb{K}^n \times \mathbb{K}^n = \mathbb{K}^{2n}$, i.e., as the zero set of the ideal

$$\mathfrak{p} + \mathfrak{q}' \subseteq \mathbb{K}[x_1, \dots, x_n, x'_1, \dots, x'_n] = S\,.$$

Since $\mathbb{K}$ is algebraically closed, $\mathfrak{p} + \mathfrak{q}'$ is prime by Exercise 15.15(4), so $V \times W$ is irreducible of dimension $\dim V + \dim W$. Let $\mathfrak{d} = (x_1 - x'_1, \dots, x_n - x'_n)$, in which case $\Delta = \operatorname{Var}(\mathfrak{d})$ is the diagonal in $\mathbb{K}^n \times \mathbb{K}^n$. Then

$$\mathbb{K}[V \cap W] = \frac{\mathbb{K}[x_1, \dots, x_n]}{\mathfrak{p} + \mathfrak{q}} \cong \frac{\mathbb{K}[x_1, \dots, x_n, x'_1, \dots, x'_n]}{\mathfrak{p} + \mathfrak{q}' + \mathfrak{d}} = \mathbb{K}[(V \times W) \cap \Delta]\,.$$

The ideal $\mathfrak{d}$ is generated by n elements, so Krull's height theorem yields

$$\begin{aligned}\dim(V \cap W) &= \dim S/(\mathfrak{p} + \mathfrak{q}' + \mathfrak{d}) \\ &\geqslant \dim S/(\mathfrak{p} + \mathfrak{q}') - n = \dim V + \dim W - n\,.\end{aligned}$$

(4) We skip the proof, but point out that an irreducible complex algebraic set of dimension d is the union of a $\mathbb{C}$-manifold of dimension d and an algebraic set of lower dimension; see [**32**, Chapter 8] for a discussion of topological dimension in this context. □

4. An extended example

Example 1.34. Consider the algebraic set V of 2×3 complex matrices of rank less than 2. Take the polynomial ring $R = \mathbb{C}[u, v, w, x, y, z]$. Then $V = \mathrm{Var}(\mathfrak{a})$, where $\mathfrak{a}$ is the ideal generated by the polynomials

$$\Delta_1 = vz - wy, \quad \Delta_2 = wx - uz, \quad \Delta_3 = uy - vx\,.$$

Exercise 1.36 below shows that $\mathfrak{a}$ is a prime ideal. We compute $\dim V$ from four different points of view.

As a topological space: The set of rank-one matrices is the union of

$$\left\{ \begin{bmatrix} a & b & c \\ ad & bd & cd \end{bmatrix} \mid (a, b, c) \in \mathbb{C}^3 \smallsetminus \{\mathbf{0}\}, d \in \mathbb{C} \right\}$$

and

$$\left\{ \begin{bmatrix} ad & bd & cd \\ a & b & c \end{bmatrix} \mid (a, b, c) \in \mathbb{C}^3 \smallsetminus \{\mathbf{0}\}, d \in \mathbb{C} \right\},$$

each of which is a copy of $(\mathbb{C}^3 \smallsetminus \{\mathbf{0}\}) \times \mathbb{C}$ and hence has dimension 8 as a topological space. The set V is the union of these and a point corresponding to the zero matrix. Hence V has topological dimension 8, so $\dim V = 4$.

Using transcendence degree: The ideal $\mathfrak{a}$ is prime, so $\dim R/\mathfrak{a}$ can be computed as $\operatorname{tr.deg}_{\mathbb{C}} \mathbb{L}$, where $\mathbb{L}$ is the fraction field of $R/\mathfrak{a}$. In the field $\mathbb{L}$ we have $v = uy/x$ and $w = uz/x$, so

$$\mathbb{L} = \mathbb{C}(u, x, y, z)$$

where u, x, y, z are algebraically independent over $\mathbb{C}$. Hence Theorem 1.21 implies that $\dim R/\mathfrak{a} = 4$.

By finding a system of parameters: Let $\mathfrak{m} = (u, v, w, x, y, z)$. In the polynomial ring R we have a chain of prime ideals

$$\mathfrak{a} \subsetneq (u, x, vz - wy) \subsetneq (u, v, x, y) \subsetneq (u, v, w, x, y) \subsetneq \mathfrak{m}\,,$$

which gives a chain of prime ideals in $R/\mathfrak{a}$ showing that $\dim R/\mathfrak{a} \geqslant 4$. Consider the four elements $u, v - x, w - y, z \in R$. Then the ideal

$$\mathfrak{a} + (u, v - x, w - y, z) = (u, v - x, w - y, z, x^2, xy, y^2)$$

contains $\mathfrak{m}^2$ and hence it is $\mathfrak{m}$-primary. This means that the image of $\mathfrak{m}$ in $R/\mathfrak{a}$ is the radical of a 4-generated ideal, so $\dim R/\mathfrak{a} \leqslant 4$. It follows that $\dim R/\mathfrak{a} = 4$ and that the images of $u, v - x, w - y, z$ in $R/\mathfrak{a}$ are a system of parameters for $R/\mathfrak{a}$.

From the Hilbert-Poincaré series: Exercise 1.36 below shows that $R/\mathfrak{a}$ is isomorphic to the ring $S = \mathbb{C}[ar, br, cr, as, bs, cs]$, under a degree preserving isomorphism, where each of the monomials ar, br, cr, as, bs, cs is assigned degree 1. The rank over $\mathbb{K}$ of S_i is the product of the number of monomials of degree i in a, b, c with the number of monomials of degree i in r, s, i.e., $\binom{i+2}{2}\binom{i+1}{1}$. It follows that

$$P(S,t) = \sum_{i \geqslant 0} \binom{i+2}{2}\binom{i+1}{1} t^i = \frac{1+2t}{(1-t)^4}\,,$$

which has a pole of order 4 at $t = 1$. Hence $\dim S = 4$.

Exercise 1.35. Let R and S be algebras over a field $\mathbb{K}$, and $\varphi\colon R \longrightarrow S$ a surjective $\mathbb{K}$-algebra homomorphism. Let $\{s_j\}$ be a $\mathbb{K}$-vector space basis for S, and $r_j \in R$ elements with $\varphi(r_j) = s_j$ for all j. Let $\mathfrak{a}$ be an ideal contained in $\ker(\varphi)$. If every element of R is congruent to an element in the $\mathbb{K}$-span of $\{r_j\}$ modulo $\mathfrak{a}$, prove that $\mathfrak{a} = \ker(\varphi)$.

Exercise 1.36. Let $R = \mathbb{K}[u, v, w, x, y, z]$ and S the subring of the polynomial ring $\mathbb{K}[a, b, c, r, s]$ generated as a $\mathbb{K}$-algebra by ar, br, cr, as, bs, cs. Consider the $\mathbb{K}$-algebra homomorphism $\varphi\colon R \longrightarrow S$ where

$$\begin{aligned} \varphi(u) &= ar, & \varphi(v) &= br, & \varphi(w) &= cr,\\ \varphi(x) &= as, & \varphi(y) &= bs, & \varphi(z) &= cs. \end{aligned}$$

Prove that $\ker(\varphi) = (vz - wy,\ wx - uz,\ uy - vx)$, and conclude that this ideal is prime. This shows that

$$\mathbb{K}[u, v, w, x, y, z]/(vz - wy, wx - uz, uy - vx) \cong \mathbb{K}[ar, br, cr, as, bs, cs]\,.$$

5. Tangent spaces and regular rings

Working over an arbitrary field, one can consider partial derivatives of polynomial functions with respect to the variables, e.g.,

$$\frac{\partial}{\partial x}(x^3 + y^3 + z^3 + xyz) = 3x^2 + yz\,.$$

Definition 1.37. Let $V = \mathrm{Var}(\mathfrak{a}) \subseteq \mathbb{K}^n$ be an algebraic set. The *tangent space* to V at a point $p = (\alpha_1, \dots, \alpha_n)$ is the algebraic set $T_p(V) \subseteq \mathbb{K}^n$ which is the solution set of the linear equations

$$\sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_p (x_i - \alpha_i) = 0 \qquad \text{for } f \in \mathfrak{a}\,.$$

An easy application of the product rule shows that to obtain the defining equations for $T_p(V)$, it is sufficient to consider the linear equations arising from a generating set for $\mathfrak{a}$.

For the rest of this section, we will work over an algebraically closed field. Let V be an irreducible algebraic set. At each point p of V, one has $\dim T_p(V) \geqslant \dim V$; see Theorem 1.40 below. When equality holds, V is said to be *nonsingular* or *smooth* at p.

Example 1.38. The circle $\operatorname{Var}(x^2+y^2-1)$ is smooth at all points so long as the field does not have characteristic 2.

Take a point (α, β) on the cusp $\operatorname{Var}(x^2-y^3)$. Then the tangent space at this point is the space defined by the polynomial equation

$$2\alpha(x-\alpha) - 3\beta^2(y-\beta) = 0\,.$$

This is a line in $\mathbb{K}^2$ if $(\alpha, \beta) \neq (0,0)$, whereas the tangent space to the cusp at the origin $(0,0)$ is all of $\mathbb{K}^2$. Hence the cusp has a unique singular point, namely the origin.

Definition 1.39. A local ring $(R, \mathfrak{m})$ of dimension d is a *regular local ring* if its maximal ideal $\mathfrak{m}$ can be generated by d elements.

Theorem 1.40. *Let V be an irreducible algebraic set over an algebraically closed field, and take a point $p \in V$. Then the dimension of $T_p(V)$ is the least number of generators of the maximal ideal of the local ring of V at p.*

Hence $\dim T_p(V) \geqslant \dim V$, *and p is a nonsingular point of V if and only if the local ring of V at p is a regular local ring.*

Proof if V is a hypersurface. Let $V = \operatorname{Var}(f)$ for $f \in \mathbb{K}[x_1, \dots, x_n]$. After a linear change of coordinates, we may assume that $p = (0, \dots, 0)$. Then $T_p(V)$ is the algebraic set in $\mathbb{K}^n$ defined by the linear equation

$$\sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_p x_i = 0\,,$$

so $\dim T_p(V) \geqslant n-1$ and V is smooth at p if and only if some partial derivative $\partial f/\partial x_i$ does not vanish at p, i.e., if and only if $f \notin (x_1, \dots, x_n)^2$. The local ring of V at p is

$$\mathbb{K}[x_1, \dots, x_n]_{(x_1,\dots,x_n)}/(f)\,,$$

and the minimal number of generators of its maximal ideal $\mathfrak{m}$ is the vector space dimension of

$$\mathfrak{m}/\mathfrak{m}^2 = (x_1, \dots, x_n)/(f + (x_1, \dots, x_n)^2)\,,$$

which equals $n-1$ precisely when $f \notin (x_1, \dots, x_n)^2$. $\square$

6. Dimension of a module

We end this lecture with a discussion of the dimension of a module, but not before sneaking in a lemma, one that richly deserves to be known as such:

> The essence of mathematics is proving theorems—and so, that is what mathematicians do: They prove theorems. But to tell the truth, what they really want to prove, once in their lifetime, is a *Lemma*, like the one by Fatou in analysis, the Lemma of Gauss in number theory, or the Burnside-Frobenius Lemma in combinatorics.
>
> Now what makes a mathematical statement a true Lemma? First, it should be applicable to a wide variety of instances, even seemingly unrelated problems. Secondly, the statement should, once you have seen it, be completely obvious. The reaction of the reader might well be one of faint envy: Why haven't I noticed this before? And thirdly, on an esthetic level, the Lemma—including its proof—should be beautiful!

–*from* [**2**, page 167].

The lemma we have in mind is, of course, Nakayama's lemma; look up [**122**, pages 212–213] for an interesting history.

Lemma 1.41 (Nakayama's lemma)**.** *If a finitely generated module M over a local ring $(R, \mathfrak{m})$ satisfies $\mathfrak{m}M = M$, then $M = 0$.*

Exercise 1.42. Let M be a finitely generated module over a local ring $(R, \mathfrak{m}, \mathbb{K})$. If $m_1, \ldots, m_n$ are elements of M whose images generate $M/\mathfrak{m}M$ as a $\mathbb{K}$-vector space, prove that $m_1, \ldots, m_n$ generate M as an R-module.

Remark 1.43. We will use $\nu_R(M)$ to denote the minimal number of generators of a finitely generated R-module M. It follows from the above exercise that over a local ring $(R, \mathfrak{m}, \mathbb{K})$, we have

$$\nu_R(M) = \operatorname{rank}_{\mathbb{K}} M/\mathfrak{m}M\,,$$

and that any minimal generating set for M has $\nu_R(M)$ elements.

Nakayama's lemma follows by taking φ to be identity in the following proposition, which is a module-theoretic version of the Cayley-Hamilton theorem; see [**6**, Proposition 2.4] or [**115**, Theorem 2.1] for a proof.

Proposition 1.44. *Let M be a finitely generated R-module. If φ is an endomorphism of M such that $\varphi(M) \subseteq \mathfrak{a}M$ for an ideal $\mathfrak{a}$ of R, then φ satisfies an equation $\varphi^n + a_1\varphi^{n-1} + \cdots + a_n = 0$ with $a_i \in \mathfrak{a}$.* $\square$

Exercise 1.45. Let M be a finitely generated R-module, and let r be an element of R. Prove that the ideals $\operatorname{ann}(M/rM)$ and $(r) + \operatorname{ann} M$ have the same radical.

Definition 1.46. Let M be a finitely generated module over a Noetherian ring R. Then the *dimension* of M, denoted $\dim M$, is $\dim(R/\operatorname{ann} M)$.

We close with a version of Krull's height theorem for modules:

Theorem 1.47. *Let M be a finitely generated module over a local ring $(R, \mathfrak{m})$. For each element r in R, one has an inequality*

$$\dim(M/rM) \geqslant \dim M - 1\,.$$

Proof. Let $n = \dim(M/rM)$ and take a system of parameters $x_1, \ldots, x_n$ for $R/\operatorname{ann}(M/rM)$. The ideal $(x_1, \ldots, x_n) + \operatorname{ann}(M/rM)$ is $\mathfrak{m}$-primary, so Exercise 1.45 implies that $(r, x_1, \ldots, x_n) + \operatorname{ann} M$ is $\mathfrak{m}$-primary, and hence that $\dim M \leqslant n + 1$. $\square$

Lecture 2

Cohomology

This lecture offers a first glimpse at sheaf theory, which involves an interplay of algebra, geometry, and topology. The goal is to introduce sheaves through examples, and to motivate their study by pointing out connections to analysis and to topology. In the process, there is little proof and much hand-waving. The reader may wish to revisit this lecture when equipped with the algebraic tools necessary to appreciate the principles outlined here.

The point of view adopted here is that sheaves are spaces of functions. This is the approach pioneered by the French school around Cartan and Leray. Functions are typically defined by local conditions, and it is of interest to determine global functions with prescribed local properties. For example, the Mittag-Leffler problem specifies the principal parts of a holomorphic function on a Riemann surface at finitely many points, and asks for the existence of a global holomorphic function with the given principal parts. Such local-to-global problems are often nontrivial to solve, and lead to the notion of sheaf cohomology.

There are many sources for reading on sheaves and their applications. For the algebraic geometry approach, we refer to Hartshorne's book [**61**]; for the differential geometric aspects, one can consult Godement [**47**] and Griffiths and Harris [**51**]. The required homological background can be found in Weibel [**161**], while Gelfand and Manin [**46**] and Iversen [**83**] show the workings of homological algebra in the context of sheaf theory. For the link between calculus and cohomology, we recommend [**14, 110**], and [**29, 88**] for connections with D-modules and singularity theory.

1. Sheaves

We assume that the reader is familiar with basic concepts of point set topology. Let us fix a space X with topology $\mathcal{T}_X$.

Example 2.1. While open and closed sets abound in some familiar spaces such as $\mathbb{R}^n$, a topology may be quite sparse. At one extreme case, the only sets in $\mathcal{T}_X$ are X and the empty set $\varnothing$. This scenario is known as the *trivial topology*. At the other extreme, all subsets of X are open, and hence all are closed as well. In that case, X is said to have the *discrete topology*.

Most topologies are somewhere between the two extremes considered in the preceding example. The case of interest to us is the spectrum of a commutative ring, with the Zariski topology, and typically this has fewer open sets than one is accustomed to in $\mathbb{R}^n$.

Example 2.2. Let $(X, \mathcal{T}_X)$ be the spectrum of the ring $R = \mathbb{C}[x]$. The points of X are prime ideals of R, which are the ideals $\{(x-c)\}_{c\in\mathbb{C}}$ together with the ideal (0). The closed sets in $\mathcal{T}_X$ are finite collections of ideals of the type $(x-c)$, and the set X. This topology is fairly coarse: for instance, it is not Hausdorff. (Prove this.)

Let F be a topological space. We attach to each open set U of X, the space of all continuous functions $C(U, F)$ from U to F. The following is the running example for this lecture.

Example 2.3. Let X be the unit circle $\mathbb{S}^1$ with topology inherited from the embedding in $\mathbb{R}^2$. Any proper open subset of $\mathbb{S}^1$ is the disjoint union of open connected arcs. Let F be $\mathbb{Z}$ with the discrete topology.

Let U be an open set of $\mathbb{S}^1$, and pick an element $f \in C(U, \mathbb{Z})$. For each $z \in \mathbb{Z}$, the preimages $f^{-1}(z)$ and $f^{-1}(\mathbb{Z} \smallsetminus \{z\})$ form a decomposition of U into disjoint open sets. Therefore, if U is connected, we deduce that $f(U) = \{z\}$ for some z, and hence $C(U, \mathbb{Z}) = \mathbb{Z}$.

In order to prepare for the definition to come, we revise our point of view. Consider the space $\mathbb{Z} \times X$ with the natural projection $\pi\colon \mathbb{Z} \times X \longrightarrow X$. For each open set U in X, elements of $C(U, \mathbb{Z})$ can be identified with continuous functions $f\colon U \longrightarrow \mathbb{Z} \times X$ such that $\pi \circ f$ is the identity on U. In this way, the elements of $C(U, \mathbb{Z})$ become *sections*, that is, continuous lifts for the projection π.

Definition 2.4. Let X be a topological space. A *sheaf* $\mathcal{F}$ on X is a topological space F, called the *sheaf space* or *espace étalé*, together with a surjective map $\pi_{\mathcal{F}}\colon F \twoheadrightarrow X$ which is locally a homeomorphism.

The sheaf $\mathcal{F}$ takes an open set $U \subseteq X$ to the set $\mathcal{F}(U) = C(U, F)$ of continuous functions $f\colon U \longrightarrow F$ for which $\pi_{\mathcal{F}} \circ f$ is the identity on U. Each inclusion $U' \subseteq U$ of open sets of X yields a *restriction map*

$$\rho_{U,U'}\colon \mathcal{F}(U) \longrightarrow \mathcal{F}(U')\,.$$

If $U'' \subseteq U' \subseteq U$ are open subsets of X, then

$$\rho_{U',U''} \circ \rho_{U,U'} = \rho_{U,U''}\,.$$

The elements of $\mathcal{F}(U)$ are *sections* of $\mathcal{F}$ over U; if $U = X$ these are known as *global sections*.

Remark 2.5. The sheaf space F of X is locally homeomorphic to X in the sense that for every point $f \in F$ there is an open neighborhood $V \subseteq F$ of f such that $\pi_{\mathcal{F}}\colon V \longrightarrow \pi_{\mathcal{F}}(V)$ is a homeomorphism.

This does not mean that F is a covering space of X. Indeed, let $X = \mathbb{R}^1$ and S a discrete space with more than one point. Construct the sheaf space F of $\mathcal{F}$ as the quotient space of $X \times S$ obtained by identifying (x, s) with (x, s') for $x \neq 0$ and $s, s' \in S$. Set $\pi_{\mathcal{F}}(x, s) = x$. Then $\mathcal{F}$ is a *skyscraper sheaf* with fiber S at $x = 0$. Evidently, $\pi_{\mathcal{F}}$ is not a covering map:

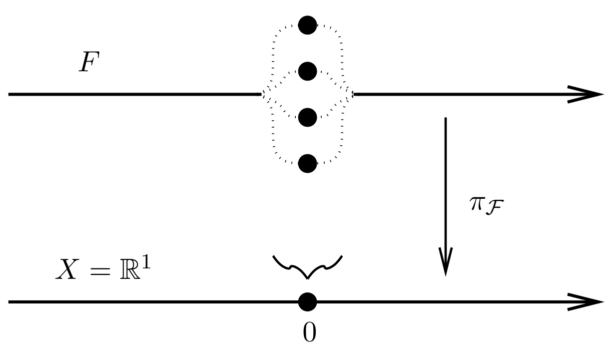


Figure 2.1. A skyscraper sheaf over the real line

An important class of sheaves, including Example 2.3, is the class of *constant sheaves*. These arise when the sheaf space F is the product of X with a space F_0 equipped with the discrete topology. In this case, sections of $\mathcal{F}$ on an open set U are identified with continuous maps from U to F_0.

Remark 2.6. We will see in Lecture 12 that one may specify a sheaf without knowing the sheaf space F. Namely, one may prescribe the sections $\mathcal{F}(U)$, so long as they satisfy certain compatibility properties. This is useful, since for many important sheaves, particularly those that arise in algebraic geometry, the sheaf space F is obscure, and its topology $\mathcal{T}_F$ complicated.

In general, the sections $\mathcal{F}(U)$ of a sheaf form a set without further structure. There are, however, more special kinds of sheaves, for example, sheaves of Abelian groups, sheaves of rings, etc., where, for each open set U, the set

$\mathcal{F}(U)$ has an appropriate algebraic structure, and the restriction maps are morphisms in the corresponding category.

Example 2.7 (The constant sheaf $\mathcal{Z}$)**.** With $X = \mathbb{S}^1$ as in Example 2.3, let $\mathcal{Z}$ be the sheaf associated to the canonical projection $\pi_{\mathcal{Z}} \colon X \times \mathbb{Z} \longrightarrow X$ where $\mathbb{Z}$ carries the discrete topology. Since $\mathbb{Z}$ is an Abelian group, $\mathcal{Z}(U)$ is an Abelian group as well under pointwise addition of maps. The restriction maps $\mathcal{Z}(U) \longrightarrow \mathcal{Z}(U')$, for $U' \subseteq U$, are homomorphisms of Abelian groups.

The sheaf $\mathcal{Z}$ can be used to show that the circle is not contractible. The remainder of this lecture is devoted to two constructions which relate $\mathcal{Z}$ to the topology of $\mathbb{S}^1$: Čech complexes and derived functors.

2. Čech cohomology

Consider the unit circle $\mathbb{S}^1$ embedded in $\mathbb{R}^2$. The sets $U_1 = \mathbb{S}^1 \smallsetminus \{-1\}$ and $U_2 = \mathbb{S}^1 \smallsetminus \{1\}$ constitute an open cover of $\mathbb{S}^1$. Set $U_{1,2} = U_1 \cap U_2$. Since U_1 and U_2 are connected, $\mathcal{Z}(U_i) = \mathbb{Z}$ for $i = 1, 2$. The restriction maps $\mathcal{Z}(\mathbb{S}^1) \longrightarrow \mathcal{Z}(U_i)$ are isomorphisms, since each of $\mathbb{S}^1$, U_1, and U_2 is a connected set. With $U_{1,2}$, however, the story is different: its connected components may be mapped to distinct integers, and hence $\mathcal{Z}(U_{1,2}) \cong \mathbb{Z}\times\mathbb{Z}$. More generally, for an open set U, we have

$$\mathcal{Z}(U) = \prod \mathbb{Z}\,,$$

where the product ranges over the connected components of U.

One has a diagram of inclusions

$$\begin{tikzcd} & U_1 \arrow[dl] & \\ \mathbb{S}^1 & & U_{1,2} \arrow[ul] \arrow[dl] \\ & U_2 \arrow[ul] & \end{tikzcd}$$

which gives rise to a commutative diagram of Abelian groups

$$\begin{tikzcd} & \mathcal{Z}(U_1) \arrow[dr] & \\ \mathcal{Z}(\mathbb{S}^1) \arrow[ur] \arrow[dr] & & \mathcal{Z}(U_{1,2})\,. \\ & \mathcal{Z}(U_2) \arrow[ur] & \end{tikzcd}$$

Consider the maps $\mathcal{Z}(U_i) \longrightarrow \mathcal{Z}(U_{1,2})$ for $i = 1, 2$. An element of $\mathcal{Z}(U_{1,2})$ is given by a pair of integers (a, b). If this element lies in the image of $\mathcal{Z}(U_i)$, then we must have $a = b$, since U_i is connected. It follows that the image of $\mathcal{Z}(U_i) \longrightarrow \mathcal{Z}(U_{1,2})$ is $\mathbb{Z} \cdot (1, 1)$ for $i = 1, 2$.

In a sense, the quotient of $\mathcal{Z}(U_{1,2})$ by the images of $\mathcal{Z}(U_i)$ measures the insufficiency of knowing the value of a section at one point in order to determine the entire section. In more fancy terms, it describes the possible $\mathbb{Z}$-bundles over $\mathbb{S}^1$. Since U_1 and U_2 are contractible, any bundle on them is given by the product of U_i with the fiber of the bundle. The question then arises how the sections on the open sets are identified along their intersection $U_{1,2}$. Choose generators for $\mathcal{Z}(U_1) = \mathbb{Z} = \mathcal{Z}(U_2)$, and suppose the two generators agree over one connected component of $U_{1,2}$. On the other connected component, these generators could agree, or could be inverses under the group law. In the former case, one gets the trivial bundle on $\mathbb{S}^1$, and in the latter case, the total space of the bundle is a "discrete Möbius band." The trivial bundle corresponds to the section $(1, 1)$ over $U_{1,2}$, while the Möbius strip is represented by $(1, -1)$. These are displayed in Figure 2.2.

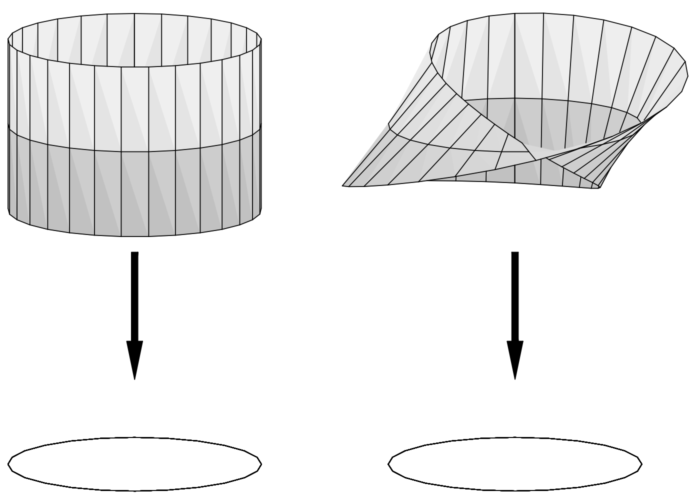


Figure 2.2. The trivial bundle and the Möbius bundle over the circle

In terms of sections, an element $(a, b) \in \mathcal{Z}(U_1)\times\mathcal{Z}(U_2)$ lifts to an element of $\mathcal{Z}(\mathbb{S}^1)$ if and only if $\rho_{U_1,U_{1,2}}(a) = \rho_{U_2,U_{1,2}}(b)$. Algebraically this can be described as follows. Consider the complex

$$0 \longrightarrow \mathcal{Z}(\mathbb{S}^1) \longrightarrow \mathcal{Z}(U_1) \times \mathcal{Z}(U_2) \xrightarrow{d^0} \mathcal{Z}(U_{1,2}) \longrightarrow 0\,,$$

where $d^0 = \rho_{U_1,U_{1,2}} - \rho_{U_2,U_{1,2}}$.

The negative sign on the second component of d^0 ensures that we have a complex, i.e., that the composition of consecutive maps is zero. The discussion above reveals that the complex is exact on the left and in the middle, and has a free group of dimension one as cohomology on the right, representing the existence of nontrivial $\mathbb{Z}$-bundles on $\mathbb{S}^1$.

It is reasonable to ask what would happen if we covered $\mathbb{S}^1$ with more than two open sets.

Exercise 2.8. Cover the circle $\mathbb{S}^1$ with the complements of the third roots of unity; call these U_i with $1 \leqslant i \leqslant 3$. The intersections $U_{i,j} = U_i \cap U_j$ are homeomorphic to disjoint pairs of intervals, and the triple intersection $U_{1,2,3}$ is the complement of the roots, hence is homeomorphic to three disjoint intervals. This gives us a diagram of restriction maps:

$$\begin{tikzcd}
 & \mathcal{Z}(U_1) \arrow[r] \arrow[dr] & \mathcal{Z}(U_{1,2}) \arrow[dr] & \\
\mathcal{Z}(\mathbb{S}^1) \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{Z}(U_2) \arrow[ur] \arrow[dr] & \mathcal{Z}(U_{1,3}) \arrow[r] & \mathcal{Z}(U_{1,2,3}) \\
 & \mathcal{Z}(U_3) \arrow[ur] \arrow[r] & \mathcal{Z}(U_{2,3}) \arrow[ur] &
\end{tikzcd}$$

which can be used to construct a complex

$$0 \longrightarrow \mathcal{Z}(\mathbb{S}^1) \longrightarrow \prod_{i=1}^{3} \mathcal{Z}(U_i) \xrightarrow{d^0} \prod_{1\leqslant i<j\leqslant 3} \mathcal{Z}(U_{i,j}) \xrightarrow{d^1} \mathcal{Z}(U_{1,2,3}) \longrightarrow 0\,,$$

where the maps d^i are built using the restriction maps, suitably signed to ensure that compositions of maps are zero. Find such a suitable choice of signs, or look ahead at Definition 2.9. Determine the various groups and the matrix representations of the homomorphisms that appear in this complex. Show that the only nonzero cohomology group is $\ker d^1 / \operatorname{image} d^0$, which is isomorphic to $\mathbb{Z}$.

The preceding examples are intended to suggest that there is a certain invariance to the computations induced by open covers. This is indeed the case, as is discussed later, but we first clarify the algebraic setup.

Definition 2.9. Let $\mathcal{F}$ be a sheaf on a topological space X, and let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover of X, where I is a totally ordered index set. Given a finite subset J of I, let $U_J = \bigcap_{j\in J} U_j$. We construct a complex $\check{C}^\bullet(\mathfrak{U};\mathcal{F})$ with t-th term

$$\check{C}^t(\mathfrak{U};\mathcal{F}) = \prod_{|J|=t+1} \mathcal{F}(U_J)\,, \qquad \text{where } t \geqslant 0\,.$$

Let J be a subset of I with $|J| = t+1$, and let $i \in I \smallsetminus J$. We define $\operatorname{sgn}(J,i)$ to be -1 raised to the number of elements of J that are greater than i. Set

$$d^t\colon \check{C}^t(\mathfrak{U};\mathcal{F}) \longrightarrow \check{C}^{t+1}(\mathfrak{U};\mathcal{F})$$

to be the product of the maps

$$\operatorname{sgn}(J,i)\cdot \rho_{U_J,U_{J\cup\{i\}}}\colon \mathcal{F}(U_J) \longrightarrow \mathcal{F}(U_{J\cup\{i\}})\,.$$

The sign choices ensure that $d^{t+1} \circ d^t = 0$, giving us a complex

$$0 \longrightarrow \check{C}^0(\mathfrak{U};\mathcal{F}) \longrightarrow \check{C}^1(\mathfrak{U};\mathcal{F}) \longrightarrow \check{C}^2(\mathfrak{U};\mathcal{F}) \longrightarrow \cdots$$

called the *Čech complex* associated to $\mathcal{F}$ and $\mathfrak{U}$. The cohomology of $\check{C}^\bullet(\mathfrak{U};\mathcal{F})$ is *Čech cohomology*, $\check{H}^\bullet(\mathfrak{U};\mathcal{F})$.

Note that $\mathcal{F}(X)$ is not part of the Čech complex. However, as the following exercise shows, $\mathcal{F}(X)$ can be read from $\check{C}^\bullet(\mathfrak{U};\mathcal{F})$.

Exercise 2.10. Let $\mathcal{F}$ be a sheaf on X, and $\mathfrak{U}$ an open cover of X. Prove that $\check{H}^0(\mathfrak{U};\mathcal{F}) = \mathcal{F}(X)$.

Next we discuss the dependence of Čech cohomology on covers.

Definition 2.11. An open cover $\mathfrak{V} = \{V_i\}_{i\in I}$ of X is a *refinement* of an open cover $\mathfrak{U} = \{U_i\}_{i\in I'}$ if there is a map $\tau\colon I \longrightarrow I'$ with $V_i \subseteq U_{\tau(i)}$ for each $i \in I$. The map τ need not be injective or surjective.

If $\mathfrak{V}$ refines $\mathfrak{U}$, one has restriction maps $\mathcal{F}(U_{\tau(i)}) \longrightarrow \mathcal{F}(V_i)$. More generally, for each finite subset $J \subseteq I$, there is a map $\mathcal{F}(U_{J'}) \longrightarrow \mathcal{F}(V_J)$ where $J' = \{\tau(j) \mid j \in J\}$. These give a morphism of complexes

$$\check{\tau}\colon \check{C}^\bullet(\mathfrak{U};\mathcal{F}) \longrightarrow \check{C}^\bullet(\mathfrak{V};\mathcal{F})\,.$$

This map depends on τ, but the induced map on cohomology does not. In fact, the morphism $\check{\tau}$ is independent up to homotopy of the choice of τ.

Consider the class of open covers of X with partial order $\mathfrak{V} \geqslant \mathfrak{U}$ if $\mathfrak{V}$ refines $\mathfrak{U}$. The class of covers may not be a set, but one can get around this problem; see [**155**, pp. 142/143] or [**47**, §5.8]. One can form the direct limit of the corresponding complexes. This is filtered, so Exercise 4.34 implies

$$H^t\big(\varinjlim_{\mathfrak{U}} \check{C}^\bullet(\mathfrak{U};\mathcal{F})\big) \cong \varinjlim_{\mathfrak{U}} \check{H}^t(\mathfrak{U};\mathcal{F})\,.$$

Definition 2.12. Set $\check{H}^t(X;\mathcal{F}) = \varinjlim_{\mathfrak{U}} \check{H}^t(\mathfrak{U};\mathcal{F})$; this is called the t-th *Čech cohomology* group of $\mathcal{F}$.

Each open cover $\mathfrak{U}$ gives rise to a natural map

$$\check{H}^t(\mathfrak{U};\mathcal{F}) \longrightarrow \check{H}^t(X;\mathcal{F})\,.$$

The next result gives conditions under which this map is an isomorphism.

Theorem 2.13. *Let $\mathcal{F}$ be a sheaf on X. Suppose X has a base $\mathfrak{V}$ for its topology that is closed under finite intersections and such that $\check{H}^t(V;\mathcal{F}) = 0$ for each $V \in \mathfrak{V}$ and $t > 0$.*

If $\mathfrak{U}$ is an open cover of X such that $\check{H}^t(U;\mathcal{F}) = 0$ for each $t > 0$ and U is a finite intersection of elements of $\mathfrak{U}$, then

$$\check{H}^t(\mathfrak{U};\mathcal{F}) \cong \check{H}^t(X;\mathcal{F}) \qquad \text{for all } t \geqslant 0\,. \qquad \square$$

The theorem follows from Théorèmes 5.4.1 and 5.9.2 in [**47**]; see also Theorem 2.26. Using this theorem seems difficult as the hypotheses involve $\mathfrak{U}$ and an auxiliary cover $\mathfrak{V}$. There are important situations where one has a cover $\mathfrak{V}$ with the desired properties. If $\mathcal{F}$ is a locally constant sheaf on a real manifold, for $\mathfrak{V}$ one can take any base such that each V is homeomorphic to a finite number of disjoint copies of $\mathbb{R}^n$. For $\mathcal{F}$ a quasi-coherent sheaf on a suitable scheme, one can take $\mathfrak{V}$ to be any affine open cover.

We close with a few remarks about cohomology of constant sheaves.

Remark 2.14. Suppose that

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

is an exact sequence of Abelian groups. Consider the induced constant sheaves $\mathcal{A}', \mathcal{A}, \mathcal{A}''$ on a space X. Suppose $\mathfrak{U}$ is an open cover of X such that for any finite intersection U_I of sets in $\mathfrak{U}$, we get an exact sequence

$$0 \longrightarrow C(U_I, A') \longrightarrow C(U_I, A) \longrightarrow C(U_I, A'') \longrightarrow 0\,.$$

One then obtains an exact sequence of Čech complexes, which, in turn, yields an exact sequence of cohomology groups

$$\cdots \longrightarrow \check{H}^t(\mathfrak{U}; \mathcal{A}') \longrightarrow \check{H}^t(\mathfrak{U}; \mathcal{A}) \longrightarrow \check{H}^t(\mathfrak{U}; \mathcal{A}'') \longrightarrow \check{H}^{t+1}(\mathfrak{U}; \mathcal{A}') \longrightarrow \cdots\,.$$

Not all open covers of X produce long exact sequences: consider Example 2.22 with the open cover $\mathfrak{U} = \{X\}$. In other words, $\check{H}^\bullet(\mathfrak{U}; -)$ need not be a δ-functor. On the other hand, $\check{H}^\bullet(X; -)$ is a δ-functor in favorable cases, such as on paracompact spaces, or for quasi-coherent sheaves on schemes.

Let $\mathbb{K}$ be a field of characteristic zero endowed with the discrete topology, and let $\mathcal{K}$ be the sheaf on $\mathbb{S}^1$ that assigns to each open U the set of continuous functions from U to $\mathbb{K}$. Then

$$\check{H}^\bullet(\mathbb{S}^1; \mathcal{K}) \cong \check{H}^\bullet(\mathbb{S}^1; \mathcal{Z}) \otimes_{\mathbb{Z}} \mathbb{K}\,,$$

since tensoring with a field of characteristic zero is an exact functor.

The situation is different if $\mathbb{K}$ has positive characteristic.

Exercise 2.15. Recall that the real projective plane $\mathbb{RP}^2$ is obtained by identifying antipodal points on the 2-sphere $\mathbb{S}^2$. Cover $\mathbb{RP}^2$ with three open hemispheres whose pairwise and triple intersections are unions of disjoint contractible sets. Use this cover to prove that $\check{H}^1(\mathbb{RP}^2; \mathcal{Z}) = 0$.

Let $\mathcal{Z}_2$ be the sheaf for which $\mathcal{Z}_2(U)$ is the set of continuous functions from U to $\mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ has the discrete topology. Prove that $\check{H}^1(\mathbb{RP}^2; \mathcal{Z}_2) \cong \mathbb{Z}/2\mathbb{Z}$. In particular, the map $\check{H}^1(\mathbb{RP}^2; \mathcal{Z}) \longrightarrow \check{H}^1(\mathbb{RP}^2; \mathcal{Z}_2)$ induced by the natural surjection $\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is not surjective. This reflects the fact that $\mathbb{RP}^2$ is non-orientable.

3. Calculus versus topology

We now switch gears and investigate what happens if instead of making the cover increasingly fine, we replace $\mathcal{F}$ by a resolution of more "flexible" sheaves. Specifically, let $\mathcal{D}$ be the sheaf on $\mathbb{S}^1$ that attaches to each open set U, the ring of real-valued smooth functions. In this case, $\mathcal{D}(U)$ is a vector space over $\mathbb{R}$.

Example 2.16. Consider the cover of $\mathbb{S}^1$ by the open sets $U_1 = \mathbb{S}^1 \smallsetminus \{-1\}$ and $U_2 = \mathbb{S}^1 \smallsetminus \{1\}$. As before, we can construct a complex of the form

$$0 \longrightarrow \mathcal{D}(\mathbb{S}^1) \longrightarrow \mathcal{D}(U_1) \times \mathcal{D}(U_2) \longrightarrow \mathcal{D}(U_{1,2}) \longrightarrow 0\,.$$

If $(f_1, f_2) \in \mathcal{D}(U_1) \times \mathcal{D}(U_2)$ maps to zero in the above complex, then the functions f_1 and f_2 agree on $U_{1,2}$. It follows that f_1 and f_2 have (at worst) a removable singularity at -1 and 1 respectively, since f_2 has no singularity at -1 and f_1 has no singularity at 1, and they are smooth outside these points. In particular, there is a function $f \in \mathcal{D}(\mathbb{S}^1)$ such that each f_i is the restriction of f to U_i, as predicted by Exercise 2.10.

Now consider the cohomology on the right. A function $f \in \mathcal{D}(U_{1,2})$ has no singularities except possibly at 1 and -1. Let u be a smooth function on $U_{1,2}$ that takes value 0 near -1, and value 1 near 1. Of course, $f = (1-u)f + uf$. Now $(1-u)f$ can be extended to a smooth function on U_1, and uf can be extended to a smooth function on U_2. It follows that f is in the image of $\mathcal{D}(U_1) \times \mathcal{D}(U_2) \longrightarrow \mathcal{D}(U_{1,2})$. Hence the complex

$$0 \longrightarrow \mathcal{D}(U_1) \times \mathcal{D}(U_2) \longrightarrow \mathcal{D}(U_{1,2}) \longrightarrow 0$$

has cohomology groups $\check{H}^0(\mathfrak{U}; \mathcal{D}) = \mathcal{D}(\mathbb{S}^1)$ and $\check{H}^1(\mathfrak{U}; \mathcal{D}) = 0$.

We remark that any other open cover of $\mathbb{S}^1$ would give us a Čech complex with a unique nonzero cohomology group in degree zero. In terms of the limit over all open covers, this says that the sheaf $\mathcal{D}$ on $\mathbb{S}^1$ has $\check{H}^0(\mathbb{S}^1; \mathcal{D}) = \mathcal{D}(\mathbb{S}^1)$ and $\check{H}^1(\mathbb{S}^1; \mathcal{D}) = 0$. The key point is the existence of partitions of unity:

Definition 2.17. Let M be a smooth manifold and $\mathfrak{U} = \{U_i\}_{i \in I}$ a locally finite open cover of M, i.e., any point $m \in M$ belongs to only finitely many of the open sets U_i. A *partition of unity subordinate to* $\mathfrak{U}$ is a collection of smooth functions $f_i \colon M \longrightarrow \mathbb{R}$ for $i \in I$, such that $f_i|_{M \smallsetminus U_i} = 0$ and $\sum f_i = 1$. Note that this is a finite sum at any point of M.

The partition in our case was $u + (1-u) = 1$, and its existence allowed a section of $\mathcal{D}$ on $U_{1,2}$ to be expressed as a sum of sections over U_1 and U_2. Partitions of unity make the sheaf $\mathcal{D}$ sufficiently fine and flexible, so that $\check{H}^t(M; \mathcal{D}) = 0$ for all $t \geqslant 1$.

Next, let $\mathcal{R}$ be the sheaf that sends an open set $U \subseteq X$ to the continuous functions $C(U, \mathbb{R})$, where $\mathbb{R}$ is endowed with the discrete topology. This is the constant sheaf associated to $\mathbb{R}$; see Example 2.7. The elements of $\mathcal{R}(U)$ are smooth functions, hence $\mathcal{R}$ may be viewed as a "subsheaf" of $\mathcal{D}$; you can make the meaning of this precise once you have seen Definition 2.19.

Example 2.18. Let U be a proper open subset of $\mathbb{S}^1$. We claim there is an exact sequence

$$0 \longrightarrow \mathcal{R}(U) \longrightarrow \mathcal{D}(U) \xrightarrow{\frac{d}{dt}} \mathcal{D}(U) \longrightarrow 0\,,$$

where the first map is inclusion, while the second map is differentiation by arclength. To see that this sequence is exact, note that U is the disjoint union of open arcs that are diffeomorphic to the real line. On such an open arc, (i) the constants are the only functions that are annihilated by differentiation, and (ii) every smooth function can be integrated to a smooth function. On the other hand, as we will discuss later, the sequence is not exact on the right if U is $\mathbb{S}^1$.

For open sets $V \subseteq U$, sequences of the form above, along with appropriate restriction maps, fit together to give a commutative diagram

$$\begin{array}{ccccc}
\mathcal{R}(U) & \longrightarrow & \mathcal{D}(U) & \xrightarrow{\frac{d}{dt}} & \mathcal{D}(U) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{R}(V) & \longrightarrow & \mathcal{D}(V) & \xrightarrow{\frac{d}{dt}} & \mathcal{D}(V)\,.
\end{array}$$

Definition 2.19. A *morphism of sheaves* $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ on X is a collection of maps $\varphi_U\colon \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ for $U \in \mathcal{T}_X$, such that for each inclusion $V \subseteq U$ of open sets, there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\
{\scriptstyle \rho_{U,V}}\big\downarrow & & \big\downarrow{\scriptstyle \rho'_{U,V}} \\
\mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V)
\end{array}$$

where ρ and ρ' are the restriction maps for the sheaves $\mathcal{F}$ and $\mathcal{G}$ respectively.

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of Abelian groups, rings, etc., a morphism of such sheaves is a collection as above with the additional property that the maps φ_U are morphisms in the appropriate category.

A natural next step would be to discuss the Abelian category structure on the category of sheaves. This then leads to a notion of a cohomology of a complex of sheaves, and hence to exactness. We take a shorter route.

Definition 2.20. Let $\mathcal{F}$ be a sheaf on a space X, and consider a point $x \in X$. The class of open sets U of X containing x forms a filtered direct system, and its limit

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$$

is the *stalk* of $\mathcal{F}$ at x. Any additional structure $\mathcal{F}$ has is usually inherited by the stalks; for instance, if $\mathcal{F}$ is a sheaf of Abelian groups or rings, then $\mathcal{F}_x$ is an Abelian group or a ring, respectively.

It follows from standard properties of direct limits that a morphism $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ of sheaves on X induces, for each point $x \in X$, a morphism $\varphi_x\colon \mathcal{F}_x \longrightarrow \mathcal{G}_x$ in the appropriate category; passage to a stalk is a functor.

Definition 2.21. A *complex* $\mathcal{F}^\bullet$ of sheaves of Abelian groups is a sequence

$$\cdots \longrightarrow \mathcal{F}^{t-1} \xrightarrow{d^{t-1}} \mathcal{F}^t \xrightarrow{d^t} \mathcal{F}^{t+1} \xrightarrow{d^{t+1}} \cdots$$

of morphisms of sheaves where $d^{t+1} \circ d^t = 0$ for all t. Such a complex $\mathcal{F}^\bullet$ is *exact* if the induced complex of Abelian groups

$$\cdots \longrightarrow \mathcal{F}_x^{t-1} \longrightarrow \mathcal{F}_x^t \longrightarrow \mathcal{F}_x^{t+1} \longrightarrow \cdots$$

is exact for each $x \in X$.

For instance, if X has a base $\mathfrak{U}$ of open sets such that for each U in $\mathfrak{U}$, the sequence of Abelian groups

$$\cdots \longrightarrow \mathcal{F}^{t-1}(U) \longrightarrow \mathcal{F}^t(U) \longrightarrow \mathcal{F}^{t+1}(U) \longrightarrow \cdots$$

is exact, then the complex of sheaves is exact; this is because computing direct limits commutes with cohomology; see Exercise 4.34. The converse is certainly not true; see the example below.

Example 2.22. It follows from the preceding discussion that the following sequence of sheaves from Example 2.18 is exact:

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{D} \xrightarrow{\frac{d}{dt}} \mathcal{D} \longrightarrow 0\,.$$

Taking global sections, we get a sequence

$$0 \longrightarrow \mathcal{R}(\mathbb{S}^1) \longrightarrow \mathcal{D}(\mathbb{S}^1) \xrightarrow{\frac{d}{dt}} \mathcal{D}(\mathbb{S}^1) \longrightarrow 0$$

which is exact at the left and in the middle, and we shall see that it is not exact on the right. If a smooth function on $\mathbb{S}^1$ arises as a derivative, then the fundamental theorem of calculus implies that its average value is 0. Consequently, nonzero constant functions on $\mathbb{S}^1$ do not arise as derivatives. On the other hand, for any smooth function $f\colon \mathbb{S}^1 \longrightarrow \mathbb{R}$, the function

$$g(t) = f(t) - \frac{1}{2\pi}\int_0^{2\pi} f(u)\,du$$

integrates to 0 on $\mathbb{S}^1$ and hence $\int_0^t g(\tau)\,d\tau$ is a function on $\mathbb{S}^1$ whose derivative is $g(t)$. It follows that the complex of global sections

$$0 \longrightarrow \mathcal{D}(\mathbb{S}^1) \longrightarrow \mathcal{D}(\mathbb{S}^1) \longrightarrow 0$$

has cohomology in degrees 0 and 1, and that both cohomology groups are isomorphic to the space of constant functions on $\mathbb{S}^1$.

This discussion leads to another type of cohomology theory.

Definition 2.23. A sheaf $\mathcal{E}$ of Abelian groups is *injective* if every injective morphism $\mathcal{E} \longrightarrow \mathcal{F}$ of sheaves of Abelian groups splits. Every sheaf $\mathcal{F}$ of Abelian groups can be embedded into an injective sheaf; see Remark 12.29.

A complex $\mathcal{G}^\bullet$ of sheaves is a *resolution* of $\mathcal{F}$ if there is an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G}^1 \longrightarrow \mathcal{G}^2 \longrightarrow \cdots .$$

It is an *injective* resolution if each $\mathcal{G}^t$ is injective. The *sheaf cohomology* $H^t(X, \mathcal{F})$ is the t-th cohomology group of the complex

$$0 \longrightarrow \mathcal{G}^0(X) \longrightarrow \mathcal{G}^1(X) \longrightarrow \mathcal{G}^2(X) \longrightarrow \cdots .$$

The cohomology groups do not depend on $\mathcal{G}^\bullet$; see Lectures 3 and 12.

Definition 2.24. A sheaf $\mathcal{F}$ is *acyclic* if $H^t(X, \mathcal{F}) = 0$ for all $t \geqslant 1$.

For the proof of the following proposition, see [**61**, Proposition III.1.2A].

Proposition 2.25 (Acyclicity principle)**.** *If $\mathcal{G}^\bullet$ is a resolution of $\mathcal{F}$ by acyclic sheaves, then $H^t(X, \mathcal{F}) = H^t(\mathcal{G}^\bullet(X))$ for each t.* □

In Example 2.16 we verified that $\mathcal{D}$ is acyclic on $\mathbb{S}^1$. As noted in Example 2.22, the constant sheaf $\mathcal{R}$ on $\mathbb{S}^1$ has an acyclic resolution

$$0 \longrightarrow \mathcal{D} \xrightarrow{\frac{d}{dt}} \mathcal{D} \longrightarrow 0 ,$$

so $H^t(\mathbb{S}^1, \mathcal{R}) = 0$ for $t \geqslant 2$. Moreover, $H^0(\mathbb{S}^1, \mathcal{R}) \cong \mathbb{R} \cong H^1(\mathbb{S}^1, \mathcal{R})$, and these agree with the singular cohomology groups of $\mathbb{S}^1$.

4. Čech cohomology and derived functors

We now compare the two approaches we have taken. The constant sheaf $\mathcal{R}$ on $\mathbb{S}^1$ has Čech cohomology groups $\check{H}^t(\mathbb{S}^1; \mathcal{R}) = 0$ if $t \geqslant 2$, and

$$\check{H}^0(\mathbb{S}^1; \mathcal{R}) \cong \mathbb{R} \cong \check{H}^1(\mathbb{S}^1; \mathcal{R}) .$$

The sheaf cohomology groups turned out to be the same as these. We take an open cover $\mathfrak{U}$ of $\mathbb{S}^1$, fine enough so that each finite intersection of the open sets is an open arc. By the discussion following Theorem 2.13, one has $\check{H}^\bullet(\mathfrak{U}; \mathcal{R}) = \check{H}^\bullet(\mathbb{S}^1; \mathcal{R})$ and $\check{H}^\bullet(\mathfrak{U}; \mathcal{D}) = \check{H}^\bullet(\mathbb{S}^1; \mathcal{D})$. Moreover, for each U_I, there is an exact sequence

$$0 \longrightarrow \mathcal{R}(U_I) \longrightarrow \mathcal{D}(U_I) \longrightarrow \mathcal{D}(U_I) \longrightarrow 0 ,$$

and hence an exact sequence of complexes

$$0 \longrightarrow \check{C}^\bullet(\mathfrak{U}; \mathcal{R}) \longrightarrow \check{C}^\bullet(\mathfrak{U}; \mathcal{D}) \longrightarrow \check{C}^\bullet(\mathfrak{U}; \mathcal{D}) \longrightarrow 0 .$$

By Remark 2.14, there exists an exact sequence of cohomology groups

$$0 \longrightarrow \check{H}^0(\mathbb{S}^1;\mathcal{R}) \longrightarrow \check{H}^0(\mathbb{S}^1;\mathcal{D}) \longrightarrow \check{H}^0(\mathbb{S}^1;\mathcal{D}) \longrightarrow \check{H}^1(\mathbb{S}^1;\mathcal{R}) \longrightarrow 0\,,$$

where the zero on the right follows from the fact that $\check{H}^t(\mathbb{S}^1;\mathcal{D}) = 0$ for all $t \geqslant 1$; see Example 2.16. We conclude that $\check{H}^t(\mathbb{S}^1;\mathcal{R}) = 0$ for all $t \geqslant 2$, and that $\check{H}^0(\mathbb{S}^1;\mathcal{R})$ and $\check{H}^1(\mathbb{S}^1;\mathcal{R})$ arise naturally as the kernel and cokernel respectively of the differentiation map

$$\frac{d}{dt}\colon \mathcal{D}(\mathbb{S}^1) \longrightarrow \mathcal{D}(\mathbb{S}^1)\,.$$

In particular, the Čech cohomology of $\mathcal{R}$ on $\mathbb{S}^1$ can be read off from the global sections of the morphism $\mathcal{D} \longrightarrow \mathcal{D}$ given by differentiation.

In Lecture 19 we revisit this theme of linking differential calculus with sheaves and topology. The main ideas are as follows: one can get topological information from the Čech approach, since fine open covers turn the computation into a triangulation of the underlying space. On the other hand, one can replace the given sheaf by a suitable complex of acyclic sheaves and consider the cohomology of the resulting complex of global sections.

The following statement summarizes the relationship between Čech cohomology and sheaf cohomology; see [**47**, Chapitre 5] for proofs.

Theorem 2.26. *Let X, $\mathcal{F}$, $\mathfrak{U}$, and $\mathfrak{V}$ be as in Theorem 2.13. Then there are isomorphisms*

$$\check{H}^t(\mathfrak{U};\mathcal{F}) \cong \check{H}^t(X;\mathcal{F}) \cong H^t(X,\mathcal{F}) \qquad \textit{for each } t\,. \qquad \square$$

Resolutions and Derived Functors

This lecture is a whirlwind introduction to, or review of, resolutions and derived functors—with a bit of tunnel vision. That is, we will give unabashed preference to topics relevant to local cohomology, and attempt to draw a straight line between the topics we cover and our final goals. At a few places along the way, we will point generally in the direction of other topics of interest, but other than that we will do our best to be single-minded.

The Appendix contains preparatory material on injective modules and Matlis theory. In this lecture, we cover roughly the same ground on the projective/flat side of the fence, followed by basics on projective and injective resolutions, and definitions and basic properties of derived functors.

Throughout this lecture, R is a commutative ring with identity.

1. Free, projective, and flat modules

In terms of module theory, fields are the simplest rings in commutative algebra, for all their modules are *free*.

Definition 3.1. An R-module F is *free* if it has a *basis*, i.e., a subset that generates F as an R-module and is linearly independent over R.

It is easy to prove that a module is free if and only if it is isomorphic to a direct sum of copies of the underlying ring. The cardinality of a basis is the *rank* of the free module. To see that the rank is well-defined, we can reduce modulo a maximal ideal of R and use the corresponding result for—what else?—fields. There is an equivalent definition of freeness in terms of a universal lifting property. It is a worthwhile exercise to formulate this property; you will know when you have the right one, for then the proof of the equivalence is trivial.

In practice, i.e., in explicit computations by hand or by computer, we usually want free modules. Theoretically, however, the properties that concern us are *projectivity* and *flatness*. For projective modules, we reverse the process above and work from the categorical definition to an elementary one.

Definition 3.2. An R-module P is *projective* if whenever there is a surjective homomorphism of R-modules $f\colon M \longrightarrow N$, and an arbitrary homomorphism of R-modules $g\colon P \longrightarrow N$, then there is a lifting $h\colon P \longrightarrow M$ so that $fh = g$. Pictorially, we have

$$\begin{tikzcd} & P \arrow[dl, dotted, "h"'] \arrow[d, "g"] & \\ M \arrow[r, "f"'] & N \arrow[r] & 0 \end{tikzcd}$$

with the bottom row forming an exact sequence of R-modules.

Here is another way to word the definition, which highlights our intended uses for projective modules. Let F be a *covariant* functor from R-modules to Abelian groups. Recall that F is *left-exact* if for each exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0\,,$$

the induced sequence

$$0 \longrightarrow \mathrm{F}(L) \longrightarrow \mathrm{F}(M) \longrightarrow \mathrm{F}(N)$$

is exact. If in addition $\mathrm{F}(M) \longrightarrow \mathrm{F}(N)$ is surjective, then F is *exact*.

Exercise 3.3. Let P be an R-module. Prove that the functor $\mathrm{Hom}_R(P, -)$ is covariant and left-exact, and that P is projective if and only if $\mathrm{Hom}_R(P, -)$ is an exact functor.

Here are the first four things that you should check about projective modules, plus one.

(1) Free modules are projective.
(2) Projective modules are precisely direct summands of free modules.
(3) Arbitrary direct sums of projective modules are projective.
(4) Freeness and projectivity both localize.
(5) Over a Noetherian local ring R, all projectives are free.

Remark 3.4. Despite their relatively innocuous definition, projective modules can be quite subtle, and are an active topic of research. For example, if

R is a polynomial ring over a field, then all finitely generated projective R-modules are free—this is the content of the Serre conjecture. Around 1978, Quillen and Suslin [**130, 153**] gave independent proofs of this conjecture; see also [**10**]. Closely related is the Bass-Quillen conjecture, which asserts that for a regular ring R, every projective $R[T]$-module is extended from R. Quillen's and Suslin's solutions of Serre's conjecture proceed by proving this statement when R is a regular ring of dimension at most 1. Popescu's celebrated theorem of general Néron desingularization [**129, 154**], together with results of Lindel [**100**], proves the Bass-Quillen conjecture for regular local rings containing a field, unramified regular local rings of mixed characteristic, and also for excellent Henselian local rings.

We next present two examples of projective modules which are not free. The first comes from number theory, and the second from topology.

Example 3.5. In the ring $R = \mathbb{Z}[\sqrt{-5}]$, the ideal $\mathfrak{a} = (2+\sqrt{-5}, 3)$ is projective but not free as an R-module. Indeed, $\mathfrak{a}$ is not principal, so cannot be free (prove this!), while the obvious surjection $R^2 \longrightarrow \mathfrak{a}$ has a splitting

$$x \longmapsto x \cdot \left(\frac{-1+\sqrt{-5}}{2+\sqrt{-5}}, \frac{2-\sqrt{-5}}{3} \right),$$

so that $\mathfrak{a}$ is a direct summand of R^2. This is directly related to the fact that R is not a UFD.

Example 3.6. Let $R = \mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$, the coordinate ring of the real 2-sphere $\mathbb{S}^2$. The homomorphism $R^3 \longrightarrow R$ defined by the row vector $v = [x,y,z]$ is surjective, so the kernel P satisfies $P \oplus R \cong R^3$. However, P is not free. Every element of R^3 gives a vector field on $\mathbb{S}^2$, with v defining the vector field pointing straight out from the origin. An element of P thus gives a vector field that is tangent to $\mathbb{S}^2$. If P were free, a basis would define two non-vanishing vector fields on $\mathbb{S}^2$ that evaluate to independent vectors in each point. But hedgehogs can't be combed, so freeness is impossible!

As we noted above, the definition of projectivity amounts to saying that some usually left-exact functor is exact. Compare this to the definition of injectivity. We next mimic this for another familiar functor.

For any given R-module M, the functor $- \otimes_R M$ is right-exact. One can give an elementary proof by chasing elements, but consider also Example 4.17 in the light of Proposition 4.18. It is clear that if we take $M = R$, then $A \otimes_R M$ is nothing but A again, so that in fact $- \otimes_R R$ is exact. With an eye towards defining the Tor functors later in this lecture, we give this property its rightful name.

Definition 3.7. An R-module M is *flat* if $- \otimes_R M$ is an exact functor.

We have observed that the free module R is flat, and it is easy to check that the direct sum of a family of flat modules is flat. Thus, free modules are flat, and it follows from the distributivity of $\otimes$ over $\oplus$ that projectives are flat as well. In fact, it is very nearly true that the only flat modules are the projectives. Specifically, we have the following theorem of Govorov [**49**] and Lazard [**96**]; you can also find a proof in [**32**, Theorem A6.6].

Theorem 3.8. *A module is flat if and only if it is a filtered direct limit of free modules. In particular, a finitely generated flat module is projective.* $\square$

2. Complexes

Having defined the classes of modules to which we will compare all others, let us move on to resolutions. The point of *resolving* a module is to measure its complexity against a given standard, such as freeness, injectivity, or flatness. Doing so calls for a quick summary of the calculus of complexes; we refer the reader to Weibel [**161**] for basics.

In what follows, complexes $K^\bullet$ are (usually) indexed cohomologically:

$$\cdots \longrightarrow K^{i-1} \xrightarrow{\partial_K^{i-1}} K^i \xrightarrow{\partial_K^i} K^{i+1} \longrightarrow \cdots .$$

The differential on $K^\bullet$ is denoted ∂_K, or sometimes simply ∂. We say that $K^\bullet$ is *bounded below* if $K^i = 0$ for $i \ll 0$, and *bounded above* if $K^i = 0$ for $i \gg 0$; if both conditions hold, then $K^\bullet$ is *bounded*. On occasion, we consider a complex as a graded module with a graded endomorphism of degree 1.

Definition 3.9. Let $K^\bullet$ be a complex of R-modules and n an integer. Then $K^\bullet[n]$ denotes the complex $K^\bullet$ with *shift* n, i.e., the complex with modules

$$(K^\bullet[n])^i = K^{n+i}\,,$$

and differential $(-1)^n\partial_K$.

Suppose $K^\bullet$ and $L^\bullet$ are complexes of R-modules. Their *tensor product* $K^\bullet \otimes_R L^\bullet$ is the complex with

$$(K^\bullet \otimes_R L^\bullet)^n = \bigoplus_{i+j=n} K^i \otimes_R L^j\,,$$

and R-linear differential defined on an element $k \otimes l \in K^i \otimes_R L^j$ by

$$\partial(k \otimes l) = \partial_K(k) \otimes l + (-1)^i k \otimes \partial_L(l)\,.$$

The *complex of homomorphisms* from $K^\bullet$ to $L^\bullet$ is a complex $\operatorname{Hom}_R(K^\bullet, L^\bullet)$ consisting of modules

$$\operatorname{Hom}_R(K^\bullet, L^\bullet)^n = \prod_i \operatorname{Hom}_R(K^i, L^{i+n})\,,$$

and R-linear differential defined on an element f in $\operatorname{Hom}_R(K^\bullet, L^\bullet)^n$ by

$$\partial(f) = \partial_L \circ f - (-1)^n f \circ \partial_K\,.$$

Note that a homomorphism $f \in \operatorname{Hom}_R(K^\bullet, L^\bullet)^0$ is a cycle if it satisfies the condition $\partial_L \circ f = f \circ \partial_K$, i.e., precisely if f is a *morphism* from $K^\bullet$ to $L^\bullet$.

Definition 3.10. A morphism $\eta\colon K^\bullet \longrightarrow L^\bullet$ is an *isomorphism* if there is a morphism $\xi\colon L^\bullet \longrightarrow K^\bullet$ such that $\eta \circ \xi$ and $\xi \circ \eta$ are identity maps.

The map η is a *quasi-isomorphism* if the induced map on cohomology $H(\eta)\colon H(K^\bullet) \longrightarrow H(L^\bullet)$ is bijective.

Let $\eta, \zeta\colon K^\bullet \longrightarrow L^\bullet$ be morphisms of complexes of R-modules. Then η and ζ are *homotopic* if there exists $\varphi \in \operatorname{Hom}(K^\bullet, L^\bullet)^{-1}$ with

$$\eta - \zeta = \partial_L \circ \varphi + \varphi \circ \partial_K\,.$$

Note that φ is not required to be a morphism of complexes. A morphism η is *null-homotopic* if it is homotopic to the zero morphism. Finally, a morphism $\eta\colon K^\bullet \longrightarrow L^\bullet$ is a *homotopy equivalence* if there is a morphism $\xi\colon L^\bullet \longrightarrow K^\bullet$ such that $\xi \circ \eta$ and $\eta \circ \xi$ are homotopic to the identity maps on $K^\bullet$ and $L^\bullet$, respectively. The terminology reflects the origin of this notion in topology; see [**32**, page 635].

Exercise 3.11. An isomorphism of complexes is a quasi-isomorphism. Give an example where the converse does not hold.

Exercise 3.12. Let $\eta, \zeta\colon K^\bullet \longrightarrow L^\bullet$ be morphisms of complexes of R-modules. Prove the following assertions.

(1) If η is homotopic to ζ, then $H(\eta) = H(\zeta)$.

(2) If the identity morphism id_K is null-homotopic, then $H(K^\bullet) = 0$.

(3) The following statements are equivalent:
  - (a) the identity morphism on $K^\bullet$ is null-homotopic;
  - (b) the natural map $0 \longrightarrow K^\bullet$ is a homotopy equivalence, where 0 is the zero complex;
  - (c) there exists $\varphi \in \operatorname{Hom}(K^\bullet, K^\bullet)^{-1}$ such that $\partial_K \circ \varphi \circ \partial_K = \partial_K$ and $K^\bullet$ is exact.

(4) The converse of (2) need not hold: the complex $K^\bullet$ of $\mathbb{Z}$-modules

$$\cdots \xrightarrow{\ 2\ } \mathbb{Z}/4\mathbb{Z} \xrightarrow{\ 2\ } \mathbb{Z}/4\mathbb{Z} \xrightarrow{\ 2\ } \mathbb{Z}/4\mathbb{Z} \xrightarrow{\ 2\ } \cdots$$

has $H(K^\bullet) = 0$, but id_K is not null-homotopic.

Exercise 3.13. Let $K^\bullet$ be a complex of R-modules, and let $E^\bullet$ be a complex of injective R-modules which is bounded below.

(1) If $K^\bullet$ is exact, prove that $\operatorname{Hom}_R(K^\bullet, E^\bullet)$ is exact. Hint: First consider the case where $E^\bullet$ is concentrated in one degree.

(2) Let $\eta\colon K^\bullet \longrightarrow L^\bullet$ be a morphism of complexes. Prove that if η is a quasi-isomorphism, then so is the induced morphism of complexes

$$\operatorname{Hom}_R(L^\bullet, E^\bullet) \longrightarrow \operatorname{Hom}_R(K^\bullet, E^\bullet)\,.$$

Hint: use mapping cones, [**32**, Proposition A3.19], and (1).

(3) State and prove dual versions of these for the projective case.

3. Resolutions

We are now ready to define resolutions.

Definition 3.14. Let M be an R-module.

An *injective resolution* of M is a complex $E^\bullet$ of injective R-modules

$$0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots,$$

equipped with a quasi-isomorphism of complexes $\iota\colon M \longrightarrow E^\bullet$; here one views M as a complex concentrated in degree 0. Said otherwise, a complex $E^\bullet$ of injective modules is a resolution of M if there is an exact sequence

$$0 \longrightarrow M \xrightarrow{\iota} E^0 \longrightarrow E^2 \longrightarrow E^2 \longrightarrow \cdots.$$

A *projective resolution* of M is a complex $P_\bullet$ of projective R-modules

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0,$$

equipped with a quasi-isomorphism of complexes $\varepsilon\colon P_\bullet \longrightarrow M$. Replacing 'projective' by 'flat' defines flat resolutions.

Remark 3.15. Any R-module M *has* an injective and a projective resolution. Another way to say this is that the category of R-modules *has enough injectives* and *enough projectives.* (Since projectives are flat, there are, of course, also enough flats.) In contrast, the category of sheaves over projective space does *not* have enough projectives, as we will see in Lecture 12.

Slightly more subtle is the question of minimality. Let us deal with injective resolutions first; see the Appendix for terminology. We say that $E^\bullet$ as above is a *minimal* injective resolution if each E^i is the injective hull of the image of $E^{i-1} \longrightarrow E^i$, and E^0 is the injective hull of M. The proof of Theorem A.20 shows that E is an injective hull for a submodule M if and only if for all $\mathfrak{p} \in \operatorname{Spec} R$, the map $\operatorname{Hom}_R(R/\mathfrak{p}, M)_\mathfrak{p} \longrightarrow \operatorname{Hom}_R(R/\mathfrak{p}, E)_\mathfrak{p}$ is an isomorphism. Therefore, $E^\bullet$ is a minimal injective resolution if and only if the result of applying $\operatorname{Hom}_R(R/\mathfrak{p}, -)_\mathfrak{p}$ to the morphism $E^i \longrightarrow E^{i+1}$ gives the zero morphism for each $i \geqslant 0$.

The *injective dimension* of M, denoted $\operatorname{injdim}_R M$, is the minimal length of an injective resolution of M. If no resolution of finite length exists, we say $\operatorname{injdim}_R M = \infty$. We have $\operatorname{injdim}_R M = 0$ if and only if M is injective. Theorem A.24 discusses numerical invariants attached to a module via injective resolutions, namely the Bass numbers of M. Observe that all this bounty springs from the structure of injective modules over Noetherian rings, Theorem A.20.

In contrast, the theory of minimal projective resolutions works best over local rings R, where projective modules are free. See Lecture 8 for more in this direction. In any case, we define the *projective dimension* of M, denoted $\operatorname{pd}_R M$, as the minimal length of a projective resolution of M.

Finally, for completeness, we mention that the *flat dimension* is the minimal length of a flat resolution. For finitely generated modules over Noetherian rings, this is the same as projective dimension, so we will not have much need for it.

One main tool for proving the existence and uniqueness of derived functors will be the following theorem; see Remark 3.17 for a proof.

Theorem 3.16 (Comparison theorem)**.** *Let $\eta\colon K^\bullet \longrightarrow L^\bullet$ be a quasi-isomorphism of complexes of R-modules.*

(1) *Let $E^\bullet$ be a complex of injective R-modules with $E^i = 0$ for $i \ll 0$. Given a diagram of solid arrows of morphisms of complexes*

$$\begin{tikzcd} E^\bullet & \\ K^\bullet \arrow[u] \arrow[r, "\simeq", "\eta"'] & L^\bullet, \arrow[ul, dotted, "\kappa"'] \end{tikzcd}$$

there is a lifting κ which makes the diagram commute up to homotopy. Furthermore, κ is unique up to homotopy.

(2) *Let $P_\bullet$ be a complex of projective R-modules with $P_i = 0$ for $i \ll 0$. Given a diagram of solid arrows of morphisms of complexes*

$$\begin{tikzcd} & P_\bullet \arrow[d] \arrow[dl, dotted, "\kappa"'] \\ K^\bullet \arrow[r, "\simeq", "\eta"'] & L^\bullet, \end{tikzcd}$$

there is a lifting κ which makes the diagram commute up to homotopy. Furthermore, κ is unique up to homotopy. □

Remark 3.17. One may rephrase part (1) of the comparison theorem as follows: for a complex $E^\bullet$ of injective R-modules with $E^i = 0$ for $i \ll 0$, the functor $\operatorname{Hom}_R(-, E^\bullet)$ preserves quasi-isomorphisms. That is to say, if $\zeta\colon K^\bullet \longrightarrow L^\bullet$ is a quasi-isomorphism, the induced morphism of complexes

$$\operatorname{Hom}_R(L^\bullet, E^\bullet) \longrightarrow \operatorname{Hom}_R(K^\bullet, E^\bullet)$$

is a quasi-isomorphism.

Part (2) translates to the statement that if $P_\bullet$ is a complex of projective R-modules with $P_i = 0$ for $i \ll 0$, then the functor $\operatorname{Hom}_R(P_\bullet, -)$ preserves quasi-isomorphisms. Thus, Theorem 3.16 is a reformulation of Exercise 3.13.

Remark 3.18. The comparison theorem implies that injective and projective resolutions are unique up to homotopy. To see this, let M be an R-module, and let $\iota_1 \colon M \longrightarrow E_1^\bullet$ and $\iota_2 \colon M \longrightarrow E_2^\bullet$ be injective resolutions of M. Part (1) of the theorem, applied with $\eta = \iota_1$ and $E^\bullet = E_2^\bullet$, yields a morphism of complexes $\kappa_1 \colon E_1^\bullet \longrightarrow E_2^\bullet$ such that $\kappa_1 \circ \iota_1$ is homotopic to ι_2. By the same token, one obtains a morphism of complexes $\kappa_2 \colon E_2^\bullet \longrightarrow E_1^\bullet$ such that $\kappa_2 \circ \iota_2$ is homotopic to ι_1.

It then follows that the morphism $\kappa_2 \circ \kappa_1$ is a lifting (up to homotopy) of ι_1. However, the identity morphism on $E_1^\bullet$ is also a lifting of ι_1, so $\kappa_2 \circ \kappa_1$ must be homotopic to it, by the uniqueness part of the theorem above. By the same logic, $\kappa_1 \circ \kappa_2$ is homotopic to the identity on $E_2^\bullet$. Thus, κ_1 and κ_2 provide a homotopy equivalence between $E_1^\bullet$ and $E_2^\bullet$.

A similar argument, using part (2) of the comparison theorem, establishes the uniqueness, up to homotopy, of projective resolutions.

4. Derived functors

At last we define derived functors:

Definition 3.19. Let F be a left-exact additive covariant functor on the category of R-modules; think, for example, of $\operatorname{Hom}_R(L, -)$. Given an R-module M, let $E^\bullet$ be an injective resolution of M and set

$$R^i \mathrm{F}(M) = H^i(\mathrm{F}(E^\bullet)) \qquad \text{for } i \geqslant 0.$$

The module $R^i \mathrm{F}(M)$ is independent of the choice of the injective resolution: injective resolutions are unique up to homotopy by Remark 3.18, and additive functors preserve homotopies. The functor $R^i \mathrm{F}(-)$ is called the *i-th derived functor* of F.

There are corresponding definitions when F is contravariant, and/or when the functor is right-exact. We leave the formulations of these concepts to the reader.

Remark 3.20. Let F be a left-exact covariant additive functor on the category of R-modules. A module A is *acyclic* for F if $R^i \mathrm{F}(A) = 0$ for $i \geqslant 1$.

Now let M be an R-module and $\iota \colon M \longrightarrow A^\bullet$ a quasi-isomorphism, where $A^\bullet$ is a bounded below complex consisting of modules which are acyclic for F. It can then be proved that there is a natural isomorphism

$$R^i \mathrm{F}(M) \cong H^i(\mathrm{F}(A^\bullet)) \qquad \text{for each } i \geqslant 0.$$

The message is thus that derived functors can be computed with acyclic resolutions; see Proposition 2.25.

Remark 3.21. Let F be a left-exact covariant additive functor. Then

(1) A homomorphism $f\colon M \longrightarrow N$ gives rise to a family of homomorphisms $R^i\,\mathrm{F}(f)\colon R^i\,\mathrm{F}(M) \longrightarrow R^i\,\mathrm{F}(N)$ for $i \geqslant 0$. In particular, if F takes multiplication by an element $r \in R$ on M to multiplication by r on $\mathrm{F}(M)$, then each $R^i\,\mathrm{F}$ has the same property.

(2) Since F is left-exact, we have $R^0\,\mathrm{F} = \mathrm{F}$. Moreover, injective modules are acyclic for F.

(3) For each exact sequence of R-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0\,,$$

there are *connecting homomorphisms* δ^i and an exact sequence

$$\cdots \longrightarrow R^i\,\mathrm{F}(L) \longrightarrow R^i\,\mathrm{F}(M) \longrightarrow R^i\,\mathrm{F}(N) \xrightarrow{\delta^i} R^{i+1}\,\mathrm{F}(L) \longrightarrow \cdots\,.$$

Similar remarks apply to functors of various types of exactness and variance.

For our purposes, there are three main examples of derived functors. We define two of them here; the third will make its grand entrance in Lecture 7.

Definition 3.22. Let M and N be R-modules. Then

(1) $\{\mathrm{Ext}^i_R(M,-)\}_{i\geqslant 0}$ are the right derived functors of $\mathrm{Hom}_R(M,-)$, and

(2) $\{\mathrm{Tor}^R_i(-,N)\}_{i\geqslant 0}$ are the left derived functors of $-\otimes_R N$.

There are other possible descriptions of Ext: while we chose to use the right derived functors of the left-exact covariant functor $\mathrm{Hom}_R(M,-)$, we could also have used the right derived functors of the left-exact *contravariant* functor $\mathrm{Hom}_R(-,N)$. It is a theorem that the two approaches agree. More precisely, if $\varepsilon\colon P_\bullet \longrightarrow M$ is a projective resolution and $\iota\colon N \longrightarrow E^\bullet$ is an injective resolution, then, by Remark 3.17, the following induced morphisms of complexes are quasi-isomorphisms:

$$\mathrm{Hom}_R(P_\bullet,N) \xrightarrow{\mathrm{Hom}_R(P_\bullet,\iota)} \mathrm{Hom}_R(P_\bullet,E^\bullet) \xleftarrow{\mathrm{Hom}_R(\varepsilon,E^\bullet)} \mathrm{Hom}_R(M,E^\bullet)\,.$$

One therefore has isomorphisms of R-modules

$$H^i(\mathrm{Hom}_R(P_\bullet,N)) \cong H^i(\mathrm{Hom}_R(P_\bullet,E^\bullet)) \cong H^i(\mathrm{Hom}_R(M,E^\bullet)).$$

Similarly, $\mathrm{Tor}^R_i(M,N)$ can be computed either by applying $M\otimes_R -$ to a flat resolution of N, or by applying $-\otimes_R N$ to a flat resolution of M. The results are again naturally isomorphic.

Here are some examples of computing Tor and Ext.

Example 3.23. Let $R = \mathbb{K}[x, y, z]$ be a polynomial ring over a field $\mathbb{K}$. We assert that

$$0 \longrightarrow R \xrightarrow{\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]} R^3 \xrightarrow{\left[\begin{smallmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{smallmatrix}\right]} R^3 \xrightarrow{[x\ y\ z]} R \longrightarrow 0$$

is a free resolution of $R/(x, y, z) \cong \mathbb{K}$. You can check this directly, or wait until Lectures 6 and 8.

From the above resolution we can calculate $\mathrm{Tor}_i^R(\mathbb{K}, \mathbb{K})$ and $\mathrm{Ext}_R^i(\mathbb{K}, R)$ for all $i \geqslant 0$. For the Tor, we apply $- \otimes_R \mathbb{K}$ to the resolution. Each free module R^b becomes $R^b \otimes_R \mathbb{K} \cong \mathbb{K}^b$, and each matrix is reduced modulo the ideal (x, y, z). The result is the complex

$$0 \longrightarrow \mathbb{K} \xrightarrow{0} \mathbb{K}^3 \xrightarrow{0} \mathbb{K}^3 \xrightarrow{0} \mathbb{K} \longrightarrow 0$$

which has zero differentials at every step. Thus

$$\mathrm{Tor}_i^R(\mathbb{K}, \mathbb{K}) \cong \mathbb{K}^{\binom{3}{i}} \qquad \text{for all } i \geqslant 0.$$

Applying $\mathrm{Hom}_R(-, R)$ has the effect of replacing each matrix in our resolution of $\mathbb{K}$ by its transpose, which yields a complex

$$0 \longleftarrow R \xleftarrow{[x\ y\ z]} R^3 \xleftarrow{\left[\begin{smallmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{smallmatrix}\right]} R^3 \xleftarrow{\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]} R \longleftarrow 0\,.$$

Noting the striking similarity of this complex to the one we started with, we conclude that

$$\mathrm{Ext}_R^i(\mathbb{K}, R) \cong \begin{cases} \mathbb{K} & \text{if } i = 3, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.24. Let $\mathbb{K}$ again be a field, and put $R = \mathbb{K}[x, y]/(xy)$. Set $M = R/(x)$ and $N = R/(y)$. To compute Tor and Ext, let us start with a projective resolution $P_\bullet$ of M. One such is the complex

$$\cdots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \longrightarrow 0\,.$$

In order to compute $\mathrm{Tor}_i^R(M, N)$, we apply $- \otimes_R N$ to $P_\bullet$. This replaces each copy of R by $N = R/(y)$, giving us a complex

$$\cdots \xrightarrow{x} R/(y) \xrightarrow{y} R/(y) \xrightarrow{x} R/(y) \longrightarrow 0\,.$$

Since y kills $R/(y)$ while x is a nonzerodivisor on $R/(y)$, computing kernels and images reveals that

$$\mathrm{Tor}_i^R(M, N) \cong \begin{cases} \mathbb{K} & \text{for } i \geqslant 0 \text{ even, and} \\ 0 & \text{for } i \geqslant 1 \text{ odd.} \end{cases}$$

Similarly, applying $\mathrm{Hom}_R(-, N)$ replaces each R by $N = R/(y)$, but this time reverses all the arrows, yielding the complex

$$\cdots \xleftarrow{x} R/(y) \xleftarrow{y} R/(y) \xleftarrow{x} R/(y) \longleftarrow 0\,.$$

Taking the cohomology of this complex, we see that

$$\operatorname{Ext}_R^i(M,N) \cong \begin{cases} 0 & \text{for } i \geqslant 0 \text{ even, and} \\ \mathbb{K} & \text{for } i \geqslant 1 \text{ odd.} \end{cases}$$

Finally, apply $\operatorname{Hom}_R(-,R)$ to find that

$$\operatorname{Ext}_R^i(M,R) \cong \begin{cases} M & \text{for } i = 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

We end this section with the properties of Ext and Tor used repeatedly in the remaining lectures. They follow directly from the definitions of injective, projective, and flat modules, and Remark 3.21.

Theorem 3.25. *Consider an exact sequence of R-modules*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0\,.$$

For each R-module N, one has exact sequences:

(1) $$\cdots \longrightarrow \operatorname{Ext}_R^i(N,M) \longrightarrow \operatorname{Ext}_R^i(N,M'') \longrightarrow \operatorname{Ext}_R^{i+1}(N,M') \longrightarrow \operatorname{Ext}_R^{i+1}(N,M) \longrightarrow \cdots\,,$$

(2) $$\cdots \longrightarrow \operatorname{Ext}_R^i(M,N) \longrightarrow \operatorname{Ext}_R^i(M',N) \longrightarrow \operatorname{Ext}_R^{i+1}(M'',N) \longrightarrow \operatorname{Ext}_R^{i+1}(M,N) \longrightarrow \cdots\,,$$

(3) $$\cdots \longrightarrow \operatorname{Tor}^R_{i+1}(M,N) \longrightarrow \operatorname{Tor}^R_{i+1}(M'',N) \longrightarrow \operatorname{Tor}^R_i(M',N) \longrightarrow \operatorname{Tor}^R_i(M,N) \longrightarrow \cdots\,. \qquad \square$$

Theorem 3.26. *Let M be an R-module.*

(1) *The module M is injective if and only if $\operatorname{Ext}_R^i(-,M) = 0$ for all $i \geqslant 1$, and this occurs if and only if $\operatorname{Ext}_R^1(-,M) = 0$.*
(2) *The module M is projective if and only if $\operatorname{Ext}_R^i(M,-) = 0$ for all $i \geqslant 1$, and this occurs if and only if $\operatorname{Ext}_R^1(M,-) = 0$.*
(3) *Lastly, M is flat if and only if $\operatorname{Tor}^R_i(-,M) = 0$ for all $i \geqslant 1$, and this is equivalent to $\operatorname{Tor}^R_1(-,M) = 0$.* $\square$

Lecture 4

Limits

In this lecture we discuss direct limits and inverse limits. These play an important—if technical—role in what follows. We are interested mainly in the category of modules over a ring and of sheaves on a space. However, the material is better presented in the context of Abelian categories.

1. An example from topology

The Seifert-van Kampen theorem in algebraic topology expresses the fundamental group of a space in terms of fundamental groups of subsets:

Theorem 4.1. *Let X be a topological space and $X = U_a \cup U_b$ an open cover with U_a, U_b, and $U_c = U_a \cap U_b$ connected. Set $G_i = \pi_1(U_i)$ for $i = a, b, c$. Let $\varphi_{c,a} \colon G_c \longrightarrow G_a$ and $\varphi_{c,b} \colon G_c \longrightarrow G_b$ be the induced homomorphisms of groups. The group $G = \pi_1(X)$ is determined by the following information.*

(1) *There are group homomorphisms $\varphi_a \colon G_a \longrightarrow G$ and $\varphi_b \colon G_b \longrightarrow G$ such that $\varphi_a \circ \varphi_{c,a} = \varphi_b \circ \varphi_{c,b}$.*

(2) *Given homomorphisms of groups ψ_a and ψ_b as below*

$$\begin{tikzcd}
 & G_a \arrow[dr, "\varphi_a"] \arrow[drr, "\psi_a"] & & \\
G_c \arrow[ur, "\varphi_{c,a}"] \arrow[dr, "\varphi_{c,b}"'] & & G \arrow[r, dashed] & H \\
 & G_b \arrow[ur, "\varphi_b"'] \arrow[urr, "\psi_b"'] & &
\end{tikzcd}$$

with $\psi_a \circ \varphi_{c,a} = \psi_b \circ \varphi_{c,b}$, there is a unique homomorphism of groups $G \dashrightarrow H$ for which the resulting diagram is commutative. $\square$

Exercise 4.2. Let $X = \mathbb{S}^1$, and let $U_a = X \smallsetminus \{-1\}$, $U_b = X \smallsetminus \{1\}$ be an open cover of X. What does Theorem 4.1 say about $\pi_1(X)$?

Exercise 4.3. A diagram of homomorphisms of groups

$$\begin{tikzcd}[row sep=small] & G_a \\ G_c \arrow[ur] \arrow[dr] & \\ & G_b \end{tikzcd}$$

as in the theorem above is known as a *pushout diagram*, and any group G satisfying conditions (1) and (2) is called the *amalgamated sum* of G_a and G_b over G_c; it is denoted $G_a *_{G_c} G_b$. Prove that the amalgamated sum is unique up to a unique isomorphism of groups.

The amalgamated sum can be constructed as follows: On the collection of words w in the alphabet consisting of the disjoint union of elements in G_a and G_b, consider the equivalence relation on the words induced by the relations: $w \sim w'$ if w arises from w' in any of the following ways:

(1) insertion of an identity element from G_a or G_b;

(2) replacing two consecutive letters from the same G_i by their product;

(3) replacing a letter $g_i \in G_i \cap \operatorname{image}(\varphi_{c,i})$ by $\varphi_{c,j}(g_c)$ where $\varphi_{c,i}(g_c) = g_i$ for $\{i, j\} = \{a, b\}$.

The quotient set is a group under composition of words; the inverse of $w = g_1 g_2 \cdots g_k$ is the word $g_k^{-1} \cdots g_2^{-1} g_1^{-1}$. This is the amalgamated sum.

Example 4.4. Set $F_1 = \mathbb{Z}$ and for $t \geqslant 1$ let $F_{t+1} = F_t *_1 \mathbb{Z}$, the amalgamated sum of F_t and the integers over the subgroup consisting of the identity. The group F_t is the *free group on t letters*; it consists of all words that can be formed from t distinct letters and their formal inverses.

The pushout of groups is an instance of a direct limit.

2. Direct limits

To begin with, let us call the underlying structure of a pushout diagram by its proper name: a partially ordered set, or *poset*. We write $\leqslant$ for the order relation on a poset.

Definition 4.5. Let I be a poset. We view it as a category whose objects are the elements of I, with a morphism from i to j whenever $i \leqslant j$; this morphism is also denoted $i \leqslant j$. Note that composition is defined when possible because $i \leqslant j$ and $j \leqslant k$ imply $i \leqslant k$, since I is a poset.

Let $\mathcal{A}$ be a category. An *I-diagram* in $\mathcal{A}$, also called a *directed system*, is a covariant functor $\Phi\colon I \longrightarrow \mathcal{A}$. In effect, Φ decorates elements of I with objects of $\mathcal{A}$ and morphisms in I with morphisms in $\mathcal{A}$ such that the resulting diagram in $\mathcal{A}$ is commutative. We often write A_i for $\Phi(i)$ and $\varphi_{i,j}$ for $\Phi(i \leqslant j)$. Note that $\varphi_{i,i} = \mathrm{id}_{A_i}$.

Definition 4.6. Let $\mathcal{A}$ be a category and Φ an I-diagram in $\mathcal{A}$. The *direct limit* of Φ is an object A of $\mathcal{A}$ together with morphisms $\varphi_i \colon A_i \longrightarrow A$ for each $i \in I$ such that the diagram

$$\begin{tikzcd} A_i \arrow[d, "\varphi_{i,j}"'] \arrow[dr, "\varphi_i"] & \\ A_j \arrow[r, "\varphi_j"'] & A \end{tikzcd}$$

commutes for each $i \leqslant j$. Moreover, A should be universal with respect to this data. In other words, given an object A' and morphisms $\psi_i \colon A_i \longrightarrow A'$ satisfying commutative diagrams akin to the ones above, there is a unique $\mathcal{A}$-morphism $\psi\colon\ A \dashrightarrow A'$ such that for each i the diagram below commutes:

$$\begin{tikzcd} A_i \arrow[dr, "\varphi_i"'] \arrow[drr, "\psi_i"] & & \\ & A \arrow[r, dashed, "\psi"'] & A' . \end{tikzcd}$$

The direct limit is denoted $\varinjlim_I \Phi$ or $\varinjlim_I A_i$; when there is no scope for confusion we sometimes drop the subscript I. The following exercise justifies the use of the phrase '*the* direct limit.'

Exercise 4.7. Prove that direct limits, when they exist, are unique up to a unique isomorphism, compatible with the structure morphisms φ_i.

An explicit construction of a direct limit typically depends on I, $\mathcal{A}$, and the functor Φ. For instance, if distinct elements of I are incomparable, then $\varinjlim A_i$ is the *coproduct* $\coprod_I A_i$. Thus, the existence of direct limits in $\mathcal{A}$ over every index set implies the existence of arbitrary coproducts; the converse holds when $\mathcal{A}$ is Abelian; see [**161**, (2.6.8)]. In particular, not all direct limits exist in the category of finite Abelian groups.

Now we focus on direct limits in the category of modules over a ring.

Exercise 4.8. Let I be a poset and Φ an I-diagram in the category of modules over a ring R. Let E be the submodule of $\bigoplus A_i$ spanned by

$$\{(\ldots, 0, a_i, 0, \ldots, 0, -\varphi_{i,j}(a_i), 0, \ldots) \mid i \leqslant j \text{ and } a_i \in A_i\}.$$

Prove that $\varinjlim A_i = \bigoplus A_i/E$, with the structure map φ_i equal to the composition $A_i \longrightarrow \bigoplus A_i \longrightarrow \varinjlim A_i$.

Exercise 4.9. Prove that if $\{A_i\}_{i\in\mathbb{N}}$ is an $\mathbb{N}$-diagram in a category of modules, where the A_i are all submodules of a given module, and the morphisms $A_i \longrightarrow A_{i+1}$ are the natural inclusions, then $\varinjlim A_i = \bigcup A_i$.

The following exercise is an example of Theorem 3.8 in action.

Exercise 4.10. Let M be an R-module and x an element in R.

(1) Prove that M_x is the direct limit of the $\mathbb{N}$-diagram
$$M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} \cdots .$$
(2) Set $M = R$. Show that $(r_i \in R_i) \times (r_{i'} \in R_{i'}) \longmapsto (r_i r_{i'} \in R_{i+i'})$ defines a map of $\mathbb{N}$-diagrams. Deduce that it yields a multiplication on the limit, which agrees with the usual multiplication of R_x.

(3) Let $\mathbb{K}$ be a field. Show that the $\mathbb{K}[x]$-module $\mathbb{K}[x, x^{-1}]$ is flat and that it is not projective.

3. The category of diagrams

Let I be a poset, and $\mathcal{A}$ an Abelian category in which all I-diagrams in $\mathcal{A}$ have a direct limit. The collection of all I-diagrams in $\mathcal{A}$ forms the object set of a category $\mathfrak{Dir}_I^{\mathcal{A}}$; its morphisms are the natural transformations between the I-diagrams. Thus, a morphism of diagrams $\eta\colon \Phi \longrightarrow \Phi'$ specifies, for each $i \leqslant j$ in I, a commutative diagram

$$\begin{tikzcd} A_i \arrow[r, "\eta_i"] \arrow[d, "\varphi_{i,j}"'] & A'_i \arrow[d, "\varphi'_{i,j}"] \\ A_j \arrow[r, "\eta_j"] & A'_j \end{tikzcd}$$

The discussion in the following paragraphs is summed up in the next result. You are free to apply it to the case where $\mathcal{A}$ is the category of modules over a ring, once you solve Exercise 4.8.

Theorem 4.11. *Let I be a poset and $\mathcal{A}$ an Abelian category in which every I-diagram has a direct limit. The following statements hold.*

(1) *The category $\mathfrak{Dir}_I^{\mathcal{A}}$ is an Abelian category.*

(2) *The map $\varinjlim\colon \mathfrak{Dir}_I^{\mathcal{A}} \longrightarrow \mathcal{A}$ is an additive functor.*

(3) *For any poset J, the category $\mathfrak{Dir}_J^{\mathcal{A}}$ has direct limits for all I-diagrams.*

(1) and (2) are justified in the discussion below, while (3) is proved in Section 5. Let us first describe the Abelian category structure on $\mathfrak{Dir}_I^{\mathcal{A}}$.

Definition 4.12. Let $\eta\colon \Phi \longrightarrow \Phi'$ be a morphism in $\mathfrak{Dir}_I^{\mathcal{A}}$.

The *kernel* of η is the I-diagram in $\mathcal{A}$ defined by $\ker(\eta)_i = \ker(\eta_i) \subseteq A_i$, for each i in I, and structure maps $\ker(\eta)_i \longrightarrow \ker(\eta)_j$ given by restricting $\varphi_{i,j}\colon A_i \longrightarrow A_j$, for each $i \leqslant j$ in I. Note that restriction does give the desired morphism because of the commutativity of the rectangle above.

The *cokernel* of η is the I-diagram in $\mathcal{A}$ defined by $\operatorname{coker}(\eta)_i = \operatorname{coker}(\eta_i)$, and structure maps $\operatorname{coker}(\eta)_i \longrightarrow \operatorname{coker}(\eta)_j$ induced by $\varphi'_{i,j} \colon A'_i \longrightarrow A'_j$.

Exercise 4.13. Prove that $\mathfrak{Dir}_I^{\mathcal{A}}$ is an Abelian category, with kernels and cokernels as defined above.

This explains the first part of Theorem 4.11. Next note that each morphism $\eta \colon \Phi \longrightarrow \Phi'$ yields a commutative diagram of solid arrows

$$\begin{tikzcd}
A_i \arrow[rr, "\eta_i"] \arrow[d, "\varphi_{i,j}"'] \arrow[drr, "\varphi_i"] & & A'_i \arrow[d, "\varphi'_{i,j}"'] \arrow[drr, "\varphi'_i"] & & \\
A_j \arrow[rr, "\eta_j"] \arrow[dr, "\varphi_j"'] & & A'_j \arrow[drr, "\varphi'_j"'] & & \\
& \varinjlim A_i \arrow[rrr, dashed, "\varinjlim \eta"] & & & \varinjlim A'_i
\end{tikzcd}$$

The composed maps $A_i \longrightarrow A'_i \longrightarrow \varinjlim A'_i$ are compatible, by commutativity of the diagrams; they induce $\varinjlim \eta$.

Given morphisms of I-diagrams $\Phi \xrightarrow{\eta} \Phi' \xrightarrow{\eta'} \Phi''$, the universal property of direct limits ensures that the induced diagram

$$\begin{tikzcd}
\varinjlim A_i \arrow[rr, "\varinjlim \eta"] \arrow[dr, "\varinjlim (\eta' \circ \eta)"'] & & \varinjlim A'_i \arrow[dl, "\varinjlim \eta'"] \\
& \varinjlim A''_i &
\end{tikzcd}$$

of morphisms in $\mathcal{A}$ commutes. Thus, $\varinjlim$ is a functor.

Exercise 4.14. Verify that $\varinjlim$ is an *additive* functor. That is, the map $\operatorname{Hom}_{\mathfrak{Dir}_I^{\mathcal{A}}}(\Phi', \Phi) \longrightarrow \operatorname{Hom}_{\mathcal{A}}\left(\varinjlim \Phi', \varinjlim \Phi\right)$ sending η to $\varinjlim \eta$ is a homomorphism of groups, natural with respect to composition of morphisms.

We now come to the central issue of this lecture.

4. Exactness

Let $\mathcal{A}$ be an Abelian category and I a poset. A diagram of morphisms

$$0 \longrightarrow \Phi \xrightarrow{\ \eta\ } \Phi' \xrightarrow{\ \eta'\ } \Phi'' \longrightarrow 0$$

in $\mathfrak{Dir}_I^{\mathcal{A}}$ is said to be *exact* if η is a monomorphism, η' is an epimorphism, η induces an isomorphism between Φ and $\ker(\eta')$, and η' induces an isomorphism between $\operatorname{coker}(\eta)$ and Φ''; note that this implies $\eta' \circ \eta = 0$. In a down-to-earth language, exactness of the sequence above is equivalent to the exactness of the sequences $0 \longrightarrow A_i \longrightarrow A'_i \longrightarrow A''_i \longrightarrow 0$ in $\mathcal{A}$.

Given such an exact sequence, applying $\varinjlim$ yields a sequence in $\mathcal{A}$:

$$0 \longrightarrow \varinjlim \Phi \longrightarrow \varinjlim \Phi' \longrightarrow \varinjlim \Phi'' \longrightarrow 0\,.$$

In general, this sequence is not exact.

Example 4.15. Let I be the pushout poset, and consider the sequence

$$0 \longrightarrow \left(\begin{tikzcd}[row sep=tiny] & \mathbb{Z} \\ 0 \arrow[ur] \arrow[dr] & \\ & \mathbb{Z}\end{tikzcd}\right) \longrightarrow \left(\begin{tikzcd}[row sep=tiny] & \mathbb{Z} \\ \mathbb{Z} \arrow[ur] \arrow[dr] & \\ & \mathbb{Z}\end{tikzcd}\right) \longrightarrow \left(\begin{tikzcd}[row sep=tiny] & 0 \\ \mathbb{Z} \arrow[ur] \arrow[dr] & \\ & 0\end{tikzcd}\right) \longrightarrow 0$$

of I-diagrams of Abelian groups, where the maps are identities whenever possible. Using Exercise 4.8 and applying $\varinjlim$ to the exact sequence of diagrams, one gets a sequence of Abelian groups

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow 0\,.$$

In the example above the limit sequence is right-exact; this holds in general. A good way to understand this phenomenon is through the notion of adjoint functors. We explain this next.

Definition 4.16. Consider functors $\mathrm{F}\colon \mathcal{B} \longrightarrow \mathcal{C}$ and $\mathrm{G}\colon \mathcal{C} \longrightarrow \mathcal{B}$. One says that (F, G) form an *adjoint pair* if there is a natural identification

$$\mathrm{Hom}_{\mathcal{B}}(B, \mathrm{G}(C)) \longleftrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathrm{F}(B), C)$$

as functors from $\mathcal{B} \times \mathcal{C}$ to the category of sets. In this situation, one says that F is the *left adjoint* of G and that G is the *right adjoint* of F.

Example 4.17. Let R be a commutative ring and M an R-module. Then $\big(M \otimes_R (-), \mathrm{Hom}_R(M, -)\big)$ is an adjoint pair of functors of R-modules.

We need the following result on adjoint pairs [**161**, Theorem A.6.2].

Proposition 4.18. *If* (F, G) *is an adjoint pair of functors between categories* $\mathcal{B}$ *and* $\mathcal{C}$*, then the following statements hold.*

(1) *There are natural transformations, called adjunction morphisms,*

$$\sigma\colon\ \mathrm{F} \circ \mathrm{G} \longrightarrow \mathrm{Id}_{\mathcal{C}} \qquad \textit{and} \qquad \tau\colon\ \mathrm{Id}_{\mathcal{B}} \longrightarrow \mathrm{G} \circ \mathrm{F}$$

where the composed maps $\mathrm{F} \xrightarrow{\mathrm{F}\,\tau} \mathrm{F\,G\,F} \xrightarrow{\sigma\,\mathrm{F}} \mathrm{F}$ *and* $\mathrm{G} \xrightarrow{\tau\,\mathrm{G}} \mathrm{G\,F\,G} \xrightarrow{\mathrm{G}\,\sigma} \mathrm{G}$ *are the identity transformations on* F *and* G *respectively.*

(2) *The adjunctions determine the identification in Definition* 4.16*.*

(3) *If in addition the categories* $\mathcal{B}$ *and* $\mathcal{C}$ *are Abelian, and the functors* F *and* G *are additive, then* F *is right-exact and* G *is left-exact.* □

Now we return to the category of diagrams.

Definition 4.19. Let I be a poset and $\mathcal{A}$ an Abelian category in which every I-diagram has a limit. For each object A of $\mathcal{A}$, let A_I denote the *constant diagram* on A; thus, $(A_I)_i = A$ and $A_I(i \leqslant j)$ is the identity on A.

Exercise 4.20. Prove that the functors $\big(\varinjlim, (-)_I\big)$ form an adjoint pair. Conversely, if β is left adjoint to $(-)_I$, then $\varinjlim \Phi = \beta(\Phi)$ for each $\Phi \in \mathfrak{Dir}_I^{\mathcal{A}}$.

The preceding exercise and Proposition 4.18 yield the following result.

Proposition 4.21. *The functor* $\varinjlim \colon \mathfrak{Dir}_I^{\mathcal{A}} \longrightarrow \mathcal{A}$ *is right-exact.* □

The question of left-exactness of direct limits is addressed later in the section. The next result gives a way to compute direct limits.

Theorem 4.22. *Let* (F, G) *be an adjoint pair of functors between categories* $\mathcal{B}$ *and* $\mathcal{C}$. *If* Φ *is an* I*-diagram in* $\mathcal{B}$ *which has a direct limit, then the direct limit of the* I*-diagram* $\mathrm{F} \circ \Phi$ *in* $\mathcal{C}$ *exists, and is equal to* $\mathrm{F}\big(\varinjlim B_i\big)$.

Proof. For each C in $\mathcal{C}$, Exercise 4.26 provides the first and the last of the following isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{Dir}_I^{\mathcal{C}}}\big(\mathrm{F} \circ \Phi, C_I\big) &\cong \mathrm{Hom}_{\mathfrak{Dir}_I^{\mathcal{B}}}\big(\Phi, \mathrm{G} \circ C_I\big) \\
&\cong \mathrm{Hom}_{\mathfrak{Dir}_I^{\mathcal{B}}}\big(\Phi, \mathrm{G}(C)_I\big) \\
&\cong \mathrm{Hom}_{\mathcal{B}}\big(\varinjlim \Phi, \mathrm{G}(C)\big) \\
&\cong \mathrm{Hom}_{\mathcal{C}}\big(\mathrm{F}(\varinjlim \Phi), C\big)\,.
\end{aligned}$$

The second isomorphism holds by inspection, while the third holds because $\big(\varinjlim, (-)_I\big)$ is an adjoint pair; see Exercise 4.20. □

Here is one application of this result.

Example 4.23. Let $\mathcal{A}$ be the category of modules over a commutative ring R, and let M be an R-module. The functor $\mathrm{F}(-) = M \otimes_R -$ is left adjoint to $\mathrm{Hom}_R(M, -)$. Therefore, by the theorem above, if $\{A_i\}_{i \in I}$ is an I-diagram in $\mathcal{A}$, then there is a natural identification

$$\varinjlim (M \otimes_R A_i) = M \otimes_R \varinjlim (A_i)\,.$$

Thus, direct limits commute with tensor products.

The preceding statement ought to be treated with some care:

Exercise 4.24. Let I be a poset. Let R be a commutative ring, and let $\{A_i\}_{i \in I}$ and $\{A'_i\}_{i \in I}$ be I-diagrams of R-modules.

(1) Prove that there is a natural homomorphism of R-modules

$$\varinjlim (A_i \otimes_R A'_i) \longrightarrow \varinjlim (A_i) \otimes_R \varinjlim (A'_i)\,.$$

(2) Find an example where the map is not bijective. Hint: pushouts!

Next we apply Theorem 4.22 to study limits in a category of diagrams.

5. Diagrams over diagrams

Let I and J be posets, and $\mathcal{A}$ a category. Observe that $I \times J$ is a poset with order $(i,j) \leqslant (i',j')$ if $i \leqslant i'$ and $j \leqslant j'$.

Exercise 4.25. Prove that there is a natural isomorphism of categories

$$\mathfrak{Dir}_I^{\mathfrak{Dir}_J^{\mathcal{A}}} \cong \mathfrak{Dir}_{I\times J}^{\mathcal{A}}\,.$$

In particular, the categories $\mathfrak{Dir}_I^{\mathfrak{Dir}_J^{\mathcal{A}}}$ and $\mathfrak{Dir}_J^{\mathfrak{Dir}_I^{\mathcal{A}}}$ are isomorphic.

Exercise 4.26. Let (F,G) be an adjoint pair of functors between categories $\mathcal{B}$ and $\mathcal{C}$. Prove that the functors

$$\begin{tikzcd} \mathfrak{Dir}_J^{\mathcal{B}} \arrow[r, shift left, "\beta"] & \mathfrak{Dir}_J^{\mathcal{C}} \arrow[l, shift left, "\zeta"] \end{tikzcd}$$

where $\beta(\Phi) = \mathrm{F}\circ\Phi$ and $\zeta(\Psi) = \mathrm{G}\circ\Psi$ form an adjoint pair.

Proof of Theorem 4.11(3). Since each I-diagram in $\mathcal{A}$ has a direct limit, there is an adjoint pair of functors:

$$\begin{tikzcd} \mathfrak{Dir}_I^{\mathcal{A}} \arrow[r, shift left, "\varinjlim"] & \mathcal{A} \arrow[l, shift left, "(-)_I"] \end{tikzcd}$$

where $(-)_I$ is the constant diagram functor; see Exercise 4.20. According to Exercise 4.26, this yields an adjoint pair

$$\begin{tikzcd} \mathfrak{Dir}_J^{\mathfrak{Dir}_I^{\mathcal{A}}} \arrow[r, shift left] & \mathfrak{Dir}_J^{\mathcal{A}} \arrow[l, shift left] \end{tikzcd}\,.$$

In view of Exercise 4.25, one thus obtains an adjoint pair

$$\begin{tikzcd} \mathfrak{Dir}_I^{\mathfrak{Dir}_J^{\mathcal{A}}} \arrow[r, shift left, "\beta"] & \mathfrak{Dir}_J^{\mathcal{A}} \arrow[l, shift left, "\alpha"] \end{tikzcd}\,.$$

Moreover, for each Ψ in $\mathfrak{Dir}_J^{\mathcal{A}}$, it follows by construction that $\alpha(\Psi)$ is equal to $(\Psi)_I$, that is to say, to the constant I-diagram in $\mathfrak{Dir}_J^{\mathcal{A}}$ with value Ψ. Since β is its left adjoint, it follows that any I-diagram Φ in $\mathfrak{Dir}_J^{\mathcal{A}}$ admits a direct limit, equal to $\beta(\Phi)$; see Exercise 4.20. $\square$

Remark 4.27. A complex in a category $\mathcal{A}$ may be viewed as a $\mathbb{Z}$-diagram in $\mathcal{A}$. It follows that if Φ is a directed system of complexes in $\mathcal{A}$, indexed by I, then there is a limit complex.

Now we discuss direct limits of diagrams of a product of posets. The gist of the following result is that direct limits commute.

Theorem 4.28. *Let I and J be posets, and let $\mathcal{A}$ be a category which has direct limits for all I-diagrams and all J-diagrams. Then any diagram Φ over $I \times J$ admits a direct limit as well, and there are natural isomorphisms*

$$\varinjlim_J \big(\varinjlim_I \Phi\big) \cong \varinjlim_{I\times J} \Phi \cong \varinjlim_I \big(\varinjlim_J \Phi\big).$$

Proof. By Theorem 4.11, the categories $\mathfrak{Dir}_I^{\mathcal{A}}$ and $\mathfrak{Dir}_J^{\mathcal{A}}$ have direct limits of J-diagrams and I-diagrams, respectively. One thus has adjoint functors

$$\mathfrak{Dir}_I^{\mathfrak{Dir}_J^{\mathcal{A}}} \underset{(-)_I}{\overset{\varinjlim_I(-)}{\rightleftarrows}} \mathfrak{Dir}_J^{\mathcal{A}} \underset{(-)_J}{\overset{\varinjlim_J(-)}{\rightleftarrows}} \mathcal{A}\,.$$

Therefore, the composition $\varinjlim_J(-)\circ\varinjlim_I(-)$ is left adjoint to the composition of functors $(-)_J$ and $(-)_I$. It remains to recall Exercise 4.25. $\square$

Because a direct sum is a direct limit over its index set, with no nontrivial relations, one has the following corollary.

Corollary 4.29. *Direct limits commute with arbitrary direct sums.* $\square$

6. Filtered posets

Recall that the direct limit functor is not left-exact. General principles of homological algebra suggest that one should study acyclic objects and left derived functors with respect to $\varinjlim$. We focus first on a property of I that forces all I-diagrams to be $\varinjlim$-acyclic.

Definition 4.30. A poset I is said to be *filtered* if for each pair of elements $i, j \in I$ there exists $k \in I$ such that $i \leqslant k$ and $j \leqslant k$.

The word 'directed' also occurs in the literature in lieu of 'filtered.'

Example 4.31. Here are some important examples of filtered posets.

(1) The poset of natural numbers $\mathbb{N}$ is filtered.

(2) Let R be a ring and M an R-module. The collection of finitely generated submodules of M, ordered by inclusion, is filtered.

(3) Let X be a topological space, and x a point in X. The poset of open sets containing x, ordered by inclusion, is filtered.

(4) Let X be a topological space. The collection of open covers is a filtered poset, where the partial order is refinement.

On the other hand, the pushout diagram is not filtered.

The following result sums up the crucial property of filtered posets. Recall the construction of direct limits in Exercise 4.8.

Lemma 4.32. *Let I be a filtered poset. Let R be a ring and let $\mathcal{A}$ denote the category of R-modules. Let Φ be an I-diagram in $\mathcal{A}$.*

Let a be an element in $\bigoplus_I A_i$ and $\overline{a}$ its image in $\varinjlim \Phi$.

(1) *There exists i in I and an element a_i in A_i such that $\varphi_i(a_i) = \overline{a}$.*

(2) *Write $a = (a_i) \in \bigoplus_I A_i$. Then $\overline{a} = 0$ if and only if there exists an index t such that $a_i = 0$ when $i \nleqslant t$ and $\sum_{j \leqslant t} \varphi_{j,t}(a_j) = 0$.*

Proof. (1) is immediate from Exercise 4.8 and the defining property of filtered posets. As to (2), it is clear that if the stated conditions hold, then $\overline{a} = 0$. Suppose $\overline{a} = 0$. Consider first the case where a is concentrated in a single degree j. Then, by the construction of direct limits, there exist finitely many $(k, l) \in I \times I$, with $k \leqslant l$, and elements b_{kl} in A_k such that

$$a_j = \sum_{(k,l)} \big(b_{kl} - \varphi_{k,l}(b_{kl})\big) \qquad \text{in } \bigoplus A_i \,.$$

Let t be an index dominating j and each of the finitely many l. Then

$$\begin{aligned}\varphi_{j,t}(a_j) = (-a_j) - \varphi_{j,t}(-a_j) &+ \sum_{(k,l)} \big(b_{kl} - \varphi_{k,t}(b_{kl})\big) \\ &+ \sum_{(k,l)} \big[\varphi_{k,l}(-b_{kl}) - \varphi_{l,t}\big(\varphi_{k,l}(-b_{kl})\big)\big] \,.\end{aligned}$$

Rearranging terms, we may rewrite this equality as

$$\varphi_{j,t}(a_j) = \sum_{i \leqslant t} \big(b_i - \varphi_{i,t}(b_i)\big)$$

where b_i is in A_i and each i appears at most once in the sum. This is a statement about elements in a direct sum, so the terms on the left and on the right are equal componentwise. Hence $b_i = 0$ for each $i \neq t$, and then the equality above simplifies to $\varphi_{j,t}(a_j) = b_t - \varphi_{t,t}(b_t) = 0$. This is as desired.

Now we tackle the general case. Only finitely many of the a_i are nonzero and I is filtered, so there is an index j such that $i \leqslant j$ for every i with $a_i \neq 0$. Set $b = \sum_{i \leqslant j} \varphi_{i,j}(a_i)$. Since

$$a - b = \sum_{i \leqslant j} \big(a_i - \varphi_{i,j}(a_i)\big) \,,$$

one has that $\overline{a} = \overline{b} = 0$ in $\varinjlim A_i$. As $b \in A_j$, the already established case of the result provides an index $t \geqslant j$ such that $\varphi_{j,t}(b) = 0$. It is easy to check that this t has the desired properties. $\square$

Theorem 4.33. *Let I be a filtered poset, R a ring, and $\mathcal{A}$ the category of R-modules. The functor $\varinjlim\colon \mathfrak{Dir}_I^{\mathcal{A}} \longrightarrow \mathcal{A}$ is exact.*

Proof. Given Proposition 4.21, it remains to prove that $\varinjlim$ preserves injectivity. Let $\Phi \xrightarrow{\eta} \Phi'$ be an injective morphism of I-diagrams. Suppose $\overline{a} = \overline{(a_i)}$ in $\varinjlim A_i$ is in the kernel of $\varinjlim \eta$. By Lemma 4.32 applied to $\eta(a)$, there is an index t in I with $t \geqslant i$ for all $a_i \neq 0$ such that

$$\eta_t\Big(\sum_i \varphi_{i,t}(a_i)\Big) = \sum_i \varphi_{i,t}(\eta_i(a_i)) = 0\,.$$

Since η_t is a monomorphism, one concludes that $\sum_i \varphi_{i,t}(a_i) = 0$. It remains to observe that this sum represents $\overline{a}$. $\square$

We present some applications of this result. The long and short of the one below is that direct limits on filtered posets commute with homology.

Exercise 4.34. Let R be a ring, I a filtered poset, and

$$\cdots \longrightarrow \Phi^{(n-1)} \xrightarrow{\partial^{(n-1)}} \Phi^{(n)} \xrightarrow{\partial^{(n)}} \Phi^{(n+1)} \longrightarrow \cdots$$

a complex of I-diagrams of R-modules. Prove the following claims.

(1) When the complex is exact, so is the resulting sequence of direct limits.

(2) Consider the homology I-diagram $H^{(n)}$ defined by

$$H_i^{(n)} = \ker(\partial^{(n)})_i / \operatorname{image}(\partial^{(n-1)})_i\,.$$

Then $\varinjlim H^{(n)}$ is naturally isomorphic to the homology in degree n of the induced complex

$$\cdots \longrightarrow \varinjlim \Phi^{(n-1)} \longrightarrow \varinjlim \Phi^{(n)} \longrightarrow \varinjlim \Phi^{(n+1)} \longrightarrow \cdots\,.$$

The following exercise complements Exercise 4.24.

Exercise 4.35. Let I be a filtered poset. Let R be a commutative ring, and let $\{A_i\}$ and $\{A_i'\}$ be I-diagrams of R-modules. Prove that the following natural homomorphism of R-modules from Exercise 4.24(1) is bijective:

$$\varinjlim(A_i \otimes_R A_i') \longrightarrow \varinjlim(A_i) \otimes_R \varinjlim(A_i')\,.$$

Hint: the target of the map coincides with $\varinjlim_{i\in I} \varinjlim_{j\in I}(A_i \otimes_R A_j')$.

This is all we have to say about filtered posets. To give an idea of complications arising in the absence of this condition, we discuss derived functors of direct limits of the pushout diagrams.

7. Diagrams over the pushout poset

We start by identifying acyclic objects in a category of diagrams with respect to arbitrary posets.

Exercise 4.36. Let I be a poset and $\mathcal{A}$ the category of Abelian groups. Let M be an Abelian group. For each i in I, let $M[i]$ be the I-diagram in $\mathcal{A}$ given by $M[i]_j = M$ whenever $j \geqslant i$ and $M[i]_j = 0$ otherwise. Let $\varphi_{j,j'}$ be the identity on M for all $j' \geqslant j \geqslant i$ and the zero map otherwise.

For example, on the pushout poset $\left\{ c \begin{smallmatrix} \nearrow a \\ \searrow b \end{smallmatrix} \right\}$ one has

$$M[c] = \left\{ M \begin{smallmatrix} \nearrow M \\ \searrow M \end{smallmatrix} \right\}, \quad M[a] = \left\{ 0 \begin{smallmatrix} \nearrow M \\ \searrow 0 \end{smallmatrix} \right\}, \quad M[b] = \left\{ 0 \begin{smallmatrix} \nearrow 0 \\ \searrow M \end{smallmatrix} \right\},$$

where every morphism is the obvious embedding.

Show that each $M_{[i]}$ is $\varinjlim$-acyclic. Hint: Consider first the case when M is free. For an arbitrary M take its free resolution and note that this gives a resolution of $M[i]$ in $\mathfrak{Dir}_I^{\mathcal{A}}$.

Exercise 4.37. Let I be the pushout poset and $\mathcal{A}$ the category of Abelian groups. Show that every I-diagram in $\mathcal{A}$ has a two-step left resolution by I-diagrams of the type $M[c] \oplus M'[a] \oplus M''[b]$ where M, M', M'' are free. Show that for two such resolutions, there is a third dominating these. Show that each exact sequence $0 \longrightarrow \{A_i\} \longrightarrow \{B_i\} \longrightarrow \{C_i\} \longrightarrow 0$ of I-diagrams in $\mathcal{A}$ gives rise to an exact sequence

$$0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow \varinjlim A_i \longrightarrow \varinjlim B_i \longrightarrow \varinjlim C_i \longrightarrow 0$$

where A', B', C' are objects of $\mathcal{A}$ that are independent of resolutions. Prove that A', B', C' are natural with respect to maps of I-diagrams. We may hence consider these groups as the first left derived functors

$$\varinjlim{}_1 \colon \mathfrak{Dir}_I^{\mathcal{A}} \longrightarrow \mathcal{A}\,.$$

Prove that the higher derived functors are zero. Compute $\varinjlim_1 \left(\mathbb{Z} \begin{smallmatrix} \nearrow 0 \\ \searrow 0 \end{smallmatrix} \right)$.

Exercise 4.38. For each integer $n \geqslant 1$, find a poset such that its n-th left derived functor $\varinjlim_n(-)$ is not the zero functor. Hint: think of the pushout as a 2-set cover of $\mathbb{S}^1$. Find a space with nonzero n-th homology, cover it with open sets, and consider the dual of the corresponding Čech complex.

In the exercises below, given a pushout diagram M in $\mathcal{A}$, let K_M be the group of elements in M_c sent to zero under both $\varphi_{c,a}$ and $\varphi_{c,b}$.

Exercise 4.39. Show that if K_M is zero then M is $\varinjlim$-acyclic. Hint: Consider the natural inclusion of the constant diagram $M_c[c]$ into M and let Q be the cokernel. Prove that one may assume $M = Q$.

Exercise 4.40. Using the preceding exercise, show that $K_M \cong \varinjlim_1 M$.

Exercise 4.41. If $0 \longrightarrow C_\bullet \longrightarrow C'_\bullet \longrightarrow C''_\bullet \longrightarrow 0$ is an exact sequence of complexes (i.e., of diagrams over the poset $\mathbb{Z}$ with its natural order), then there is a homology exact sequence. Use this to prove: $K_M \cong \varinjlim_1 M$.

The rest of this lecture is a brief introduction to inverse limits.

8. Inverse limits

Definition 4.42. Let I be a poset, $\mathcal{A}$ a category, and Φ an I-diagram in $\mathcal{A}$. Recall the constant diagram functor from $\mathcal{A}$ to $\mathfrak{Dir}_I^{\mathcal{A}}$; see Definition 4.19. The *inverse limit* of Φ, denoted $\varprojlim_I \Phi$, is a right adjoint to this functor; compare Exercise 4.20.

We leave it to the reader to translate this definition to one akin to Definition 4.6. An important example is the *pullback diagram*, in which case the inverse limit in the category of Abelian groups is the usual pullback.

In the rest of this lecture, we focus on the case $I = \mathbb{N}^{\text{opp}}$, the opposite category of $\mathbb{N}$. Let $\mathcal{A}$ be the category of modules over a ring R. Note that an I-diagram in $\mathcal{A}$ is the same as a *contravariant* functor from $\mathbb{N}$ to $\mathcal{A}$. Thus, an $\mathbb{N}^{\text{opp}}$-diagram can be visualized as

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\varphi_{n+1}} A_n \xrightarrow{\varphi_n} \cdots \longrightarrow A_1 \xrightarrow{\varphi_1} A_0 \,.$$

In this case, $\varphi_{j,i} = \varphi_{i+1} \circ \cdots \circ \varphi_j$ for $j > i$.

Exercise 4.43. Let Φ be an $\mathbb{N}^{\text{opp}}$-diagram in $\mathcal{A}$. Consider the R-module

$$A = \Big\{ (\ldots, a_2, a_1, a_0) \in \prod_{n \in I} A_n \mid \varphi_n(a_n) = a_{n-1} \text{ for each } n \Big\}\,.$$

Prove that $\varprojlim \Phi = A$, where $A \longrightarrow A_n$ is the projection.

Example 4.44. Let $\mathfrak{a}$ be an ideal in a commutative ring R, and consider the I-diagram

$$\cdots \longrightarrow R/\mathfrak{a}^3 \longrightarrow R/\mathfrak{a}^2 \longrightarrow R/\mathfrak{a}$$

where every map is the natural projection. Its inverse limit is the *completion* $\widehat{R}$ of R with respect to $\mathfrak{a}$. It can be identified with the set of formal power series $\sum a_i$ where $a_i \in \mathfrak{a}^i$. Similarly, for any R-module M, the inverse limit of the diagram $\{M/\mathfrak{a}^i M\}$ is the $\mathfrak{a}$-adic completion of M.

For example, if $R = \mathbb{Z}$ and $\mathfrak{a} = (2)$ then in $\widehat{R}$, $1 + 2 + 4 + 8 + \cdots = -1$. Also, $(1+2)(1 - 2 + 2^2 - 2^3 + \cdots) = 1$, so 3 is a unit in $\widehat{R}$.

Inverse limits are left-exact since they are right adjoints to the constant diagram functor; see Proposition 4.18. They are not right-exact in general.

Exercise 4.45. Consider the exact sequence of pullback diagrams obtained by applying $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ to the exact sequence of pushout diagrams in Example 4.15. Show that the resulting sequence of Abelian groups obtained by applying $\varprojlim$ is not exact on the right.

This phenomenon also occurs when the poset is filtered.

Exercise 4.46. Consider the following $\mathbb{N}^{\mathrm{opp}}$-diagrams of Abelian groups:

$$\begin{array}{rcccccccc} \Phi = & \cdots & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z}\,, \\ \Phi' = & \cdots & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z}\,, \\ \Phi'' = & \cdots & \longrightarrow & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0\,, \end{array}$$

where each map in Φ'' is the natural projection. Consider the exact sequence

$$0 \longrightarrow \Phi_i \xrightarrow{2^i} \Phi'_i \longrightarrow \Phi''_i \longrightarrow 0\,.$$

Show that the resulting sequence of inverse limits is

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \widehat{\mathbb{Z}} \longrightarrow 0\,,$$

where $\widehat{\mathbb{Z}}$ is the (2)-adic completion of $\mathbb{Z}$. This sequence is not exact.

A diagram $\{M_i\}_{i\in\mathbb{N}}$ of R-modules satisfies the *Mittag-Leffler condition* if for each k in $\mathbb{N}$ the sequence of submodules $\{\varphi_{j,k}(M_j) \subseteq M_k\}_{j\geqslant k}$ stabilizes.

Exercise 4.47. Let $0 \longrightarrow \Phi \longrightarrow \Phi' \longrightarrow \Phi'' \longrightarrow 0$ be an exact sequence of $\mathbb{N}^{\mathrm{opp}}$-diagrams of R-modules. Prove that if Φ satisfies the Mittag-Leffler condition, then the induced sequence of inverse limits is exact.

If needed, consult [**161**, Proposition 3.5.7].

Lecture 5

Gradings, Filtrations, and Gröbner Bases

In this lecture we introduce filtrations and associated graded rings. With a view towards applications in Lectures 17, 23, and 24, some notions are presented for non-commutative algebras.

1. Filtrations and associated graded rings

In what follows, C is a commutative semigroup with identity element 0.

Definition 5.1. A ring R is C*-graded* if $R = \bigoplus_{i \in C} R_i$ as an Abelian group and $R_i \cdot R_j \subseteq R_{i+j}$ for each $i, j \in C$. When R is an algebra over a field $\mathbb{K}$, we assume that the direct sum decomposition is one of $\mathbb{K}$-vector spaces.

An R-module M is *graded* if $M = \bigoplus_{i \in C} M_i$ with $R_i \cdot M_j \subseteq M_{i+j}$. Observe that R_0 is a subring of R, and each M_i is an R_0-module.

Example 5.2. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{K}$. The *standard* grading on R is the $\mathbb{N}$-grading where $\deg x_i = 1$ for each i. The *fine grading* is the $\mathbb{N}^n$-grading where $\deg x_i$ is the i-th basis vector in $\mathbb{N}^n$.

Example 5.3. Let R be a C-graded algebra over a field $\mathbb{K}$. The n-th tensor power of R over $\mathbb{K}$, denoted $R^{\otimes n}$, is again C-graded with

$$(R^{\otimes n})_i = \sum_{j_1 + \cdots + j_n = i} (R_{j_1} \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} R_{j_n}) \qquad \text{for } i \in C\,.$$

Alternatively, one can grade $R^{\otimes n}$ by the product semigroup C^n. For example, if $R = \mathbb{K}[x]$ with $\deg x = 1$, then $R^{\otimes n} \cong \mathbb{K}[x_1, \dots, x_n]$ and the induced $\mathbb{N}$-grading on the polynomial ring is the standard grading, while the $\mathbb{N}^n$-grading is the fine grading.

Definition 5.4. Let R be a ring. An *increasing filtration* on R is a family $F = \{F_i\}_{i \geqslant 0}$ of additive subgroups of R satisfying the conditions:

(1) $F_i \subseteq F_{i+1}$ for each i;

(2) $F_i \cdot F_j \subseteq F_{i+j}$ for each i, j, and the identity element of R is in F_0.

The filtration is *exhaustive* if $\bigcup_i F_i = R$. Similarly one has a notion of a *decreasing filtration*, where the containment relation in (1) is reversed. Such a filtration is *separated* if $\bigcap_i F_i = 0$.

It is convenient to set $F_i = 0$ for $i < 0$ for F an increasing filtration, and $F_i = R$ for $i < 0$ when F is decreasing. If R is an algebra over a field $\mathbb{K}$, the F_i should be $\mathbb{K}$-vector spaces.

Example 5.5. If R is $\mathbb{N}$-graded, $F_i = \sum_{j \leqslant i} R_j$ defines an exhaustive increasing filtration of R, while $F_i = \sum_{j \geqslant i} R_j$ is a separated decreasing filtration.

Example 5.6. Each two-sided ideal $\mathfrak{a}$ in a ring R gives a decreasing filtration $\{\mathfrak{a}^i\}$ called the $\mathfrak{a}$-*adic filtration*. It is separated precisely when $\bigcap_i \mathfrak{a}^i = 0$.

The ring R in the following example is a Weyl algebra; see Lecture 17.

Exercise 5.7. Let $R = \mathbb{K}\langle x, y\rangle/(xy - yx - 1)$, the tensor algebra on x and y modulo the two-sided ideal generated by $xy - yx - 1$. Prove that R has nontrivial $\mathbb{Z}$-gradings, but no nontrivial $\mathbb{N}$-grading. Find an increasing exhaustive filtration F on R with $\operatorname{rank}_{\mathbb{K}}(F_i)$ finite for each i.

Definition 5.8. Let R be a ring and let F be an increasing filtration. The *associated graded ring* with respect to F is the graded Abelian group

$$\operatorname{gr}_F(R) = \bigoplus_{i \geqslant 0} F_i/F_{i-1}$$

with an $\mathbb{N}$-graded ring structure given by the canonical multiplication. If F is a decreasing filtration, we set $\operatorname{gr}_F(R) = \bigoplus_i F_i/F_{i+1}$.

For example, the associated graded ring of a graded ring with respect to either filtration in Example 5.5 is isomorphic to R.

Exercise 5.9. Determine the associated ring of $\mathbb{K}[x]$ with its (x)-adic filtration and of $\mathbb{Z}$ with its (2)-adic filtration.

Definition 5.10. Let F be an increasing filtration on R. A *filtration* on an R-module M is a family $G = \{G_i\}_{i \in \mathbb{Z}}$ of additive subgroups satisfying:

(1) $G_i \subseteq G_{i+1}$ for each i, and $G_i = 0$ for $i \ll 0$;

(2) $F_i \cdot G_j \subseteq G_{i+j}$ for each i, j.

We say that G is *exhaustive* if $\bigcup_i G_i = M$. One constructs an associated graded module $\operatorname{gr}_G(M)$ in the obvious way.

There are analogous notions for decreasing filtrations.

Exercise 5.11. Let R be a ring with filtration F and M an R-module with filtration G. Check that $\mathrm{gr}_G(M)$ is a graded $\mathrm{gr}_F(R)$-module.

Example 5.12. Let R be a ring, $\mathfrak{a}$ a two-sided ideal, and F the $\mathfrak{a}$-adic filtration. Each R-module M has a decreasing filtration with $G_i = \mathfrak{a}^i M$.

Exercise 5.13. For the filtration on $\mathbb{K}[x]$ with F_i spanned by monomials of degree at most $i \cdot \sqrt{2}$, prove that $\mathrm{gr}_F(\mathbb{K}[x])$ is not Noetherian.

Our interest is in filtrations with Noetherian associated graded rings. We focus on increasing exhaustive filtrations.

Exercise 5.14. Let F be an increasing exhaustive filtration such that the associated graded ring $\mathrm{gr}_F(R)$ is Noetherian. If an R-module M has an exhaustive filtration G with $\mathrm{gr}_G(M)$ a finitely generated $\mathrm{gr}_F(R)$-module, then M is Noetherian. In particular, R is Noetherian.

Exercise 5.15. Let F be an increasing exhaustive filtration on R such that the ring $\mathrm{gr}_F(R)$ is Noetherian. Let M be an R-module with generators $m_1, \dots, m_s$, and fix integers $k_1, \dots, k_s$. Verify that

$$G_i = \sum_{j=1}^{s} F_{i-k_j} m_j \,, \qquad \text{for } i \in \mathbb{Z} \,,$$

is an exhaustive filtration on M and $\mathrm{gr}_G(M)$ is a Noetherian $\mathrm{gr}_F(R)$-module.

Such a filtration on M is said to be *induced* by F.

Theorem 5.16. *Let F be an increasing exhaustive filtration on R such that the ring $\mathrm{gr}_F(R)$ is Noetherian, and let M be an R-module.*

If there exists an exhaustive filtration G on M with $\mathrm{gr}_G(M)$ a Noetherian $\mathrm{gr}_F(R)$-module, then it is induced by F. Furthermore, if G' is an exhaustive filtration on M, there exists an integer s such that $G_i \subseteq G'_{i+s}$ for each i.

Proof. Let $m_1, \dots, m_s$ in M be elements whose images generate $\mathrm{gr}_G(M)$ as a $\mathrm{gr}_F(R)$-module. Set k_j to be the degree of the image of m_j in $\mathrm{gr}_G(M)$. The rest of the proof is left as an exercise. □

2. Hilbert polynomials

In this section, R is a ring with a filtration F satisfying the following conditions. We use $\ell(-)$ to denote length.

(a) F is increasing and exhaustive;

(b) $A = \mathrm{gr}_R(F)_0$ is an Artinian ring and $\ell_A(\mathrm{gr}_R(F)_1)$ is finite;

(c) $\mathrm{gr}_F(R)$ is commutative and generated, as an A-algebra, by $\mathrm{gr}_F(R)_1$.

Exercise 5.17. Let M be an R-module with filtration G. If $\operatorname{gr}_G(M)$ is finitely generated over $\operatorname{gr}_F(R)$, prove that $\ell_A(\operatorname{gr}_G(M)_t)$ is finite for each t. Show that the converse need not hold.

The first part of the result below is a consequence of [**6**, Theorem 11.1]; the second is a corollary of Theorem 5.16.

Theorem 5.18. *Let M be an R-module with a filtration G for which the $\operatorname{gr}_R(F)$-module $\operatorname{gr}_G(M)$ is finitely generated. Then $\ell_A(\operatorname{gr}_G(M)_t)$ agrees with a polynomial in t for $t \gg 0$, and its leading term does not depend on G.* □

Definition 5.19. The polynomial in Theorem 5.18 is the *Hilbert polynomial* of the filtered module M, denoted $H_G(M,t)$. The number

$$\dim_F M = \deg H_G(M,t) + 1$$

is the *F-dimension* of M, while the *F-multiplicity*, denoted $e_F(M)$, is the leading coefficient of $H_G(M,t)$ multiplied by $(d-1)!$. By Theorem 5.18, these numbers do not depend on the filtration G.

When R and M are graded and the filtrations are induced by the gradings, we omit F and G from the notation. When R is *standard graded*, i.e., finitely generated over R_0 by degree 1 elements, the dimension of M in the sense defined here agrees with the one in Lecture 1.

Exercise 5.20. Consider an exact sequence of filtered R-modules

$$0 \longrightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$$

such that $G''_i = \psi(G_i)$ and $\varphi(G'_i) = \ker(\psi) \cap G_i$. Prove that the induced sequence of associated graded modules is exact, and so one has an equality

$$H_G(M,t) = H_{G'}(M',t) + H_{G''}(M'',t)\,.$$

Hence $\dim_F M = \max\{\dim_F M', \dim_F M''\}$. When, in addition, one has $\dim M' = \dim M = \dim M''$, then $e_F(M) = e_F(M') + e_F(M'')$.

Exercise 5.21. Let $R = \mathbb{K}[x,y]$ and let F be the filtration induced by the standard grading. Compute $H_G(M,t)$ for $M = R/\mathfrak{a}$ and G the induced filtration, in the following cases: $\mathfrak{a} = (x^2 - y^2)$ and $\mathfrak{a} = (x^2 - y^3)$.

Definition 5.22. Let R and M be as in Theorem 5.18. The *Hilbert-Poincaré series* of M with respect to G is the Laurent series

$$P_G(M,t) = \sum_{i\in\mathbb{Z}} \ell_A(\operatorname{gr}_G(M)_i) t^i \in \mathbb{Z}[[t]][t^{-1}]\,.$$

Note that $\operatorname{gr}_G(M)_i = 0$ for $i \ll 0$, since M is finitely generated.

When R and M are graded, we write $P(M,t)$ for the Hilbert-Poincaré series of the induced filtration; see Lecture 1.

Remark 5.23. Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ an $\mathfrak{m}$-primary ideal, and M a finitely generated R-module. The $\mathfrak{a}$-adic filtration on M is decreasing and separated. The ring $\mathrm{gr}_{\mathfrak{a}}(R)$ is Noetherian, and $\mathrm{gr}_{\mathfrak{a}}(M)$ is a finitely generated $\mathrm{gr}_{\mathfrak{a}}(R)$-module. For $t \gg 0$, the length of $\mathfrak{a}^t M/\mathfrak{a}^{t+1}M$ as an $R/\mathfrak{a}$-module agrees with a polynomial in t of degree $\dim M - 1$. Its leading coefficient multiplied by $(\dim M - 1)!$ is the *multiplicity* of M with respect to $\mathfrak{a}$.

Each submodule $M' \subseteq M$ has an $\mathfrak{a}$-adic filtration, as well as one induced by the $\mathfrak{a}$-adic filtration on M. The Artin-Rees lemma [**6**, Corollary 10.10] implies the existence of an integer s such that

$$(\mathfrak{a}^j M) \cap M' \subseteq \mathfrak{a}^{j-s} M' \qquad \text{for all } j\,.$$

Compare this result with Theorem 5.16.

3. Monomial orders and initial forms

For the remainder of this lecture, let $\mathbb{K}$ be a field and $R = \mathbb{K}[x_1, \ldots, x_n]$, the polynomial ring with its standard grading. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we write $\boldsymbol{x}^\alpha$ for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

The proof of the following result is straightforward.

Lemma 5.24. *If $\mathfrak{a}$ is a homogeneous ideal in R and f a homogeneous element of degree d, then*

$$P(R/\mathfrak{a}, t) = P(R/(\mathfrak{a}, f), t) + t^d P(R/(\mathfrak{a} :_R f), t)\,. \qquad \square$$

Exercise 5.25. Let $\mathfrak{a} \subseteq R$ be a monomial ideal. Using Lemma 5.24 describe an algorithm for computing $P(R/\mathfrak{a}, t)$. Compute it when $\mathfrak{a} = (x^4, x^2y^3, xy^3)$.

In order to make use of this exercise, one needs a procedure to pass from arbitrary ideals to monomial ones, while preserving Hilbert polynomials. The rest of this lecture is concerned with exactly that.

Definition 5.26. A *monomial order* is a partial order $\geqslant$ on the multiplicative semigroup $\{\boldsymbol{x}^\alpha \mid \alpha \in \mathbb{N}^n\}$ such that if $\boldsymbol{x}^\alpha \geqslant \boldsymbol{x}^\beta$, then $\boldsymbol{x}^{\alpha+\gamma} \geqslant \boldsymbol{x}^{\beta+\gamma}$ for each $\gamma \in \mathbb{N}^n$, and $\boldsymbol{x}^\alpha > 1$ when $\boldsymbol{x}^\alpha \neq 1$.

A monomial order is a *term order* if it is a well-order.

Example 5.27. In the *lexicographic order* monomials are ordered as in a dictionary: $\boldsymbol{x}^\alpha \geqslant_{\mathrm{lex}} \boldsymbol{x}^\beta$ if $\alpha = \beta$ or the first nonzero entry in $\alpha - \beta$ is positive.

Definition 5.28. For a monomial order $\geqslant$, we write $\mathrm{in}_\geqslant(f)$ for the *initial form* of f, which is the sum of the terms $c_\alpha \boldsymbol{x}^\alpha$ of maximal order. The *initial ideal* of an ideal $\mathfrak{a}$ of R is the ideal

$$\mathrm{in}_\geqslant(\mathfrak{a}) = \big(\mathrm{in}_\geqslant(f) \mid f \in \mathfrak{a}\big)\,.$$

When $\geqslant$ is a term order, $\operatorname{in}_{\geqslant}(\mathfrak{a})$ is a monomial ideal. In this case, the monomials not belonging to $\operatorname{in}_{\geqslant}(\mathfrak{a})$ are the *standard monomials*, denoted $\operatorname{std}_{\geqslant}(\mathfrak{a})$. Evidently, the standard monomials form a $\mathbb{K}$-basis for $R/\mathfrak{a}$.

For a term order $\geqslant$ the *support* of $f(\boldsymbol{x}) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha \boldsymbol{x}^\alpha$ is

$$\operatorname{Supp}(f) = \{\alpha \mid c_\alpha \neq 0\}\,.$$

The monomial $\boldsymbol{x}^\alpha$ of highest order in $\operatorname{Supp}(f)$ is the *leading monomial*, denoted $\operatorname{lm}(f)$. The *leading term* of f is $\operatorname{lt}(f) = c_\alpha \boldsymbol{x}^\alpha$.

Remark 5.29. Note that $\operatorname{in}_{\mathrm{lex}}(x) + \operatorname{in}_{\mathrm{lex}}(1) = x + 1 \neq x = \operatorname{in}_{\mathrm{lex}}(x+1)$, so $\operatorname{in}_{\geqslant}(-)$ is not a homomorphism of groups in general.

The following theorem reduces the computation of the Hilbert-Poincaré series of ideals to those of monomial ideals; see [**28**, Proposition 9.3.4]. It is best explained from the point of view of the next section.

Theorem 5.30. *Fix a term order $\geqslant$ on R refining the partial order by degree. For each ideal $\mathfrak{a} \subseteq R$, one has $P(R/\mathfrak{a}, t) = P(R/\operatorname{in}_{\geqslant}(\mathfrak{a}), t)$.*

In particular, one has $\dim(R/\mathfrak{a}) = \dim(R/\operatorname{in}_{\geqslant}(\mathfrak{a}))$. □

Example 5.31. Given a vector $\omega \in \mathbb{R}^n$, the *weight order* $\geqslant_\omega$ is defined by

$$\boldsymbol{x}^\alpha \geqslant_\omega \boldsymbol{x}^\beta \qquad \text{if } \langle \alpha, \omega\rangle \geqslant \langle \beta, \omega\rangle\,,$$

where $\langle -, -\rangle$ is the inner product on $\mathbb{R}^n$.

One may have ties: $\boldsymbol{x}^\alpha =_\omega \boldsymbol{x}^\beta$ with $\alpha \neq \beta$. Moreover, if $\omega_i < 0$, then $1 >_\omega x_i >_\omega x_i^2 >_\omega \cdots$ is an infinite descending sequence, so a weight order may not be a term order; it is one if there are no ties and $\omega \in \mathbb{N}^n$.

Exercise 5.32. For nonzero $\omega \in \mathbb{N}^n$, consider the $\mathbb{K}$-subspaces of R with

$$F_{\omega,r} = \sum_{\langle \alpha,\omega\rangle \leqslant r} \mathbb{K}\cdot \boldsymbol{x}^\alpha\,, \qquad \text{where } r \in \mathbb{N}\,.$$

Prove that $\{F_{\omega,r}\}_{r\geqslant 0}$ is a filtration on R; we write $\operatorname{gr}_\omega R$ for the associated graded ring. Prove that the ring $\operatorname{gr}_\omega(R)$ is Noetherian.

Example 5.33. Consider the ideal $\mathfrak{a} = (x^4 + x^2y^3, y^4 - y^2x^3)$ in $\mathbb{K}[x,y]$. We consider lexicographic orders $\geqslant_{\mathrm{lex}\{x,y\}}$ and $\geqslant_{\mathrm{lex}\{y,x\}}$, and the weight order $\geqslant_{(1,2)}$. While $\geqslant_{(1,2)}$ is not a term order, the ideal $\operatorname{in}_{(1,2)}(\mathfrak{a})$ is monomial.

A monomial ideal $\mathfrak{a}$ can be viewed as a staircase in $\mathbb{N}^n$ by plotting α for $\boldsymbol{x}^\alpha \in \mathfrak{a}$; see [**118**]. Figure 5.1 displays the staircases of the initial ideals corresponding to the orders in the previous example. Why is the area under the staircase independent of the order?

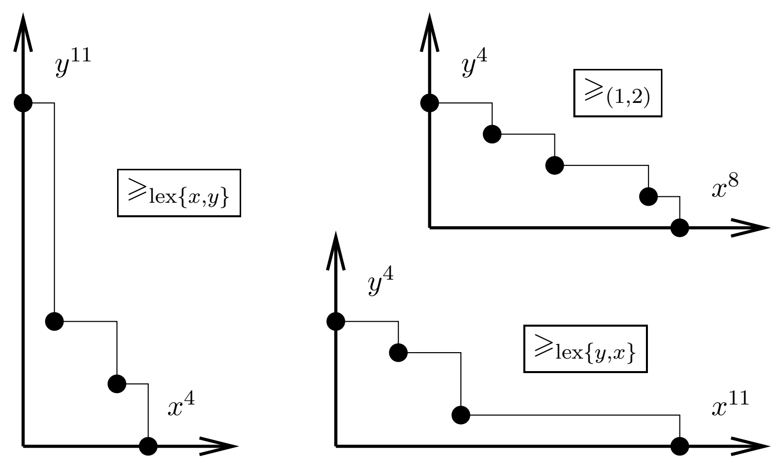


Figure 5.1. Staircases

4. Weight vectors and flat families

Let $\omega \in \mathbb{Z}^n$ be a weight on $R = \mathbb{K}[x_1, \dots, x_n]$, and fix an element f in R. Its *degree* $\deg_\omega(f)$ with respect to ω is the maximum of $\langle \omega, \alpha \rangle$ for $\alpha \in \mathrm{Supp}(f)$. Set $d = \deg_\omega(f)$ and consider the homogenization

$$\widetilde{f} = \sum_{\alpha \in \mathrm{Supp}(f)} x_0^{d - \langle \omega, \alpha \rangle} \boldsymbol{x}^\alpha \,;$$

this is a polynomial of weight d in $R[x_0]$, where x_0 is assigned weight 1. For any ideal $\mathfrak{a} \subseteq R$, we write $\widetilde{\mathfrak{a}}_\omega$ for the ideal of $R[x_0]$ generated by $\{\widetilde{f} \mid f \in \mathfrak{a}\}$. Note that for each t in $\mathbb{K}$, setting $x_0 = t$ gives an ideal $\mathfrak{a}_{\omega,t}$ in R.

Exercise 5.34. Verify that $\mathrm{in}_\omega(\mathfrak{a})$ is the *special fiber* $\mathfrak{a}_{\omega,0}$, while $\mathfrak{a} = \mathfrak{a}_{\omega,1}$.

The next exercise illustrates that the homogenization of a set of generators of $\mathfrak{a}$ may not generate $\widetilde{\mathfrak{a}}_\omega$. However, look ahead to Exercise 5.45.

Exercise 5.35. Set $\mathfrak{a} = (x^2, xy - y^2)$, an ideal in $\mathbb{K}[x, y]$, and $\omega = (3, 1)$. Prove that y^3 is in $\mathfrak{a}_{\omega,0}$, and deduce that $\widetilde{\mathfrak{a}}_\omega$ is not generated by homogenizations of x^2 and $xy - y^2$.

Exercise 5.36. Set $\mathfrak{a} = (x^2 z - y^3)$. Discuss how the projective variety of $\mathfrak{a}_{\omega,t}$ changes with t, for ω: $(1,1,1)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Definition 5.37. Fix $\omega \in \mathbb{Z}^n$ and a basis $e_1, \dots, e_m$ for R^m.

Let N be a submodule of R^m. To each element $f = \sum_{i=1}^m f_i e_i$ in N with $f_i = \sum_{\alpha \in \mathrm{Supp}(f_i)} c_{i,\alpha} \boldsymbol{x}^\alpha$ assign the ω-degree d, defined to be the maximum

ω-degree of its components. The *ω-homogenization* of f is the element

$$\widetilde{f} = \sum_{i,\alpha} c_{i,\alpha} x_0^{d-\langle\omega,\alpha\rangle} \boldsymbol{x}^\alpha e_i \,.$$

We write $\widetilde{N}$ for the $R[x_0]$-submodule of $R[x_0]^m$ generated by $\{\widetilde{f} \mid f \in N\}$. One has thus a family $N_{\omega,t}$ of R-modules with $N_{\omega,1} = N$ and $N_{\omega,0} = \mathrm{in}_\omega(N)$.

The *homogenization* of $M = R^m/N$ is the $R[x_0]$-module $\widetilde{M} = R[x_0]^m/\widetilde{N}$. We set $M_{\omega,t} = R^m/N_{\omega,t}$.

Exercise 5.38. The $\mathbb{K}[x_0]$-module $\widetilde{M}$ is torsion-free, hence flat. Hint: for homogeneous f in $\widetilde{N}$, the homogenization of $f(1, x_1, \ldots, x_n)$ divides f.

Let $\omega \in \mathbb{N}^n$. Verify that one can refine $\geqslant_\omega$ to a term order $\succeq$, and so speak about standard monomials of ideals. Generalize to the module case and verify that for each $N \subseteq R^m$ there is a $\mathbb{K}$-splitting

$$N \oplus \mathrm{std}_\succeq(N) \cong R^m \,,$$

which, under homogenization, gives a $\mathbb{K}[x_0]$-splitting

$$\widetilde{N} \oplus \big(\mathbb{K}[x_0] \otimes_\mathbb{K} \mathrm{std}_\succeq(N)\big) \cong R[x_0]^m \,.$$

Conclude that $\widetilde{M} \cong \mathbb{K}[x_0] \otimes_\mathbb{K} \mathrm{std}_\succeq(N)$ is a free $\mathbb{K}[x_0]$-module.

Discuss fiber dimensions of the flat $\mathbb{K}[x_0]$-module $\mathbb{K}[x_0, x_0^{-1}]$. This is an example of a non-free ω-homogenization if $M = \mathbb{K}[x_1]/(x_1 - 1)$ and $\deg_\omega(x_1) = -1$. This suggests the role of the nonnegativity of ω.

Flatness ensures that the dimension and multiplicity are preserved when passing to initial ideals; this reduces the proof of Theorem 5.30 to the homogeneous case, where one proceeds by working with graded components.

Exercise 5.39. Prove that when the length of M is finite, the length of $M_{\omega,t}$ is independent of t and of ω.

When M does not have finite length, the multiplicity depends on ω. However, it is the same for each fiber $M_{\omega,t}$ when $t \neq 0$.

Exercise 5.40. For $M = \mathbb{K}[x_1, x_2]/(x_1^3 - x_2^2)$ and $\omega = (1,1)$ and $\omega = (1,2)$, compute multiplicities of $M_{\omega,t}$.

In the flat families in Definition 5.37, all the non-special fibers are isomorphic (Exercise!). This need not be the case for arbitrary flat families.

5. Buchberger's algorithm

As before, $R = \mathbb{K}[x_1, \ldots, x_n]$ where $\mathbb{K}$ is a field.

Definition 5.41. Fix a monomial order on R. A generating set G for an ideal $\mathfrak{a}$ of R is a *Gröbner basis* if $\mathrm{in}(\mathfrak{a}) = (\mathrm{in}(g) \mid g \in G)$.

Obviously, $\mathfrak{a}$ itself is a Gröbner basis, albeit not a very useful one.

Exercise 5.42. Let $\geqslant$ be a term order and G a subset of $\mathfrak{a}$. Prove that if the elements $\{\mathrm{in}(g) \mid g \in G\}$ generate the ideal $\mathrm{in}(\mathfrak{a})$, then G generates $\mathfrak{a}$. Show that this need not be the case for monomial orders.

Example 5.43. The Gröbner bases for Example 5.33 can be computed using the `gb` command in *Macaulay 2*, [**50**].

```
i1 : Rxy = QQ[x,y, MonomialOrder=>Lex];

i2 : a = ideal(x^4+x^2*y^3, y^4-y^2*x^3);

o2 : Ideal of Rxy

i3 : gb a

o3 = | y11+y6 xy4-y8 x3y2-y4 x4+x2y3 |

o3 : GroebnerBasis

i4 : R12 = QQ[x,y, Weights=>{1,2}];

i4 : gb substitute(a,R12)

o5 = | y4-x3y2 x2y3+x4 x4y2-x7 x8+x4y x7y+x6 |

o5 : GroebnerBasis
```

Since $\mathrm{in}(\mathfrak{a})$ is generated by the initial forms of the Gröbner basis elements, it is easy to verify that the staircases in Figure 5.1 are correct.

We focus on the case $\mathfrak{a} = (b_1, \ldots, b_c)$, where the b_i are binomials. Let ω be a positive weight vector and m a monomial in R.

Suppose we are allowed to swap a monomial $\boldsymbol{x}^\alpha$ for a monomial $\boldsymbol{x}^\beta$ as long as $\boldsymbol{x}^\alpha - \boldsymbol{x}^\beta$ is in $\mathfrak{a}$, and that we are interested in finding monomials of least degree. An obvious strategy would be to modify m by $\boldsymbol{x}^\alpha b_i$ to decrease its degree; this is the *greedy method*. It is far from clear that this is optimal.

Example 5.44. Let $\mathfrak{a} = (x^2 - y, x^3 - z) \subseteq \mathbb{K}[x, y, z]$ and $\omega = (1, 1, 1)$. Note that $y^3 = z^2 \bmod \mathfrak{a}$, but $\deg_\omega(z^2) < \deg_\omega(y^3)$. Unfortunately, neither $\mathrm{in}(x^2 - y)$ nor $\mathrm{in}(x^3 - z)$ divides y^3, and so, starting with y^3, the greedy algorithm comes to a screeching halt. The problem is that $y^3 \in \mathrm{in}(\mathfrak{a})$ but neither $b_1 = x^2 - y$ nor $b_2 = x^3 - z$ warned us about this.

Since a Gröbner basis can be used to recover the ideal of initial forms, it guarantees that the greedy method succeeds.

Exercise 5.45. In the context of Exercise 5.35, let $\geqslant$ be a term order that refines the ω-degree on R, where $\omega \in \mathbb{N}^n$. Show that if the f_i form a Gröbner basis for $\mathfrak{a}$, then $\widetilde{\mathfrak{a}}_\omega$ is generated by the $\widetilde{f_i}$.

We now discuss a method to get Gröbner bases.

One could have predicted trouble in the example above by noticing that the initial form of $b_3 = b_2 - xb_1 = xy - z \in \mathfrak{a}$ is not in the ideal $(\mathrm{in}(b_1), \mathrm{in}(b_2))$. Similarly, $b_4 = xb_3 - yb_1 = -xz + y^2$ is obtained. At this point we need to break ties, since $\deg(xz) = \deg(y^2)$. Say we use lex to break ties; then $xz > y^2$. One further obtains $b_5 = yb_2 - x^2b_3 = x^2z - yz$. But $b_5 = zb_1$.

Fix a term order $\geqslant$ on R and a set $G \subseteq R$. The algorithm below takes $f \in R$ and outputs its *normal form* $\mathrm{NF}(f, G)$ where $\mathrm{NF}(f, G) \equiv f \bmod (G)$ and $\mathrm{lm}(\mathrm{NF}(f, G))$ is not divisible by $\mathrm{lm}(g)$ for any $g \in G$. If f is in the ideal generated by a Gröbner basis G, then $\mathrm{NF}(f, G) = 0$.

Algorithm 5.46. $\overline{f} = \mathrm{NF}(f, G)$

$f' := f$
while $\mathrm{lm}(f')$ is divisible by $\mathrm{lm}(g_k)$ for some $g_k \in G$ **do**

$$f' := f' - \frac{\mathrm{lt}(f')}{\mathrm{lt}(g_k)} \cdot g_k$$

end while
$\overline{f} := f'$

Note that the normal form algorithm gives an expression

$$\mathrm{NF}(f, G) = f - \sum r_k g_k$$

where r_k is the sum of all the terms $\mathrm{lt}(f')/\,\mathrm{lt}(g_k)$ occurring in the algorithm.

The choice of g_k in the **while** loop is not deterministic, so the output is not unique. Algorithm 5.46 terminates since the leading monomials of f' at each step form a decreasing sequence and a term order is a well-order.

Definition 5.47. The *S-polynomial* of $f, g \in R$ is

$$S(f, g) = \frac{\boldsymbol{x}^\alpha}{\mathrm{lt}(f)} f - \frac{\boldsymbol{x}^\alpha}{\mathrm{lt}(g)} g\,, \qquad \text{where } \boldsymbol{x}^\alpha = \mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g))\,.$$

See [**28**, §2.7] for a proof of Buchberger's criterion:

Theorem 5.48. *Let $\mathfrak{a}$ be an ideal and $\geqslant$ a term order. A generating set G of $\mathfrak{a}$ is a Gröbner basis if and only if $NF(S(f, g), G) = 0$ for all $f, g \in G$.* □

The algorithm constructs a Gröbner basis G from a generating set F:

Algorithm 5.49. Buchberger's algorithm

$G := F, \quad Q := \{(f, g) \mid f, g \in F\}$
while $Q \neq \varnothing$ **do**
  Pick a pair $(f, g) \in Q$
  $h := \mathrm{NF}(S(f, g), G)$

if $h \neq 0$ **then**
$Q := Q \cup \{(f, h) \mid f \in G\}$
$G := G \cup \{h\}$
end if
$Q := Q \smallsetminus (f, g)$
end while

This algorithm terminates, so $\mathfrak{a}$ has a finite Gröbner basis. A key point is that $\geqslant$ is a term order; see [**28**, §2.6].

Exercise 5.50. Show that if $\omega \in \mathbb{N}^n$ is a weight vector and $\geqslant$ a term order, then the order below, refining ω, is again a term order: $\boldsymbol{x}^\alpha \succeq \boldsymbol{x}^\beta$ if

$$\langle \omega, \alpha \rangle \geqslant \langle \omega, \beta \rangle\,, \qquad \text{or} \qquad \langle \omega, \alpha \rangle = \langle \omega, \beta \rangle \text{ and } \boldsymbol{x}^\alpha \geqslant \boldsymbol{x}^\beta\,.$$

Moreover, a Gröbner basis with respect to $\succeq$ is also one with respect to ω.

6. Gröbner bases and syzygies

One can extend the theory of Gröbner bases and Buchberger's algorithm to the case of modules; see [**32**, Chapter 15].

Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring with $\deg x_i \geqslant 1$, and M a finitely generated graded R-module. Then M admits a free resolution in which the modules are graded and the morphisms respect the grading. Hilbert's syzygy theorem says that there is one that is finite. Schreyer gave a proof of this using Gröbner bases; see [**32**, §15.5].

The first part of the proof is that, writing $M = R^m/N$, where N is a graded submodule of R^m, and choosing a Gröbner basis $G = \{m_i\}$ for N, one can start the free resolution of M as follows:

$$\bigoplus_{m_i, m_j \in G} Re_{i,j} \xrightarrow{\partial_2} \bigoplus_{m_i \in G} Re_i \xrightarrow{\partial_1} R^m \longrightarrow 0\,.$$

Here $\partial_1(e_i) = m_i$ and ∂_2 is defined by the normal form reduction of $S(m_i, m_j)$ relative to G. Since $\mathrm{NF}(S(m_i, m_j), G) = 0$ by the Gröbner property, the normal form algorithm produces a syzygy between elements of G, that is to say, an element in the kernel of ∂_1. The syzygies so obtained generate $\ker(\partial_1)$. One then iterates this construction to obtain a free resolution of M:

$$\cdots \longrightarrow F_i \xrightarrow{\partial_i} F_{i-1} \xrightarrow{\partial_{i-1}} \cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0\,.$$

The second part of Schreyer's proof is that for a suitable order on the F_i, the initial forms of the generators of $\ker(\partial_i)$ do not use any of the variables $x_1, \dots, x_i$. This implies that F_{n+1} can be chosen to be zero.

We discuss similar finiteness theorems for local rings in Lecture 8, and for certain non-commutative filtered rings in Lecture 17.

Complexes from a Sequence of Ring Elements

In Lecture 3 we postulated or proved the existence of several examples of exact sequences: projective, free, and injective resolutions, as well as long exact sequences in (co)homology. Most of these were abstract, and the concrete examples came out of thin air. The problem of actually producing any one of these kinds of resolutions for a given module was essentially ignored. In this lecture, we will give a few concrete constructions of complexes, which we use to obtain resolutions. The complexes we construct will also carry information relevant to our long-term goal of understanding local cohomology.

In this lecture, R is once again a commutative Noetherian ring.

1. The Koszul complex

When we are handed a single element of a ring, there is one complex simply crying out to be constructed.

Definition 6.1. Let x be an element in R. The *Koszul complex on* x, denoted $K^\bullet(x; R)$, is the complex

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0\,,$$

with R in degrees -1 and 0. Suppose we are given a sequence $\boldsymbol{x} = x_1, \dots, x_d$ of elements in R. Recall the construction of tensor products of complexes from Definition 3.9. The *Koszul complex* on $\boldsymbol{x}$ is the complex

$$K^\bullet(\boldsymbol{x}; R) = K^\bullet(x_1; R) \otimes_R \cdots \otimes_R K^\bullet(x_d; R)\,.$$

The nonzero modules in this complex are situated in degrees $-d$ to 0. Note that, up to an isomorphism of complexes, $K^\bullet(\boldsymbol{x}; R)$ does not depend on the order of the elements x_i.

Example 6.2. Let $x, y \in R$. The Koszul complexes on x and y are

$$0 \longrightarrow R \xrightarrow{\ x\ } R \longrightarrow 0$$

and

$$0 \longrightarrow R \xrightarrow{\ y\ } R \longrightarrow 0\,.$$

Their tensor product $K^\bullet(x, y; R)$ is the complex

$$0 \longrightarrow R \xrightarrow{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]} R^2 \xrightarrow{[x\ y]} R \longrightarrow 0\,,$$

where the three nonzero modules are in degrees -2, -1, and 0. Observe that $K^\bullet(x, y; R)$ is indeed a complex.

Remark 6.3. Let $\boldsymbol{x} = x_1, \ldots, x_d$. A simple count using Definition 3.9 reveals that for each nonnegative integer r, one has

$$K^{-r}(\boldsymbol{x}; R) \cong R^{\binom{d}{r}}\,,$$

where $\binom{d}{r}$ is the appropriate binomial coefficient.

Exercise 6.4. Construct the Koszul complex on a sequence of three elements of R. Compare with Example 3.23.

Exercise 6.5. For a sequence $\boldsymbol{x} = x_1, \ldots, x_d$, identify the maps ∂^{-d} and ∂^{-1} in the complex $K^\bullet(\boldsymbol{x}; R)$.

The Koszul complex, as defined above, holds an enormous amount of information about the ideal generated by $x_1, \ldots, x_d$. In future lectures we will see some of this information laid bare. For the best applications, however, we will want more relative information about $\boldsymbol{x}$ and its impact on various R-modules. We therefore define the Koszul complex on a module and introduce Koszul cohomology.

Definition 6.6. Let $\boldsymbol{x} = x_1, \ldots, x_d$ be a sequence of elements of R, and M an R-module. The *Koszul complex of* $\boldsymbol{x}$ *on* M is the complex

$$K^\bullet(\boldsymbol{x}; M) = K^\bullet(\boldsymbol{x}; R) \otimes_R M\,.$$

The *Koszul cohomology of* $\boldsymbol{x}$ *on* M is

$$H^j(\boldsymbol{x}; M) = H^j(K^\bullet(\boldsymbol{x}; M)) \qquad \text{for } j \in \mathbb{Z}\,.$$

Example 6.7. Let x be an element of R, and let M be an R-module. The tininess of the Koszul complex $K^\bullet(x; M)$ makes computing the Koszul cohomology trivial in this case:

$$H^{-1}(x; M) = (0 :_M x) \quad \text{and} \quad H^0(x; M) = M/xM\,.$$

In particular, we can make two immediate observations.

(1) If $xm \neq 0$ for each nonzero element m in M, then $H^{-1}(x; M) = 0$.

(2) If $xM \neq M$, then $H^0(x; M) \neq 0$. In particular, if the module M is finitely generated over a local ring $(R, \mathfrak{m})$, and $x \in \mathfrak{m}$, then Nakayama's lemma implies that $H^0(x; M)$ is nonzero.

Exercise 6.8. Let $\boldsymbol{x} = x_1, \dots, x_d$ be a sequence of elements in R. Prove that for each R-module M one has

$$H^{-d}(\boldsymbol{x}; M) = (0 :_M \boldsymbol{x}) \quad \text{and} \quad H^0(\boldsymbol{x}; M) = M/\boldsymbol{x}M\,.$$

Exercise 6.9. Prove that the Koszul complex is self-dual: if $\boldsymbol{x} = x_1, \dots, x_d$ are elements of R, then the complex $K^\bullet(\boldsymbol{x}; M) = K^\bullet(\boldsymbol{x}; R) \otimes_R M$ is naturally isomorphic to $\operatorname{Hom}_R(K^\bullet(\boldsymbol{x}; R), M)[d]$ for each R-module M.

2. Regular sequences and depth: a first look

In order to put Example 6.7 in its proper context, we introduce *regular sequences* and *depth*, two concepts central to local cohomology theory.

Definition 6.10. Let R be a ring and M an R-module. An element x of R is a *nonzerodivisor on M* if $xm \neq 0$ for all nonzero $m \in M$; if in addition $xM \neq M$, then the element x is said to be *regular on M*, or *M-regular*.

Remark 6.11. From Example 6.7 we see that x is a nonzerodivisor on M if and only if $H^{-1}(x; M) = 0$, while x is M-regular precisely if $H^{-1}(x; M) = 0$ and $H^0(x; M) \neq 0$.

Definition 6.12. Let M be an R-module. A prime ideal of R is *associated* to M if it is the annihilator of an element of M. We write $\operatorname{Ass} M$ for the set of associated primes of M.

The following lemma is left as an exercise; see [**115**, Theorem 6.1].

Lemma 6.13. *Let M be a module over a Noetherian ring. The set of zerodivisors on M is the union of the associated primes of M.* □

Definition 6.14. A sequence $\boldsymbol{x} = x_1, \dots, x_d$ of elements of R is a *regular sequence* on an R-module M if

(1) x_1 is M-regular, and

(2) x_i is regular on $M/(x_1, \dots, x_{i-1})M$ for each $i = 2, \dots, d$.

An equivalent definition is that for $1 \leqslant i \leqslant d$, the element x_i is a nonzerodivisor on $M/(x_1, \dots, x_{i-1})M$ and $\boldsymbol{x}M \neq M$. If one drops the assumption that $\boldsymbol{x}M \neq M$, one obtains the notion of a *weakly regular* sequence.

Remark 6.15. In the definition of a regular sequence, the order of the elements is relevant. For example, the sequence $x - 1$, xy, xz is regular on $\mathbb{K}[x, y, z]$, while xy, xz, $x - 1$ is not. (Check this!) If, however, $\boldsymbol{x}$ consists of elements in the maximal ideal of a local ring, then we shall see that the order of the x_i is immaterial.

Exercise 6.16. If $x_1, \dots, x_d$ is an M-regular sequence, prove that for positive integers $a_1, \dots, a_d$, the sequence $x_1^{a_1}, \dots, x_d^{a_d}$ is M-regular as well.

We end this section by smuggling in one more definition.

Definition 6.17. Let $\mathfrak{a} \subseteq R$ be an ideal and M an R-module. The *depth of* $\mathfrak{a}$ *on* M, denoted $\operatorname{depth}_R(\mathfrak{a}, M)$, is the maximal length of an M-regular sequence in $\mathfrak{a}$. Exercise 6.16 implies $\operatorname{depth}_R(\mathfrak{a}, M) = \operatorname{depth}_R(\operatorname{rad}\mathfrak{a}, M)$.

3. Back to the Koszul complex

Let us return now to the Koszul complex. We computed above the Koszul cohomology of a single element, and now recognize it as determining regularity. We have high hopes for the case of two elements.

Example 6.18. Let x, y be elements of R, and let M be an R-module. Consider the complex $K^\bullet(x, y; M)$, i.e.,

$$0 \longrightarrow M \xrightarrow{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]} M^2 \xrightarrow{[x\ y]} M \longrightarrow 0\,.$$

The cohomology of this complex at the ends can be computed easily:

$$\begin{aligned} H^{-2}(x, y; M) &= (0 :_M (x, y))\,, \\ H^0(x, y; M) &= M/(x, y)M\,. \end{aligned}$$

The cohomology in the middle is

$$H^{-1}(x, y; M) = \frac{\{(a, b) \in M^2 \mid xa + yb = 0\}}{\{(-ym, xm) \in M^2 \mid m \in M\}}\,,$$

which may be viewed as relations on x, y in M modulo the trivial relations. When x is regular on M, the map

$$(xM :_M y) \longrightarrow H^{-1}(x, y; M)\,,$$

which sends c in $(xM :_M y)$ to the class of $(cy/x, -c)$, is well-defined. We leave it as an exercise to verify that it is surjective, with kernel xM. Thus

$$H^{-1}(x, y; M) \cong (xM :_M y)/xM$$

when x is M-regular.

Exercise 6.19. Let M be a finitely generated module over a local ring. Prove that elements x, y of R form an M-regular sequence if and only if $H^{-2}(x, y; M) = 0 = H^{-1}(x, y; M)$ and $H^0(x, y; M) \neq 0$.

Remark 6.11 and Exercise 6.19 suggest that the Koszul complex detects regular sequences, and this is indeed the case, as we will see in Theorem 6.21. Its proof uses an observation on annihilators of Koszul cohomology modules.

Proposition 6.20. *Let $\boldsymbol{x} = x_1, \dots, x_d$ be elements of R and let M be an R-module. Then, for each j, the ideal $(\boldsymbol{x})$ annihilates $H^j(\boldsymbol{x}; M)$.*

Proof. It suffices to prove that $x_i \cdot H^j(\boldsymbol{x}; M) = 0$ for each i; without loss of generality, we may assume $i = 1$. We claim that multiplication by x_1 is null-homotopic on $K^\bullet(\boldsymbol{x}; M)$, which implies the desired result.

Indeed, for each element $y \in R$, the morphism of complexes of R-modules $\lambda \colon K^\bullet(y, R) \longrightarrow K^\bullet(y, R)$ given by multiplication by y is null-homotopic, via the homotopy given by the dashed arrows in the diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "y"] \arrow[d, "y"'] \arrow[dl, dashed, "0"'] & R \arrow[r] \arrow[d, "y"] \arrow[dl, dashed, "1"'] & 0 \\
0 \arrow[r] & R \arrow[r, "y"'] & R \arrow[r] & 0
\end{tikzcd}$$

Therefore, for any complex L of R-modules, the morphism of complexes

$$\lambda \otimes_R L \colon K^\bullet(y; R) \otimes_R L \longrightarrow K^\bullet(y; R) \otimes_R L\,,$$

which is again multiplication by y, is null-homotopic. Setting $y = x_1$ and $L = K^\bullet(x_2, \dots, x_d; M)$, one obtains that multiplication by x_1 is null-homotopic on $K^\bullet(\boldsymbol{x}; M)$, as desired. $\square$

The next result is known as the "depth-sensitivity" of Koszul complexes.

Theorem 6.21. *Let M be a nonzero R-module, $\mathfrak{a}$ an ideal in R, and let $\boldsymbol{x} = x_1, \dots, x_d$ be a sequence of elements generating the ideal $\mathfrak{a}$.*

(1) *If $\mathfrak{a}$ contains a weakly M-regular sequence of length t, then one has $H^{j-d}(\boldsymbol{x}; M) = 0$ for $j < t$.*

(2) *If the R-module M is finitely generated and $\mathfrak{a}M \neq M$, then*

$$\operatorname{depth}_R(\mathfrak{a}, M) = \min\{j \mid H^{j-d}(\boldsymbol{x}; M) \neq 0\}\,.$$

Proof. The crucial point in the proof is the following: suppose $y \in \mathfrak{a}$ is a nonzerodivisor on M, so that one has an exact sequence of R-modules

$$0 \longrightarrow M \xrightarrow{\ y\ } M \longrightarrow M/yM \longrightarrow 0\,.$$

Since the complex $K^\bullet(\boldsymbol{x}; R)$ consists of projective modules, tensoring the preceding exact sequence with it yields an exact sequence

$$0 \longrightarrow K^\bullet(\boldsymbol{x}; M) \xrightarrow{\ y\ } K^\bullet(\boldsymbol{x}; M) \longrightarrow K^\bullet(\boldsymbol{x}; M/yM) \longrightarrow 0$$

of complexes of R-modules. Its homology long exact sequence reads, in part,

$$\begin{aligned}\longrightarrow H^j(\boldsymbol{x}; M) \xrightarrow{\ y\ } H^j(\boldsymbol{x}; M) \longrightarrow H^j(\boldsymbol{x}; M/yM)& \\ \longrightarrow H^{j+1}(\boldsymbol{x}; M) \xrightarrow{\ y\ } H^{j+1}(\boldsymbol{x}; M) \longrightarrow &\ .\end{aligned}$$

Since y annihilates each $H^j(\boldsymbol{x};M)$ by Proposition 6.20, this sequence splits into short exact sequences

$$0 \longrightarrow H^j(\boldsymbol{x};M) \longrightarrow H^j(\boldsymbol{x};M/yM) \longrightarrow H^{j+1}(\boldsymbol{x};M) \longrightarrow 0\,.$$

(1) We proceed by induction on t. When $t = 0$, there is nothing to prove. Suppose $t \geqslant 1$, and that the result holds for $t-1$. Let $y_1, \dots, y_t$ be a weakly M-regular sequence in $\mathfrak{a}$. Then the sequence $y_2, \dots, y_t$ is weakly regular on M/y_1M, so the induction hypothesis applies to M/y_1M, and yields $H^{j-d}(\boldsymbol{x};M/y_1M) = 0$ for $j < t-1$. The exact sequence above, applied with $y = y_1$, implies the desired vanishing.

(2) Set $t = \operatorname{depth}_R(\mathfrak{a}, M)$. Since $\mathfrak{a}M \neq M$, any weakly M-regular sequence is M-regular. Therefore, given (1), it suffices to prove $H^{t-d}(\boldsymbol{x};M)$ is nonzero. Again, we induce on t. One has $t = 0$ if and only if $\mathfrak{a}$ is contained in the union of the associated primes of M, by Lemma 6.13. Since M is finitely generated, there are only finitely many of those, and then by prime avoidance [**6**, Proposition 1.11], $\mathfrak{a}$ must be contained in one of them, say $\mathfrak{p}$. By definition, there exists an $m \in M$ such that $(0 :_R m) = \mathfrak{p}$; in particular, $m \neq 0$ and $m\mathfrak{a} = 0$. Therefore, $m \in H^{-d}(\boldsymbol{x};M)$ by Exercise 6.8.

Suppose that $t \geqslant 1$ and that the desired result holds whenever the depth is at most $t-1$. Let $y_1, \dots, y_t$ be an M-regular sequence in $\mathfrak{a}$. Note that $\operatorname{depth}_R(\mathfrak{a}, M/y_1M) = t-1$. From the exact sequence above, specialized to $y = y_1$ and $j = t-1-d$, one obtains an exact sequence

$$0 \longrightarrow H^{t-1-d}(\boldsymbol{x};M) \longrightarrow H^{t-1-d}(\boldsymbol{x};M/y_1M) \longrightarrow H^{t-d}(\boldsymbol{x};M) \longrightarrow 0\,.$$

The module on the left is zero by (1), and the one in the middle is nonzero by the induction hypothesis. Thus, $H^{t-d}(\boldsymbol{x};M) \neq 0$, as desired. □

A special case of (1) of the preceding result is worth noting.

Corollary 6.22. *If $\boldsymbol{x}$ is a weakly M-regular sequence, then $H^j(\boldsymbol{x};M) = 0$ for all integers $j \neq 0$.* □

Theorem 6.23. *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated module. If $\boldsymbol{x} = x_1, \dots, x_d$ in $\mathfrak{m}$ satisfies $H^{-1}(\boldsymbol{x};M) = 0$, then $\boldsymbol{x}$ is M-regular.*

Proof. We induce on d; the result being clear for $d = 1$.

Observe that one has an exact sequence of complexes

$$0 \longrightarrow M \longrightarrow K^\bullet(x_d;M) \longrightarrow M[1] \longrightarrow 0\,.$$

Setting $\boldsymbol{y} = x_1, \dots, x_{d-1}$ and tensoring the sequence above with $K^\bullet(\boldsymbol{y};M)$ gives an exact sequence of complexes

$$0 \longrightarrow K^\bullet(\boldsymbol{y};M) \longrightarrow K^\bullet(\boldsymbol{x};M) \longrightarrow K^\bullet(\boldsymbol{y};M)[1] \longrightarrow 0\,.$$

Taking homology, one gets an exact sequence of R-modules

$$\longrightarrow H^{-1}(\boldsymbol{y};M) \xrightarrow{-x_d} H^{-1}(\boldsymbol{y};M) \longrightarrow H^{-1}(\boldsymbol{x};M)$$
$$\longrightarrow H^0(\boldsymbol{y};M) \xrightarrow{x_d} H^0(\boldsymbol{y};M) \longrightarrow .$$

The hypothesis implies that x_d in a nonzerodivisor on $H^0(\boldsymbol{y};M) = M/\boldsymbol{y}M$. Moreover, one also obtains $H^{-1}(\boldsymbol{y};M) = 0$, using Nakayama's lemma. The induction hypothesis yields that $\boldsymbol{y}$ is M-regular. Hence, $\boldsymbol{x}$ is M-regular. □

Theorems 6.21(2) and 6.23 yield the following result.

Corollary 6.24. *Let M be a finitely generated module over a local ring $(R,\mathfrak{m})$, and let $\mathfrak{a} \subseteq \mathfrak{m}$ be an ideal of R. If $\operatorname{depth}_R(\mathfrak{a},M) \geqslant \nu_R(\mathfrak{a})$, then any minimal system of generators for $\mathfrak{a}$ is an M-regular sequence.* □

This implies the following corollary, which we mentioned in Remark 6.15.

Corollary 6.25. *Let M be a finitely generated module over a local ring. If $\boldsymbol{x}$ is an M-regular sequence, then so is any permutation of $\boldsymbol{x}$.* □

4. The Čech complex

Given a single element x in a ring R, it may seem like the only complex we can build from such meager information is the Koszul complex. If we insist on clinging to the world of finitely generated R-modules, this is essentially true. If, however, we allow some small amount of infinite generation, new vistas open to us.

The Čech complex attached to a sequence of ring elements, like the Koszul complex, is built inductively by tensoring together short complexes. Recall that for $x \in R$, the localization R_x, also sometimes written $R[1/x]$, is obtained by inverting the multiplicatively closed set $\{1, x, x^2, \dots\}$.

Definition 6.26. Let x be an element of a ring R. The *Čech complex on x* is the complex

$$0 \longrightarrow R \xrightarrow{\iota} R_x \longrightarrow 0\,,$$

denoted $\check{C}^\bullet(x;R)$, with modules R in degree 0 and R_x in degree 1, and where ι is the canonical map sending each $r \in R$ to the class of the fraction $r/1 \in R_x$. For a sequence $\boldsymbol{x} = x_1, \dots, x_d$ in R, the *Čech complex on $\boldsymbol{x}$* is

$$\check{C}^\bullet(\boldsymbol{x};R) = \check{C}^\bullet(x_1;R) \otimes_R \cdots \otimes_R \check{C}^\bullet(x_d;R)\,.$$

Given an R-module M, set $\check{C}^\bullet(\boldsymbol{x};M) = \check{C}^\bullet(\boldsymbol{x};R) \otimes_R M$. The R-module

$$\check{H}^j(\boldsymbol{x};M) = H^j(\check{C}^\bullet(\boldsymbol{x};M))$$

is the *j-th Čech cohomology of $\boldsymbol{x}$ on M.*

The complex we have defined here is sometimes called the "stable Koszul complex." It is also known as the "extended Čech complex," by way of contrast with the topological Čech complex introduced in Lecture 2. We will justify this collision of nomenclature in Lecture 12.

Example 6.27. As with Koszul cohomology, Čech cohomology is easy to describe for short sequences. We first compute Čech cohomology in the case of one element, i.e., the cohomology of $\check{C}^\bullet(x;R)$. Note that

$$\begin{aligned}\check{H}^0(x;R) &= \{r \in R \mid r/1 = 0 \text{ in } R_x\}\\ &= \{r \in R \mid x^a r = 0 \text{ for some } a \geqslant 0\}\\ &= \bigcup_{a\geqslant 0} (0 :_R x^a)\end{aligned}$$

is the union of annihilators of x^a, sometimes written $(0 :_R x^\infty)$. Meanwhile, we see that $\check{H}^1(x;R) \cong R_x/R$, though this description is perhaps not immediately illuminating.

Suppose that $R = \mathbb{K}[x]$ is the univariate polynomial ring over a field $\mathbb{K}$. Then $R_x \cong \mathbb{K}[x,x^{-1}]$ is the ring of *Laurent polynomials*. The quotient $\mathbb{K}[x,x^{-1}]/\mathbb{K}[x]$ has a $\mathbb{K}$-basis consisting of the elements x^{-c} for $c \geqslant 1$, and has R-module structure dictated by

$$x^a x^{-c} = \begin{cases} x^{a-c} & \text{if } a < c, \text{ and}\\ 0 & \text{otherwise.}\end{cases}$$

Exercise 6.28. Let $R = \mathbb{K}[x]$, considered as an $\mathbb{N}$-graded ring where x has degree 1. The map

$$\mathbb{K} \hookrightarrow \frac{R_x}{R}(-1) = E$$

sending 1 to the image of $1/x$ is a graded (i.e., degree preserving) injection of R-modules. Prove the following:

(1) The module E above is the injective hull of R/xR in the category of graded R-modules.

(2) The module R_x/R is the injective hull of R/xR in the category of all R-modules.

(3) The injective hull of R in the category of graded R-modules is $R_x = \mathbb{K}[x,x^{-1}]$, whereas in the category of all R-modules, it is the fraction field $\mathbb{K}(x)$.

Example 6.29. For a sequence x, y, the Čech complex $\check{C}^\bullet(x,y;R)$ is the tensor product of the complexes

$$0 \longrightarrow R \xrightarrow{r \longmapsto r/1} R_x \longrightarrow 0$$

and

$$0 \longrightarrow R \xrightarrow{r \mapsto r/1} R_y \longrightarrow 0\,,$$

which is

$$0 \longrightarrow R \otimes R \xrightarrow{\alpha} (R_x \otimes R) \oplus (R \otimes R_y) \xrightarrow{\beta} R_x \otimes R_y \longrightarrow 0\,.$$

The map α sends $1 \otimes 1$ to $(1 \otimes 1, 1 \otimes 1)$ and for β we have

$$\beta(1 \otimes 1, 0) = -1 \otimes 1 \quad \text{and} \quad \beta(0, 1 \otimes 1) = 1 \otimes 1\,.$$

Simplified, this shows that the complex $\check{C}^\bullet(x, y; R)$ is better written as

$$0 \longrightarrow R \xrightarrow{1 \mapsto (1,1)} R_x \oplus R_y \xrightarrow{\substack{(1,0) \,\mapsto\, -1 \\ (0,1) \,\mapsto\, 1}} R_{xy} \longrightarrow 0\,.$$

Let us try to compute the cohomology $\check{H}^j(x, y; R)$ for $j = 0, 2$. If $r \in R$ maps to 0 in $R_x \oplus R_y$, so that $(r/1, r/1) = 0$, then there exist integers $a, b \geqslant 0$ such that $x^a r = y^b r = 0$. Equivalently, the ideal $(x, y)^c$ kills r for $c \gg 0$. Thus

$$\check{H}^0(x, y; R) \cong \bigcup_{c \geqslant 0} \left(0 :_R (x, y)^c\right)$$

is the union of annihilators of the ideals $(x, y)^c$. On the other hand, it is clear that $\check{H}^2(x, y; R) \cong R_{xy}/(R_x + R_y)$, though this is a less than completely satisfactory answer. Here is a more useful one:

Exercise 6.30. An element η of $\check{H}^2(x, y; R)$ can be written as $\eta = \left[\frac{r}{(xy)^c}\right]$, that is, as an equivalence class of fractions in R_{xy}. Prove that $\eta = 0$ if and only if there exists k in $\mathbb{N}$ such that

$$r(xy)^k \in (x^{c+k}, y^{c+k})\,.$$

If x, y is a regular sequence, conclude that $\eta = 0$ if and only if $r \in (x^c, y^c)$.

State and prove similar statements about $\check{H}^i(x_1, \ldots, x_d; R)$ for $i = 0, d$.

Remark 6.31. Unlike the Koszul complex $K^\bullet(\boldsymbol{x}; R)$, in which all modules are free, the Čech complex $\check{C}^\bullet(\boldsymbol{x}; R)$ has direct sums of localizations of R. Specifically, $\check{C}^0(\boldsymbol{x}; R) = R$, while $\check{C}^1(\boldsymbol{x}; R) = R_{x_1} \oplus \cdots \oplus R_{x_d}$, and in general

$$\check{C}^k(\boldsymbol{x}; R) = \bigoplus_{1 \leqslant i_1 < \cdots < i_k \leqslant d} R_{x_{i_1} \cdots x_{i_k}}\,.$$

Note that, typically, $\check{C}^k(\boldsymbol{x}; R)$ is not finitely generated over R, but that it is *flat*, since each R-module $R_{x_{i_1} \cdots x_{i_k}}$ is flat.

Local Cohomology

In this lecture we introduce, at long last, the local cohomology functors and discuss various methods of calculating them.

Let R be a Noetherian commutative ring and $\mathfrak{a}$ an ideal of R.

1. The torsion functor

Definition 7.1. For each R-module M, set

$$\Gamma_{\mathfrak{a}}(M) = \{m \in M \mid \mathfrak{a}^t m = 0 \text{ for some } t \in \mathbb{N}\}\,.$$

Evidently, $\Gamma_{\mathfrak{a}}(M)$ is a submodule of M. When they coincide, M is said to be $\mathfrak{a}$-*torsion*. Each homomorphism of R-modules $\varphi\colon M \longrightarrow M'$ induces a homomorphism of R-modules

$$\Gamma_{\mathfrak{a}}(\varphi)\colon \Gamma_{\mathfrak{a}}(M) \longrightarrow \Gamma_{\mathfrak{a}}(M')\,,$$

that is to say, $\Gamma_{\mathfrak{a}}(-)$ is a functor on the category of R-modules; it is called the $\mathfrak{a}$-*torsion functor*. It extends to a functor on the category of complexes of R-modules: for each complex $I^\bullet$ of R-modules, $\Gamma_{\mathfrak{a}}(I^\bullet)$ is the complex with $\Gamma_{\mathfrak{a}}(I^j)$ in degree j and differential induced by $I^\bullet$.

Exercise 7.2. Check that the $\mathfrak{a}$-torsion functor is left-exact.

The j-th right derived functor of $\Gamma_{\mathfrak{a}}(-)$ is denoted $H^j_{\mathfrak{a}}(-)$, that is to say,

$$H^j_{\mathfrak{a}}(M) = H^j(\Gamma_{\mathfrak{a}}(I^\bullet))\,,$$

where $I^\bullet$ is an injective resolution of M. The R-module $H^j_{\mathfrak{a}}(M)$ is called the *j-th local cohomology of M with support in* $\mathfrak{a}$.

Here are a few basic properties of local cohomology.

Proposition 7.3. *Let M be an R-module.*

(1) *One has $H^0_{\mathfrak{a}}(M) \cong \Gamma_{\mathfrak{a}}(M)$, and $H^j_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion for each j.*

(2) *If $\operatorname{rad}\mathfrak{b} = \operatorname{rad}\mathfrak{a}$, then $H^j_{\mathfrak{b}}(M) \cong H^j_{\mathfrak{a}}(M)$ for each j.*

(3) *Let $\{M_\lambda\}$ be a family of R-modules. For each integer j, one has*

$$H^j_{\mathfrak{a}}\Big(\bigoplus_\lambda M_\lambda\Big) \cong \bigoplus_\lambda H^j_{\mathfrak{a}}(M_\lambda)\,.$$

(4) *An exact sequence of R-modules $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ induces an exact sequence in local cohomology*

$$\cdots \longrightarrow H^j_{\mathfrak{a}}(M') \longrightarrow H^j_{\mathfrak{a}}(M) \longrightarrow H^j_{\mathfrak{a}}(M'') \longrightarrow H^{j+1}_{\mathfrak{a}}(M') \longrightarrow \cdots\,.$$

Proof. (1) Let $I^\bullet$ be an injective resolution of M. The left-exactness of $\Gamma_{\mathfrak{a}}(-)$ implies that

$$\Gamma_{\mathfrak{a}}(M) \cong H^0(\Gamma_{\mathfrak{a}}(I^\bullet)) = H^0_{\mathfrak{a}}(M)\,.$$

Furthermore, the R-module $\Gamma_{\mathfrak{a}}(I^j)$ is $\mathfrak{a}$-torsion, so the same property carries over to its subquotient $H^j_{\mathfrak{a}}(M)$.

(2) This is immediate from the equality $\Gamma_{\mathfrak{b}}(-) = \Gamma_{\mathfrak{a}}(-)$, which holds because R is Noetherian.

(3) Let $I^\bullet_\lambda$ be an injective resolution of M_λ, in which case $\bigoplus_\lambda I^\bullet_\lambda$ is an injective resolution of $\bigoplus_\lambda M_\lambda$; see Proposition A.18. It is not hard to verify that $\Gamma_{\mathfrak{a}}\big(\bigoplus_\lambda I^\bullet_\lambda\big) = \bigoplus_\lambda \Gamma_{\mathfrak{a}}(I^\bullet_\lambda)$. Since homology commutes with direct sums, the desired result follows.

(4) This is a special case of Remark 3.21, and may be proved as follows. Let $I'^\bullet$ and $I''^\bullet$ be injective resolutions of M' and M'', respectively. One can then construct (do it!) an injective resolution $I^\bullet$ of M which fits in an exact sequence of complexes of R-modules

$$0 \longrightarrow I'^\bullet \longrightarrow I^\bullet \longrightarrow I''^\bullet \longrightarrow 0\,.$$

Since $I'^\bullet$ consists of injective modules, this exact sequence is split in each degree, so it induces an exact sequence of complexes of R-modules

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(I'^\bullet) \longrightarrow \Gamma_{\mathfrak{a}}(I^\bullet) \longrightarrow \Gamma_{\mathfrak{a}}(I''^\bullet) \longrightarrow 0\,.$$

The homology exact sequence of this sequence is the one announced. □

Example 7.4. Let $R = \mathbb{Z}$ and let p be a prime number. We want to compute the local cohomology with support in the ideal (p) of a finitely generated R-module. Thanks to Proposition 7.3(3), it suffices to consider the case where M is indecomposable. By the fundamental theorem of Abelian groups, M is isomorphic to $\mathbb{Z}/d\mathbb{Z}$ where either $d = 0$ or d is a prime power. In either case, by Exercise A.4, the complex

$$0 \longrightarrow \mathbb{Q}/d\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

is an injective resolution of $\mathbb{Z}/d\mathbb{Z}$. In computing local cohomology, there are three cases to consider; in what follows, $\mathbb{Z}_p$ denotes $\mathbb{Z}$ with p inverted.

Case 1. If $M = \mathbb{Z}/p^e\mathbb{Z}$ for some integer $e \geqslant 1$, then applying $\Gamma_{(p)}(-)$ to the resolution above yields the complex

$$0 \longrightarrow \mathbb{Z}_p/p^e\mathbb{Z} \longrightarrow \mathbb{Z}_p/\mathbb{Z} \longrightarrow 0\,.$$

The sole nonzero differential is evidently surjective. Hence

$$H^0_{(p)}(M) = \mathbb{Z}/p^e\mathbb{Z} = M \qquad \text{and} \qquad H^1_{(p)}(M) = 0\,.$$

Case 2. If $M = \mathbb{Z}/d\mathbb{Z}$ with d nonzero and relatively prime to p, then applying $\Gamma_{(p)}(-)$ to the given injective resolution of $\mathbb{Z}/d\mathbb{Z}$ yields the complex

$$0 \longrightarrow d\mathbb{Z}_p/d\mathbb{Z} \longrightarrow \mathbb{Z}_p/\mathbb{Z} \longrightarrow 0\,.$$

Therefore, one obtains that

$$H^0_{(p)}(M) = 0 \qquad \text{and} \qquad H^1_{(p)}(M) = \mathbb{Z}_p/(d\mathbb{Z}_p + \mathbb{Z}) = 0\,.$$

Case 3. If $M = \mathbb{Z}$, then the p-torsion part of the given injective resolution of $\mathbb{Z}$ is the complex

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z}_p/\mathbb{Z} \longrightarrow 0\,.$$

Thus we conclude that

$$H^0_{(p)}(M) = 0 \qquad \text{and} \qquad H^1_{(p)}(M) = \mathbb{Z}_p/\mathbb{Z}\,.$$

The calculation of the local cohomology with support in any ideal in $\mathbb{Z}$ is equally elementary; see also Theorem 7.13. One noteworthy feature of this example is that $H^j_{\mathfrak{a}}(-) = 0$ for $j \geqslant 2$ for any ideal $\mathfrak{a}$; confer Corollary 7.14.

Example 7.5. Let R be a ring and $\mathfrak{a}$ an ideal in R. If $\mathfrak{a}$ is *nilpotent*, i.e., if $\mathfrak{a}^e = 0$ for some integer $e \geqslant 0$, then $\Gamma_{\mathfrak{a}}(-)$ is the identity functor and so

$$H^0_{\mathfrak{a}}(M) = M, \qquad \text{while} \qquad H^j_{\mathfrak{a}}(M) = 0 \quad \text{for } j \geqslant 1\,.$$

For some purposes, as in the proof of Theorem 7.11 below, it is useful to know the local cohomology of injective modules. By Theorem A.21, an injective R-module is a direct sum of modules $E_R(R/\mathfrak{p})$ for primes $\mathfrak{p}$ of R. Thus, by Proposition 7.3(3), it suffices to focus on $E_R(R/\mathfrak{p})$ for various $\mathfrak{p}$.

Example 7.6. Let R be a ring and $\mathfrak{a}$ an ideal in R. For each prime ideal $\mathfrak{p}$ in R, one has $H^j_{\mathfrak{a}}(E_R(R/\mathfrak{p})) = 0$ for $j \geqslant 1$, and

$$H^0_{\mathfrak{a}}(E_R(R/\mathfrak{p})) = \begin{cases} E_R(R/\mathfrak{p}) & \text{if } \mathfrak{a} \subseteq \mathfrak{p}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

This follows from the definition and Theorem A.20.

Here is one application; keep in mind Corollary A.32.

Exercise 7.7. Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. Prove that $H^j_{\mathfrak{m}}(M)$ is Artinian for each integer j.

2. Direct limit of Ext modules

Next we describe a method for computing local cohomology as a direct limit of suitable Ext functors; see Lecture 3 for a discussion of the Ext functors, and Lecture 4 for an introduction to direct limits.

Theorem 7.8. *For each R-module M, there are natural isomorphisms*

$$\varinjlim_t \operatorname{Ext}^j_R(R/\mathfrak{a}^t, M) \cong H^j_{\mathfrak{a}}(M) \qquad \text{for each } j \geqslant 0\,.$$

Proof. For each R-module I and integer t, one has a functorial identification

$$\begin{aligned} \operatorname{Hom}_R(R/\mathfrak{a}^t, I) &\xrightarrow{\cong} \{x \in I \mid \mathfrak{a}^t x = 0\} \subseteq \Gamma_{\mathfrak{a}}(I)\,, \\ f &\longmapsto f(1)\,. \end{aligned}$$

With this identification, one has a directed system

$$\operatorname{Hom}_R(R/\mathfrak{a}, I) \subseteq \cdots \subseteq \operatorname{Hom}_R(R/\mathfrak{a}^t, I) \subseteq \operatorname{Hom}_R(R/\mathfrak{a}^{t+1}, I) \subseteq \cdots$$

of submodules of $\Gamma_{\mathfrak{a}}(I)$. It is evident that its limit (that is to say, its union) equals $\Gamma_{\mathfrak{a}}(I)$; in other words, one has

$$\varinjlim_t \operatorname{Hom}_R(R/\mathfrak{a}^t, I) = \Gamma_{\mathfrak{a}}(I)\,.$$

Let $I^\bullet$ be an injective resolution of M. The construction of the directed system above is functorial, so

$$\varinjlim_t \operatorname{Hom}_R(R/\mathfrak{a}^t, I^\bullet) = \Gamma_{\mathfrak{a}}(I^\bullet)\,.$$

Since $H^j(-)$ commutes with filtered direct limits, the preceding identification results in a natural isomorphism

$$\varinjlim_t H^j(\operatorname{Hom}_R(R/\mathfrak{a}^t, I^\bullet)) \cong H^j(\Gamma_{\mathfrak{a}}(I^\bullet)) = H^j_{\mathfrak{a}}(M)\,.$$

It remains to note that $H^j(\operatorname{Hom}_R(R/\mathfrak{a}^t, I^\bullet)) = \operatorname{Ext}^j_R(R/\mathfrak{a}^t, M)\,.$ □

Remark 7.9. In the context of Theorem 7.8, let $\{\mathfrak{a}_t\}_{t\geqslant 0}$ be a decreasing chain of ideals *cofinal* with the chain $\{\mathfrak{a}^t\}_{t\geqslant 0}$, that is to say, for each integer $t \geqslant 0$, there exist positive integers c, d such that $\mathfrak{a}^{t+c} \subseteq \mathfrak{a}_t$ and $\mathfrak{a}_{t+d} \subseteq \mathfrak{a}^t$. Then there is a functorial isomorphism

$$\varinjlim_t \operatorname{Ext}^j_R(R/\mathfrak{a}_t, M) \cong H^j_{\mathfrak{a}}(M)\,.$$

This assertion can be proved along the lines of Theorem 7.8.

These considerations apply when R is a ring of prime characteristic p. Then the system $\{\mathfrak{a}^{[p^e]}\}_{e\geqslant 0}$ of Frobenius powers of $\mathfrak{a}$—see Definition 21.26—is cofinal with $\{\mathfrak{a}^t\}_{t\geqslant 0}$, so one obtains a functorial isomorphism of R-modules

$$\varinjlim_{e} \operatorname{Ext}_R^j(R/\mathfrak{a}^{[p^e]}, M) \cong H_{\mathfrak{a}}^j(M)\,.$$

This expression for local cohomology was exploited by Peskine and Szpiro in their work on the intersection theorems [**128**]; see Lectures 21 and 22.

3. Direct limit of Koszul cohomology

Next we express local cohomology in terms of Koszul complexes, which were introduced in Lecture 6.

Construction 7.10. Let x be an element in R. For each integer t, consider the Koszul complex $K^\bullet(x^t;R)$, which is

$$0 \longrightarrow R \xrightarrow{x^t} R \longrightarrow 0\,,$$

with the nonzero modules in cohomological degrees -1 and 0. The complexes $\{K^\bullet(x^t;R)\}_{t\geqslant 1}$ form an inverse system, with structure morphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{x^{t+1}} & R & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle x} & & \Big\| & & \\ 0 & \longrightarrow & R & \xrightarrow{x^t} & R & \longrightarrow & 0\,. \end{array} \tag{7.10.1}$$

Let $\boldsymbol{x} = x_1,\dots,x_d$ be elements in R, and set $\boldsymbol{x}^t = x_1^t,\dots,x_d^t$. The Koszul complex on $\boldsymbol{x}^t$ is the complex of R-modules

$$K^\bullet(\boldsymbol{x}^t;R) = K^\bullet(x_1^t;R)\otimes_R \cdots \otimes_R K^\bullet(x_d^t;R)$$

concentrated in degrees $[-d,0]$. This is equipped with a morphism of complexes $\epsilon_t\colon K^\bullet(\boldsymbol{x}^t;R) \longrightarrow R/\boldsymbol{x}^tR$, where $R/\boldsymbol{x}^tR$ is viewed as a complex concentrated in degree 0. Associated to each x_i, there is an inverse system as in (7.10.1). Tensoring these componentwise, we obtain an inverse system of complexes of R-modules

$$\cdots \longrightarrow K^\bullet(\boldsymbol{x}^{t+1};R) \longrightarrow K^\bullet(\boldsymbol{x}^t;R) \longrightarrow \cdots \longrightarrow K^\bullet(\boldsymbol{x};R)\,, \tag{7.10.2}$$

compatible with the morphisms ϵ_t and natural maps $R/\boldsymbol{x}^{t+1}R \longrightarrow R/\boldsymbol{x}^tR$.

Let M be an R-module and $\eta\colon M \longrightarrow I^\bullet$ an injective resolution of M. The maps ϵ_t and η induce morphisms of complexes of R-modules

$$\operatorname{Hom}_R(R/\boldsymbol{x}^tR, I^\bullet) \longrightarrow \operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^t;R), I^\bullet) \longleftarrow \operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^t;R), M)\,,$$

where the map on the left is $\operatorname{Hom}_R(\epsilon_t, I^\bullet)$, and the map on the right is $\operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^t;R), \eta)$. The latter is a quasi-isomorphism by Remark 3.17

because $K^\bullet(\boldsymbol{x}^t;R)$ is a bounded complex of free R-modules. Thus, passing to homology yields, for each integer j, a homomorphism

$$\theta_t^j\colon \operatorname{Ext}_R^j(R/\boldsymbol{x}^tR, M) \longrightarrow H^j(\operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^t;R),M))\,.$$

It is not hard to verify that for varying t this homomorphism is compatible with the maps induced through (7.10.2). As $\operatorname{Hom}_R(-,M)$ is contravariant, one gets a compatible directed system of R-modules and R-linear maps

$$\begin{CD}
\operatorname{Ext}_R^j(R/\boldsymbol{x}^tR, M) @>{\theta_t^j}>> H^j(\operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^t;R),M))\\
@VVV @VVV\\
\operatorname{Ext}_R^j(R/\boldsymbol{x}^{t+1}R, M) @>{\theta_{t+1}^j}>> H^j(\operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^{t+1};R),M))\,.
\end{CD}$$

In the limit, this gives us a homomorphism of R-modules

$$\theta^j(M)\colon \varinjlim_t \operatorname{Ext}_R^j(R/\boldsymbol{x}^tR, M) \longrightarrow \varinjlim_t H^j(\operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^t;R),M))\,.$$

The module on the left is $H^j_{\boldsymbol{x}R}(M)$; this follows from Remark 7.9, because the system of ideals $\{(\boldsymbol{x}^t)\}_{t\geqslant 1}$ is cofinal with the system $\{(\boldsymbol{x})^t\}_{t\geqslant 1}$. It is an important point that $\theta^j(M)$ is functorial in M and also compatible with connecting homomorphisms: each exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of R-modules induces a commutative diagram of R-modules

$$\begin{CD}
H^j_{\boldsymbol{x}R}(M') @>>> H^j_{\boldsymbol{x}R}(M) @>>> H^j_{\boldsymbol{x}R}(M'') @>>> H^{j+1}_{\boldsymbol{x}R}(M') @>>> \\
@V{\theta^j(M')}VV @V{\theta^j(M)}VV @V{\theta^j(M'')}VV @V{\theta^{j+1}(M')}VV \\
F^j_{\boldsymbol{x}}(M') @>>> F^j_{\boldsymbol{x}}(M) @>>> F^j_{\boldsymbol{x}}(M'') @>>> F^{j+1}_{\boldsymbol{x}}(M') @>>>
\end{CD}$$

where $F^j_{\boldsymbol{x}}(-) = \varinjlim_t H^j(\operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^t;R),-))$. This is easy to verify given the construction of $F^j_{\boldsymbol{x}}$ and θ^j. In category-theory language, what this says is that $\{\theta^j\}_{j\geqslant 0}$ is a natural transformation between δ-functors.

Theorem 7.11. *Let $\boldsymbol{x} = x_1,\dots,x_c$ be a set of generators for an ideal $\mathfrak{a}$. For each R-module M and integer j, the homomorphism*

$$\theta^j : H^j_{\mathfrak{a}}(M) \longrightarrow \varinjlim_t H^j(\operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^t;R),M))$$

constructed above is bijective.

Proof. As above, set $F^j_{\boldsymbol{x}}(-) = \varinjlim_t H^j(\operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^t;R),-))$. The argument is a standard one for proving that a natural transformation between δ-functors is an equivalence. The steps are as follows:

(1) $\theta^0(M)$ is an isomorphism for any R-module M;

(2) $H^j_{\mathfrak{a}}(I) = 0 = F^j_{\boldsymbol{x}}(I)$ for any injective R-module I and $j \geqslant 1$;

(3) Use induction on j to verify that each $\theta^j(M)$ is an isomorphism.

Here are the details:

Step 1. We claim that

$$F^0_{\boldsymbol{x}}(M) = \varinjlim_t \operatorname{Hom}_R(R/\boldsymbol{x}^t R, M) = \Gamma_{\mathfrak{a}}(M)\,.$$

The first equality holds because $\operatorname{Hom}_R(-, M)$ is left-exact; the second holds because the system of ideals $\{(\boldsymbol{x}^t)\}_{t\geqslant 1}$ is cofinal with the system $\{(\boldsymbol{x})^t\}_{t\geqslant 1}$.

Step 2. Since I is injective, $H^j_{\mathfrak{a}}(I) = 0$ for $j \geqslant 1$. As to the vanishing of $F^j_{\boldsymbol{x}}(I)$, one can check easily that $F^j_{\boldsymbol{x}}(-)$ commutes with direct sums. So it suffices to verify the claim for an indecomposable injective $E = E_R(R/\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of R. In this case, one has

$$\operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^t; R), E) \cong \operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}} \otimes_R K^\bullet(\boldsymbol{x}^t; R), E)\,,$$

since E is naturally an $R_{\mathfrak{p}}$-module.

If $\mathfrak{p} \not\supseteq (\boldsymbol{x})$, one of the x_i must be invertible in $R_{\mathfrak{p}}$, so $H^\bullet(R_{\mathfrak{p}} \otimes_R K^\bullet(\boldsymbol{x}^t; R))$ vanishes for each t; see Proposition 6.20. Hence $F^j_{\boldsymbol{x}}(E) = 0$ for all $j \geqslant 1$.

In the case $\mathfrak{p} \supseteq (\boldsymbol{x})$, the R-module E, being $\mathfrak{p}$-torsion, is also $(\boldsymbol{x})$-torsion. In particular, any homomorphism $L \longrightarrow E$, where L is a finitely generated R-module, is $(\boldsymbol{x})$-torsion, when viewed as an element in $\operatorname{Hom}_R(L, E)$. For $u \geqslant t \geqslant 1$, the homomorphism

$$\alpha^j_{ut}\colon \operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^t; R), E)^j \longrightarrow \operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^u; R), E)^j$$

in the directed system defining $F^j_{\boldsymbol{x}}(E)$ is induced by

$$K^{-j}(\boldsymbol{x}^u; R) \xrightarrow{\wedge^j A} K^{-j}(\boldsymbol{x}^t; R)\,,$$

where $A = (a_{pq})$ is the $c \times c$ diagonal matrix with $a_{pp} = x_p^{u-t}$, and $\wedge^j A$ is the j-th exterior power of A; see [**161**, page 41]. Therefore, for $j \geqslant 1$, the matrix $\wedge^j A$ has coefficients in the ideal $(\boldsymbol{x}^{u-t})$. The upshot is that for a fixed integer $t \geqslant 1$ and homomorphism $f\colon K^{-j}(\boldsymbol{x}^t; R) \longrightarrow E$, since f is $(\boldsymbol{x})$-torsion, there exists $u \geqslant t$ with $\alpha^j_{ut}(f) = 0$. Thus cycles in $\operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^t; R), E)^j$ do not survive in the limit, so $F^j_{\boldsymbol{x}}(E) = 0$, as claimed.

Step 3. We argue by induction on j that $\theta^j(-)$ is an isomorphism for all R-modules; the basis of the induction is Step 1. Assume that $\theta^{j-1}(-)$ is an isomorphism for some integer $j \geqslant 1$. Given an R-module M, embed it into an injective module I to get an exact sequence

$$0 \longrightarrow M \longrightarrow I \longrightarrow N \longrightarrow 0\,.$$

The functoriality of the $\theta^j(-)$ yields a commutative diagram

$$\begin{tikzcd}
H^{j-1}_{\boldsymbol{x}R}(I) \arrow[r] \arrow[d, "\cong"', "\theta^{j-1}(I)"] & H^{j-1}_{\boldsymbol{x}R}(N) \arrow[r] \arrow[d, "\cong"', "\theta^{j-1}(N)"] & H^{j}_{\boldsymbol{x}R}(M) \arrow[r] \arrow[d, "\theta^{j}(M)"] & H^{j}_{\boldsymbol{x}R}(I) = 0 \\
F^{j-1}_{\boldsymbol{x}}(I) \arrow[r] & F^{j-1}_{\boldsymbol{x}}(N) \arrow[r] & F^{j}_{\boldsymbol{x}}(M) \arrow[r] & F^{j}_{\boldsymbol{x}}(I) = 0\,,
\end{tikzcd}$$

where the isomorphisms are given by the induction hypothesis, and the vanishing is by Step 2. Thus by the three-lemma, if there is such a thing, $\theta^j(M)$ is bijective. □

4. Return of the Čech complex

Next we provide an alternative formulation of Theorem 7.11.

Construction 7.12. As before, let $\boldsymbol{x} = x_1, \ldots, x_c$ be elements in R, and $t \geqslant 1$ an integer. There is a canonical isomorphism

$$\mathrm{Hom}_R(K^\bullet(\boldsymbol{x}^t; R), M) \cong \big(K^\bullet(\boldsymbol{x}^t; R) \otimes_R M\big)[-c]$$

of complexes of R-modules; see Exercise 6.9. Set

$$\check{C}^\bullet(\boldsymbol{x}; M) = \varinjlim_t \big(K^\bullet(\boldsymbol{x}^t; R) \otimes_R M\big)[-c]\,.$$

We will see shortly that this is precisely the Čech complex encountered in Definition 6.26, and the notation anticipates this discovery. Since direct limits commute with tensor products, we have

$$\check{C}^\bullet(\boldsymbol{x}; M) = \varinjlim_t K^\bullet(\boldsymbol{x}^t; R)[-c] \otimes_R M = \check{C}^\bullet(\boldsymbol{x}; R) \otimes_R M\,. \tag{7.12.1}$$

Next, we want to analyze $\check{C}^\bullet(\boldsymbol{x}; R)$. The tensor product decomposition of $K^\bullet(\boldsymbol{x}^t; R)$ implies that

$$\check{C}^\bullet(\boldsymbol{x}; R) \cong \big(\varinjlim_t K^\bullet(x_1^t; R)[-1]\big) \otimes_R \cdots \otimes_R \big(\varinjlim_t K^\bullet(x_c^t; R)[-1]\big), \tag{7.12.2}$$

so it suffices to examine $\check{C}^\bullet(x; R)$ for an element $x \in R$. The directed system is obtained by applying $\mathrm{Hom}_R(-, R)$ to the one in (7.10.1) and looks like

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "x^t"] \arrow[d, equal] & R \arrow[r] \arrow[d, "x"] & 0 \\
0 \arrow[r] & R \arrow[r, "x^{t+1}"] & R \arrow[r] & 0\,.
\end{tikzcd}$$

The direct limit of the system $R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots$ is R_x by Exercise 4.10. With this in hand, the direct limit $\check{C}^\bullet(x; R)$ of the system above is

$$0 \longrightarrow R \longrightarrow R_x \longrightarrow 0$$

sitting in degrees 0 and 1, where the map $R \longrightarrow R_x$ is localization. Feeding this into (7.12.2), one sees that $\check{C}^\bullet(\boldsymbol{x};R)$ is the complex with

$$\check{C}^j(\boldsymbol{x};R) = \bigoplus_{1\leqslant i_1<\cdots<i_j\leqslant c} R_{x_{i_1}\cdots x_{i_j}},$$

and differential $\partial^j_{\boldsymbol{x}}\colon \check{C}^j(\boldsymbol{x};R) \longrightarrow \check{C}^{j+1}(\boldsymbol{x};R)$ defined to be an alternating sum of maps

$$\partial\colon R_{x_{i_1}\cdots x_{i_j}} \longrightarrow R_{x_{i'_1}\cdots x_{i'_{j+1}}}$$

which is the localization map if $\{i_1,\dots,i_j\} \subseteq \{i'_1,\dots,i'_{j+1}\}$, and zero otherwise. This is the Čech complex on $\boldsymbol{x}$ encountered in Definition 6.26, and it has the form

$$0 \longrightarrow R \longrightarrow \bigoplus_{1\leqslant i\leqslant c} R_{x_i} \longrightarrow \bigoplus_{1\leqslant i<j\leqslant c} R_{x_ix_j} \longrightarrow \cdots \longrightarrow R_{x_1\cdots x_c} \longrightarrow 0\,.$$

By (7.12.1) and the description of $\check{C}^\bullet(\boldsymbol{x};R)$, Theorem 7.11 translates to:

Theorem 7.13. *Let $\boldsymbol{x} = x_1,\dots,x_c$ be a set of generators for an ideal $\mathfrak{a}$. For each R-module M and integer j, there is a natural isomorphism*

$$H^j_{\mathfrak{a}}(M) \cong H^j\left(\check{C}^\bullet(\boldsymbol{x};R)\otimes_R M\right) = \check{H}^j(\boldsymbol{x};M)\,. \qquad \square$$

One can now describe the last conceivably nonzero local cohomology module; compare this result with Example 6.29.

Corollary 7.14. *If $\boldsymbol{x} = x_1,\dots,x_c$ is a set of generators for an ideal $\mathfrak{a}$, then, for each R-module M, one has $H^j_{\mathfrak{a}}(M) = 0$ for $j \geqslant c+1$, and*

$$H^c_{\mathfrak{a}}(M) = M_{x_1\cdots x_c}\Big/\sum_{i=1}^{c} \operatorname{image}(M_{x_1\cdots x_{i-1}x_{i+1}\cdots x_c})\,. \qquad \square$$

The next item adds to the list of properties of local cohomology stated in Proposition 7.3. These are all straightforward applications of Theorem 7.13 and are left as exercises. Thought exercise: can you verify these without taking recourse to Theorem 7.13 ?

Proposition 7.15. *Let $\mathfrak{a}$ be an ideal of a ring R, and M an R-module.*

(1) *If U is a multiplicatively closed subset of R, then*

$$H^j_{\mathfrak{a}}(U^{-1}M) \cong U^{-1}H^j_{\mathfrak{a}}(M)\,.$$

(2) *If $R \longrightarrow S$ is a ring homomorphism and N is an S-module, then*

$$H^j_{\mathfrak{a}}(N) \cong H^j_{\mathfrak{a}S}(N)\,.$$

(3) *If $R \longrightarrow S$ is flat, then there is a natural isomorphism of S-modules*

$$S\otimes_R H^j_{\mathfrak{a}}(M) \cong H^j_{\mathfrak{a}S}(S\otimes_R M)\,. \qquad \square$$

The following calculation illustrates Corollary 7.14:

Example 7.16. Let $R = \mathbb{K}[x_1, \ldots, x_c]$ be a polynomial ring over a field $\mathbb{K}$, and let $\mathfrak{a} = (x_1, \ldots, x_c)$. Then

$$H^j_{\mathfrak{a}}(R) = \begin{cases} y_1 \cdots y_c \mathbb{K}[y_1, \ldots, y_c] & \text{if } j = c\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where $\mathbb{K}[y_1, \ldots, y_c]$ is the polynomial ring over $\mathbb{K}$, in variables $y_1, \ldots, y_c$ of degree -1, and the R-module structure on it is defined by

$$x_i \cdot (y_1^{a_1} \cdots y_c^{a_c}) = \begin{cases} y_1^{a_1} \cdots y_i^{a_i - 1} \cdots y_c^{a_c} & \text{if } a_i \geqslant 2\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Indeed, the claim about $H^j_{\mathfrak{a}}(R)$ for $j \geqslant c$ follows from Corollary 7.14. As to the other values of j, for each t the sequence $x_1^t, \ldots, x_c^t$ is R-regular. Thus, the depth sensitivity of Koszul complexes implies $H^j(\boldsymbol{x}^t; R) = 0$ for $-c \leqslant j \leqslant -1$. Since $\check{C}^\bullet(\boldsymbol{x}; R)$ is a direct limit of these Koszul complexes shifted c steps to the right (see Construction 7.12), and homology commutes with direct limits, one obtains $\check{H}^j(\boldsymbol{x}; R) = 0$ for $j < c$; see also Theorem 9.1.

Exercise 7.17. Find a $\mathbb{K}$-basis for the following local cohomology modules:

(1) $H^1_{(xy,xz)}(\mathbb{K}[x,y,z])$.

(2) $H^2_{(x,y,z)}(R)$, where $R = \mathbb{K}[x,y,z]/(xz - y^2)$.

(3) $H^1_{(x,z)}(R)$, where $R = \mathbb{K}[w,x,y,z]/(wx - yz)$.

Auslander-Buchsbaum Formula and Global Dimension

This lecture revisits regular sequences in local rings. One of the results is a characterization of depth in terms of the vanishing of Ext-modules, and another is the Auslander-Buchsbaum formula that relates depth and projective dimension. We will prove that regular local rings are precisely those local rings over which all modules have finite projective resolutions.

1. Regular sequences and depth redux

In Lecture 6 we briefly encountered regular sequences and depth. The basic question we wish to address here is how to establish the existence of a regular sequence, short of actually specifying elements. For example, given a ring R, an ideal $\mathfrak{a}$, and a module M, how can we tell whether $\mathfrak{a}$ contains an M-regular element? More generally, can we get a lower bound for $\operatorname{depth}_R(\mathfrak{a}, M)$?

Lemma 8.1. *Let R be a Noetherian ring and let M, N be finitely generated modules such that $M \otimes_R N$ is nonzero.*

Then $\operatorname{ann} N$ *has an M-regular element if and only if* $\operatorname{Hom}_R(N, M) = 0$.

Proof. Assume that $\operatorname{ann} N$ consists of zerodivisors on M. By Lemma 6.13, $\operatorname{ann} N$ is contained in the union of the associated primes of M and hence, by prime avoidance, $\operatorname{ann} N$ is contained in some associated prime $\mathfrak{p}$ of M. Since N is finitely presented, we have $\operatorname{Hom}_R(N, M)_{\mathfrak{p}} = \operatorname{Hom}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}})$ and it suffices to show that this module is nonzero. Localize at $\mathfrak{p}$ and assume that

$(R, \mathfrak{m})$ is a local ring and $\mathfrak{m}$ is in $\operatorname{Ass} M$. This implies that $R/\mathfrak{m} \hookrightarrow M$, and composing with the surjection $N \longrightarrow N/\mathfrak{m}N \longrightarrow R/\mathfrak{m}$, we get a nonzero homomorphism $N \longrightarrow M$. We leave the converse as an exercise. $\square$

The following result and Lemma 8.1 will allow us to compute depth.

Proposition 8.2. *Let R be a ring and let M and N be R-modules. If there is an M-regular sequence $\boldsymbol{x} = x_1, \dots, x_d$ in* $\operatorname{ann} N$*, then*

$$\operatorname{Ext}^j_R(N, M) = \begin{cases} 0 & \text{if } j < d\,, \\ \operatorname{Hom}_R(N, M/\boldsymbol{x}M) & \text{if } j = d\,. \end{cases}$$

Proof. The case $d = 0$ is trivial, so assume $d \geqslant 1$ and proceed by induction on d. Observe that $N \xrightarrow{x_1} N$ is the zero map, and hence, by functoriality, $\operatorname{Ext}^\bullet_R(N, -) \xrightarrow{x_1} \operatorname{Ext}^\bullet_R(N, -)$ is zero as well. Thus the exact sequence of Ext-modules obtained from the exact sequence

$$0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0$$

splits into exact sequences

$$0 \longrightarrow \operatorname{Ext}^t_R(N, M) \longrightarrow \operatorname{Ext}^t_R(N, M/x_1M) \longrightarrow \operatorname{Ext}^{t+1}_R(N, M) \longrightarrow 0\,.$$

Since $x_2, \dots, x_d$ in $\operatorname{ann} N$ is a regular sequence on M/x_1M, the induction hypothesis implies that

$$\operatorname{Ext}^j_R(N, M/x_1M) = \begin{cases} 0 & \text{if } j < d-1\,, \\ \operatorname{Hom}_R(N, M/\boldsymbol{x}M) & \text{if } j = d-1\,. \end{cases}$$

The desired result now follows from the exact sequence above. $\square$

Definition 8.3. A *maximal* M-regular sequence in $\mathfrak{a}$ is one which cannot be extended to a longer M-regular sequence in $\mathfrak{a}$.

It is easy to see that in a Noetherian ring every regular sequence can be extended to a maximal one. It is perhaps surprising that all maximal regular sequences have the same length:

Theorem 8.4 (Rees' theorem)**.** *Let R be a Noetherian ring, M a finitely generated R-module, and $\mathfrak{a}$ an ideal such that $\mathfrak{a}M \neq M$.*

Maximal M-regular sequences in $\mathfrak{a}$ have the same length, which is

$$\min\{i \mid \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\}\,.$$

Proof. Let $\boldsymbol{x} = x_1, \dots, x_d$ be a maximal M-regular sequence in $\mathfrak{a}$. Then, by Proposition 8.2, $\operatorname{Ext}^i_R(R/\mathfrak{a}, M) = 0$ for $i < d$ and

$$\operatorname{Ext}^d_R(R/\mathfrak{a}, M) = \operatorname{Hom}_R(R/\mathfrak{a}, M/\boldsymbol{x}M)\,.$$

As $\boldsymbol{x}$ is maximal, $\operatorname{Hom}_R(R/\mathfrak{a}, M/\boldsymbol{x}M) \neq 0$ by Lemma 8.1. $\square$

One special case arises so often that we single it out. When $(R, \mathfrak{m})$ is a local ring, we simply write $\operatorname{depth}_R M$ for $\operatorname{depth}_R(\mathfrak{m}, M)$.

Corollary 8.5. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring. If M is a finitely generated R-module, then* $\operatorname{depth}_R M = \min\{i \mid \operatorname{Ext}^i_R(\mathbb{K}, M) \neq 0\}$. □

Computing $\operatorname{Ext}^\bullet_R(R/\mathfrak{a}, M)$ by a projective resolution of $R/\mathfrak{a}$, Rees' theorem yields the following corollary.

Corollary 8.6. *For a nonzero finitely generated module M over a Noetherian ring R, we have* $\operatorname{depth}_R(\mathfrak{a}, M) \leqslant \operatorname{pd}_R(R/\mathfrak{a})$. □

The fortuitous coincidence of the property that $\mathfrak{a}$ contains an M-regular sequence, with the homological property that certain Ext-modules vanish, accounts for the power of the concept of depth. A further such coincidence is awaiting us in the form of Theorem 9.1.

The reader is invited to prove the next result using Rees' theorem.

Lemma 8.7. *Let $\mathfrak{a}$ be an ideal in a Noetherian ring R.*

For any exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ of finitely generated modules, one has the following equalities:

(1) $\operatorname{depth}_R(\mathfrak{a}, M) \geqslant \min\{\operatorname{depth}_R(\mathfrak{a}, M'), \operatorname{depth}_R(\mathfrak{a}, M'')\}$
(2) $\operatorname{depth}_R(\mathfrak{a}, M') \geqslant \min\{\operatorname{depth}_R(\mathfrak{a}, M), \operatorname{depth}_R(\mathfrak{a}, M'') + 1\}$
(3) $\operatorname{depth}_R(\mathfrak{a}, M'') \geqslant \min\{\operatorname{depth}_R(\mathfrak{a}, M') - 1, \operatorname{depth}_R(\mathfrak{a}, M)\}$. □

2. Global dimension

Recall that the projective dimension of a module M is the minimal length of an R-projective resolution of M. Motivated by Corollary 8.6, we study homological characterizations of projective dimension. Over local rings, projective modules are free: for finitely generated modules, this is an application of Nakayama's lemma, and the general case is a theorem of Kaplansky; see [**115**, Theorem 2.5] for a proof. Therefore, over local rings, projective resolutions are free resolutions.

Definition 8.8. Let $(R, \mathfrak{m})$ be a local ring. Each finitely generated module has a free resolution $F_\bullet$ of the form

$$\cdots \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

with each F_i free of finite rank. By choosing bases for each F_i, we can write each φ_i as a matrix with entries from R. The resolution $F_\bullet$ is *minimal* if $\varphi_i(F_i) \subseteq \mathfrak{m}F_{i-1}$ for each i; equivalently, if the entries of the matrices representing the maps φ_i are in $\mathfrak{m}$.

Proposition 8.9. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and M a finitely generated R-module. Then every minimal free resolution of M has the same length, which equals the projective dimension of M. Specifically,*

$$\operatorname{pd}_R M = \sup\{i \mid \operatorname{Tor}_i^R(\mathbb{K}, M) \neq 0\}\,.$$

Proof. Consider a minimal free resolution $F_\bullet$ of M,

$$\cdots \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0\,,$$

and let β_i be the rank of F_i. We claim that $\beta_i = \operatorname{rank}_{\mathbb{K}} \operatorname{Tor}_i^R(\mathbb{K}, M)$. To see this, note that the maps in the complex $\mathbb{K} \otimes_R F_\bullet$ are zero. Consequently $\operatorname{Tor}_i^R(\mathbb{K}, M)$ is the i-th homology of the complex

$$\cdots \xrightarrow{0} \mathbb{K}^{\beta_n} \xrightarrow{0} \mathbb{K}^{\beta_{n-1}} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{K}^{\beta_1} \xrightarrow{0} \mathbb{K}^{\beta_0} \longrightarrow 0\,,$$

which is $\mathbb{K}^{\beta_i}$ as claimed. □

In the context of the theorem above, it turns out that minimal resolutions of M are, in fact, isomorphic as complexes of R-modules.

Definition 8.10. Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and M a finitely generated R-module. The *i-th Betti number of M over R* is the number

$$\beta_i = \beta_i^R(M) = \operatorname{rank}_{\mathbb{K}} \operatorname{Tor}_i^R(\mathbb{K}, M)\,.$$

The *global dimension* of R is the supremum of projective dimensions of finitely generated R-modules, which by Proposition 8.9 is equal to

$$\sup\{i \mid \beta_R^i(M) \neq 0 \text{ for some finitely generated module } M\}\,.$$

Corollary 8.11. *The global dimension of a local ring $(R, \mathfrak{m}, \mathbb{K})$ is $\operatorname{pd}_R \mathbb{K}$. In particular, $\operatorname{pd}_R(M) \leqslant \operatorname{pd}_R(\mathbb{K})$ for each finitely generated R-module M.*

Proof. Compute $\operatorname{Tor}_i^R(\mathbb{K}, M)$ from a free resolution of $\mathbb{K}$. Since $\operatorname{Tor}_i^R(\mathbb{K}, M)$ vanishes for $i > \operatorname{pd}_R \mathbb{K}$, it follows that $\operatorname{pd}_R M \leqslant \operatorname{pd}_R \mathbb{K}$. □

After these preparations, we are ready to study regular rings. Recall from Lecture 1 that a local ring $(R, \mathfrak{m})$ is *regular* if $\mathfrak{m}$ can be generated by $\dim R$ elements. It turns out that a minimal generating set for the maximal ideal of a regular local ring is a regular sequence. The regularity of this sequence is the key to the homological characterization of regular local rings.

Remark 8.12. The *embedding dimension* of a local ring $(R, \mathfrak{m})$ is $\nu_R(\mathfrak{m})$, the minimal number of generators of $\mathfrak{m}$. Krull's height theorem implies $\nu(\mathfrak{m}) \geqslant \operatorname{height} \mathfrak{m} = \dim R$; regular rings are those for which equality holds.

3. Auslander-Buchsbaum formula

In Corollary 8.6, we saw that $\operatorname{depth}_R(\mathfrak{a}, R) \leqslant \operatorname{pd}_R(R/\mathfrak{a})$; ideals for which equality is attained are called *perfect*. Our next result provides a substantial sharpening of this corollary when $\mathfrak{a} = \mathfrak{m}$.

Theorem 8.13 (Auslander-Buchsbaum formula)**.** *Let R be a local ring. If M is a nonzero finitely generated module of finite projective dimension, then*

$$\operatorname{pd}_R M + \operatorname{depth} M = \operatorname{depth} R\,.$$

Proof. If $\operatorname{pd}_R M = 0$, then M is free, and $\operatorname{depth} M = \operatorname{depth} R$. We may therefore assume that $h = \operatorname{pd}_R M \geqslant 1$. If $h = 1$, let

$$0 \longrightarrow R^s \xrightarrow{\ \varphi\ } R^r \longrightarrow M \longrightarrow 0$$

be a minimal presentation of M. We consider φ as an $r \times s$ matrix over R, with entries in the maximal ideal $\mathfrak{m}$. Let $\mathbb{K}$ be the residue field of R, and apply $\operatorname{Hom}_R(\mathbb{K}, -)$ to obtain the exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i_R(\mathbb{K}, R^s) \xrightarrow{\operatorname{Ext}^i_R(\mathbb{K},\varphi)} \operatorname{Ext}^i_R(\mathbb{K}, R^r) \longrightarrow \operatorname{Ext}^i_R(\mathbb{K}, M) \longrightarrow \cdots\,.$$

Using the identification $\operatorname{Ext}^i_R(\mathbb{K}, R^m) \cong \operatorname{Ext}^i_R(\mathbb{K}, R)^m$, we see that the map $\operatorname{Ext}^i_R(\mathbb{K}, \varphi)$ is represented by the reduction modulo $\mathfrak{m}$ of φ. Since φ is minimal, this is the zero map. Then, for each i, we have an exact sequence

$$0 \longrightarrow \operatorname{Ext}^i_R(\mathbb{K}, R)^r \longrightarrow \operatorname{Ext}^i_R(\mathbb{K}, M) \longrightarrow \operatorname{Ext}^{i+1}_R(\mathbb{K}, R)^s \longrightarrow 0\,.$$

It follows that $\operatorname{depth} M = \operatorname{depth} R - 1$, and we are done in this case.

If $h \geqslant 2$, consider an exact sequence $0 \longrightarrow M' \longrightarrow R^r \longrightarrow M \longrightarrow 0$. Then $\operatorname{pd}_R M' = \operatorname{pd}_R M - 1$. By induction, $\operatorname{depth} M' = \operatorname{depth} R - h + 1$. Since $\operatorname{depth} M' = \operatorname{depth} M + 1$ by Lemma 8.7, the result follows. $\square$

Corollary 8.14. *Let R be a local ring. If M is a finitely generated module of finite projective dimension, then* $\operatorname{pd}_R M \leqslant \operatorname{depth} R$. $\square$

Remark 8.15. The corollary states that there is a global bound on the projective dimension of modules of finite projective dimension. For non-commutative rings, there is no obvious analogue. It is an open question in the theory of noncommutative Artinian rings, the *finitistic dimension conjecture*, whether the number

$$\sup\{\operatorname{pd}_R M \mid M \text{ is finitely generated and } \operatorname{pd}_R M < \infty\}$$

is necessarily finite.

4. Regular local rings

We now investigate the singular world of regular local rings. To get an idea where we are headed, observe the following: on the regular local ring $R = \mathbb{K}[[x_1, \dots, x_n]]$, the sequence $\boldsymbol{x} = x_1, \dots, x_n$ is regular and hence the Koszul complex on $\boldsymbol{x}$ is a resolution of $R/\boldsymbol{x}R = \mathbb{K}$. In particular, $\mathbb{K}$ has finite projective dimension. From Corollary 8.11 it follows that R has finite global dimension. We shall see that this property characterizes regularity.

Lemma 8.16. *Let $\boldsymbol{x} = x_1, \dots, x_t$ be elements in a local ring R. Consider the statements*

(1) *$\boldsymbol{x}$ is a regular sequence.*

(2) $\operatorname{height}(x_1, \dots, x_i) = i$ *for* $i = 1, \dots, t$.

(3) $\operatorname{height}(x_1, \dots, x_t) = t$.

(4) *$\boldsymbol{x}$ is part of a system of parameters for R.*

Then (1) $\implies$ (2) $\iff$ (3) $\iff$ (4). *In particular,* $\operatorname{depth}_R(\mathfrak{a}, R) \leqslant \operatorname{height} \mathfrak{a}$ *for each ideal* $\mathfrak{a}$.

We use $\operatorname{Min}(\mathfrak{a})$ to denote the set of primes minimal over $\mathfrak{a}$.

Proof. (1) $\implies$ (2). Lemma 6.13 implies the inequalities

$$0 < \operatorname{height}(x_1) < \operatorname{height}(x_1, x_2) < \cdots < \operatorname{height}(\boldsymbol{x}),$$

and the claim then follows from Krull's height theorem.

(2) $\implies$ (3). This is obvious.

(3) $\implies$ (2). This implication again follows from Krull's height theorem: if $\operatorname{height}(x_1, \dots, x_i) < i$, then $\operatorname{height}(x_1, \dots, x_t) < t$.

(3) $\implies$ (4). If R has dimension t, we are done. If $\dim R > t$, then $\mathfrak{m}$ is not a minimal prime of $(\boldsymbol{x})$. It follows from prime avoidance applied to $\mathfrak{m}$ and $\operatorname{Min}(\boldsymbol{x})$ that there exists $x_{t+1} \in \mathfrak{m} \smallsetminus \operatorname{Min}(\boldsymbol{x})$, so $(x_1, \dots, x_t, x_{t+1})$ has height $t+1$. Continuing this way, we obtain a system of parameters for R.

(4) $\implies$ (3). Let $\boldsymbol{x}, \boldsymbol{y}$ be a system of parameters. Theorem 1.17 yields $\operatorname{height}(\boldsymbol{x}, \boldsymbol{y}) = \dim R$, so $\operatorname{height}(\boldsymbol{x}) = t$ by the implication (3) $\implies$ (2) applied to $\boldsymbol{x}, \boldsymbol{y}$.

The last claim is obvious. □

The implication (2) $\implies$ (1) fails in the following example.

Example 8.17. Let $R = \mathbb{K}[[x, y, z]]/(xy, xz)$. Then $x - y, z$ is not a regular sequence, though the ideal $(x - y, z)$ has height two.

Theorem 8.18. *A regular local ring is a domain.*

Proof. Let $(R, \mathfrak{m})$ be a regular local ring of dimension d. We induce on d. If $d = 0$, then R is a field. Assume that $d > 0$ and that the desired result holds for rings of dimension at most $d - 1$. By prime avoidance,[1] there exists an element $x \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ which is not in any minimal prime of R. By Nakayama's lemma, the maximal ideal of R/xR is generated by $d - 1$ elements, so Theorems 1.17 and 1.47 imply that $\dim R/xR = d - 1$. Hence R/xR is regular, and the induction hypothesis implies that it is a domain. Consequently xR is a prime ideal of R and is not minimal. Let $\mathfrak{p}$ be a minimal prime of R contained in xR and let $y \in \mathfrak{p}$. Then $y = ax$ for some $a \in R$, and since $x \notin \mathfrak{p}$, one has $a \in \mathfrak{p}$. Hence $x\mathfrak{p} = \mathfrak{p}$, which implies that $\mathfrak{p} = (0)$ by Nakayama's lemma. □

Corollary 8.19. *Let $(R, \mathfrak{m})$ be a local ring and $x \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$. If R is regular, then so is R/xR. The converse holds if x is not in any minimal prime.*

Proof. If R is regular, then the maximal ideal of R/xR is generated by $\dim R - 1$ elements and so Theorems 1.17 and 1.47 imply $\dim R/xR = d - 1$. Consequently R/xR is regular. The converse is clear from the definition. □

The next result shows that regular local rings are Cohen-Macaulay; see Lecture 10 for the definition.

Proposition 8.20. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a regular local ring. Each system of parameters is a regular sequence; in particular, $\operatorname{pd}_R \mathbb{K} = \dim R$.*

Proof. It suffices to prove that for any module M of finite length, we have

$$\operatorname{Ext}_R^j(M, R) = 0 \qquad \text{for } j < \dim R\,;$$

indeed then if $\boldsymbol{x}$ is a system of parameters, taking $M = R/\boldsymbol{x}R$ one obtains $\operatorname{depth}_R(\boldsymbol{x}R, R) \geqslant \dim R$ by Theorem 8.4, and hence $\boldsymbol{x}$ is a regular sequence by Corollary 6.24.

We prove the desired vanishing by induction on $\ell(M)$, the length of M. If $\ell(M) = 1$, then $M \cong \mathbb{K}$. Let $\boldsymbol{x}$ be a minimal generating set for $\mathfrak{m}$. By Theorem 8.18 the ring R is a domain, so x_1 is a nonzerodivisor. As R/x_1R is again regular by Corollary 8.19, an iteration yields that $\boldsymbol{x}$ is a regular sequence and Theorem 8.4 gives the result we seek.

When $\ell(M) > 1$ there is an exact sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow M \longrightarrow M' \longrightarrow 0\,.$$

The induction hypothesis and the induced exact sequence of Ext-modules complete the proof. □

[1] We are using a general form of prime avoidance where up to two of the ideals need not be prime; see [**87**, Theorem 81].

Putting the pieces together, we have seen that if $(R, \mathfrak{m})$ is a regular local ring, then $\mathfrak{m}$ is generated by a regular sequence. In particular, $R/\mathfrak{m}$ has finite projective dimension equal to the number of minimal generators of $\mathfrak{m}$. Before we can prove that finite global dimension implies regularity, we need one more ingredient.

Lemma 8.21. *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated module. If $x \in \mathfrak{m}$ is a nonzerodivisor on R and on M, then*

$$\operatorname{pd}_{R/xR} M/xM = \operatorname{pd}_R M\,.$$

Proof. Let $F_\bullet$ be a minimal free resolution of M and set $S = R/xR$. The complex $S \otimes_R F_\bullet$ of free S-modules is minimal, and its homology in degree i equals $\operatorname{Tor}_i^R(S, M)$. Since x is a nonzerodivisor, the complex

$$0 \longrightarrow R \xrightarrow{\;x\;} R \longrightarrow 0$$

is a free resolution of S, so $\operatorname{Tor}_i^R(S, M)$ can also be computed as the homology of the complex

$$0 \longrightarrow M \xrightarrow{\;x\;} M \longrightarrow 0\,.$$

When x is a nonzerodivisor on M, the only nonzero homology module of this complex is in degree 0, where it is M/xM, so $S \otimes_R F_\bullet$ is a minimal free resolution of M/xM over S. This implies the desired equality. $\square$

The following result is due to Auslander, Buchsbaum, and Serre.

Theorem 8.22. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring. The following are equivalent.*

(1) *R is regular.*

(2) *The global dimension of R equals* $\dim R$.

(3) $\operatorname{pd}_R \mathbb{K}$ *is finite.*

Proof. The implication (1) $\Longrightarrow$ (2) follows from Proposition 8.20 along with Corollary 8.11, while (2) $\Longrightarrow$ (3) is clear. For (3) $\Longrightarrow$ (1), we induce on the minimal number of generators t of $\mathfrak{m}$. If $t = 0$, then the zero ideal is maximal, so R is a field. Assume that $t \geqslant 1$ and that $\operatorname{pd}_R \mathbb{K}$ is finite.

Note that $\mathfrak{m} \notin \operatorname{Ass}(R)$; this follows from the Auslander-Buchsbaum formula as $\mathbb{K}$ has finite projective dimension. Alternatively, take a minimal free resolution of $\mathbb{K}$. Since the last map in this resolution is injective and it can be represented by a matrix with entries in $\mathfrak{m}$, it follows that $(0 :_R \mathfrak{m}) = 0$.

By prime avoidance, there is an element $x \in \mathfrak{m}$ not in $\mathfrak{m}^2$ or in any associated prime of R. Set $S = R/xR$; it suffices to show that S is regular, for then R is regular by Corollary 8.19.

To show that S is regular, it suffices by induction to show that its residue field $\mathbb{K}$ has finite projective dimension or, equivalently, that the maximal ideal of S has finite projective dimension. By Lemma 8.21, the projective dimension of $\mathfrak{m}/x\mathfrak{m}$ over S is finite. We claim that the sequence of S-modules

$$0 \longrightarrow xR/x\mathfrak{m} \longrightarrow \mathfrak{m}/x\mathfrak{m} \longrightarrow \mathfrak{m}/xR \longrightarrow 0$$

is split-exact, which implies the result.

Indeed, it is clear that the sequence above is exact. The image of $xR/x\mathfrak{m}$ under the natural surjection $\mathfrak{m}/x\mathfrak{m} \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ is nonzero since $x \notin \mathfrak{m}^2$. But

$$R/\mathfrak{m} \cong xR/x\mathfrak{m} \longrightarrow \mathfrak{m}/\mathfrak{m}^2$$

is an injective map of vector spaces, and is hence split. $\square$

It is now easy to prove the following fact, which, before Auslander, Buchsbaum, and Serre pioneered the use of homological methods in local algebra, was known only for certain classes of regular rings.

Corollary 8.23. *Let $\mathfrak{p}$ be a prime ideal in a local ring R. If R is regular, then so is $R_\mathfrak{p}$.*

Proof. When R is regular, $\operatorname{pd}_R R/\mathfrak{p}$ is finite by Theorem 8.22, and hence $\operatorname{pd}_{R_\mathfrak{p}} R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is finite as well. Since $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is the residue field of $R_\mathfrak{p}$, another use of Theorem 8.22 yields the desired conclusion. $\square$

Exercise 8.24. Let $\mathfrak{p}$ be a prime ideal in a Noetherian ring R. Prove that if $R/\mathfrak{p}$ has finite projective dimension, then $R_\mathfrak{p}$ is a domain.

Exercise 8.25. Let R be a regular local ring of dimension d, and $\mathfrak{a}$ an ideal. If $R/\mathfrak{a}$ has depth $d-1$, prove that $\mathfrak{a}$ is principal.

If $R/\mathfrak{a}$ has depth and dimension equal to $d-2$ and $\beta_2^R(R/\mathfrak{a}) = 1$, prove that $\mathfrak{a}$ is generated by a regular sequence of length two.

Remark 8.26. We have clung to the case of local rings but we note the similarities with the graded case, discussed in Lecture 5. A little care allows one to extend everything in this lecture to graded rings, homogeneous elements, and homogeneous resolutions; see also [**118**, Proposition 8.18]. In particular, we have the following theorem, for which an argument could be made that it is the second[2] theorem of commutative algebra. Section 5.6 outlines a proof which, like Hilbert's original proof, produces a free resolution rather than a projective one.

Theorem 8.27 (Hilbert's syzygy theorem)**.** *Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a ring of polynomials over a field $\mathbb{K}$. Then every finitely generated R-module has a free resolution of length at most n. If M is a finitely generated graded module with respect to an arbitrary grading on R, then the resolution of M can be chosen to be graded as well.* $\square$

[2] Hilbert's proof of the syzygy theorem (1890) uses his basis theorem (1888), so there is at least one that is older.

5. Complete local rings

Let $(R, \mathfrak{m})$ be a local ring and $\widehat{R}$ its $\mathfrak{m}$-adic completion. Since the dimension of R as well as the number of generators of $\mathfrak{m}$ are unaffected by completion, it follows that R is regular if and only if $\widehat{R}$ is regular. Complete regular local rings are described by Theorem 8.28 below.

Every ring R admits a unique ring homomorphism $\varphi \colon \mathbb{Z} \longrightarrow R$. The kernel of φ is generated by a nonnegative integer $\operatorname{char}(R)$, the *characteristic* of R. A local domain $(R, \mathfrak{m}, \mathbb{K})$ is *equicharacteristic* if $\operatorname{char}(R) = \operatorname{char}(\mathbb{K})$, and has *mixed characteristic* otherwise. The possible values of $\operatorname{char}(\mathbb{K})$ and $\operatorname{char}(R)$ for a local domain are:

(1) $\operatorname{char}(\mathbb{K}) = \operatorname{char}(R) = 0$, in which case $\mathbb{Q} \subseteq R$;

(2) $\operatorname{char}(\mathbb{K}) = \operatorname{char}(R) = p > 0$, in which case $\mathbb{F}_p \subseteq R$;

(3) $\operatorname{char}(\mathbb{K}) = p > 0$ and $\operatorname{char}(R) = 0$, for example, $R = \mathbb{Z}_{(p)}$.

The following theorem is due to Cohen; see [**115**, §29] for a proof.

Theorem 8.28 (Cohen's structure theorem)**.** *Every complete local ring is a homomorphic image of a complete regular local ring.*

Let R be a complete regular local ring. If R is equicharacteristic, then $R \cong \mathbb{K}[[x_1, \dots, x_d]]$ for $\mathbb{K}$ a field. Otherwise, R contains a complete discrete valuation ring V and $R \cong V[[x_1, \dots, x_d]]$, or $R \cong V[[x_1, \dots, x_d]]/(p - f)$ where p is a prime number and f is in $(x_1, \dots, x_d)^2$. □

Lecture 9

Depth and Cohomological Dimension

Given a cohomology theory, a basic problem is to relate its vanishing to properties of the object under consideration. For example, given a module M over a ring R, the functor $\mathrm{Ext}^1_R(M,-)$ is zero if and only if M is projective, while $\mathrm{Tor}^R_1(M,-)$ is zero if and only if M is flat. This lecture provides partial answers in the case of local cohomology.

1. Depth

Recall from Definition 6.17 that $\mathrm{depth}_R(\mathfrak{a}, M)$ denotes the length of the longest M-regular sequence contained in $\mathfrak{a}$. In Theorem 8.4 it was proved that when M is finitely generated, $\mathrm{depth}_R(\mathfrak{a}, M)$ can be measured in terms of the vanishing of $\mathrm{Ext}^\bullet_R(R/\mathfrak{a}, M)$. One consequence of the following theorem is that depth is detected also by local cohomology modules.

Theorem 9.1. *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal, and $K^\bullet$ the Koszul complex on a finite generating set for $\mathfrak{a}$. For each R-module M, the numbers*

$$\begin{aligned}
&\inf\{n \mid \mathrm{Ext}^n_R(R/\mathfrak{a}, M) \neq 0\},\\
&\inf\{n \mid H^n_{\mathfrak{a}}(M) \neq 0\}, \textit{ and}\\
&\inf\{n \mid H^n(\mathrm{Hom}_R(K^\bullet, M)) \neq 0\}
\end{aligned}$$

coincide. In particular, when M is finitely generated, one has

$$\mathrm{depth}_R(\mathfrak{a}, M) = \inf\{n \mid H^n_{\mathfrak{a}}(M) \neq 0\}\,.$$

Sketch of proof. Denote the three numbers in question e, l, and k, respectively; assume that each of these is finite, that is to say, the cohomology modules in question are nonzero in some degree. The argument is more delicate when we do not assume a priori that these numbers are all finite; see the proof of [**41**, Theorem 2.1].

Let $I^\bullet$ be an injective resolution of M, in which case $H^\bullet_{\mathfrak{a}}(M)$ is the cohomology of the complex $\Gamma_{\mathfrak{a}}(I^\bullet)$. It is not hard to verify that

$$H^n\big(\operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(I^\bullet))\big) = \begin{cases} 0 & \text{for } n < l\,, \\ \operatorname{Hom}_R(R/\mathfrak{a}, H^l_{\mathfrak{a}}(M)) & \text{for } n = l\,. \end{cases}$$

By Proposition 7.3(1), the R-module $H^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion, and consequently $\operatorname{Hom}_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$ is nonzero whenever $H^n_{\mathfrak{a}}(M)$ is nonzero. On the other hand, it is clear that

$$\operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(I^\bullet)) = \operatorname{Hom}_R(R/\mathfrak{a}, I^\bullet)\,.$$

The preceding displays imply $e = l$.

To prove that $l = k$, one first proves that the inclusion $\Gamma_{\mathfrak{a}}(I^\bullet) \subseteq I^\bullet$ induces a quasi-isomorphism $\operatorname{Hom}_R(K^\bullet, \Gamma_{\mathfrak{a}}(I^\bullet)) \simeq \operatorname{Hom}_R(K^\bullet, I^\bullet)$. Since $H^n_{\mathfrak{a}}(M) = H^n(\Gamma_{\mathfrak{a}}(I^\bullet))$ is $\mathfrak{a}$-torsion for each n, the desired result follows from a repeated application of the following claim:

Let $C^\bullet$ be a complex of R-modules such that $H^n(C^\bullet)$ is $\mathfrak{a}$-torsion for each integer n, and zero for $n \ll 0$. For each element $x \in \mathfrak{a}$, one has

$$\inf\{n \mid H^n(\operatorname{Hom}_R(K^\bullet(x;R), C^\bullet)) \neq 0\} = \inf\{n \mid H^n(C^\bullet) \neq 0\}\,.$$

Indeed, this is immediate from the long exact sequence that results when we apply $\operatorname{Hom}_R(-, C^\bullet)$ to the exact sequence of complexes

$$0 \longrightarrow R \longrightarrow K^\bullet(x;R) \longrightarrow R[1] \longrightarrow 0$$

and pass to homology. □

The preceding result suggests that when M is not necessarily finitely generated, the right notion of depth is the one introduced via any one of the formulae in the theorem. Such an approach has the merit that it immediately extends to the case where M is a complex. Foxby and Iyengar [**41**] have proved that Theorem 9.1 extends to complexes, with no restrictions on their homology. Thus, all homological notions of depth lead to the same invariant.

Remark 9.2. It turns out that for $d = \operatorname{depth}_R(\mathfrak{a}, M)$, one has

$$\operatorname{Ass}_R \operatorname{Ext}^d_R(R/\mathfrak{a}, M) = \operatorname{Ass}_R H^d_{\mathfrak{a}}(M) = \operatorname{Ass}_R H^d(\operatorname{Hom}_R(K^\bullet, M))\,.$$

See [**53**, §5] or the discussion in [**84**, page 564]. When M is finitely generated, so is $H^d(\operatorname{Hom}_R(K^\bullet, M))$, and hence the latter has only finitely many associated primes. The equalities above now imply that $H^d_{\mathfrak{a}}(M)$ has only finitely many associated primes. This suggests the question: does each local cohomology module have finitely many associated primes? This is not the case, as we will see in Lecture 22.

Now we know in which degree the nonzero local cohomology modules of an R-module M begin to appear. Corollary 7.14 tells us that they disappear eventually, so the next natural step is to determine in which degree the last nonvanishing cohomology module occurs; what one has in mind is a statement akin to Theorem 9.1. A basic result in this direction is the following theorem due to Grothendieck. Its proof uses the local duality theorem which will be covered in Lecture 11.

Theorem 9.3. *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. Then*

$$\sup\{n \mid H^n_{\mathfrak{m}}(M) \neq 0\} = \dim_R M\,.$$

Proof. We claim that $H^n_{\mathfrak{m}}(M) = 0$ for $n > \dim M$. Since M is a module over $R/\operatorname{ann} M$, replacing R by $R/\operatorname{ann} M$ and using Proposition 7.15(2), it suffices to prove that $H^n_{\mathfrak{m}}(M) = 0$ for $n > \dim R$. But $\mathfrak{m}$ is generated up to radical by $\dim R$ elements, so the claim follows from Corollary 7.14.

Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R; it is a local ring with maximal ideal $\mathfrak{m}\widehat{R}$ and residue field $\mathbb{K}$. The $\widehat{R}$-module $\widehat{M} = \widehat{R} \otimes_R M$ is finite, with

$$\dim_{\widehat{R}} \widehat{M} = \dim_R M \qquad \text{and} \qquad H^\bullet_{\mathfrak{m}\widehat{R}}(\widehat{M}) \cong H^\bullet_{\mathfrak{m}}(M)\,,$$

where the first equality is essentially [**6**, Corollary 11.19] and the second follows from Proposition 7.15(3). Thus, substituting $\widehat{R}$ and $\widehat{M}$ for R and M respectively, we may assume that R is $\mathfrak{m}$-adically complete. Cohen's result, Theorem 8.28, now provides a surjective homomorphism $Q \longrightarrow R$ with $(Q, \mathfrak{n})$ a regular local ring. According to Proposition 7.15(2), viewing M as a Q-module through R, one has $H^\bullet_{\mathfrak{n}}(M) \cong H^\bullet_{\mathfrak{m}}(M)$, so we may replace R by Q and assume that R is a complete regular local ring.

We are now in a position to apply Theorem 11.29, which yields, for each integer n, an isomorphism of R-modules

$$H^n_{\mathfrak{m}}(M) \cong \operatorname{Ext}_R^{\dim R - n}(M, R)^\vee\,,$$

with $(-)^\vee = \operatorname{Hom}_R(-, E)$, where E is the injective hull of $\mathbb{K}$. Since $(-)^\vee$ is a faithful functor, Rees' Theorem 8.4 implies the first inequality below:

$$\begin{aligned}\sup\{n \mid H^n_{\mathfrak{m}}(M) \neq 0\} &\geqslant \dim R - \operatorname{depth}_R(\operatorname{ann} M, R)\\ &\geqslant \dim R - \operatorname{height}(\operatorname{ann} M)\\ &\geqslant \dim(R/\operatorname{ann} M)\\ &= \dim M\,,\end{aligned}$$

where the second inequality is given by Lemma 8.16. $\square$

Remark 9.4. Let $(R, \mathfrak{m})$ be a local ring and M a nonzero finitely generated module. By Theorems 9.1 and 9.3 one has

$$H^n_{\mathfrak{m}}(M) = 0 \qquad \text{for } n \notin [\operatorname{depth} M, \dim M]\,,$$

and it is nonzero for n equal to $\operatorname{depth} M$ and $\dim M$. In general, nothing can be said about the vanishing of local cohomology for intermediate values of n: given any sequence of nonnegative integers $n_0 < \cdots < n_s$, there exists a local ring $(R, \mathfrak{m})$ with $\operatorname{depth} R = n_0$ and $\dim R = n_s$ such that $H^n_{\mathfrak{m}}(R)$ is nonzero exactly when n is one of the integers n_i [**34**].

2. Cohomological dimension

Definition 9.5. Let R be a Noetherian ring and $\mathfrak{a}$ an ideal in R. For each R-module M, set

$$\operatorname{cd}_R(\mathfrak{a}, M) = \sup\{n \in \mathbb{Z} \mid H^n_{\mathfrak{a}}(M) \neq 0\}\,.$$

The *cohomological dimension* of $\mathfrak{a}$ is

$$\operatorname{cd}_R(\mathfrak{a}) = \sup\{\operatorname{cd}_R(\mathfrak{a}, M) \mid M \text{ is an } R\text{-module}\}\,.$$

For a local ring $(R, \mathfrak{m})$, one has $\operatorname{cd}_R(\mathfrak{m}) = \dim R$ by Theorem 9.3.

It turns out that the cohomological dimension has a test module.

Theorem 9.6. *Let R be a Noetherian ring. Then $\operatorname{cd}_R(\mathfrak{a}) = \operatorname{cd}_R(\mathfrak{a}, R)$ for each ideal $\mathfrak{a}$.*

Proof. We may assume that $\mathfrak{a} \neq R$. Set $d = \operatorname{cd}_R(\mathfrak{a})$; this number is finite by, for example, Theorem 9.1. Thus, $H^n_{\mathfrak{a}}(-) = 0$ for $n \geqslant d+1$, and there is an R-module M with $H^d_{\mathfrak{a}}(M) \neq 0$. Pick a surjective homomorphism $F \twoheadrightarrow M$, with F a free R-module, and complete to an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0\,.$$

From the resulting long exact sequence, Proposition 7.3(4), one obtains an exact sequence

$$H^d_{\mathfrak{a}}(K) \longrightarrow H^d_{\mathfrak{a}}(F) \longrightarrow H^d_{\mathfrak{a}}(M) \longrightarrow H^{d+1}_{\mathfrak{a}}(K) = 0\,.$$

We conclude that $H^d_{\mathfrak{a}}(F) \neq 0$, and so $H^d_{\mathfrak{a}}(R) \neq 0$ by Proposition 7.3(3). $\square$

The result above can be enhanced to a precise expression relating the local cohomology of M and R at the cohomological dimension:

Exercise 9.7. Let R be a Noetherian ring, $\mathfrak{a}$ an ideal, and set $d = \operatorname{cd}_R(\mathfrak{a}, R)$. Prove that for any R-module M, one has a natural isomorphism

$$H^d_{\mathfrak{a}}(M) \cong H^d_{\mathfrak{a}}(R) \otimes_R M\,.$$

Hint: Theorem 9.6 implies that the functor $H^d_{\mathfrak{a}}(-)$ is right-exact. By the way, what are its left derived functors?

Exercise 9.8. Let $\mathfrak{a}$ be an ideal in a Noetherian ring R, and let M and N be finitely generated modules. Prove the following statements.

(1) $\operatorname{cd}_R(\mathfrak{a}, M) = \operatorname{cd}_R(\mathfrak{a}, R/\operatorname{ann}_R M)$.

(2) If $\operatorname{Supp}_R(M) \subseteq \operatorname{Supp}_R(N)$, then $\operatorname{cd}_R(\mathfrak{a}, M) \leqslant \operatorname{cd}_R(\mathfrak{a}, N)$.

(3) Show that the analogues of (1) and (2) fail for $\operatorname{depth}_R(\mathfrak{a}, -)$.

One way to approach these exercises is via the following exercise:

Exercise 9.9. Let R be a Noetherian ring and M a finitely generated module. If $\operatorname{Supp}_R(M) = \operatorname{Spec} R$, show that for each nonzero module H, the module $H \otimes_R M$ is nonzero.

3. Arithmetic rank

Recall that if ideals $\mathfrak{a}$ and $\mathfrak{b}$ have the same radical, then $H^n_{\mathfrak{a}}(-) = H^n_{\mathfrak{b}}(-)$ by Proposition 7.3(2). This suggests the following definition.

Definition 9.10. Let $\mathfrak{a}$ be an ideal in a Noetherian ring R. The *arithmetic rank* of $\mathfrak{a}$ is the number

$$\operatorname{ara}\mathfrak{a} = \inf\{\nu(\mathfrak{b}) \mid \mathfrak{b} \text{ is an ideal with } \operatorname{rad}\mathfrak{b} = \operatorname{rad}\mathfrak{a}\}\,,$$

where $\nu(\mathfrak{b})$ stands for the minimal number of generators of the ideal $\mathfrak{b}$. Evidently, $\operatorname{ara}\mathfrak{a} \leqslant \nu(\mathfrak{a})$, though the arithmetic rank of $\mathfrak{a}$ could be quite a bit smaller than $\nu(\mathfrak{a})$. For example, $\operatorname{ara}(\mathfrak{a}^n) = \operatorname{ara}\mathfrak{a}$ for each integer $n \geqslant 1$.

Remark 9.11. The arithmetic rank is also pertinent from a geometric perspective. For instance, when R is a polynomial ring over an algebraically closed field, $\operatorname{ara}\mathfrak{a}$ equals the minimal number of hypersurfaces needed to cut out the algebraic set $\operatorname{Var}(\mathfrak{a})$ in affine space.

We return to cohomological dimension.

Proposition 9.12. *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of height h. Then $H^h_{\mathfrak{a}}(R)$ is nonzero and*

$$\operatorname{height}\mathfrak{a} \leqslant \operatorname{cd}_R(\mathfrak{a}) \leqslant \operatorname{ara}\mathfrak{a}\,.$$

Proof. For any ideal $\mathfrak{b}$ with $\operatorname{rad}\mathfrak{b} = \operatorname{rad}\mathfrak{a}$, we have $\operatorname{cd}_R(\mathfrak{a}) = \operatorname{cd}_R(\mathfrak{b})$ by Proposition 7.3(2), and $\operatorname{cd}_R(\mathfrak{b}) \leqslant \nu(\mathfrak{b})$ by Corollary 7.14. This proves the inequality on the right.

Let $\mathfrak{p}$ be a prime containing $\mathfrak{a}$ with $\operatorname{height}\mathfrak{p} = h$. Then

$$H^h_{\mathfrak{a}}(R)_{\mathfrak{p}} \cong H^h_{\mathfrak{a}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \cong H^h_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})\,,$$

where the first isomorphism is by Proposition 7.15(3), and the second by Proposition 7.3(2). But $H^h_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \neq 0$ by Theorem 9.3, so the above isomorphisms imply that $H^h_{\mathfrak{a}}(R) \neq 0$, which settles the inequality on the left. $\square$

This leads to an important result on the arithmetic rank of ideals in local rings; in its original form, it is due to Kronecker [**93**], and has been improved on by several people, notably Forster [**39**]; see also [**102, 103**].

Theorem 9.13. *If* $\mathfrak{a}$ *is a proper ideal in a local ring* R*, then* $\operatorname{ara}\mathfrak{a} \leqslant \dim R$.

Proof. Set $\mathcal{P} = \operatorname{Spec} R \smallsetminus V(\mathfrak{a})$, and for each $n \geqslant 0$ consider the sets

$$\mathcal{P}(n) = \{\mathfrak{p} \in \mathcal{P} \mid \operatorname{height}\mathfrak{p} = n\}, \qquad \text{so that} \qquad \mathcal{P} = \bigcup_{n=0}^{d-1} \mathcal{P}(n)$$

where $d = \dim R$. The idea of the proof is to pick elements $r_0, \dots, r_{d-1}$ in $\mathfrak{a}$ such that, for $0 \leqslant i \leqslant d-1$, the ideal $\mathfrak{b}_i = (r_0, \dots, r_i)$ satisfies the condition

$$\text{if } \mathfrak{p} \in \mathcal{P}(i), \text{ then } \mathfrak{b}_i \not\subseteq \mathfrak{p}\,. \tag{9.13.1}$$

Once this is accomplished, we have $\operatorname{rad}\mathfrak{b}_{d-1} = \operatorname{rad}\mathfrak{a}$ giving the desired result.

Since $\mathcal{P}(0)$ is a subset of the minimal primes of R, its cardinality is finite. Prime avoidance lets us pick an element r_0 in $\mathfrak{a}$ and not in $\bigcup_{\mathfrak{p}\in\mathcal{P}(0)} \mathfrak{p}$. Evidently, $\mathfrak{b}_0$ satisfies (9.13.1) for $i = 0$.

Suppose that for some $0 \leqslant i \leqslant d-2$, elements $r_0, \dots, r_i$, have been chosen such that the ideals $\mathfrak{b}_0, \dots, \mathfrak{b}_i$ satisfy condition (9.13.1). Another use of prime avoidance allows us to pick an element

$$r_{i+1} \in \mathfrak{a} \setminus \bigcup_{\mathfrak{p}\in\operatorname{Min}(\mathfrak{b}_i)\cap\mathcal{P}(i+1)} \mathfrak{p}\,.$$

We claim that the ideal $\mathfrak{b}_{i+1} = \mathfrak{b}_i + (r_{i+1})$ satisfies condition (9.13.1). If not, there exists $\mathfrak{p}$ in $\mathcal{P}(i+1)$ containing $\mathfrak{b}_{i+1}$, and hence also $\mathfrak{b}_i$. If $\mathfrak{p}$ is not minimal over $\mathfrak{b}_i$, there exists a prime ideal $\mathfrak{p}'$ with $\mathfrak{b}_i \subseteq \mathfrak{p}' \subsetneq \mathfrak{p}$. But then $\mathfrak{p}' \in \mathcal{P}(\operatorname{height}\mathfrak{p}')$, which contradicts condition (9.13.1). Thus $\mathfrak{p}$ is minimal over $\mathfrak{b}_i$, that is to say, $\mathfrak{p} \in \operatorname{Min}(\mathfrak{b}_i) \cap \mathcal{P}(i+1)$. But then $r_{i+1} \in \mathfrak{b}_{i+1} \subseteq \mathfrak{p}$ gives a contradiction. □

Remark 9.14. In the situation of the previous theorem, suppose R is a Noetherian ring of dimension d, but is not local. Following the construction of ideals $\mathfrak{b}_i$, we obtain an ideal $\mathfrak{b}_{d-1} \subseteq \mathfrak{a}$ such that $\mathfrak{b}_{d-1}$ is not contained in primes of $\operatorname{Spec} R \smallsetminus V(\mathfrak{a})$ of height less than $\dim R$. There is at most a finite set of prime ideals of height d in $\operatorname{Spec} R \smallsetminus V(\mathfrak{a})$ which contain $\mathfrak{b}_{d-1}$. Picking an element r_d in $\mathfrak{a}$ but outside these finitely many prime ideals, gives an ideal $\mathfrak{b} = (r_0, \dots, r_d) \subseteq \mathfrak{a}$ with the same radical as $\mathfrak{a}$. In particular, algebraic sets in affine d-space can be defined by $d+1$ equations.

Moreover, if R is a standard graded polynomial ring in d variables over a local ring of dimension zero, and if $\mathfrak{a}$ is homogeneous, then the construction in the proof of Theorem 9.13 yields a homogeneous ideal $\mathfrak{b}_{d-1}$ that satisfies condition (9.13.1) for all primes of height less than d. In particular, algebraic sets in $\mathbb{P}^{d-1}$ can be defined by d equations.

Proposition 9.15. *Let $\mathfrak{a}$ be an ideal in a Noetherian ring R. For each finitely generated module M, one has $H^n_{\mathfrak{a}}(M) = 0$ for $n \geqslant \dim M + 1$. In particular,* $\operatorname{cd}_R(\mathfrak{a}) \leqslant \dim R$.

Proof. For each prime ideal $\mathfrak{p}$, one has $H^n_{\mathfrak{a}}(M)_{\mathfrak{p}} = H^n_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for all n. Since $\dim M_{\mathfrak{p}} \leqslant \dim M$, it suffices to consider the case where R is local. Moreover, M is a module over the ring $S = R/\operatorname{ann}_R M$, and Proposition 7.15(2) implies that $H^n_{\mathfrak{a}}(M) \cong H^n_{\mathfrak{a}S}(M)$ for each n. It remains to note that $\dim S = \dim M$ and that $\operatorname{cd}_S(\mathfrak{a}S) \leqslant \dim S$ by Theorem 9.13 and Proposition 9.12. $\square$

Proposition 9.15 raises the question: what is the import of the nonvanishing of $H^{\dim R}_{\mathfrak{a}}(R)$? In the lectures ahead we will encounter a number of answers, which cover different contexts. Here is a prototype, due to Hartshorne and Lichtenbaum; its proof is given in Lecture 14.

Theorem 9.16. *Let $(R, \mathfrak{m})$ be a complete local domain, and $\mathfrak{a}$ an ideal. Then $\operatorname{cd}_R(\mathfrak{a}) = \dim R$ if and only if $\mathfrak{a}$ is $\mathfrak{m}$-primary.*

Proposition 9.12 may also be used to obtain lower bounds on arithmetic rank. Here is a beautiful example, due to Hartshorne, that illustrates this particular use of cohomological dimension:

Example 9.17. Let $\mathbb{K}$ be a field and consider the ring $\mathbb{K}[x, y, u, v]$. The ideal $\mathfrak{a} = (x, y) \cap (u, v)$ has height two; we claim that $\operatorname{ara} \mathfrak{a} \geqslant 3$.

Indeed, set $\mathfrak{b} = (x, y)$ and $\mathfrak{c} = (u, v)$ and consider the Mayer-Vietoris exact sequence to be established later in Theorem 15.1,

$$\longrightarrow H^3_{\mathfrak{b}}(R) \oplus H^3_{\mathfrak{c}}(R) \longrightarrow H^3_{\mathfrak{a}}(R) \longrightarrow H^4_{\mathfrak{b}+\mathfrak{c}}(R) \longrightarrow H^4_{\mathfrak{b}}(R) \oplus H^4_{\mathfrak{c}}(R) \longrightarrow .$$

Corollary 7.14 implies that the first and the last displayed terms are zero, and from Theorem 9.3 one obtains that

$$H^4_{\mathfrak{b}+\mathfrak{c}}(R) = H^4_{\mathfrak{m}}(R) = H^4_{\mathfrak{m}}(R_{\mathfrak{m}})$$

is nonzero, where $\mathfrak{m} = \mathfrak{b} + \mathfrak{c}$. Thus, $H^3_{\mathfrak{a}}(R)$ is nonzero, so that $\operatorname{ara} \mathfrak{a} \geqslant 3$ by Proposition 9.12. The following exercise implies that $\operatorname{ara} \mathfrak{a} = 3$.

Exercise 9.18. Let $R = \mathbb{K}[x, y, u, v]$. Find elements $f, g, h \in R$ with

$$\operatorname{rad}(f, g, h) = (x, y) \cap (u, v)\,.$$

The ideal $\mathfrak{a}$ in Example 9.17 is not a set-theoretic complete intersection: the algebraic set that it defines has codimension two, but it cannot be defined by two equations. Cowsik and Nori [**26**] proved that in characteristic p, every irreducible curve is a set-theoretic complete intersection. This is an open question in characteristic 0, even for curves in $\mathbb{P}^3$.

Cohen-Macaulay Rings

> Life is really worth living in a Noetherian ring R when all the local rings have the property that every s.o.p. is an R-sequence. Such a ring is called Cohen-Macaulay.
>
> –*from* [**69**, page 887].[1]

The aim of this lecture is to illustrate how Cohen-Macaulay rings arise naturally; each section is devoted to a different point of view. The material is adapted from Hochster's beautiful survey article [**69**] where he discusses the question "What does it really mean for a ring to be Cohen-Macaulay?"

Definition 10.1. A local ring is *Cohen-Macaulay* if some system of parameters is a regular sequence. A ring R is Cohen-Macaulay if $R_{\mathfrak{m}}$ is Cohen-Macaulay for every maximal ideal $\mathfrak{m}$ of R.

Remark 10.2. A local ring $(R, \mathfrak{m})$ is Cohen-Macaulay if and only if every system of parameters is a regular sequence; equivalently, $\operatorname{depth} R = \dim R$.

Indeed, if R is Cohen-Macaulay, then $\operatorname{depth} R = \dim R$. If this equality holds and $\boldsymbol{x}$ is a system of parameters, then $\operatorname{depth}_R(\boldsymbol{x}R, R) = \dim R$ by Theorem 9.1. Thus $\boldsymbol{x}$ is a regular sequence by Corollary 6.24.

If $\mathfrak{a}$ is of height n in a Cohen-Macaulay ring R, then there exist elements $\boldsymbol{y} = y_1, \dots, y_n$ in $\mathfrak{a}$ which form part of a system of parameters for R. But then $\boldsymbol{y}$ is a regular sequence, which implies $\operatorname{depth}_R(\mathfrak{a}, R) \geqslant n$. Lemma 8.16 gives the reverse inequality, and so we have

$$\operatorname{depth}_R(\mathfrak{a}, R) = \operatorname{height} \mathfrak{a}\,.$$

[1] Hochster has subsequently revised this to: Life is worth living. Period.

If R is an $\mathbb{N}$-graded ring, finitely generated over a field R_0, it turns out that R is Cohen-Macaulay if and only if some (equivalently, every) homogeneous system of parameters is a regular sequence.

Example 10.3. Rings of dimension zero are trivially Cohen-Macaulay. A reduced ring of dimension one is Cohen-Macaulay.

Example 10.4. A regular ring is Cohen-Macaulay by Proposition 8.20.

Example 10.5. Let A be a regular local ring, and $\boldsymbol{x} = x_1, \dots, x_r$ elements of A such that the ring $R = A/\boldsymbol{x}A$ has dimension $\dim A - r$. If a ring R (or, more generally, its completion) has this form, then R is a *complete intersection* ring; see Definition 11.18. The sequence $\boldsymbol{x}$ can be extended to a system of parameters $\boldsymbol{x}, \boldsymbol{y}$ for A, in which case the image of $\boldsymbol{y}$ in R is a system of parameters for R. Since A is Cohen-Macaulay, $\boldsymbol{x}, \boldsymbol{y}$ is a regular sequence on A, but then the sequence $\boldsymbol{y}$ is regular on $R = A/\boldsymbol{x}A$. Hence complete intersections are Cohen-Macaulay.

Example 10.6. The ring $R = \mathbb{K}[x, y]/(x^2, xy)$ has dimension one and y is a homogeneous parameter. Since y is a zerodivisor, R is not Cohen-Macaulay.

Example 10.7. Let R be the subring of the polynomial ring $\mathbb{K}[s, t]$ generated, as a $\mathbb{K}$-algebra, by the monomials s^4, s^3t, st^3, t^4. Then R has dimension two, and s^4, t^4 is a homogeneous system of parameters for R. Since R is a domain, s^4 is a nonzerodivisor. However t^4 is a zerodivisor on R/s^4R since

$$t^4(s^3t)^2 = s^4(st^3)^2$$

and $(s^3t)^2 \notin s^4R$. It follows that R is not Cohen-Macaulay.

Example 10.8. Let R be a subring of $\mathbb{K}[x_1, \dots, x_d]$ which is generated, as a $\mathbb{K}$-algebra, by monomials in the variables. Such *affine semigroup rings* are discussed in Lecture 20. Hochster [**67**, Theorem 1] proved that if R is normal, then it is Cohen-Macaulay; see Exercise 20.32 for a proof using polyhedral geometry. Hochster's theorem is also a special case of Theorem 10.30.

1. Noether normalization

We recall the Noether normalization theorem in its graded form:

Theorem 10.9. *Let $\mathbb{K}$ be a field and R an $\mathbb{N}$-graded ring, finitely generated over $R_0 = \mathbb{K}$. If $x_1, \dots, x_d$ is a homogeneous system of parameters for R, then the elements $x_1, \dots, x_d$ are algebraically independent over $\mathbb{K}$, and R is a finitely generated module over the subring $\mathbb{K}[x_1, \dots, x_d]$.* $\square$

In the situation above, a natural question arises: when is R a free module over the polynomial ring $\mathbb{K}[x_1, \dots, x_d]$? We start with a few examples.

Example 10.10. Let S_n be the symmetric group on n symbols acting on $R = \mathbb{K}[x_1, \dots, x_n]$ by permuting the variables. The ring of invariants is $R^{S_n} = \mathbb{K}[e_1, \dots, e_n]$, where e_i is the elementary symmetric function of degree i in $\boldsymbol{x}$. By [**5**, Chapter II.G], the ring R is a free R^{S_n}-module with basis

$$x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \qquad \text{where } 0 \leqslant m_i \leqslant i-1 \text{ for } 1 \leqslant i \leqslant n\,.$$

Example 10.11. Fix a positive integer d, and let R be the subring of $\mathbb{K}[x, y]$ generated, as a $\mathbb{K}$-algebra, by the monomials of degree d, i.e., by the elements $x^d, x^{d-1}y, \dots, xy^{d-1}, y^d$. As x^d, y^d is a homogeneous system of parameters for R, Theorem 10.9 implies that R is a finitely generated module over $A = \mathbb{K}[x^d, y^d]$. Indeed, the element 1 and the monomials $x^i y^j$ with $1 \leqslant i, j \leqslant d-1$ and $i + j = d$ generate R as an A-module. It is a straightforward exercise to prove that R is a free A-module on this set.

Example 10.12. Let $R = \mathbb{K}[s^4, s^3t, st^3, t^4]$, the ring in Example 10.7. The elements s^4, t^4 form a homogeneous system of parameters, so R is a finitely generated module over the polynomial subring $A = \mathbb{K}[s^4, t^4]$. The monomials $1, s^3t, st^3, s^6t^2, s^2t^6$ are a minimal generating set for R as an A-module. However R is not a free module since we have the relation

$$t^4(s^6t^2) = s^4(s^2t^6)\,,$$

which was used earlier to demonstrate that R is not Cohen-Macaulay.

Theorem 10.13. *Let $\mathbb{K}$ be a field, R an $\mathbb{N}$-graded ring finitely generated over $R_0 = \mathbb{K}$, and let $x_1, \dots, x_d$ be a homogeneous system of parameters. Then R is Cohen-Macaulay if and only if it is a free module over $\mathbb{K}[x_1, \dots, x_d]$.*

Proof. By Hilbert's syzygy theorem, R has finite projective dimension over $A = \mathbb{K}[x_1, \dots, x_d]$. The Auslander-Buchsbaum formula yields

$$\operatorname{depth} R + \operatorname{pd}_A R = \operatorname{depth} A\,,$$

where we note that the depth of the ring R equals its depth as an A-module. Since $\operatorname{depth} A = d = \dim R$, the ring R is Cohen-Macaulay if and only if $\operatorname{pd}_A R = 0$, i.e., if and only if R is a projective A-module. Since R is a finitely generated graded A-module, it is projective if and only if it is free. $\square$

Exercise 10.14. Let $\mathbb{K}$ be a field. Find homogeneous systems of parameters for the rings below and determine whether they are Cohen-Macaulay.

(1) $R = \mathbb{K}[x, y, z]/(xy, yz)$.

(2) $R = \mathbb{K}[x, y, z]/(xy, yz, zx)$.

(3) $R = \mathbb{K}[s, t, x, y]/(sx, sy, tx, ty)$.

Exercise 10.15. Let R be an $\mathbb{N}$-graded ring finitely generated over a field $R_0 = \mathbb{K}$. If R is Cohen-Macaulay with homogeneous system of parameters $x_1, \ldots, x_d$, prove that the Hilbert-Poincaré series of R has the form

$$P(R,t) = \frac{g(t)}{(1-t^{e_1}) \cdots (1-t^{e_d})},$$

where $\deg x_i = e_i$ and $g(t) \in \mathbb{Z}[t]$ has nonnegative coefficients.

2. Intersection multiplicities

Let $f, g \in \mathbb{K}[x,y]$ be polynomials without a common factor. Then $\operatorname{Var}(f)$ and $\operatorname{Var}(g)$ are plane curves with isolated points of intersection. Suppose that the origin $p = (0,0)$ is one of these intersection points, and we wish to compute the *intersection multiplicity* or *order of tangency* of the curves at the point p. This can be achieved by working in the local ring $R = \mathbb{K}[x,y]_{(x,y)}$ and taking the length of the module

$$R/(f,g) \cong R/(f) \otimes_R R/(g).$$

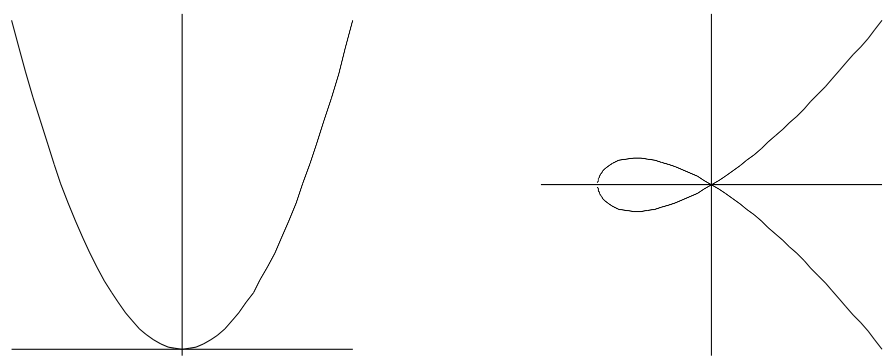


Figure 10.1. The curves $y = x^2$ and $y^2 = x^2 + x^3$

Example 10.16. The intersection multiplicities of the parabola $\operatorname{Var}(y - x^2)$ with the x-axis and the y-axis, at the origin, are, respectively,

$$\ell\left(\frac{\mathbb{K}[x,y]_{(x,y)}}{(y-x^2, y)}\right) = 2 \quad \text{and} \quad \ell\left(\frac{\mathbb{K}[x,y]_{(x,y)}}{(y-x^2, x)}\right) = 1.$$

Example 10.17. The intersection multiplicity of $\operatorname{Var}(y^2 - x^2 - x^3)$ with the x-axis, at the origin, is

$$\ell\left(\frac{\mathbb{K}[x,y]_{(x,y)}}{(y^2 - x^2 - x^3, y)}\right) = 2.$$

This illustrates the need to work with the *local* ring $\mathbb{K}[x,y]_{(x,y)}$ to measure the intersection multiplicity at $(0,0)$; the length of $\mathbb{K}[x,y]/(y^2-x^2-x^3,y)$ is 3, since it also counts the other intersection point of the curves, $(-1,0)$.

Remark 10.18. For plane curves $\mathrm{Var}(f)$ and $\mathrm{Var}(g)$, the length of

$$\mathbb{C}[x,y]_{(x,y)}/(f,g)$$

gives the correct intersection multiplicity of the curves at the origin, in the sense that for typical small complex numbers ϵ, this intersection multiplicity is precisely the number of distinct intersection points of $\mathrm{Var}(f)$ and $\mathrm{Var}(g+\epsilon)$ that lie in a small neighborhood of the origin.

To generalize from plane curves to arbitrary algebraic sets, let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of $\mathbb{K}[x_1,\ldots,x_n]$ defining algebraic sets $\mathrm{Var}(\mathfrak{a})$ and $\mathrm{Var}(\mathfrak{b})$ with an isolated point of intersection, $p=(0,\ldots,0)$. Working in the local ring $R=\mathbb{K}[x_1,\ldots,x_n]_{(x_1,\ldots,x_n)}$ and taking the length of the module

$$R/(\mathfrak{a}+\mathfrak{b}) \cong R/\mathfrak{a} \otimes_R R/\mathfrak{b}$$

may not give the same answer as perturbing the equations and counting distinct intersection points. Serre's definition from [**143**] does:

$$\chi(R/\mathfrak{a},R/\mathfrak{b}) = \sum_{i=0}^{\dim R} (-1)^i \ell(\mathrm{Tor}_i^R(R/\mathfrak{a},R/\mathfrak{b}))\,.$$

Example 10.19. Let $\mathfrak{a}=(x^3-w^2y, x^2z-wy^2, xy-wz, y^3-xz^2)$ and $\mathfrak{b}=(w,z)$ be ideals of the polynomial ring $\mathbb{C}[w,x,y,z]$. Then the ideal $\mathfrak{a}+\mathfrak{b}=(w,z,x^3,xy,y^3)$ has radical (w,x,y,z), so the algebraic sets $\mathrm{Var}(\mathfrak{a})$ and $\mathrm{Var}(\mathfrak{b})$ have a unique point of intersection, namely the origin in $\mathbb{C}^4$. Let $R=\mathbb{C}[w,x,y,z]_{(w,x,y,z)}$. Note that

$$\ell(R/(\mathfrak{a}+\mathfrak{b})) = \ell(R/(w,z,x^3,xy,y^3)) = 5\,.$$

However, we claim that the intersection multiplicity of $\mathrm{Var}(\mathfrak{a})$ and $\mathrm{Var}(\mathfrak{b})$ should be 4. To see this, we perturb the linear space $\mathrm{Var}(w,z)$ and count the number of points in the intersection

$$\mathrm{Var}(\mathfrak{a}) \cap \mathrm{Var}(w-\delta, z-\epsilon)$$

for typical small complex numbers δ and ϵ, i.e., we determine the number of elements of $\mathbb{C}^4$ which are solutions of the equations

$$w=\delta,\ z=\epsilon,\ x^3-w^2y=0,\ x^2z-wy^2=0,\ xy-wz=0,\ y^3-xz^2=0\,.$$

It is easily seen that x is a fourth root of $\delta^3\epsilon$, and that x uniquely determines y. Hence for nonzero δ and ϵ, there are four distinct intersection points.

To determine the Serre intersection multiplicity $\chi(R/\mathfrak{a},R/\mathfrak{b})$, first note that the Koszul complex

$$0 \longrightarrow R \xrightarrow{\left[\begin{smallmatrix}-z\\ w\end{smallmatrix}\right]} R^2 \xrightarrow{[w\ z]} R \longrightarrow 0$$

gives a projective resolution of $R/\mathfrak{b}$. To compute $\operatorname{Tor}_i^R(R/\mathfrak{a}, R/\mathfrak{b})$, we tensor this complex with

$$R/\mathfrak{a} \cong \mathbb{C}[s^4, s^3t, st^3, t^4]\,,$$

and take the homology of the resulting complex

$$0 \longrightarrow R/\mathfrak{a} \xrightarrow{\left[\begin{smallmatrix}-t^4\\ s^4\end{smallmatrix}\right]} (R/\mathfrak{a})^2 \xrightarrow{\left[\begin{smallmatrix}s^4 & t^4\end{smallmatrix}\right]} R/\mathfrak{a} \longrightarrow 0\,.$$

In this notation, the module $\operatorname{Tor}_1^R(R/\mathfrak{a}, R/\mathfrak{b})$ is the $\mathbb{C}$-vector space spanned by the element

$$\begin{pmatrix} -s^2t^6 \\ s^6t^2 \end{pmatrix} \in (R/\mathfrak{a})^2\,.$$

Now $\operatorname{Tor}_2^R(R/\mathfrak{a}, R/\mathfrak{b}) = 0$ and $\ell(\operatorname{Tor}_0^R(R/\mathfrak{a}, R/\mathfrak{b})) = \ell(R/(\mathfrak{a}+\mathfrak{b})) = 5$, so

$$\chi(R/\mathfrak{a}, R/\mathfrak{b}) = 5 - 1 + 0 = 4\,.$$

The issue is precisely that the ring $R/\mathfrak{a} \cong \mathbb{C}[s^4, s^3t, st^3, t^4]$ is not Cohen-Macaulay, as follows from the next theorem [**143**, page 111].

Theorem 10.20. *Let $(R, \mathfrak{m})$ be an unramified regular local ring, and $\mathfrak{a}, \mathfrak{b}$ ideals such that $\mathfrak{a}+\mathfrak{b}$ is $\mathfrak{m}$-primary. Then $\chi(R/\mathfrak{a}, R/\mathfrak{b})$ equals*

$$\ell(R/(\mathfrak{a}+\mathfrak{b})) = \ell(\operatorname{Tor}_0^R(R/\mathfrak{a}, R/\mathfrak{b}))$$

if and only if $R/\mathfrak{a}$ and $R/\mathfrak{b}$ are Cohen-Macaulay rings. □

The naïve estimate $\ell(R/(f,g))$ is the right intersection multiplicity for plane curves $\operatorname{Var}(f)$ and $\operatorname{Var}(g)$ since $R/(f)$ and $R/(g)$ are Cohen-Macaulay.

Exercise 10.21. Let $R = \mathbb{C}[w, x, y, z]$, and consider the ideals

$$\mathfrak{a} = (w^3 - x^2, wy - xz, y^2 - wz^2, w^2z - xy) \quad \text{and} \quad \mathfrak{b} = (w, z)\,.$$

(1) Check that the ring $R/\mathfrak{a} \cong \mathbb{C}[s^2, s^3, st, t]$ is not Cohen-Macaulay.
(2) Compute the length of $R/(\mathfrak{a}+\mathfrak{b})$.
(3) Compute the intersection multiplicity of the algebraic sets $\operatorname{Var}(\mathfrak{a})$ and $\operatorname{Var}(\mathfrak{b})$ at the origin in $\mathbb{C}^4$.

3. Invariant theory

Let G be a group acting on a polynomial ring T. We use T^G to denote the *ring of invariants*, i.e., the subring

$$T^G = \{x \in T \mid g(x) = x \text{ for all } g \in G\}\,.$$

Example 10.22. Let $T = \mathbb{K}[a, b, c, r, s]$ be a polynomial ring over an infinite field $\mathbb{K}$. Consider the action of the multiplicative group $G = \mathbb{K} \smallsetminus \{0\}$ on T under which $\lambda \in G$ sends a polynomial $f(a, b, c, r, s)$ to

$$f(\lambda a, \lambda b, \lambda c, \lambda^{-1} r, \lambda^{-1} s)\,.$$

Under this action, every monomial is taken to a scalar multiple. Let f be a polynomial fixed by the group action. If $a^i b^j c^k r^m s^n$ occurs in f with nonzero coefficient, comparing coefficients of this monomial in f and $\lambda(f)$ gives us

$$\lambda^{i+j+k-m-n} = 1 \qquad \text{for all } \lambda \in G\,.$$

Since G is infinite, we must have $i + j + k = m + n$. It follows that the ring of invariants is the monomial ring

$$T^G = \mathbb{K}[ar, br, cr, as, bs, cs]\,.$$

Let $R = \mathbb{K}[u, v, w, x, y, z]$, and consider the surjection $\varphi \colon R \longrightarrow T^G$ as in Exercise 1.36. We saw that $\ker(\varphi)$ is the prime ideal $\mathfrak{p} = (\Delta_1, \Delta_2, \Delta_3)$ where $\Delta_1 = vz - wy$, $\Delta_2 = wx - uz$, $\Delta_3 = uy - vx$.

We write $R(m)$ for the graded module with $[R(m)]_n = [R]_{m+n}$. A graded resolution of $R/\mathfrak{p} \cong T^G$ over R is

$$0 \longrightarrow R^2(-3) \xrightarrow{\left[\begin{smallmatrix} u & x \\ v & y \\ w & z \end{smallmatrix}\right]} R^3(-2) \xrightarrow{[\,\Delta_1\ \Delta_2\ \Delta_3\,]} R \longrightarrow 0\,.$$

Such a resolution can be used to compute the Hilbert-Poincaré series of $R/\mathfrak{p}$ as follows. For each integer n, we have an exact sequence of $\mathbb{K}$-vector spaces

$$0 \longrightarrow [R^2(-3)]_n \longrightarrow [R^3(-2)]_n \longrightarrow [R]_n \longrightarrow [R/\mathfrak{p}]_n \longrightarrow 0\,.$$

The alternating sum of the vector space dimensions must be zero, so

$$P(R/\mathfrak{p}, t) = P(R, t) - 3P(R(-2), t) + 2P(R(-3), t)\,.$$

Since $P(R(-m), t) = t^m P(R, t)$ and $P(R, t) = (1-t)^{-6}$, we see that

$$P(R/\mathfrak{p}, t) = \frac{1 - 3t^2 + 2t^3}{(1-t)^6} = \frac{1 + 2t}{(1-t)^4}\,,$$

which is precisely what we obtained earlier in Example 1.34.

Remark 10.23. Given an action of a group G on a polynomial ring T, the *first fundamental problem of invariant theory*, according to Weyl [**162**], is to find generators for the ring of invariants T^G, in other words to find a polynomial ring R with a surjection $\varphi \colon R \longrightarrow T^G$. The *second fundamental problem* is to find relations amongst these generators, i.e., to find a free R-module R^{b_1} with a surjection $R^{b_1} \longrightarrow \ker \varphi$. Continuing this sequence of fundamental problems, one would like to determine the resolution of T^G as an R-module, i.e., to find an exact complex

$$\cdots \longrightarrow R^{b_3} \longrightarrow R^{b_2} \longrightarrow R^{b_1} \longrightarrow R \xrightarrow{\varphi} T^G \longrightarrow 0\,.$$

In Example 10.22, we obtained this for the given group action, and saw how the resolution provides information such as the Hilbert-Poincaré series (and hence the dimension, multiplicity, etc.) of the ring of invariants T^G. Another fundamental question then arises: what is the length of the minimal resolution of T^G as an R-module, i.e., what is its projective dimension? The Cohen-Macaulay property appears once again:

Theorem 10.24. *Let T be a polynomial ring over a field $\mathbb{K}$, and G a group acting on T by degree preserving $\mathbb{K}$-algebra automorphisms. Assume that T^G is a finitely generated $\mathbb{K}$-algebra, and let R be a polynomial ring mapping onto T^G. Then*

$$\operatorname{pd}_R T^G \geqslant \dim R - \dim T^G \,,$$

and equality holds precisely if T^G is Cohen-Macaulay.

Proof. By the Auslander-Buchsbaum formula,

$$\operatorname{pd}_R T^G = \operatorname{depth} R - \operatorname{depth} T^G \,.$$

The ring R is Cohen-Macaulay and $\operatorname{depth} T^G \leqslant \dim T^G$, so we get the asserted inequality. Equality holds if and only if $\operatorname{depth} T^G = \dim T^G$, i.e., precisely when T^G is Cohen-Macaulay. □

Exercise 10.25. Let p be a prime and $T = \mathbb{F}_p[a, b, c, r, s]$ a polynomial ring. Consider the action of the multiplicative group $G = \mathbb{F}_p \smallsetminus \{0\}$ on T under which $\lambda \in G$ sends a polynomial $f(a, b, c, r, s)$ to

$$f(\lambda a, \lambda b, \lambda c, \lambda^{-1} r, \lambda^{-1} s) \,.$$

Determine the ring of invariants T^G.

Exercise 10.26. Let $T = \mathbb{C}[x_1, \ldots, x_d]$ be a polynomial ring, n a positive integer, and σ the $\mathbb{C}$-linear automorphism of T with

$$\sigma(x_j) = e^{2\pi i/n} x_j \qquad \text{for all } 1 \leqslant j \leqslant d \,.$$

Determine the ring T^G where G is the cyclic group generated by σ.

The following theorem is due to Hochster and Eagon [**72**].

Theorem 10.27. *Let T be a polynomial ring over a field $\mathbb{K}$, and G a finite group acting on T by degree preserving $\mathbb{K}$-algebra automorphisms. If $|G|$ is invertible in $\mathbb{K}$, then T^G is Cohen-Macaulay.*

Proof. Consider the map $\rho \colon T \longrightarrow T^G$ given by the *Reynolds operator*

$$\rho(t) = \frac{1}{|G|} \sum_{g \in G} g(t) \,.$$

It is easily verified that $\rho(t) = t$ for all $t \in T^G$ and that ρ is a T^G-module homomorphism. Hence T^G is a direct summand of T as a T^G-module, i.e., $T \cong T^G \oplus M$ for some T^G-module M.

Let $\boldsymbol{x}$ be a homogeneous system of parameters for T^G. Since G is finite, T is an integral extension of T^G, so $\boldsymbol{x}$ is a system of parameters for T as well. The ring T is Cohen-Macaulay, so $\boldsymbol{x}$ is a regular sequence on T. But then it is also a regular sequence on its direct summand T^G. $\square$

The proof of Theorem 10.27 shows, more generally, that a direct summand S of a Cohen-Macaulay ring T is Cohen-Macaulay, provided that a system of parameters for S forms part of a system of parameters for T. In general, a direct summand of a Cohen-Macaulay ring need not be Cohen-Macaulay, as we see in the next example.

Example 10.28. Let $\mathbb{K}$ be an infinite field, and let T be the hypersurface

$$T = \mathbb{K}[a, b, c, r, s]/(a^3 + b^3 + c^3)\,.$$

The multiplicative group $G = \mathbb{K} \smallsetminus \{0\}$ acts $\mathbb{K}$-linearly on T where $\lambda \in G$ sends a polynomial $f(a, b, c, r, s)$ to

$$f(\lambda a, \lambda b, \lambda c, \lambda^{-1} r, \lambda^{-1} s)\,.$$

As in Example 10.22, the ring of invariants is $T^G = \mathbb{K}[ar, br, cr, as, bs, cs]$. The ring T is a complete intersection, and hence is Cohen-Macaulay. Also, it is easy to see that T^G is a direct summand of T. However T^G is not Cohen-Macaulay: the elements ar, bs, $as + br$ form a homogeneous system of parameters for T^G (verify!) and

$$crcs(as + br) = (cs)^2 ar + (cr)^2 bs\,,$$

which shows that $as + br$ is a zerodivisor on $T^G/(ar, bs)T^G$. For a different proof that T^G is not Cohen-Macaulay, see Example 22.5.

Remark 10.29. A *linear algebraic group* is a Zariski closed subgroup of a general linear group $GL_n(\mathbb{K})$. A linear algebraic group G is *linearly reductive* if every finite-dimensional G-module is a direct sum of irreducible G-modules, equivalently, if every G-submodule has a G-stable complement. Linearly reductive groups in characteristic zero include finite groups, algebraic tori (i.e., products of copies of the multiplicative group of the field), and the classical groups $GL_n(\mathbb{K})$, $SL_n(\mathbb{K})$, $Sp_{2n}(\mathbb{K})$, $O_n(\mathbb{K})$, and $SO_n(\mathbb{K})$.

If a linearly reductive group G acts on a finitely generated $\mathbb{K}$-algebra T by degree preserving $\mathbb{K}$-algebra automorphisms, then there is a T^G-linear Reynolds operator $\rho\colon T \longrightarrow T^G$, making T^G a direct summand of T.

The following theorem is due to Hochster and J. Roberts [**75**].

Theorem 10.30. *Let G be a linearly reductive group acting linearly on a polynomial ring T. Then the ring of invariants T^G is Cohen-Macaulay.*

More generally, a direct summand of a polynomial ring over a field is Cohen-Macaulay. $\square$

We record a few examples of rings of invariants which, by the Hochster-Roberts theorem, are Cohen-Macaulay.

Example 10.31. Let $n \leqslant d$ be positive integers, $X = (x_{ij})$ an $n \times d$ matrix of variables over a field $\mathbb{K}$, and consider the polynomial ring in nd variables, $T = \mathbb{K}[X]$. Let $G = SL_n(\mathbb{K})$ act on T as follows:

$$M\colon x_{ij} \longmapsto (MX)_{ij}\,,$$

i.e., an element $M \in G$ sends x_{ij}, the (i,j) entry of the matrix X, to the (i,j) entry of the matrix MX. Since $\det M = 1$, it follows that the size n minors of X are fixed by the group action. It turns out that if $\mathbb{K}$ is infinite, T^G is the $\mathbb{K}$-algebra generated by these size n minors. The ring T^G is the homogeneous coordinate ring of the Grassmann variety of n-dimensional subspaces of a d-dimensional vector space. The relations between the minors are the well-known Plücker relations.

Example 10.32. Let $X = (x_{ij})$ and $Y = (y_{jk})$ be $r \times n$ and $n \times s$ matrices of variables over an infinite field $\mathbb{K}$, and $T = \mathbb{K}[X,Y]$ the polynomial ring of dimension $rn + ns$. Let $G = GL_n(\mathbb{K})$ be the general linear group acting on T where $M \in G$ maps the entries of X to corresponding entries of XM^{-1} and the entries of Y to those of MY. Then T^G is the $\mathbb{K}$-algebra generated by the entries of the product matrix XY. If $Z = (z_{ij})$ is an $r \times s$ matrix of new variables mapping onto the entries of XY, the kernel of the induced $\mathbb{K}$-algebra surjection $\mathbb{K}[Z] \longrightarrow T^G$ is the ideal generated by the size $n{+}1$ minors of the matrix Z. These *determinantal rings* are the subject of [**21**]. The case where $r = 2$, $s = 3$, $n = 1$ was earlier encountered in Example 10.22.

Exercise 10.33. Let $X = (x_{ij})$ be an $n \times n$ matrix of variables over a field $\mathbb{K}$, and take the polynomial ring

$$A = \mathbb{K}[x_{ij} \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n]\,.$$

Consider the hypersurface $R = A/(\det X)$. If Δ is any size $(n-1)$ minor of X, show that R_Δ, the localization of R at the element Δ, is a regular ring.

Exercise 10.34. Let G be a group acting by ring automorphisms on a domain R.

(1) Show that the action of G on R extends to an action of G on the fraction field $\mathbb{L}$ of R.

(2) If R is normal, show that the ring of invariants R^G is normal.

(3) If G is finite, prove that $\mathbb{L}^G$ is the fraction field of R^G.

4. Local cohomology

We have seen how Cohen-Macaulay rings come up in the study of intersection multiplicities and in studying rings of invariants. They also arise naturally when considering local cohomology, but first a definition:

Definition 10.35. Let R be a local ring. A finitely generated R-module M is *Cohen-Macaulay* if $\operatorname{depth} M = \dim M$. Observe that R is a Cohen-Macaulay ring if and only if it is a Cohen-Macaulay R-module.

Theorem 10.36. *Let $(R, \mathfrak{m})$ be a local ring. A finitely generated R-module M is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq \dim M$.*

In particular, the ring R is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(R) = 0$ for all $i \neq \dim R$.

Proof. Theorem 9.1 implies that $H^i_{\mathfrak{m}}(M) = 0$ for $i < \operatorname{depth} M$ and nonzero for $i = \operatorname{depth} M$. Moreover, $H^i_{\mathfrak{m}}(M) = 0$ for $i > \dim M$ by Theorem 9.3. $\square$

Gorenstein Rings

Gorenstein rings were introduced by Bass in his influential paper [**9**]. Among the class of commutative rings, Gorenstein rings are remarkable for their duality properties. The definition we adopt, however, is not too illuminating.

Definition 11.1. A Noetherian ring R is *Gorenstein* if $\operatorname{injdim}_R R$ is finite.

When $\operatorname{injdim}_R R$ is finite, so is $\operatorname{injdim}_{R_\mathfrak{p}} R_\mathfrak{p}$ for each prime $\mathfrak{p}$ in R. Hence when R is Gorenstein, so is $R_\mathfrak{p}$. As to the converse:

Exercise 11.2. Prove that a Noetherian ring R is Gorenstein if and only if $\dim R$ is finite and $R_\mathfrak{m}$ is Gorenstein for each maximal ideal $\mathfrak{m}$ in R.

In [**9**], and in some other places in the literature, you will find that a Noetherian ring is defined to be 'Gorenstein' if it is locally Gorenstein; the finiteness of the Krull dimension has been sacrificed. However, from the perspective of these lectures, where we focus on Gorenstein rings for their duality properties, the more stringent definition is the better one.

Here is one source of Gorenstein rings.

Proposition 11.3. *Regular local rings are Gorenstein.*

Proof. Let R be a regular local ring. Then R has finite global dimension by Theorem 8.22. Therefore $\operatorname{Ext}^i_R(-, R) = 0$ for $i \gg 0$, and it follows that R has finite injective dimension. $\square$

When R is Gorenstein, the modules in the minimal injective resolution of R can be described completely; see Remark 11.25. This is a consequence of general results concerning the structure of injective resolutions.

1. Bass numbers

Let R be a Noetherian ring and let M be a finitely generated R-module. Let $I^\bullet$ be the minimal injective resolution of M. For a prime ideal $\mathfrak{p}$ and integer i, the number

$$\mu^i_R(\mathfrak{p}, M) = \text{number of copies of } E_R(R/\mathfrak{p}) \text{ in } I^i$$

is the *i-th Bass number* of M with respect to $\mathfrak{p}$; see Definition A.23. When R is local with maximal ideal $\mathfrak{m}$, we sometimes write $\mu^i_R(M)$ for $\mu^i_R(\mathfrak{m}, M)$. By Theorem A.24, the Bass numbers can be calculated as

$$\mu^i_R(\mathfrak{p}, M) = \operatorname{rank}_{\mathbb{K}(\mathfrak{p})} \operatorname{Hom}_{R_\mathfrak{p}}(\mathbb{K}(\mathfrak{p}), I^i_\mathfrak{p}) = \operatorname{rank}_{\mathbb{K}(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(\mathbb{K}(\mathfrak{p}), M_\mathfrak{p}),$$

where $\mathbb{K}(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. These formulae also show that the Bass numbers are finite, and that they can be calculated locally, that is to say, if U is a multiplicatively closed subset of R with $U \cap \mathfrak{p} = \varnothing$, then

$$\mu^i_R(\mathfrak{p}, M) = \mu^i_{U^{-1}R}(U^{-1}\mathfrak{p}, U^{-1}M).$$

The following exercise is a first step towards understanding the structure of minimal injective resolutions.

Exercise 11.4. Let M be an R-module. Prove that

$$\operatorname{Ass}_R M = \{\mathfrak{p} \mid \mu^0_R(\mathfrak{p}, M) \neq 0\} = \operatorname{Ass} E_R(M).$$

Bass numbers propagate along chains of prime ideals:

Lemma 11.5. *Let R be a Noetherian ring and M a finitely generated module. If $\mu^i_R(\mathfrak{p}, M) \neq 0$ for a prime $\mathfrak{p}$, then for each prime $\mathfrak{q} \supset \mathfrak{p}$ with* $\operatorname{height}(\mathfrak{q}/\mathfrak{p}) = 1$ *we have $\mu^{i+1}_R(\mathfrak{q}, M) \neq 0$.*

Proof. Localizing at $\mathfrak{q}$, we may assume that $(R, \mathfrak{m}, \mathbb{K})$ is local and that $\dim R/\mathfrak{p} = 1$. Suppose $\mu^{i+1}_R(M) = 0$, that is to say,

$$\operatorname{Ext}^{i+1}_R(\mathbb{K}, M) = 0.$$

An induction on length shows that $\operatorname{Ext}^{i+1}_R(L, M) = 0$ for modules L of finite length. Thus $\operatorname{Ext}^{i+1}_R(R/(\mathfrak{p} + xR), M) = 0$ for any element $x \in \mathfrak{m} \smallsetminus \mathfrak{p}$, since the length of the R-module $R/(\mathfrak{p} + xR)$ is finite. Now the exact sequence

$$0 \longrightarrow R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p} + xR) \longrightarrow 0$$

yields an exact sequence of finitely generated modules

$$\cdots \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{p}, M) \xrightarrow{x} \operatorname{Ext}^i_R(R/\mathfrak{p}, M) \longrightarrow \operatorname{Ext}^{i+1}_R(R/(\mathfrak{p} + xR), M) = 0.$$

Nakayama's lemma yields $\operatorname{Ext}^i_R(R/\mathfrak{p}, M) = 0$, and hence $\mu^i_R(\mathfrak{p}, M) = 0$. $\square$

Corollary 11.6. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring, and let M be a finitely generated R-module. Then*

$$\operatorname{injdim}_R M = \sup\{i \mid \operatorname{Ext}^i_R(\mathbb{K}, M) \neq 0\}\,. \qquad \square$$

We state without proof a better result, proved by Fossum, Foxby, Griffith, and Reiten [**40**], and also by P. Roberts [**131**]:

Theorem 11.7. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and let M be a finitely generated R-module. Then $\operatorname{Ext}^i_R(\mathbb{K}, M) \neq 0$ for $\operatorname{depth}_R M \leqslant i \leqslant \operatorname{injdim}_R M$.* $\square$

Using Exercise 11.4 and Lemma 11.5, solve the following:

Exercise 11.8. Every finitely generated R-module M satisfies

$$\operatorname{injdim}_R M \geqslant \dim M\,.$$

Prove that if R admits a nonzero finitely generated injective module, then R is Artinian. See also Remark 11.11.

The following proposition shows that the injective dimension of M is either infinite, or depends only on R.

Proposition 11.9. *Let R be a local ring and M a finitely generated R-module. If $\operatorname{injdim}_R M$ is finite, then*

$$\operatorname{injdim}_R M = \operatorname{depth} R\,.$$

Proof. Set $d = \operatorname{depth} R$ and $e = \operatorname{injdim}_R M$. Choose a maximal R-sequence $\boldsymbol{x} = x_1, \ldots, x_d$. Computing via the Koszul resolution of $R/\boldsymbol{x}R$ one sees that

$$\operatorname{Ext}^d_R(R/\boldsymbol{x}R, M) \cong M/\boldsymbol{x}M\,,$$

which is nonzero by Nakayama's lemma. Thus $e \geqslant d$. As to the reverse inequality, since $\operatorname{depth} R/\boldsymbol{x}R = 0$, there is an exact sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow R/\boldsymbol{x}R \longrightarrow C \longrightarrow 0\,.$$

Since $\operatorname{Ext}^{e+1}_R(-, M) = 0$, the induced exact sequence has the form

$$\cdots \longrightarrow \operatorname{Ext}^e_R(R/\boldsymbol{x}R, M) \longrightarrow \operatorname{Ext}^e_R(\mathbb{K}, M) \longrightarrow 0\,.$$

Corollary 11.6 yields $\operatorname{Ext}^e_R(\mathbb{K}, M) \neq 0$, hence $\operatorname{Ext}^e_R(R/\boldsymbol{x}R, M) \neq 0$. Therefore, $e \leqslant \operatorname{pd}_R R/\boldsymbol{x}R = d$. $\square$

The proof of the following result is now clear:

Corollary 11.10. *Gorenstein rings are Cohen-Macaulay.* $\square$

Remark 11.11. If a Noetherian local ring R has a nonzero finitely generated module of finite injective dimension, then R is Cohen-Macaulay. This result, which was conjectured by Bass, is a theorem due to Peskine and Szpiro [**128**], Hochster [**68**], and P. Roberts [**134**].

2. Recognizing Gorenstein rings

Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring. The *socle* of an R-module M, denoted $\operatorname{soc} M$, is the submodule $(0 :_M \mathfrak{m})$, which may be identified with $\operatorname{Hom}_R(\mathbb{K}, M)$. It is thus a $\mathbb{K}$-vector space, and, indeed, the largest $\mathbb{K}$-vector space in M. Convince yourself that the terminology is particularly well-chosen, [**127**].

Theorem 11.12. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a zero-dimensional local ring. The following conditions are equivalent:*

(1) *R is Gorenstein;*

(2) *R is injective as an R-module;*

(3) $\operatorname{rank}_{\mathbb{K}} \operatorname{soc} R = 1$*;*

(4) $E_R(\mathbb{K}) \cong R$*;*

(5) (0) *is irreducible, i.e., it is not an intersection of two nonzero ideals.*

Proof. The equivalence of (1) and (2) is immediate from Proposition 11.9, and that of (2) and (3) was proved in Theorem A.29. The equivalence of these with (4) and (5) follows from basic properties of essential extensions proved in the Appendix. □

Condition (3) in the result above provides a particularly simple test for identifying zero-dimensional Gorenstein rings.

Example 11.13. The ring $\mathbb{K}[x, y]/(x^2, y^2)$ is Gorenstein since its socle is generated by a single element xy, while the ring $\mathbb{K}[x, y]/(x, y)^2$ is not, since its socle is minimally generated by the elements x and y.

This test applies only to zero-dimensional rings: $\mathbb{K}[[x, y]]/(x^2, xy)$ is not Gorenstein though its socle has rank-one.

How can one identify higher-dimensional Gorenstein rings? One way is by using the following result.

Proposition 11.14. *Let $\boldsymbol{x}$ be a regular sequence on a ring R. If R is Gorenstein, then so is $R/\boldsymbol{x}R$. The converse holds if R is local.*

The proof of this result is almost trivial, once we use Corollary 11.6, and the following theorem due to Rees.

Theorem 11.15. *Let M and N be R-modules. If an element $x \in \operatorname{ann}_R N$ is a nonzerodivisor on R and on M, then for each i one has*

$$\operatorname{Ext}^i_R(N, M) \cong \operatorname{Ext}^{i-1}_{R/xR}(N, M/xM)\,.$$

Proof. Let $I^\bullet$ be an injective resolution of M, and set $S = R/xR$. One has an isomorphism of complexes

$$\operatorname{Hom}_R(N, I^\bullet) \cong \operatorname{Hom}_S(N, \operatorname{Hom}_R(S, I^\bullet)) .$$

The complex on the left computes $\operatorname{Ext}_R^\bullet(N, M)$. The desired isomorphism follows once we verify that the complex of injective S-modules $\operatorname{Hom}_R(S, I^\bullet)$ has cohomology M/xM concentrated in degree 1.

The cohomology of $\operatorname{Hom}_R(S, I^\bullet)$ is $\operatorname{Ext}_R^\bullet(S, M)$, and a free resolution of S is $0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0$, so $\operatorname{Ext}_R^\bullet(S, M)$ is the cohomology of the complex

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow 0$$

concentrated in degrees 0 and 1, as desired. □

Here is one use of Proposition 11.14.

Example 11.16. Let $\mathbb{K}$ be a field and let $R = \mathbb{K}[x^3, x^5, x^7]$, viewed as a subring of $\mathbb{K}[x]$. Evidently R is a domain of dimension one, and hence Cohen-Macaulay. However, $\operatorname{soc} R/x^3R$ has rank two. Therefore R/x^3R is not Gorenstein, and so neither is R itself.

Lemma 11.17. *A local ring $(R, \mathfrak{m})$ is Gorenstein if and only if its $\mathfrak{m}$-adic completion $\widehat{R}$ is Gorenstein.*

Proof. This follows from Corollary 11.6 and the isomorphism

$$\operatorname{Ext}^i_{\widehat{R}}(\mathbb{K}, \widehat{R}) \cong \operatorname{Ext}^i_R(\mathbb{K}, R) \otimes_R \widehat{R} .$$ □

We now enlarge our supply of Gorenstein rings; first, a definition.

Definition 11.18. Let $(R, \mathfrak{m})$ be a local ring. By Theorem 8.28, the $\mathfrak{m}$-adic completion $\widehat{R}$ has a presentation $Q \twoheadrightarrow \widehat{R}$ where Q is a regular local ring. One says that R is a *complete intersection* ring if $\ker(Q \twoheadrightarrow \widehat{R})$ is generated by a regular sequence; this does not depend on the choice of the presentation. If $R \cong Q/\mathfrak{a}$ for a regular ring Q, then R is complete intersection if and only if $\mathfrak{a}$ is generated by a regular sequence; see [**115**, §21].

Proposition 11.19. *Complete intersection local rings are Gorenstein.*

Proof. If R is a complete intersection local ring, then so is $\widehat{R}$. Lemma 11.17 allows us to assume that R is complete. Then $R \cong Q/\boldsymbol{x}Q$ for a regular sequence $\boldsymbol{x}$ in a regular ring Q. Now Q is Gorenstein by Proposition 11.3, hence so is R by Proposition 11.14. □

The next example is a Gorenstein ring which is not complete intersection.

Example 11.20. Let $\mathbb{K}$ be a field, and let

$$R = \mathbb{K}[x, y, z]/(x^2 - y^2, y^2 - z^2, xy, yz, xz)\,.$$

Note that R is 0-dimensional; its socle is generated by the class of x^2, and hence it is Gorenstein. The ideal of relations defining R is minimally generated by the elements listed, which do not form a $\mathbb{K}[x, y, z]$-regular sequence.

Determinantal rings are another (potential) source of Gorenstein rings.

Example 11.21. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a field $\mathbb{K}$. Let $\mathbb{K}[X]$ denote the polynomial ring in the x_{ij}. Fix an integer $r \geqslant 1$, and let $I_r(X)$ be the ideal generated by the $r \times r$ minors of X. Then

$$R = \mathbb{K}[X]/I_r(X)$$

is the coordinate ring of the algebraic set of $m \times n$ matrices over $\mathbb{K}$ of rank less than r; see Example 1.5. The ring R is a Cohen-Macaulay normal domain of dimension $(m+n-r+1)(r-1)$, [**20**, Theorem 7.3.1]. However, for $r > 1$, the ring R is Gorenstein if and only if $m = n$, [**20**, Theorem 7.3.6].

Let us verify this last statement when $m = 2$, $n = 3$, and $r = 2$. Then

$$R = \mathbb{K}[u, v, w, x, y, z]/(vz - wy, wx - uz, uy - vx)\,.$$

Recall from Example 1.34 that the elements u, $v - x$, $w - y$, z form a system of parameters for R. Now

$$R/(u, v - x, w - y, z) \cong \mathbb{K}[x, y]/(x^2, xy, y^2)$$

is not Gorenstein (look at its socle), and hence neither is R.

We now extend Theorem 11.12 to arbitrary local rings.

Theorem 11.22. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a d-dimensional local ring. The following conditions are equivalent:*

(1) *R is Gorenstein;*
(2) $\operatorname{injdim}_R R = d$*;*
(3) *R is Cohen-Macaulay and* $\operatorname{rank}_{\mathbb{K}} \operatorname{Ext}^d_R(\mathbb{K}, R) = 1$*;*
(4) *R is Cohen-Macaulay and some (equivalently, every) system of parameters for R generates an irreducible ideal.*

Proof. (1) and (2) are equivalent by Proposition 11.9 and Corollary 11.10.

(1) $\iff$ (3). By Corollary 11.10, we may assume that R is Cohen-Macaulay. Let $x_1, \dots, x_d$ be a regular sequence. Theorem 11.15 yields

$$\operatorname{Ext}^d_R(\mathbb{K}, R) \cong \operatorname{Ext}^{d-1}_R(\mathbb{K}, R/x_1R) \cong \cdots \cong \operatorname{Hom}_R(\mathbb{K}, R/\boldsymbol{x}R)\,.$$

Now apply Proposition 11.14 and Theorem 11.12.

(1) $\iff$ (4). Reduce to dimension zero and apply Theorem 11.12. $\square$

Remark 11.23. The non-Gorenstein ring $\mathbb{K}[[x,y]]/(x^2,xy)$ shows that the Cohen-Macaulay hypothesis cannot be dropped from (4) above. On the other hand, one can drop the requirement that R is Cohen-Macaulay in (3): P. Roberts [**133**] proved that $\operatorname{Ext}_R^d(\mathbb{K},R)\cong\mathbb{K}$ implies that R is Gorenstein.

The *type* of a finitely generated R-module M is the number

$$\operatorname{type}_R M = \operatorname{rank}_{\mathbb{K}} \operatorname{Ext}_R^n(\mathbb{K},M)\,,$$

where $n=\operatorname{depth}_R M$. The equivalence of (1) and (3) is the statement that Gorenstein rings are precisely Cohen-Macaulay rings of type one.

3. Injective resolutions of Gorenstein rings

We now turn to the structure of injective resolutions of Gorenstein rings.

Theorem 11.24. *Let R be a Noetherian ring and $\mathfrak{p}$ a prime ideal in R. The following conditions are equivalent:*

(1) *$R_\mathfrak{p}$ is Gorenstein;*
(2) *$\mu_R^i(\mathfrak{p},R)=0$ for each integer $i>\operatorname{height}\mathfrak{p}$;*
(3) *$\mu_R^i(\mathfrak{p},R)=0$ for some integer $i>\operatorname{height}\mathfrak{p}$;*
(4) $\mu_R^i(\mathfrak{p},R)=\begin{cases}0 & \text{if } i<\operatorname{height}\mathfrak{p}\,,\\ 1 & \text{if } i=\operatorname{height}\mathfrak{p}\,.\end{cases}$

Proof. We may assume that R is local with $\mathfrak{p}=\mathfrak{m}$. In that case, the equivalence of (1) and (4) follows from Theorem 11.22. Proposition 11.9 yields (1) $\Longrightarrow$ (2), while it is clear that (2) $\Longrightarrow$ (3). Finally, (3) $\Longrightarrow$ (1) is an immediate consequence of Theorem 11.7. $\square$

Remark 11.25. Let R be a Gorenstein ring and $I^\bullet$ a minimal injective resolution of R. For each prime $\mathfrak{p}$ the ring $R_\mathfrak{p}$ is Gorenstein and $I^\bullet_\mathfrak{p}$ is a minimal resolution of $R_\mathfrak{p}$ by Proposition A.22. Thus, Theorem 11.24 implies that for each i one has an isomorphism

$$I^i \cong \bigoplus_{\operatorname{height}\mathfrak{p}=i} E_R(R/\mathfrak{p})\,.$$

Therefore, $E_R(R/\mathfrak{p})$ appears exactly once in the complex $I^\bullet$, namely in cohomological degree equal to $\operatorname{height}\mathfrak{p}$.

4. Local duality

The main result of this section is the Grothendieck duality theorem for Gorenstein rings, Theorem 11.29; ‘Grothendieck’ because it was proved by the man himself, and ‘duality’ for reasons explained in Lecture 18.

Theorem 11.26. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring of dimension d. Then R is Gorenstein if and only if*

$$H^i_{\mathfrak{m}}(R) = \begin{cases} 0 & \text{for } i \neq d\,, \\ E_R(\mathbb{K}) & \text{for } i = d\,. \end{cases}$$

Proof. When R is Gorenstein, its local cohomology is evident, given Example 7.6 and the injective resolution of R; see Remark 11.25.

For the converse, $H^i_{\mathfrak{m}}(R) = 0$ for $i \neq d$ implies that R is Cohen-Macaulay by Theorem 10.36. Let $\boldsymbol{x}$ be a maximal regular sequence. Then $R/\boldsymbol{x}R$ is zero-dimensional, so Exercise 11.27 yields the first of the isomorphisms:

$$R/\boldsymbol{x}R \cong \operatorname{Hom}_R(R/\boldsymbol{x}R, H^d_{\mathfrak{m}}(R)) \cong \operatorname{Hom}_R(R/\boldsymbol{x}R, E_R(\mathbb{K})) \cong E_{R/\boldsymbol{x}R}(\mathbb{K})\,.$$

The second isomorphism follows from the hypothesis, while the last is implied by Theorem A.25. Therefore $R/\boldsymbol{x}R$ is Gorenstein by Theorem 11.12, and hence Proposition 11.14 yields the desired conclusion. □

Exercise 11.27. Let $(R, \mathfrak{m})$ be a local ring, M a finitely generated module, and $\boldsymbol{x} = x_1, \ldots, x_n$ a regular sequence on M. Set $d = \operatorname{depth} M$. Prove that there is a natural isomorphism

$$H^{d-n}_{\mathfrak{m}}(M/\boldsymbol{x}M) \cong \operatorname{Hom}_R(R/\boldsymbol{x}R, H^d_{\mathfrak{m}}(M))\,.$$

Hint: induce on n.

Using the preceding theorem and Theorem 7.11, one obtains a description of $E_R(\mathbb{K})$ for any Gorenstein local ring.

Remark 11.28. Let $x_1, \ldots, x_d$ be a system of parameters for a Gorenstein ring R and, for $t \geqslant 1$, set $\boldsymbol{x}^t = x_1^t, \ldots, x_d^t$. Then

$$H^d(\operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^t; R), R)) = R/\boldsymbol{x}^t R\,,$$

and in the direct system in Theorem 7.11, the induced homomorphism

$$R/\boldsymbol{x}^t R = H^d(\operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^t), R)) \longrightarrow H^d(\operatorname{Hom}_R(K^\bullet(\boldsymbol{x}^{t+1}), R)) = R/\boldsymbol{x}^{t+1} R$$

is given by multiplication by the element $x = x_1 \cdots x_d$. Theorem 11.26 yields

$$E_R(\mathbb{K}) = \varinjlim \left(R/\boldsymbol{x}R \xrightarrow{x} R/\boldsymbol{x}^2 R \xrightarrow{x} R/\boldsymbol{x}^3 R \xrightarrow{x} \cdots \right).$$

Now we come to one of the principal results on Gorenstein rings: local duality. There is a version of the local duality theorem for Cohen-Macaulay rings with a canonical module—see Theorem 11.44—and, better still, for any local ring with a dualizing complex. For the moment, however, we focus on Gorenstein rings. The connection to Serre duality on projective space is explained in Lecture 18. It is an important point that the isomorphisms below are 'natural'; once again, this is explained in Lecture 18.

Theorem 11.29. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a Gorenstein local ring of dimension d, and M a finitely generated module. For $0 \leqslant i \leqslant d$, there are isomorphisms*

$$H^i_{\mathfrak{m}}(M) \cong \operatorname{Ext}^{d-i}_R(M, R)^\vee$$

where $(-)^\vee = \operatorname{Hom}_R(-, E_R(\mathbb{K}))$.

Proof. Set $E = E_R(\mathbb{K})$, let $x_1, \dots, x_d$ be a system of parameters for R, and let $\check{C}^\bullet(\boldsymbol{x}; R)$ be the Čech complex on $\boldsymbol{x}$. By Theorem 7.13, the R-module $H^i_{\mathfrak{m}}(M)$ is the degree i cohomology of the complex $\check{C}^\bullet(\boldsymbol{x}; R) \otimes_R M$. Since R is Gorenstein, Theorem 11.26 yields

$$H^i(\check{C}^\bullet(\boldsymbol{x}; R)) = \begin{cases} 0 & \text{if } i < d\,, \\ E & \text{if } i = d\,. \end{cases}$$

Thus $\check{C}^\bullet(\boldsymbol{x}; R)$ is a flat resolution of E shifted d steps to the right, so

$$H^i_{\mathfrak{m}}(M) = H^i(\check{C}^\bullet(\boldsymbol{x}; R) \otimes_R M) = \operatorname{Tor}^R_{d-i}(E, M)\,.$$

We claim that the module on the right is precisely $\operatorname{Ext}^{d-i}_R(M, R)^\vee$. To see this, let $I^\bullet$ be a minimal injective resolution of R; note that $I^n = 0$ for $n > d$ since R is Gorenstein. A standard argument now shows that the canonical morphism of complexes

$$\operatorname{Hom}_R(I^\bullet, E) \otimes_R M \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(M, I^\bullet), E) \tag{11.29.1}$$

is bijective. Moreover, since E is injective, one has that

$$H_n(\operatorname{Hom}_R(I^\bullet, E)) = \begin{cases} 0 & \text{if } n \neq 0\,, \\ E & \text{if } n = 0\,. \end{cases}$$

Thus, since $\operatorname{Hom}_R(I^\bullet, E)$ is a bounded complex of flat R-modules, it is a flat resolution of E. Hence taking homology in (11.29.1) yields, for each integer n, an isomorphism of R-modules

$$\operatorname{Tor}^R_n(E, M) \cong \operatorname{Hom}_R(\operatorname{Ext}^n_R(M, R), E)\,.$$

This completes the proof of the theorem. □

The proof is admittedly terse; fill in the details. The preceding theorem gives a description of local cohomology over any complete local ring, for such a ring is a homomorphic image of a regular local ring by Theorem 8.28, and so the following result applies.

Corollary 11.30. *If $(R, \mathfrak{m}, \mathbb{K})$ is a homomorphic image of a Gorenstein local ring Q of dimension c, and M is a finitely generated R-module, then*

$$H^i_{\mathfrak{m}}(M) \cong \operatorname{Ext}^{c-i}_Q(M, Q)^\vee \qquad \textit{for each } i \geqslant 0$$

where $(-)^\vee = \operatorname{Hom}_Q(-, E_Q(\mathbb{K}))$. □

This is somewhat unsatisfactory: one would like a duality involving only cohomology groups over R. This is given in Theorem 11.44, for which one needs canonical modules.

5. Canonical modules

We focus here on the local case. The graded and non-local cases are covered in Lectures 13 and 18 respectively.

Definition 11.31. Let R be a Cohen-Macaulay local ring. A *canonical module* is a finitely generated R-module ω with the properties

(1) $\operatorname{depth} \omega = \dim R$,
(2) $\operatorname{type}_R \omega = 1$, and
(3) $\operatorname{injdim}_R \omega$ is finite.

Note that (1) implies that ω is Cohen-Macaulay with $\dim \omega = \dim R$, that is to say, ω is a *maximal Cohen-Macaulay* module. Evidently, a finitely generated R-module ω is a canonical module for R if and only if

$$\mu_R^i(\omega) = \begin{cases} 0 & \text{for } i \neq \dim R\,, \\ 1 & \text{for } i = \dim R\,. \end{cases}$$

We have already seen examples of canonical modules:

Remark 11.32. Let R be a local ring. Then R is Gorenstein if and only if R is itself a canonical module. When R is Artinian, $E_R(\mathbb{K})$ is the unique canonical module for R. It turns out that a Cohen-Macaulay local ring has, up to isomorphism, at most one canonical module; see Theorem 11.46.

Proposition 11.33. *Let R be a Cohen-Macaulay local ring and ω a canonical module. Let M be a finitely generated module and set $t = \dim R - \dim M$.*

If M is Cohen-Macaulay, then $\operatorname{Ext}_R^i(M,\omega) = 0$ for $i \neq t$, and the module $\operatorname{Ext}_R^t(M,\omega)$ is Cohen-Macaulay of dimension $\dim M$.

Proof. Suppose $t = \dim R$. Then $\dim M = 0$ so $\operatorname{ann} M$ is primary to the maximal ideal. Since ω is maximal Cohen-Macaulay, Theorem 8.4 implies $\operatorname{Ext}_R^i(M,\omega) = 0$ for $i < t$. On the other hand, $\operatorname{injdim}_R \omega = t$ by Proposition 11.9, so we obtain the desired vanishing for $i > t$. It remains to note that $\operatorname{ann} M$ annihilates $\operatorname{Ext}_R^t(M,\omega)$, which therefore has dimension zero.

Suppose $t < \dim R$, so that $\dim M \geqslant 1$. Let x be an M-regular element. We may assume by iteration that the desired result holds for M/xM since it is Cohen-Macaulay of dimension $\dim M - 1$. The exact sequence induced by $0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$ reads

$$\cdots \longrightarrow \operatorname{Ext}_R^i(M,\omega) \xrightarrow{x} \operatorname{Ext}_R^i(M,\omega) \longrightarrow \operatorname{Ext}_R^{i+1}(M/xM,\omega) \longrightarrow \cdots .$$

The induction hypothesis and Nakayama's lemma give the vanishing of $\operatorname{Ext}^i_R(M,\omega)$ for $i \neq t$ and hence an exact sequence

$$0 \longrightarrow \operatorname{Ext}^t_R(M,\omega) \xrightarrow{x} \operatorname{Ext}^t_R(M,\omega) \longrightarrow \operatorname{Ext}^{t+1}_R(M/xM,\omega) \longrightarrow 0\,.$$

Thus, x is regular on $\operatorname{Ext}^t_R(M,\omega)$ and its dimension is as claimed. □

Theorem 11.34. *Let R and S be Cohen-Macaulay local rings and $R \longrightarrow S$ a local homomorphism such that S is a finitely generated R-module. Let $t = \dim R - \dim S$. If ω is a canonical module for R, then $\operatorname{Ext}^t_R(S,\omega)$ is a canonical module for S.*

Proof. Proposition 11.33 yields that the S-module $\operatorname{Ext}^t_R(S,\omega)$ is maximal Cohen-Macaulay. Let $I^\bullet$ be a minimal injective resolution of ω. Then $\operatorname{Hom}_R(S,I^\bullet)$ is a bounded complex of injective S-modules, and its cohomology is concentrated in degree t, again by Proposition 11.33. Thus, after shifting t steps to the left, $\operatorname{Hom}_R(S,I^\bullet)$ is a finite injective resolution of the S-module $\operatorname{Ext}^t_R(S,\omega)$.

It remains to check that $\operatorname{Ext}^t_R(S,\omega)$ has type 1. Let $\mathbb{K}$ and $\mathbb{L}$ denote the residue fields of R and S respectively. Consider the natural isomorphisms of $\mathbb{L}$-vector spaces

$$\begin{aligned}\operatorname{Hom}_S(\mathbb{L},\operatorname{Hom}_R(S,I^\bullet)) &\cong \operatorname{Hom}_R(\mathbb{L},I^\bullet)\\ &\cong \operatorname{Hom}_{\mathbb{K}}(\mathbb{L},\mathbb{K}) \otimes_{\mathbb{K}} \operatorname{Hom}_{\mathbb{K}}(\mathbb{K},I^\bullet)\,.\end{aligned}$$

Passing to cohomology yields isomorphisms

$$\operatorname{Ext}^i_S(\mathbb{L},\operatorname{Ext}^t_R(S,\omega)) \cong \operatorname{Hom}_{\mathbb{K}}(\mathbb{L},\mathbb{K}) \otimes_{\mathbb{K}} \operatorname{Ext}^{i+t}_{\mathbb{K}}(\mathbb{K},\omega)\,.$$

Setting $i = \dim S$ implies the desired result. □

Corollary 11.35. *Let R be a Cohen-Macaulay local ring and $\boldsymbol{x}$ a regular sequence in R. If ω is a canonical module for R, then $\omega/\boldsymbol{x}\omega$ is a canonical module for the ring $R/\boldsymbol{x}R$.*

Proof. The theorem implies that $\operatorname{Ext}^d_R(R/\boldsymbol{x}R,\omega)$ is a canonical module for $R/\boldsymbol{x}R$, where $\boldsymbol{x}$ has length d. Since $\boldsymbol{x}$ is a regular sequence, $K^\bullet(\boldsymbol{x};R)$ gives a resolution of $R/\boldsymbol{x}R$. Calculate Ext using this resolution. □

Exercise 11.36. Let R be a Cohen-Macaulay ring and let M be a finitely generated maximal Cohen-Macaulay module. Prove that any R-regular sequence is also regular on M. Is the converse true?

Exercise 11.37. Let R be a local ring and M,N finitely generated modules. Let x be an N-regular element of R. Prove the statements below.

(1) If $\operatorname{Ext}^1_R(M,N)=0$, then the following natural map is an isomorphism:

$$R/xR \otimes_R \operatorname{Hom}_R(M,N) \longrightarrow \operatorname{Hom}_{R/xR}(M/xM,N/xN)\,.$$

(2) Let $\theta\colon M \longrightarrow N$ be a homomorphism. If $R/xR \otimes_R \theta$ is an isomorphism, then so is θ.

Lemma 11.38. *Let R be a Cohen-Macaulay ring, and ω, ω' canonical modules for R. Then the evaluation map*

$$\theta\colon \omega' \otimes_R \operatorname{Hom}_R(\omega', \omega) \longrightarrow \omega$$

is an isomorphism.

Proof. We induce on $\dim R$. When this is zero, ω and ω' are the injective hull of the residue field by Remark 11.32, and the isomorphism is clear.

Suppose $\dim R > 0$, let x be a nonzerodivisor on R and set $S = R/xR$. Note that x is a nonzerodivisor on ω. Consider the commutative diagram

$$\begin{tikzcd}
(S \otimes_R \omega') \otimes_S (S \otimes_R \operatorname{Hom}_R(\omega', \omega)) \arrow[r, "S\otimes\theta"] \arrow[d, "\cong"'] & (S \otimes_R \omega) \\
(S \otimes_R \omega') \otimes_S \operatorname{Hom}_S(\omega'/x\omega', \omega/x\omega) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

where the vertical isomorphism is induced by the one in Exercise 11.37. The diagonal map is an isomorphism by the induction hypothesis since $\omega/x\omega$ and $\omega'/x\omega'$ are canonical modules for S; see Corollary 11.35. Thus $S \otimes \theta$ is an isomorphism, and hence so is θ by Exercise 11.37. $\square$

Theorem 11.39. *Let R be a Cohen-Macaulay local ring. If ω is a canonical module for R, then the homothety map*

$$\lambda\colon R \longrightarrow \operatorname{Hom}_R(\omega, \omega) \qquad \text{with } r \longmapsto (w \longmapsto rw)$$

is an isomorphism and $\operatorname{Ext}^i_R(\omega, \omega) = 0$ for $i \geqslant 1$.

Proof. The vanishing of Ext modules is given by Proposition 11.33. For the proof that λ is an isomorphism, we induce on $\dim R$. When this is zero, ω is the injective hull of the residue field and the result follows by Theorem A.31. If $\dim R > 0$, let x be a nonzerodivisor on R and consider the maps

$$R/xR \xrightarrow{R/xR\otimes\lambda} R/xR \otimes_R \operatorname{Hom}_R(\omega, \omega) \xrightarrow{\cong} \operatorname{Hom}_{R/xR}(\omega/x\omega, \omega/x\omega)$$

where the isomorphism is by Exercise 11.37. By the induction hypothesis, the composed map is an isomorphism, hence so is $R/xR \otimes \lambda$. Exercise 11.37 implies the desired result. $\square$

An argument similar to the one in the preceding proof can be used to prove that $\operatorname{Hom}_R(\omega, \omega') \cong R$ for canonical modules ω and ω', and hence that they are isomorphic by Lemma 11.38. We obtain the same result from the local duality theorem; see Theorem 11.46.

Corollary 11.43 below gives the definitive result on the existence of canonical modules. It uses the following construction.

Remark 11.40. Let R be a ring and M an R-module. The *trivial extension* of R by M, written $R \ltimes M$, is the R-algebra formed by endowing the direct sum $R \oplus M$ with the following multiplication:

$$(r, m)(s, n) = (rs, rn + sm)\,.$$

The submodule $0 \oplus M$ is an ideal whose square is zero, and taking the quotient by this ideal yields R again. If R is Noetherian and M is finitely generated, then $R \ltimes M$ is Noetherian and $\dim(R \ltimes M) = \dim R$. If R is local with maximal ideal $\mathfrak{m}$, then $R \ltimes M$ is local with maximal ideal $\mathfrak{m} \oplus M$, and $\operatorname{depth}(R \ltimes M) = \min\{\operatorname{depth} R, \operatorname{depth} M\}$.

Exercise 11.41. Let R be a local ring and let M be a finitely generated R-module. Prove that

$$\operatorname{soc}(R \ltimes M) = \{(r, m) \mid r \in \operatorname{soc}(R) \cap \operatorname{ann}_R M, \text{ and } m \in \operatorname{soc}(M)\}\,.$$

Theorem 11.42. *Let R be a Cohen-Macaulay local ring with canonical module ω. Set $S = R \ltimes \omega$ and let $S \longrightarrow R$ be the canonical surjection.*

Then S is a Gorenstein local ring, and

$$\operatorname{Ext}^i_S(R, S) = \begin{cases} \omega & \textit{for } i = 0\,, \\ 0 & \textit{for } i \geqslant 1\,. \end{cases}$$

Proof. Let $R \longrightarrow S$ be the canonical injection. We claim there is a natural isomorphism of S-modules, $S \cong \operatorname{Hom}_R(S, \omega)$. Indeed, one has natural maps

$$\begin{aligned} S = R \oplus \omega &\xrightarrow{\cong} \operatorname{Hom}_R(\omega, \omega) \oplus \operatorname{Hom}_R(R, \omega) \\ &\xrightarrow{\cong} \operatorname{Hom}_R(\omega \oplus R, \omega) = \operatorname{Hom}_R(S, \omega)\,. \end{aligned}$$

One can check that the composed map, which sends an element $(r, w) \in S$ to the homomorphism $(s, v) \longmapsto sw + rv$, is S-linear.

Given the isomorphism above, Theorem 11.34 yields that S is a canonical module for S, and hence S is Gorenstein by Remark 11.32. Moreover, one has natural isomorphisms

$$\operatorname{Hom}_S(R, S) \cong \operatorname{Hom}_S(R, \operatorname{Hom}_R(S, \omega)) \cong \operatorname{Hom}_R(R, \omega) \cong \omega\,.$$

The vanishing for $i \geqslant 1$ follows from Proposition 11.33. □

Corollary 11.43. *A Cohen-Macaulay local ring has a canonical module if and only if it is a homomorphic image of a Gorenstein local ring.* □

We now prove the local duality theorem for Cohen-Macaulay rings, which extends Theorem 11.29.

Theorem 11.44. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a d-dimensional Cohen-Macaulay local ring with a canonical module ω. Let M be a finitely generated R-module. Then for $0 \leqslant i \leqslant d$, there are isomorphisms*

$$H^i_{\mathfrak{m}}(M) \cong \operatorname{Ext}^{d-i}_R(M, \omega)^\vee$$

where $(-)^\vee = \operatorname{Hom}_R(-, E_R(\mathbb{K}))$. In particular, $H^d_{\mathfrak{m}}(R) \cong \omega^\vee$.

Proof. Set $S = R \ltimes \omega$; this is a Gorenstein ring of dimension d. Thus Theorem 11.29 gives the first isomorphism below

$$\begin{aligned} H^i_{\mathfrak{m}}(M) &\cong \operatorname{Hom}_S(\operatorname{Ext}^{d-i}_S(M, S), E_S(\mathbb{K})) \\ &\cong \operatorname{Hom}_R(\operatorname{Ext}^{d-i}_S(M, S), E_R(\mathbb{K}))\,, \end{aligned}$$

and the second isomorphism holds because $\operatorname{Ext}^{d-i}_S(M, S)$ is an R-module. It remains to note that one has an isomorphism

$$\operatorname{Ext}^{d-i}_S(M, S) \cong \operatorname{Ext}^{d-i}_R(M, \omega)\,.$$

To see this, let $I^\bullet$ be an injective resolution of S. Then $\operatorname{Hom}_S(R, I^\bullet)$ is a complex of injective R-modules which is a resolution of ω by Theorem 11.42. The isomorphism above is thus induced by the isomorphism of complexes

$$\operatorname{Hom}_S(M, I^\bullet) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_S(R, I^\bullet))\,. \qquad \square$$

Exercise 11.45. Calculate $H^n_{\mathfrak{m}}(\omega)$, for each $n \in \mathbb{Z}$.

Theorem 11.46. *Let R be a Cohen-Macaulay local ring. If ω and ω' are canonical modules for R, then they are isomorphic.*

Proof. Let $d = \dim R$. Theorem 11.44 gives the first isomorphism below:

$$\operatorname{Hom}(\omega', \omega)^\vee \cong H^d_{\mathfrak{m}}(\omega') \cong E_R(\mathbb{K})\,,$$

whereas the second follows from the fact that $\mu^d_R(\omega') = 1$ and $\mu^i_R(\omega') = 0$ for $i \neq d$. By Theorem A.31 the $\mathfrak{m}$-adic completion of $\operatorname{Hom}(\omega', \omega)$ is isomorphic to $\widehat{R}$. Therefore $\operatorname{Hom}(\omega', \omega)$ is isomorphic to R since it is a finitely generated R-module. (Exercise!) Given this, Lemma 11.38 implies that $\omega' \cong \omega$. $\square$

Henceforth, we use ω_R to denote *the* canonical module of R. We conclude with properties of the canonical module; the proofs are left as an exercise.

Theorem 11.47. *Let R be a Cohen-Macaulay local ring with canonical module ω_R. Then $\omega_{\widehat{R}} \cong \widehat{\omega_R}$ as $\widehat{R}$-modules, and for each prime ideal $\mathfrak{p}$, there is an isomorphism $\omega_{R_\mathfrak{p}} \cong (\omega_R)_\mathfrak{p}$.* $\square$

Lecture 12

Connections with Sheaf Cohomology

Our goal in this lecture is to reconcile the two types of Čech complexes encountered earlier: the sheaf-theoretic version in Lecture 2 and the ring-theoretic version in Lecture 6. This calls for an interlude on scheme theory and cohomology with supports.

1. Sheaf theory

Sheaves were introduced in Lecture 2 through sheaf spaces. Here is an equivalent definition, better suited to algebraic geometry. For details the reader may consult [**55, 61**].

Definition 12.1. Let X be a topological space. A *sheaf* $\mathcal{F}$ of Abelian groups consists of an Abelian group $\mathcal{F}(U)$ for each open set $U \subseteq X$, and a homomorphism of groups

$$\rho_{U,V} \colon \mathcal{F}(U) \longrightarrow \mathcal{F}(V)\,,$$

for each open subset $V \subseteq U$, called the *restriction map*. We sometimes write $s|_V$ for $\rho_{U,V}(s)$, where $s \in \mathcal{F}(U)$. These are subject to the following axioms:

(1) $\rho_{U,U} = \mathrm{id}_U$ for each open set U, and $\mathcal{F}(\varnothing) = 0$.

(2) Given open sets $U \supseteq V \supseteq W$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[rr, "\rho_{U,W}"] \arrow[dr, "\rho_{U,V}"'] & & \mathcal{F}(W) \\ & \mathcal{F}(V) \arrow[ur, "\rho_{V,W}"'] & \end{tikzcd}$$

(3) If s and t in $\mathcal{F}(U)$ coincide upon restriction to each V_α in an open cover $U = \bigcup_\alpha V_\alpha$, then $s = t$.

(4) For each open cover $U = \bigcup_\alpha V_\alpha$ and elements $\{s_\alpha \in \mathcal{F}(V_\alpha)\}$ satisfying

$$s_\alpha|_{V_\alpha \cap V_\beta} = s_\beta|_{V_\alpha \cap V_\beta} \qquad \text{for all } \alpha, \beta\,,$$

there exists an $s \in \mathcal{F}(U)$ such that $s|_{V_\alpha} = s_\alpha$ for each α.

Axioms (1) and (2) say that $\mathcal{F}$ is a contravariant functor from the topology on X to Abelian groups. Axioms (3) and (4) mean that sheaves are determined by their local data. Sometimes one writes $\Gamma(U, \mathcal{F})$ for $\mathcal{F}(U)$. The elements of $\Gamma(U, \mathcal{F})$ are the *sections* of $\mathcal{F}$ over U; the ones in $\Gamma(X, \mathcal{F})$ are the *global sections*.

Analogously, one has a notion of a sheaf of rings and of a sheaf of modules over a sheaf of rings; see [**61**, Chapter 2] for details. A sheaf of modules over a sheaf of rings $\mathcal{O}$ is usually referred to as an *$\mathcal{O}$-module*.

Example 12.2. Let X be a topological space, and A an Abelian group with the discrete topology. The *constant sheaf* $\mathcal{A}$ is the sheaf with $\mathcal{A}(U)$ the group of continuous functions from U to A and obvious restriction maps.

When X is *irreducible*, i.e., the intersection of each pair of nonempty open subsets is nonempty, $\mathcal{A}(U) = A$ for each nonempty open set U.

Since an Abelian group is the same as a $\mathbb{Z}$-module, a sheaf of Abelian groups is a sheaf of modules over the constant sheaf $\mathcal{Z}$ associated to $\mathbb{Z}$.

Example 12.3. Let R be a commutative ring and set $X = \operatorname{Spec} R$, with the Zariski topology, so that X has a base of *distinguished open sets*

$$U_f = \operatorname{Spec} R \smallsetminus V(f) \qquad \text{for } f \in R\,.$$

The *structure sheaf* on X, denoted $\mathcal{O}_X$, is the sheaf of rings with

$$\mathcal{O}_X(U_f) = R_f\,.$$

In particular, $\mathcal{O}_X(X) = R$. For elements $f, g \in R$, if $U_f \supseteq U_g$ then there exists an integer $k \geqslant 1$ such that $f \mid g^k$, so localization gives a homomorphism of rings $R_f \longrightarrow R_g$; this is the restriction map. Using the following exercise, check that this defines a sheaf on X.

Exercise 12.4. Let $\mathfrak{U}$ be a base for the topology of the space X. Suppose $\mathcal{F}$ is defined on $\mathfrak{U}$ consistent with the axioms in Definition 12.1. Prove that it extends uniquely to a sheaf on X.

Example 12.5. For $\mathbb{K}$ a field, $X = \operatorname{Spec} \mathbb{K}$ is a point and $\mathcal{O}_X(X) = \mathbb{K}$.

Example 12.6. The space $\operatorname{Spec} \mathbb{Z}$ consists of a closed point (p) for each prime number p, and the generic point (0), whose closure is $\operatorname{Spec} \mathbb{Z}$. The distinguished open sets are the cofinite sets of primes

$$U_n = \{(p) \in \operatorname{Spec} \mathbb{Z} \mid p \text{ does not divide } n\}$$

for nonnegative integers n. Note that $\mathcal{O}_{\operatorname{Spec}\mathbb{Z}}(U_n) = \mathbb{Z}[1/n]$.

One may have an abundance of local sections but few global sections:

Example 12.7. Let $X = \mathbb{C}$ with the usual Hausdorff topology, and $\mathcal{F}$ the sheaf of bounded holomorphic functions. A global section of $\mathcal{F}$ is a bounded entire function on $\mathbb{C}$ and hence it is a constant, by Liouville's theorem. However, there are many non-constant sections on bounded open sets.

Example 12.8. Let $\mathbb{K}$ be a field and $R = \mathbb{K}[s^4, s^3t, st^3, t^4]$. Set $X = \operatorname{Spec} R$ and let U be the open set $X \smallsetminus \{(s^4, s^3t, st^3, t^4)\}$. Then $\Gamma(X, \mathcal{O}_X) = R$ so s^2t^2 is not a global section. However, it belongs to $\Gamma(U, \mathcal{O}_X)$ since it is a section on each of the open sets U_{s^4} and U_{t^4} which cover U.

Note that X is a surface, so each point has codimension two. In general, the poles of a rational function define a hypersurface, so one would not expect new sections on the complement of a codimension two closed subset.

Definition 12.9. Given a morphism of sheaves $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$, for each open set U of X, let

$$\ker \varphi(U) = \ker \left(\mathcal{F}(U) \xrightarrow{\varphi(U)} \mathcal{G}(U)\right).$$

It is not hard to check that $\ker \varphi$ is a sheaf on X called the *kernel sheaf*.

On the other hand, the assignment

$$U \longmapsto \operatorname{image}\left(\mathcal{F}(U) \longrightarrow \mathcal{G}(U)\right)$$

satisfies axioms (1) and (2) of Definition 12.1, but not necessarily the others; thus it is not a sheaf, but a *presheaf*. In the same vein, the map

$$U \longmapsto \operatorname{coker}(\mathcal{F}(U) \longrightarrow \mathcal{G}(U))$$

defines a presheaf which need not be a sheaf; see Example 12.11 below.

Verify that the class of presheaves is closed under kernels and cokernels.

Example 12.10. Let $X = \{a, b\}$ with the discrete topology. Set $\mathcal{F}(X) = \{0\}$ and $\mathcal{F}(\{a\}) = \mathbb{Z} = \mathcal{F}(\{b\})$. Check that $\mathcal{F}$ is a presheaf but not a sheaf.

Example 12.11. Let $X = \mathbb{C} \smallsetminus \{0\}$. For each open set U, let $\mathcal{F}(U)$ be the additive group of holomorphic functions, and $\mathcal{G}(U)$ the multiplicative group of nowhere-vanishing holomorphic functions, on U. These define sheaves $\mathcal{F}$ and $\mathcal{G}$ on X. Define $\exp\colon \mathcal{F} \longrightarrow \mathcal{G}$ by $\exp(f) = e^{2\pi i f}$.

The image of exp is not a sheaf: for any choice of a branch of the logarithm function, $\exp(\log z) = z$ is in the image of exp on the open subset of X defined by the branch. These open sets cover X, but $f(z) = z$ has no global preimage on all of X, so it is not in the image of exp. In other words, $e^{2\pi i z}$ is locally invertible, but has no analytic inverse on all of X.

Example 12.12. Let $R = \mathbb{C}[x_1, \dots, x_n]$, where $n \geqslant 2$, and let X be its *punctured spectrum*, $\operatorname{Spec} R \smallsetminus \{(x_1, \dots, x_n)\}$. Let $\mathcal{O}_X$ be the structure sheaf of R restricted to X. It follows from Corollary 12.42 that $\Gamma(X, \mathcal{O}_X) = R$.

Define a morphism $\varphi\colon \mathcal{O}_X^n \longrightarrow \mathcal{O}_X$ by $\varphi(s_1, \dots, s_n) = \sum_i s_i x_i$; verify that this is a morphism of $\mathcal{O}_X$-modules. The image $\mathcal{F}$ of φ is not a sheaf. To see this, consider the cover of X by $U_i = X \smallsetminus V(x_i)$. On each U_i one has $\varphi(0, \dots, 1/x_i, \dots, 0) = 1$, hence $1 \in \mathcal{F}(U_i)$ for each i, but $1 \notin \mathcal{F}(X)$.

One overcomes the failure of the category of sheaves to be closed under images and cokernels by a process of sheafification, described further below.

Definition 12.13. Let $\mathcal{F}$ be a presheaf. The *stalk* of $\mathcal{F}$ at a point x is

$$\mathcal{F}_x = \varinjlim \mathcal{F}(U)\,,$$

where the limit is taken over the filtered direct system of open sets U containing x; see Example 4.31.

Exercise 12.14. Check that for $\mathfrak{p} \in \operatorname{Spec} R$ the stalk of the structure sheaf at $\mathfrak{p}$ is the local ring $R_\mathfrak{p}$.

Exercise 12.15. Prove that a morphism of presheaves $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ induces a morphism on stalks $\varphi_x\colon \mathcal{F}_x \longrightarrow \mathcal{G}_x$ for each $x \in X$, and that φ is injective if and only if φ_x is injective for each x.

Example 12.16. Let $\mathcal{O}_X$ be a sheaf of rings on X. For each $x \in X$, the stalk $\mathcal{O}_{X,x}$ is then a ring. Given an $\mathcal{O}_{X,x}$-module M, its *skyscraper sheaf* is the sheaf of $\mathcal{O}_X$-modules with $\mathcal{F}(U) = M$ if $U \ni x$, and $\mathcal{F}(U) = 0$ otherwise.

Check that for any $\mathcal{O}_X$-module $\mathcal{G}$, the passage to stalks gives an isomorphism $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, M)$.

Proposition 12.17. *Let $\mathcal{F}$ be a presheaf on X. Then there exists a sheaf $\widetilde{\mathcal{F}}$ and a morphism $\iota\colon \mathcal{F} \longrightarrow \widetilde{\mathcal{F}}$ such that for any morphism $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ with $\mathcal{G}$ a sheaf, there is a unique morphism $\widetilde{\varphi}\colon \widetilde{\mathcal{F}} \longrightarrow \mathcal{G}$ satisfying $\varphi = \widetilde{\varphi}\iota$.*

Proof. For each open set $U \subseteq X$, define $\widetilde{\mathcal{F}}(U)$ to be the set of functions

$$\sigma\colon U \longrightarrow \bigcup_{x \in U} \mathcal{F}_x$$

such that for each $x \in U$, one has $\sigma(x) \in \mathcal{F}_x$ and a neighborhood $V \subseteq U$ of x and an element s in $\mathcal{F}(V)$ whose image in $\mathcal{F}_y$ equals $\sigma(y)$ for each $y \in V$.

We leave it to the reader to verify that $\widetilde{\mathcal{F}}$ has the required properties. □

Definition 12.18. It follows from Proposition 12.17 that the sheaf $\widetilde{\mathcal{F}}$ is unique up to a unique isomorphism; it is called the *sheafification* of $\mathcal{F}$.

Exercise 12.19. Prove that $\iota_x\colon \mathcal{F}_x \longrightarrow \widetilde{\mathcal{F}}_x$ is an isomorphism for each x.

Definition 12.20. Let $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. The *image* sheaf and the *cokernel* sheaf of φ are the sheafifications of the respective presheaves. Using the universal property of sheafifications, one identifies $\operatorname{image}(\varphi)$ with a subsheaf of $\mathcal{G}$. A sequence $\mathcal{E} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ of morphisms of sheaves is *exact* at $\mathcal{F}$ if $\operatorname{image}(\psi) = \ker(\varphi)$.

Exercise 12.21. Prove that a morphism of sheaves $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ on X is surjective, i.e., $\operatorname{coker}\varphi$ is the zero sheaf, if and only if $\varphi_x\colon \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is surjective for each $x \in X$. In particular, exactness of a sequence can be checked at stalks.

Exercise 12.22. Let $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ be a surjective morphism of sheaves. While φ need not give a surjective map on sections, prove that for each open set U and each s in $\Gamma(U, \mathcal{G})$, there is an open cover $U = \bigcup_\alpha V_\alpha$ such that $s|_{V_\alpha}$ is in the image of $\varphi(V_\alpha)$.

Are the morphisms in Examples 12.11 and 12.12 surjective?

Remark 12.23. The morphism $\varphi\colon \mathcal{O}_X^n \longrightarrow \mathcal{O}_X$ in Example 12.12 is surjective, but it does not split. Hence $\mathcal{O}_X$ is not a projective $\mathcal{O}_X$-module.

Definition 12.24. Let R be a Noetherian ring and set $X = \operatorname{Spec} R$. Each R-module M gives rise to a sheaf $\widetilde{M}$ of $\mathcal{O}_X$-modules with

$$\widetilde{M}(U_f) = M_f \qquad \text{for } f \in R\,;$$

recall that $U_f = X \smallsetminus V(f)$. Using Exercise 12.4 it is not hard to verify that $\widetilde{M}$ is indeed a sheaf. The $\mathcal{O}_X$-module structure is evident.

A sheaf of $\mathcal{O}_X$-modules is *quasi-coherent* if it is of the form $\widetilde{M}$ for an R-module M; it is *coherent* if M is finitely generated.

Exercise 12.25. Let R be a Noetherian ring, $X = \operatorname{Spec} R$, and M an R-module. Prove the following assertions.

(1) The stalk of $\widetilde{M}$ at a point $\mathfrak{p} \in X$ is isomorphic to $M_\mathfrak{p}$.

(2) Quasi-coherent sheaves and morphisms between them are determined by global sections.

Remark 12.26. The gist of the preceding exercise is that the assignment $M \longmapsto \widetilde{M}$ is a fully faithful functor from R-modules to $\mathcal{O}_X$-modules. It is also exact, for one can test this property at stalks; see Exercise 12.21.

Here is one way to construct sheaves that are not quasi-coherent.

Exercise 12.27. With notation as in Exercise 12.25, let $U \subseteq X$ be open and define a presheaf by $\mathcal{F}(V) = \widetilde{M}(U)$ if $V \subseteq U$, and $\mathcal{F}(V) = 0$ otherwise.

Check that $\mathcal{F}$ is an $\mathcal{O}_X$-module, with $\mathcal{F}_x$ equal to $\widetilde{M}_x$ for $x \in U$ and zero otherwise. Show that $\mathcal{F}$ need not be quasi-coherent.

We now focus on exactness properties of the global sections functor. It is easy to check that this functor is left-exact.

Proposition 12.28. *Each exact sequence of sheaves of Abelian groups*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

induces an exact sequence of Abelian groups:

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H}) \,. \qquad \square$$

We are nearly ready to define sheaf cohomology.

Remark 12.29. Let $\mathcal{O}_X$ be a sheaf of rings. An $\mathcal{O}_X$-module $\mathcal{E}$ is *injective* if $\operatorname{Hom}_{\mathcal{O}_X}(-, \mathcal{E})$ is exact. The category of $\mathcal{O}_X$-modules has enough injectives.

Indeed, given an $\mathcal{O}_X$-module $\mathcal{F}$, the stalk $\mathcal{F}_x$ at each point x embeds in an injective $\mathcal{O}_{X,x}$-module; let $\mathcal{E}_x$ be the associated skyscraper sheaf. Set $\mathcal{E} = \prod_{x \in X} \mathcal{E}_x$. The natural morphism $\mathcal{F} \longrightarrow \mathcal{E}$ is an embedding and $\mathcal{E}$ is an injective $\mathcal{O}_X$-module since each $\mathcal{E}_x$ is injective and

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{E}) = \prod_{x \in X} \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, \mathcal{E}_x) \,.$$

Once we know that each $\mathcal{O}_X$-module embeds into an injective one, it is clear that injective resolutions exist. Fill in the details in this remark.

The category of $\mathcal{O}_X$-modules does not have enough projectives.

Note that there are enough injectives in the category of sheaves of Abelian groups since these are the same as $\mathcal{Z}$-modules; see Example 12.2.

Definition 12.30. Let X be a space, $\mathcal{F}$ a sheaf of Abelian groups, and let

$$0 \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{E}^1 \longrightarrow \mathcal{E}^2 \longrightarrow \cdots$$

be an injective resolution of $\mathcal{F}$. Taking global sections gives a complex

$$0 \longrightarrow \Gamma(X, \mathcal{E}^0) \longrightarrow \Gamma(X, \mathcal{E}^1) \longrightarrow \Gamma(X, \mathcal{E}^2) \longrightarrow \cdots$$

of Abelian groups. Its cohomology, $H^\bullet(X, \mathcal{F})$, is the *sheaf cohomology* of $\mathcal{F}$. By the left-exactness of $\Gamma(X, -)$, one has $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

Remark 12.31. Sheaf cohomology is independent of the choice of an injective resolution. This is justified by arguments akin to those in Remark 3.18. As in Remark 3.21, one has a long exact sequence for sheaf cohomology.

Let $\mathcal{O}_X$ be a sheaf of rings. An injective $\mathcal{O}_X$-module $\mathcal{E}$ is *acyclic*, i.e., $H^j(X, \mathcal{E}) = 0$ for each $j \geqslant 1$; see Example 12.34 and Proposition 12.36. Thus, in Definition 12.30, one can take an injective resolution of $\mathcal{F}$ as a sheaf of Abelian groups, or as an $\mathcal{O}_X$-module. The advantage of the former is that it shows that sheaf cohomology is intrinsic to X and $\mathcal{F}$. The advantage of the latter is that one obtains a $\Gamma(X, \mathcal{O}_X)$-module structure on cohomology.

2. Flasque sheaves

Injective sheaves are acyclic; however, these tend to be enormous. For instance, the injective quasi-coherent modules on $\operatorname{Spec} R$ arise from injective R-modules, and hence are rarely coherent. One would thus like to find other classes of acyclic sheaves.

Definition 12.32. A sheaf $\mathcal{F}$ on X is *flasque* or *flabby* if for every pair of open sets $V \subseteq U$, the restriction map $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is surjective. In particular, sections on open subsets extend to sections on X.

Example 12.33. Skyscraper sheaves are flasque, and, when the space is irreducible, so are the constant sheaves.

Example 12.34. Let $\mathcal{O}_X$ be a sheaf of rings. If $\mathcal{E}$ is an injective $\mathcal{O}_X$-module, then it is flasque.

To see this, let $V \subseteq U$ be open sets, and let $\mathcal{O}'_V$, $\mathcal{O}'_U$ be the structure sheaves extended by zero outside V and U respectively; see Exercise 12.27. Then $\mathcal{O}'_V \longrightarrow \mathcal{O}'_U$ is a monomorphism of $\mathcal{O}_X$-modules, and so it induces a surjective map $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}'_U, \mathcal{E}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}'_V, \mathcal{E})$. It remains to verify that $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}'_U, \mathcal{E})$ is naturally isomorphic to $\Gamma(U, \mathcal{E})$.

Remark 12.35. Here are some properties of flasque sheaves.

(1) If a sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ of sheaves is exact and $\mathcal{F}$ is flasque, then for each open set U the sequence below is exact:

$$0 \longrightarrow \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{G}) \longrightarrow \Gamma(U, \mathcal{H}) \longrightarrow 0\,.$$

(2) Quotients of flasque sheaves by flasque subsheaves are flasque.

(3) On Noetherian spaces, filtered direct limits of flasque sheaves are flasque. Direct sums of flasque sheaves are flasque.

(4) When E is an injective module over a Noetherian ring R, the sheaf $\widetilde{E}$ on $\operatorname{Spec} R$ is flasque.

We only sketch the proofs. For (1), as Γ is left-exact, it remains to check surjectivity. Note that $\mathcal{G} \longrightarrow \mathcal{H}$ is surjective on stalks, so any section $s \in \Gamma(U, \mathcal{H})$ is an image on small neighborhoods. Zorn's lemma provides a maximal such neighborhood, which by the flasqueness condition must be U.

For (2), use part (1) to extend a section of $\mathcal{H}$ to a global section of $\mathcal{G}$. The main point in (3) is that $\varinjlim(\mathcal{F}_i(U)) = (\varinjlim \mathcal{F}_i)(U)$ when X is Noetherian; see Exercise 12.39 below. For (4), see [**61**, Proposition III.3.4].

Proposition 12.36. *Let $\mathcal{O}_X$ be a sheaf of rings and $\mathcal{F}$ an $\mathcal{O}_X$-module. If $\mathcal{F}$ is flasque, then $H^j(U, \mathcal{F}) = 0$ for all $j \geqslant 1$ and open sets U. Consequently, resolutions by flasque sheaves can be used to compute sheaf cohomology.*

Proof. Since $H^i(U, \mathcal{F}) = H^i(U, \mathcal{F}|_U)$, it suffices to consider $U = X$. Embed $\mathcal{F}$ into an injective sheaf $\mathcal{E}$ with cokernel $\mathcal{Q} = \mathcal{E}/\mathcal{F}$. Note that $\mathcal{E}$ is flasque by Example 12.34, so $\mathcal{Q}$ is flasque by Remark 12.35. The cohomology exact sequence gives $H^1(X, \mathcal{F}) = 0$ and $H^j(X, \mathcal{F}) = H^{j-1}(X, \mathcal{Q})$ for all $j > 1$. Since $\mathcal{Q}$ is flasque, an iteration gives $H^j(X, \mathcal{F}) = 0$ for all $j \geqslant 1$.

The last assertion follows from the acyclicity principle. □

Example 12.37. Let $\mathbb{K}$ be an algebraically closed field and $X = \mathbb{P}^1_{\mathbb{K}}$; see Lecture 13. Let $\mathcal{O}_X$ be the structure sheaf of X, and $\mathcal{L}$ the constant sheaf associated to the function field $\mathbb{L}$ of X. Then $\mathcal{O}_X$ embeds naturally in $\mathcal{L}$. The quotient sheaf $\mathcal{L}/\mathcal{O}_X$ can be thought of as the direct sum of skyscraper sheaves $\mathcal{L}_x/\mathcal{O}_{X,x}$ at x in X, so it is flasque. Thus

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}/\mathcal{O}_X \longrightarrow 0$$

is a flasque resolution of $\mathcal{O}_X$. Passing to global sections yields a sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathbb{L} \longrightarrow \bigoplus_{x \in X} (\mathbb{L}/\mathcal{O}_{X,x}) \longrightarrow 0\,.$$

Verify that this sequence is exact. Thus, $H^j(X, \mathcal{O}_X) = 0$ for all $j \geqslant 1$, since sheaf cohomology can be computed via flasque resolutions.

The following result is one direction of a theorem of Serre [**142**].

Proposition 12.38. *Let R be a Noetherian ring. For each R-module M, one has $H^j(\operatorname{Spec} R, \widetilde{M}) = 0$ for $j \geqslant 1$.*

Proof. Let $E^\bullet$ be an injective resolution of M. The complex $\widetilde{E^\bullet}$ is exact by Remark 12.26. Since each $\widetilde{E^j}$ is flasque by Remark 12.35,

$$H^j(\operatorname{Spec} R, \widetilde{M}) = H^j(\Gamma(\widetilde{E^\bullet})) = H^j(E^\bullet)\,. \qquad \square$$

Exercise 12.39. Let $\{\mathcal{F}_i\}$ be a direct system of presheaves on a space X.

The *direct limit presheaf* is the presheaf with $(\varinjlim_i \mathcal{F}_i)(U) = \varinjlim_i (\mathcal{F}_i(U))$. It is not hard to check that it is indeed a presheaf. Using Definition 12.13, identify the stalk $(\varinjlim_i \mathcal{F}_i)_x$ with $\varinjlim_i (\mathcal{F}_i)_x$. Hint: use Theorem 4.28.

When the $\mathcal{F}_i$ are sheaves, the *direct limit sheaf* is the sheafification of the direct limit presheaf. Prove that this sheaf is the direct limit of $\{\mathcal{F}_i\}$ in the category of sheaves of Abelian groups.

When the index set is filtered and X is Noetherian, prove as follows that the direct limit presheaf is a sheaf.

(1) Let $s \in (\varinjlim_\alpha \mathcal{F}_\alpha)(U)$ and assume that $s|_{V_\alpha} = 0$ for all V_α constituting an open cover of U; replace $\{V_\alpha\}$ by a finite subcover, and use the filter property of the index set to show that $s = 0$.

(2) Suppose $\{V_\alpha\}$ is an open cover of U and $s_\alpha \in (\varinjlim_i \mathcal{F}_i)(V_\alpha)$ are sections which agree on the overlaps. Replace the cover by a finite subcover and use the filter property again to patch the sections together into a section in $\mathcal{F}_i(U)$ for some i, hence a gluing of the s_α in $(\varinjlim_i \mathcal{F}_i)(U)$.

Prove also that the cohomology of the direct limit sheaf is the direct limit of the constituent cohomology groups.

Remark 12.40. A space X with a sheaf of rings $\mathcal{F}$ is *affine* if there is a commutative ring R such that $(X, \mathcal{F})$ is isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$. A *scheme* is a sheaf of rings $(X, \mathcal{O}_X)$ with an open cover $\mathfrak{U}$ such that each $U \in \mathfrak{U}$ is an affine scheme, whose structure sheaf is $\mathcal{O}_X|_U$.

3. Local cohomology and sheaf cohomology

Let R be a Noetherian ring and $\mathfrak{a} = (x_1, \dots, x_d)$ an ideal. Set $X = \operatorname{Spec} R$ and $U = X \smallsetminus V(\mathfrak{a})$. Let $U_i = X \smallsetminus V(x_i)$, so $\mathfrak{U} = \{U_i\}$ is an open cover of U.

Let M be an R-module and $\widetilde{M}$ the associated sheaf on X. One then has $\Gamma(U_i, \widetilde{M}) \cong M_{x_i}$. There are two complexes associated to these data: the Čech complex $\check{C}^\bullet(\boldsymbol{x}; M)$, which in degree k is the R-module

$$\bigoplus_{1 \leqslant i_1 < \dots < i_k \leqslant d} M_{x_{i_1} x_{i_2} \cdots x_{i_k}},$$

and the topological Čech complex $\check{C}^\bullet(\mathfrak{U}; \widetilde{M})$, which in degree k is

$$\prod_{1 \leqslant i_1 < \dots < i_{k+1} \leqslant d} \Gamma(U_{i_1} \cap \dots \cap U_{i_{k+1}}, \widetilde{M}).$$

Recall that $\Gamma(U, \widetilde{M})$ does not appear in this complex; it is, however, naturally isomorphic to the degree zero cohomology, by Exercise 2.10.

Theorem 12.41. *With notation as above, one has an exact sequence*

$$0 \longrightarrow H^0_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow \Gamma(U, \widetilde{M}) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0$$

and, for each $j \geqslant 1$, an isomorphism $H^j(U, \widetilde{M}) \cong H^{j+1}_{\mathfrak{a}}(M)$.

Proof. For each $k \geqslant 0$ one has identifications

$$U_{i_1} \cap \dots \cap U_{i_{k+1}} = \bigcap_{j=1}^{k+1} \operatorname{Spec}(R_{x_{i_j}}) = \operatorname{Spec}(R_{x_{i_1} \cdots x_{i_{k+1}}}).$$

Thus, $U_{i_1} \cap \dots \cap U_{i_{k+1}}$ is affine and the restriction of $\widetilde{M}$ is a quasi-coherent sheaf. Proposition 12.38 implies $H^j(U_{i_1} \cap \dots \cap U_{i_{k+1}}, \widetilde{M}) = 0$ for $j \geqslant 1$, and hence Theorem 2.26 yields, for each j, an isomorphism

$$H^j(U, \widetilde{M}) \cong \check{H}^j(\mathfrak{U}; \widetilde{M}).$$

Computing $H^\bullet_{\mathfrak{a}}(M)$ from the Čech complex completes the proof. $\square$

The result below follows from Theorems 9.1 and 12.41.

Corollary 12.42. *If* $\operatorname{depth}_R(\mathfrak{a}, M) \geqslant 2$*, then each section of the sheaf* $\widetilde{M}$ *over* $U = \operatorname{Spec} R \smallsetminus V(\mathfrak{a})$ *extends uniquely to a global section.* $\square$

One interpretation of Theorem 12.41 is that the sheaf cohomology of $\widetilde{M}$ on $\operatorname{Spec} R$ away from $V(\mathfrak{a})$ controls the local cohomology of M with support in $\mathfrak{a}$. In other words, only the support of $\mathfrak{a}$ matters in computing $H^j_{\mathfrak{a}}(M)$. Following this idea leads naturally to cohomology with supports.

Definition 12.43. Let $Z \subseteq X$ be a closed set. For a sheaf $\mathcal{F}$ of Abelian groups on X, the *sections with support* in Z are the elements of the subgroup

$$\Gamma_Z(X, \mathcal{F}) = \ker(\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X \smallsetminus Z, \mathcal{F}))\,.$$

For $Z = V(\mathfrak{a})$ in $\operatorname{Spec} R$, one has $\Gamma_Z(\widetilde{M}) = \Gamma_{\mathfrak{a}}(M)$ for any R-module M.

Exercise 12.44. Check that, as with global sections, the functor Γ_Z is left-exact. The snake lemma should come in handy.

Definition 12.45. The *j-th local cohomology of $\mathcal{F}$ with support in Z* is

$$H^j_Z(X, \mathcal{F}) = R^j\Gamma_Z(X, \mathcal{F})\,.$$

In other words, to compute the cohomology of $\mathcal{F}$ with support in Z, take an injective resolution of $\mathcal{F}$, apply $\Gamma_Z = H^0_Z$, and take cohomology.

Remark 12.46. Let X, Z, U, and $\mathcal{F}$ be as in Definition 12.45. There is a natural exact sequence

$$0 \longrightarrow \Gamma_Z(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F})\,.$$

If $\mathcal{F}$ is flasque, each section of $\mathcal{F}$ over U extends to a global section, so the map on the right is surjective. It follows by taking flasque resolutions that for any $\mathcal{F}$, flasque or otherwise, there is an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0_Z(X, \mathcal{F}) & \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & H^0(U, \mathcal{F}|_U) & \longrightarrow & \\ \cdots & \longrightarrow & H^j_Z(X, \mathcal{F}) & \longrightarrow & H^j(X, \mathcal{F}) & \longrightarrow & H^j_Z(U, \mathcal{F}|_U) & \longrightarrow & \cdots\,. \end{array}$$

Proposition 12.38 and Theorem 12.41 thus yield the following result.

Theorem 12.47. *Let $\mathfrak{a}$ be an ideal in a Noetherian ring R. Set $X = \operatorname{Spec} R$ and $Z = V(\mathfrak{a})$. For each R-module M and integer j, one has*

$$H^j_{\mathfrak{a}}(M) \cong H^j_Z(X, \widetilde{M})\,. \qquad \square$$

Projective Varieties

We discuss quasi-coherent sheaves on projective varieties. The main topic is a connection between graded local cohomology and sheaf cohomology, akin to the one established in Lecture 12. We focus on the $\mathbb{Z}$-graded case which was first investigated in depth in [**48**]. A detailed treatment of $\mathbb{Z}$-graded local cohomology can also be found in [**19**, §§12, 13] and [**20**, §3.6].

1. Graded local cohomology

In this lecture, $\mathbb{K}$ is a field, S is an $\mathbb{N}$-graded ring finitely generated over $S_0 = \mathbb{K}$, and $\mathfrak{m} = S_{\geqslant 1}$ is the *irrelevant ideal* of S. The category of $\mathbb{Z}$-graded S-modules is an Abelian category admitting arbitrary direct sums, products, as well as direct limits.

Definition 13.1. For each $a \in \mathbb{Z}$, we write $M(a)$ for the graded S-module with $M(a)_b = M_{a+b}$ and the same underlying module structure as M.

Let M, N be graded S-modules. We write ${}^*\mathrm{Hom}_S(M,N)$ for the graded S-module of homogeneous maps from M to N. Its component in degree a is the set of degree a homomorphisms from M to N; equivalently, the degree 0 homomorphisms from $M(-a)$ to N; equivalently, the degree 0 homomorphisms from M to $N(a)$. We write ${}^*\mathrm{Ext}^i_S(M,N)$ for Ext computed in this category. The category has enough injectives by [**20**, Theorem 3.6.2].

Let $\mathfrak{a}$ be an ideal generated by homogeneous elements $\boldsymbol{f}$. The modules $H^i_{\mathfrak{a}}(M)$ have a natural $\mathbb{Z}$-grading, obtained in the following equivalent ways:

(1) The Čech complex $\check{C}^\bullet(\boldsymbol{f}; M)$ is a complex of graded S-modules, and the maps are degree preserving, so each $H^i_{\mathfrak{a}}(M)$ is graded.

(2) The derived functors of $\Gamma_{\mathfrak{a}}(-)$ in this category coincide with $H^\bullet_{\mathfrak{a}}(-)$.

Definition 13.2. A finitely generated graded S-module C is a **canonical module* for S if

$$ {}^*\mathrm{Ext}^i_S(\mathbb{K}, C) \cong \begin{cases} 0 & \text{for } i \neq \dim S\,, \\ \mathbb{K} & \text{for } i = \dim S\,. \end{cases} $$

The next result links canonical modules over S to those over $S_{\mathfrak{m}}$.

Lemma 13.3. *Let C be a finitely generated graded S-module such that $C_{\mathfrak{m}}$ is a canonical module for the Cohen-Macaulay local ring $S_{\mathfrak{m}}$. Then there exists a unique integer a such that $C(a)$ is a *canonical module for S.* □

The result above implies that when S is Gorenstein, $S(a)$ is a *canonical module of S for a unique $a \in \mathbb{Z}$.

Example 13.4. Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \dots, x_d]$ the ring of polynomials. By Proposition 11.3 and Lemma 13.3, there exists $a \in \mathbb{Z}$ such that $S(a)$ is a *canonical module for S. The Koszul complex on $\boldsymbol{x}$ is a free resolution of $\mathbb{K}$, and using it to compute ${}^*\mathrm{Ext}^{\bullet}_S(\mathbb{K}, S(a))$ yields $a = -\sum \deg x_i$.

The next result is the graded version of local duality, Theorem 11.44. We omit its proof, and refer the reader to [**20**, Theorem 3.6.19].

Theorem 13.5. *Let C be a *canonical module for S, and M a finitely generated graded S-module. For each integer i, there is a natural isomorphism*

$$ {}^*\mathrm{Hom}_{\mathbb{K}}\left(H^i_{\mathfrak{m}}(M), \mathbb{K}\right) \cong {}^*\mathrm{Ext}^{d-i}_S(M, C)\,. $$ □

2. Sheaves on projective varieties

Definition 13.6. Let $\mathrm{Proj}\, S$ be the set of graded prime ideals of S not containing the irrelevant ideal. The *projective variety* defined by a graded ideal $\mathfrak{a}$ of S is the set $\{\mathfrak{p} \in \mathrm{Proj}\, S \mid \mathfrak{a} \subseteq \mathfrak{p}\}$. These are the closed sets for the Zariski topology on $\mathrm{Proj}\, S$. That this defines a topology follows from arguments similar to those in Definition 1.12. As S is Noetherian, so is $\mathrm{Proj}\, S$: descending chains of closed sets stabilize.

For each homogeneous element $f \in \mathfrak{m}$, the ring S_f is $\mathbb{Z}$-graded. We set $D_+(f) = \mathrm{Proj}\, S \smallsetminus V(f)$. These are the distinguished open sets. Note that $D_+(f) \subseteq D_+(g)$ if and only if g divides a power of f if and only if there is a natural map $S_g \longrightarrow S_f$.

Exercise 13.7. Let $f \in \mathfrak{m}$ be a homogeneous element. Show that $D_+(f)$ can be naturally identified with $\mathrm{Spec}\,[S_f]_0$.

Prove that $\{D_+(f_i)\}_i$ cover $\mathrm{Proj}\, S$ if and only if $\mathfrak{m} = \mathrm{rad}(\boldsymbol{f})$.

Definition 13.8. The *structure sheaf* on $X = \operatorname{Proj} S$ is the sheaf $\mathcal{O}_X$ with the property that for each homogeneous element $f \in \mathfrak{m}$, one has

$$\Gamma\big(D_+(f), \mathcal{O}_X\big) = [S_f]_0 .$$

Using Exercise 12.4, one verifies that this assignment and the natural restriction maps determine the sheaf $\mathcal{O}_X$.

Exercise 13.9. Let $\{U_i\}_{i=1}^r$ be distinguished open sets in $\operatorname{Proj} S$. Prove that $\bigcap_{i=1}^r U_i$ is *affine*, i.e., isomorphic to the spectrum of a ring.

Recall that S is standard graded if, as a $\mathbb{K}$-algebra, it is generated by S_1. The standard grading on $\mathbb{K}[x_0, \ldots, x_n]$ is the one where $\deg x_i = 1$.

Definition 13.10. Let $S = \mathbb{K}[x_0, \ldots, x_n]$ be a standard graded polynomial ring. The space $\mathbb{P}^n_{\mathbb{K}} = \operatorname{Proj} S$, with the sheaf of rings from Definition 13.8, is *projective n-space* over $\mathbb{K}$.

Remark 13.11. Let S be a standard graded ring. Expressing S as a quotient of $\mathbb{K}[y_0, \ldots, y_n]$ for some n gives an embedding of $\operatorname{Proj} S$ in $\mathbb{P}^n_{\mathbb{K}}$.

If the irrelevant ideal of S is generated by $\boldsymbol{x}$, then $\operatorname{Proj} S$ is covered by the affine open sets

$$D_+(x_i) = \operatorname{Spec} [S_{x_i}]_0 .$$

Note that $D_+(x_i) \cap D_+(x_j)$ is $\operatorname{Spec} [S_{x_i x_j}]_0$.

The smooth affine varieties $\operatorname{Spec} \mathbb{K}[x_0/x_i, \ldots, x_n/x_i]$ give a covering of $\mathbb{P}^n_{\mathbb{K}}$, hence it is smooth. This property depends on the grading; indeed, if $S = \mathbb{K}[x, y, z]$ with $\deg(x, y, z) = (1, 1, 2)$, then $\operatorname{Proj} S$ is not smooth as

$$[S_z]_0 = \mathbb{K}[x^2/z,\ xy/z,\ y^2/z]$$

is not a regular ring.

Definition 13.12. Set $X = \operatorname{Proj} S$ and let M be a graded S-module. Using Exercise 12.4, one verifies that the assignment

$$D_+(f) \longmapsto [M_f]_0$$

defines a unique sheaf of $\mathcal{O}_X$-modules, denoted $\widetilde{M}$. A sheaf of $\mathcal{O}_X$-modules is *quasi-coherent* if it is of the form $\widetilde{M}$ for some graded S-module M.

In Lecture 12 we saw that the functor $M \longmapsto \widetilde{M}$ is an equivalence between the category of R-modules and the category of quasi-coherent sheaves on $\operatorname{Spec} R$. For graded modules one has:

Exercise 13.13. Show that $M \longmapsto \widetilde{M}$ is an exact functor from the category of graded S-modules to the category of quasi-coherent sheaves on $\operatorname{Proj} S$.

3. Global sections and cohomology

Set $X = \operatorname{Proj} S$. For each integer a, the *twisted* sheaf $\widetilde{S(a)}$ is denoted $\mathcal{O}_X(a)$. For each sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, set

$$\mathcal{F}(a) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(a)\,.$$

Note that $\widetilde{M}(a) = \widetilde{M(a)}$ for each graded S-module M.

Example 13.14. Let $S = \mathbb{K}[x, y]$ with the standard grading, and set

$$X = \operatorname{Proj} S\,, \qquad U = D_+(x)\,, \qquad \text{and} \qquad V = D_+(y)\,.$$

Since $\deg x = 1 = \deg y$, one has that

$$\Gamma(U, \mathcal{O}_X) = \mathbb{K}[y/x], \quad \Gamma(V, \mathcal{O}_X) = \mathbb{K}[x/y], \quad \Gamma(U \cap V, \mathcal{O}_X) = \mathbb{K}[y/x, x/y]\,.$$

It is not hard to check that $\Gamma(X, \mathcal{O}_X) = \mathbb{K}$.

For the twisted sheaf $\mathcal{O}_X(-1)$ one has

$$\Gamma(U, \mathcal{O}_X(-1)) = [S_x \otimes_S S(-1)]_0 = [S_x]_{-1}\,.$$

Therefore, $\Gamma(U, \mathcal{O}_X(-1))$ is the free $\Gamma(U, \mathcal{O}_X)$-module on $1/x$. Similarly, $\Gamma(V, \mathcal{O}_X(-1))$ is the free $\Gamma(V, \mathcal{O}_X)$-module on $1/y$, so

$$\Gamma(X, \mathcal{O}_X(-1)) \;=\; \Gamma(U, \mathcal{O}_X(-1)) \;\cap\; \Gamma(V, \mathcal{O}_X(-1)) \;=\; 0\,.$$

In the same vein, $\Gamma(U, \mathcal{O}_X(1))$ is the free $\Gamma(U, \mathcal{O}_X)$-module on x and $\Gamma(V, \mathcal{O}_X(1))$ is the free $\Gamma(V, \mathcal{O}_X)$-module on y. It follows that the global sections of $\mathcal{O}_X(1)$ are the linear forms in S:

$$\Gamma(X, \mathcal{O}_X(1)) = \mathbb{K}x \oplus \mathbb{K}y\,.$$

Exercise 13.15. For $X = \mathbb{P}^n_{\mathbb{K}}$, compute $\Gamma(X, \mathcal{O}_X(a))$ for each $a \in \mathbb{Z}$.

Next, we describe a functor from the category of quasi-coherent sheaves on $\operatorname{Proj} S$ to the category of graded S-modules.

Definition 13.16. For each sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on $X = \operatorname{Proj} S$, set

$$\Gamma_*(\mathcal{F}) = \bigoplus_{a \in \mathbb{Z}} \Gamma(X, \mathcal{F}(a))\,.$$

Verify that $\Gamma_*(\mathcal{O}_X)$ is a graded ring and $\Gamma_*(\mathcal{F})$ is a graded $\Gamma_*(\mathcal{O}_X)$-module. It follows from the next exercise that $\Gamma_*(\mathcal{F})$ is a graded S-module.

Let M be a graded S-module. Each element of M_0 gives an element of $\Gamma(X, \widetilde{M})$. Hence there is a map $M_a \longrightarrow \Gamma(X, \widetilde{M}(a))$, which gives a map

$$\zeta(M)\colon M \longrightarrow \Gamma_*(\widetilde{M})\,.$$

Exercise 13.17. Prove that $\zeta(S)$ is a homomorphism of graded rings and that $\zeta(M)$ is S-linear.

The following exercise implies that $\big(\widetilde{(-)}, \Gamma_*(-)\big)$ is an adjoint pair of functors with adjunction morphism $\zeta(-)$; see Proposition 4.18.

Exercise 13.18. Let M be a graded S-module. Verify that the S-modules $M_{\geqslant k}$ define the same sheaf on X. Hence, if $M_a = 0$ for $a \gg 0$, then $\widetilde{M} = 0$.

If $\mathcal{F}$ is a quasi-coherent sheaf on $\operatorname{Proj} S$, prove that the sheaf induced by the S-module $\Gamma_*(\mathcal{F})$ is $\mathcal{F}$.

We wish to calculate the cohomology of quasi-coherent sheaves. The idea is to take a cover by distinguished open sets, form the Čech complex, and take cohomology.

Example 13.19. Let $S = \mathbb{K}[x, y]$ and take the cover $\mathfrak{U} = \{D_+(x), D_+(y)\}$ of $\operatorname{Proj} S$. Let M be a graded S-module. One then has

$$\Gamma(D_+(x), \widetilde{M}) = [M_x]_0\,,\ \Gamma(D_+(y), \widetilde{M}) = [M_y]_0\,,\ \Gamma(D_+(xy), \widetilde{M}) = [M_{xy}]_0\,.$$

The Čech complex $\check{C}^\bullet(\mathfrak{U}; \widetilde{M})$ is the complex of $\mathbb{K}$-vector spaces

$$0 \longrightarrow [M_x]_0 \oplus [M_y]_0 \longrightarrow [M_{xy}]_0 \longrightarrow 0\,.$$

On the other hand, the complex $\check{C}^\bullet(x, y; M)$ is

$$0 \longrightarrow M \longrightarrow M_x \oplus M_y \longrightarrow M_{xy} \longrightarrow 0\,.$$

Evidently, $\check{C}^\bullet(\mathfrak{U}; \widetilde{M})[-1]$ is a subcomplex of $\check{C}^\bullet(x, y; M)$.

Remark 13.20. The calculation in the preceding example generalizes as follows. Let S be an $\mathbb{N}$-graded Noetherian ring, and $\boldsymbol{x}$ a set of generators for the ideal $S_{\geqslant 1}$. Set $U_i = \operatorname{Spec}\,[S_{x_i}]_0$ and $\mathfrak{U} = \{U_0, \dots, U_n\}$. For each graded S-module M, one has an exact sequence of complexes

$$0 \longrightarrow \bigoplus_{a \in \mathbb{Z}} \check{C}^\bullet(\mathfrak{U}; \widetilde{M}(a))[-1] \longrightarrow \check{C}^\bullet(\boldsymbol{x}; M) \longrightarrow M \longrightarrow 0\,.$$

Observe that $\mathfrak{U}$ is an open cover of $\operatorname{Proj} S$. Moreover, all intersections of sets in $\mathfrak{U}$ are affine by Exercise 13.9. Hence, $H^\bullet(\operatorname{Proj} S, \widetilde{M}(a))$ can be computed from the Čech complex $\check{C}^\bullet(\mathfrak{U}; \widetilde{M}(a))$ by Theorem 2.26 and Proposition 12.38. Passing to cohomology in the exact sequence of complexes above yields the following theorem; compare with Theorem 12.41.

Theorem 13.21. *For each graded S-module M, one has an exact sequence*

$$0 \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow M \xrightarrow{\zeta(M)} \Gamma_*(\widetilde{M}) \longrightarrow H^1_{\mathfrak{m}}(M) \longrightarrow 0\,.$$

Moreover, for each $i \geqslant 1$ one has natural isomorphisms of graded S-modules

$$\bigoplus_{a \in \mathbb{Z}} H^i\big(\operatorname{Proj} S, \widetilde{M}(a)\big) \cong H^{i+1}_{\mathfrak{m}}(M)\,. \qquad \square$$

The corollary below parallels Corollary 12.42.

Corollary 13.22. *The map $\zeta(M)\colon M \longrightarrow \Gamma_*(\widetilde{M})$ is injective if and only if* $\operatorname{depth}_S(\mathfrak{m}, M) \geqslant 1$*; it is bijective if and only if* $\operatorname{depth}_S(\mathfrak{m}, M) \geqslant 2$. □

Remark 13.23. Let $R \longrightarrow S$ be a finite morphism of graded rings. Each graded S-module is naturally a graded R-module, and hence defines sheaves on $X = \operatorname{Proj} S$ and $Y = \operatorname{Proj} R$. By Theorem 13.21 and Proposition 7.15(2), for each $i \geqslant 1$, we have

$$H^i(X, \widetilde{M}) = H^{i+1}_{\mathfrak{m}_S}(M)_0 = H^{i+1}_{\mathfrak{m}_R}(M)_0 = H^i(Y, \widetilde{M})\,.$$

Using standard arguments involving direct limits, parts of the next result can be extended to the case when M is an arbitrary graded S-module.

Corollary 13.24. *Let $\mathbb{K}$ be a field, S a standard graded $\mathbb{K}$-algebra, and set $X = \operatorname{Proj} S$. For each finitely generated graded S-module M, one has*

(1) $H^i(X, \widetilde{M}(a)) = 0$ *when $i \geqslant 1$ and $a \gg 0$;*

(2) $H^i(X, \widetilde{M}(a)) = 0$ *for $i \geqslant \dim M$ and $a \in \mathbb{Z}$;*

(3) *For each i, the $\mathbb{K}$-vector space $H^i(X, \widetilde{M})$ has finite rank.*

Proof. (1) For $i \geqslant 1$ and each $a \in \mathbb{Z}$, one has $H^i(X, \widetilde{M}(a)) = H^{i+1}_{\mathfrak{m}}(M)_a$ by Theorem 13.21. Since the ideal $\mathfrak{m} = S_{\geqslant 1}$ is maximal and M is Noetherian, the module $H = H^{i+1}_{\mathfrak{m}}(M)$ is Artinian by Exercise 7.7. Thus, the sequence $H_{\geqslant 0} \supseteq H_{\geqslant 1} \supseteq \cdots$ stabilizes. Part (2) is immediate from Grothendieck's result, Theorem 9.3, and (3) is left as an exercise. □

Using Theorem 13.21 and Example 7.16 one obtains the following:

Theorem 13.25. *Let $\mathbb{K}$ be a field and set $X = \mathbb{P}^n_{\mathbb{K}}$. For each a, one has*

(1) $H^0(X, \mathcal{O}_X(a)) \cong \mathbb{K}[x_0, \ldots, x_n]_a$*,*

(2) $H^i(X, \mathcal{O}_X(a)) = 0$ *for $i \neq 0, n$, and*

(3) $H^n(X, \mathcal{O}_X(a)) \cong \big[(x_0 \cdots x_n)^{-1}\mathbb{K}[x_0^{-1}, \ldots, x_n^{-1}]\big]_a$. □

In Lecture 18, we discuss a duality theorem which gives a pairing between $H^0(\mathbb{P}^n_{\mathbb{K}}, \mathcal{O}_X(a))$ and $H^n(\mathbb{P}^n_{\mathbb{K}}, \mathcal{O}_X(-n-1-a))$, via the product:

$$\mathbb{K}[x_0, \ldots, x_n]_a \times \big[(x_0 \cdots x_n)^{-1}\mathbb{K}[x_0^{-1}, \ldots, x_n^{-1}]\big]_{-n-1-a} \longrightarrow \mathbb{K}\,.$$

Exercise 13.26. Let $R = \mathbb{K}[x, y, z]$ and $f \in R$ a homogeneous element of degree d. For $X = \operatorname{Proj}(R/(f))$, determine $H^i(X, \mathcal{O}_X(a))$ for each a.

The Hartshorne-Lichtenbaum Vanishing Theorem

We return to the study of cohomological dimension, begun in Lecture 9. For an ideal $\mathfrak{a}$ in a Noetherian ring R, Proposition 9.15 implies $\operatorname{cd}_R(\mathfrak{a}) \leqslant \dim R$. Moreover, this upper bound is sharp: when R is local with maximal ideal $\mathfrak{m}$, and $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, Theorem 9.3 implies that $\operatorname{cd}_R(\mathfrak{a}) = \dim R$.

The main result of this lecture, the Hartshorne-Lichtenbaum vanishing theorem [**57**], provides a better upper bound on cohomological dimension when $\mathfrak{a}$ is not primary to a maximal ideal:

Theorem 14.1. *Let $(R, \mathfrak{m})$ be a complete local domain. If an ideal $\mathfrak{a}$ in R is not $\mathfrak{m}$-primary, then $\operatorname{cd}_R(\mathfrak{a}) \leqslant \dim R - 1$.*

The proof is presented later in this lecture; it draws on simplifications due to Brodmann and Huneke [**16**].

Remark 14.2. Let X be a scheme of finite type over a field $\mathbb{K}$. The *cohomological dimension of* X, denoted $\operatorname{cd}(X)$, is the number

$$\sup\{n \mid H^n(X, \mathcal{F}) \neq 0 \text{ for some quasi-coherent sheaf } \mathcal{F}\}\,.$$

This number is nonnegative. Serre proved that $\operatorname{cd}(X) = 0$ if and only if X is affine; see Proposition 12.38. Lichtenbaum's theorem [**91**] states that for X separated and irreducible, X is proper over $\mathbb{K}$ if and only if $\operatorname{cd}(X) = \dim X$. In view of Theorem 12.41, the Hartshorne-Lichtenbaum vanishing theorem implies that if Y is a closed, connected subset of $\mathbb{P}^d$ of dimension at least 1, then $\operatorname{cd}(\mathbb{P}^d - Y) \leqslant d - 2$.

The proof of Theorem 14.1 uses a number of interesting results. One such is Chevalley's theorem:

Theorem 14.3. *Let $(R, \mathfrak{m})$ be a complete local ring, M a finitely generated R-module, and let $\{M_t\}_{t\in\mathbb{Z}}$ be a non-increasing filtration of M.*

One then has $\bigcap_{t\in\mathbb{Z}} M_t = 0$ if and only if, for each integer $t \geqslant 0$, there exists an integer k_t such that $M_{k_t} \subseteq \mathfrak{m}^t M$.

Proof. The 'if' part follows from the Krull intersection theorem [**6**, Corollary 10.19]. Let E be the injective hull of the residue field of R. We write $(-)^\vee$ for the Matlis dual $\operatorname{Hom}_R(-, E)$. For each nonnegative integer t, set

$$C_t = (M/\mathfrak{m}^t M)^\vee \quad \text{and} \quad D_t = (M/M_t)^\vee\,,$$

viewed as submodules of $M^\vee$. Observe that the R-module $M^\vee$ is Artinian and $C_t = (0 :_{M^\vee} \mathfrak{m}^t)$, and hence the R-module C_t has finite length. Moreover, $D_t \subseteq D_{t+1}$ for each t. It thus suffices to prove that

$$\bigcup_{t\geqslant 0} D_t = M^\vee\,,$$

for then for each integer t there exists an integer k_t such that $C_t \subseteq D_{k_t}$, and this translates to $M_{k_t} \subseteq \mathfrak{m}^t M$, as desired.

Note that $\bigcup_{t\geqslant 0} D_t = \varinjlim_t D_t$. To establish the displayed equality, it suffices to prove that the natural map $\varinjlim_t D_t \longrightarrow M^\vee$ is surjective; equivalently its Matlis dual is injective. This fits in the diagram

$$\begin{tikzcd}
M^{\vee\vee} \arrow[r] & (\varinjlim_t D_t)^\vee \arrow[r, "\cong"] & \varprojlim_t D_t^\vee \\
M \arrow[u, "\cong"] \arrow[rr] & & \varprojlim_t M/M_t \arrow[u, "\cong"']
\end{tikzcd}$$

It is easy to verify the isomorphism in the top row; the other isomorphisms in the diagram are by Matlis duality, since R is complete. It remains to note that the map in the bottom row is injective, since its kernel is $\bigcap_t M_t$. $\square$

The next exercise provides another input in the proof of Theorem 14.1.

Exercise 14.4. Let $\mathfrak{a}$ be an ideal and x an element in R. For each R-module M, there is an exact sequence

$$\cdots \longrightarrow H^i_{\mathfrak{a}+Rx}(M) \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}_x}(M_x) \longrightarrow H^{i+1}_{\mathfrak{a}+Rx}(M) \longrightarrow \cdots\,.$$

Hint: Compare Čech complexes or look at injective modules.

Remark 14.5. Let $\mathfrak{p}$ be a prime ideal in a local ring R, and let $h = \operatorname{height} \mathfrak{p}$. One can choose elements $x_1, \dots, x_h$ in $\mathfrak{p}$, forming part of a system of parameters for R, such that $\operatorname{height}(x_1, \dots, x_h) = h$.

Indeed, choose x_1 not in any minimal prime ideal of R. Assume that $x_1, \dots, x_i$ have been chosen with $\operatorname{height}(x_1, \dots, x_i) = i < h$. If $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ are the minimal primes of $(x_1, \dots, x_i)$, then these have height i by Krull's height theorem, so $\mathfrak{p} \not\subseteq \bigcup \mathfrak{p}_j$. Choose an element x_{i+1} in $\mathfrak{p} \smallsetminus \bigcup \mathfrak{p}_j$; then the height of $(x_1, \dots, x_{i+1})$ is at least $i+1$ and so exactly $i+1$, by Krull's theorem. Since x_{i+1} is not in any minimal prime of $(x_1, \dots, x_i)$, one has

$$\dim R/(x_1, \dots, x_i, x_{i+1}) = \dim R/(x_1, \dots, x_i) - 1\,.$$

Therefore the sequence $x_1, \dots, x_h$ is part of a system of parameters for R.

Proof of Theorem 14.1. Set $d = \dim R$. In view of Theorem 9.6 we have only to prove that $H^d_{\mathfrak{a}}(R) = 0$.

It suffices to consider the case where $\mathfrak{a}$ is a prime ideal with $\dim R/\mathfrak{a} = 1$. Indeed, let $\mathfrak{a}$ be maximal with respect to the property that $\dim R/\mathfrak{a} \geqslant 1$ and $H^d_{\mathfrak{a}}(R) \neq 0$. If $\mathfrak{a}$ is not prime or $\dim R/\mathfrak{a} \geqslant 2$, then there is an element x not in $\mathfrak{a}$ such that $\dim R/(\mathfrak{a} + Rx) \geqslant 1$. Exercise 14.4 yields an exact sequence

$$\cdots \longrightarrow H^d_{\mathfrak{a}+Rx}(R) \longrightarrow H^d_{\mathfrak{a}}(R) \longrightarrow H^d_{\mathfrak{a}_x}(R_x) \longrightarrow \cdots\,.$$

Since R is local, $\dim R_x \leqslant d-1$ and hence $H^d_{\mathfrak{a}_x}(R_x) = 0$. Thus, $H^d_{\mathfrak{a}}(R) \neq 0$ implies $H^d_{\mathfrak{a}+Rx}(R) \neq 0$, contradicting the maximality of $\mathfrak{a}$.

For the rest of the proof, we fix a prime ideal $\mathfrak{p}$ with $\dim R/\mathfrak{p} = 1$ and prove that $H^d_{\mathfrak{p}}(R) = 0$.

First we settle the case when R is Gorenstein. For each integer $t \geqslant 1$, primary decomposition yields that $\mathfrak{p}^t = \mathfrak{p}^{(t)} \cap J_t$, where J_t is either $\mathfrak{m}$-primary or equal to R; here $\mathfrak{p}^{(t)} = \mathfrak{p}^t R_{\mathfrak{p}} \cap R$ is the t-th symbolic power of t. The Krull intersection theorem yields

$$\Big(\bigcap_t \mathfrak{p}^{(t)}\Big) R_{\mathfrak{p}} \subseteq \bigcap_t \mathfrak{p}^t R_{\mathfrak{p}} = (0)\,.$$

Therefore $\bigcap \mathfrak{p}^{(t)} = 0$, as the ring R is a domain. Given that $\operatorname{rad}(J_t)$ is either $\mathfrak{m}$ or R, it then follows from Chevalley's theorem that for each integer $t \geqslant 1$, there exists an integer k_t with $\mathfrak{p}^{(k_t)} \subseteq J_t$. Consequently, $\mathfrak{p}^{(\max(t,k_t))} \subseteq \mathfrak{p}^t$, so the sequences $\{\mathfrak{p}^t\}$ and $\{\mathfrak{p}^{(t)}\}$ are cofinal. Hence for any R-module M, Remark 7.9 implies that

$$\varinjlim_t \operatorname{Ext}^d_R(R/\mathfrak{p}^{(t)}, M) \cong H^d_{\mathfrak{p}}(M)\,.$$

Evidently, the dimension of the R-module $R/\mathfrak{p}^{(t)}$ is one; its depth is also one because the only prime associated to it is $\mathfrak{p}$. Thus, combining Theorems 9.1 and 9.3 and local duality, Theorem 11.29, one deduces that

$$\operatorname{Ext}^n_R(R/\mathfrak{p}^{(t)}, R) = 0 \qquad \text{for } n \neq d-1\,.$$

Hence $H^d_{\mathfrak{p}}(R) = 0$, which is the desired result.

This completes the proof of the theorem when R is Gorenstein. The general case reduces to this, as follows.

The ring R is complete, so by Cohen's structure theorem R has a coefficient ring $V \subseteq R$ which is a field or a discrete valuation ring (DVR) with uniformizing parameter q, where q is the characteristic of the residue field.

Cohen's structure theorem implies that R is a quotient of a regular local ring, and hence it is catenary [**115**, Theorem 17.4]. Since R is a domain, and $\dim R/\mathfrak{p} = 1$, one now deduces that $\operatorname{height}\mathfrak{p} = d-1$. There are thus elements $x_1, \ldots, x_{d-1}$ in $\mathfrak{p}$ which are part of a system of parameters for R with $\operatorname{height}(x_1, \ldots, x_{d-1}) = d-1$; when V is a DVR with $q \in \mathfrak{p}$ we choose $x_1 = q$; see Remark 14.5. Next choose an element $x_d \in \mathfrak{p}$ not in any of the other minimal primes (if any) of $x_1, \ldots, x_{d-1}$. Then $\mathfrak{p}$ is the only minimal prime of $x_1, \ldots, x_d$ and hence

$$\operatorname{rad}(x_1, \ldots, x_d) = \mathfrak{p}\,.$$

Furthermore, when V is a field or a DVR with $q \in \mathfrak{p}$, choose any $y \notin \mathfrak{p}$. The elements $x_1, \ldots, x_{d-1}, y$ form a system of parameters for R. When V is a DVR with $q \notin \mathfrak{p}$, the elements $x_1, \ldots, x_{d-1}, q$ form a system of parameters for R. In this case, set $y = 0$.

At the end of the day, one has inclusions of rings

$$A = V[[x_1, \ldots, x_{d-1}, y]] \subseteq B = A[x_d] \subseteq R\,.$$

The complete local ring A is regular and, by Cohen's structure theorem, R, and hence B, is finite over A. In particular, the element x_d is integral over A; let $f(x)$ be its minimal polynomial. Then $B \cong A[x]/(f(x))$ and so B is a hypersurface and thus Gorenstein.

Set $\mathfrak{a} = \mathfrak{p}R \cap B$. Since $B \subseteq R$ and $B/\mathfrak{a} \subseteq R/\mathfrak{p}$ are finite extensions, one has $\dim B = \dim R = d$ and $\dim B/\mathfrak{a} = \dim R/\mathfrak{p} \geqslant 1$. Therefore, B being Gorenstein, we know $\operatorname{cd}_B(\mathfrak{a}) \leqslant d-1$. Since $(x_1, \ldots, x_d)R \subseteq \mathfrak{a}R \subseteq \mathfrak{p}$, one deduces that $\operatorname{rad}(\mathfrak{a}R) = \mathfrak{p}$. Therefore,

$$H^d_{\mathfrak{p}}(R) = H^d_{\mathfrak{a}}(R) = 0\,,$$

where the first equality is by Propositions 7.3(2) and 7.15(2), and the second follows by viewing R as a B-module. This completes the proof. □

The next result is an enhancement of the Hartshorne-Lichtenbaum theorem; its derivation from Theorem 14.1 is routine.

Theorem 14.6. *Let R be a local ring and $\widehat{R}$ its completion at the maximal ideal. Let $\mathfrak{a}$ be an ideal of R.*

Then $\operatorname{cd}_R(\mathfrak{a}) \leqslant \dim R - 1$ *if and only if* $\operatorname{height}(\mathfrak{a}\widehat{R} + \mathfrak{p}) < \dim R$ *for each prime ideal $\mathfrak{p}$ in* $\operatorname{Spec}\widehat{R}$ *with* $\dim(\widehat{R}/\mathfrak{p}) = \dim R$. □

The following theorem was proved by Hartshorne [**57**, Theorem 7.5] in the projective situation, by Ogus [**126**, Corollary 2.11] in the characteristic zero case, and by Peskine and Szpiro [**128**, Theorem III.5.5] in positive characteristic. In [**147**] the number of connected components is determined in terms of the Frobenius action on local cohomology.

Theorem 14.7. *Let* $(R, \mathfrak{m})$ *be a complete regular local ring containing a separably closed coefficient field, and* $\mathfrak{a}$ *an ideal. Then* $\operatorname{cd}_R(\mathfrak{a}) \leqslant \dim R - 2$ *if and only if* $\dim R/\mathfrak{a} \geqslant 2$ *and* $\operatorname{Spec}(R/\mathfrak{a}) \smallsetminus \{\mathfrak{m}\}$ *is connected.*

Huneke and Lyubeznik gave a characteristic-free proof of the above theorem in [**78**] using a generalization of a result of Faltings [**35**, Satz 1], and obtained other bounds for cohomological dimension; see also [**109**].

Connectedness

This lecture deals with connections between cohomological dimension and connectedness of varieties. An important ingredient is a local cohomology version of the Mayer-Vietoris theorem encountered in topology.

1. Mayer-Vietoris sequence

Let $\mathfrak{a}' \supseteq \mathfrak{a}$ be ideals in R and M an R-module. The inclusion $\Gamma_{\mathfrak{a}'}(-) \subseteq \Gamma_{\mathfrak{a}}(-)$ induces, for each n, an R-module homomorphism

$$\theta^n_{\mathfrak{a}',\mathfrak{a}}(M)\colon H^n_{\mathfrak{a}'}(M) \longrightarrow H^n_{\mathfrak{a}}(M)\,.$$

This homomorphism is functorial in M, that is to say, $\theta^n_{\mathfrak{a}',\mathfrak{a}}(-)$ is a natural transformation from $H^n_{\mathfrak{a}'}(-)$ to $H^n_{\mathfrak{a}}(-)$. Given ideals $\mathfrak{a}$ and $\mathfrak{b}$ in R, for each integer n we set

$$\begin{aligned}
\iota^n_{\mathfrak{a},\mathfrak{b}}(M)\colon H^n_{\mathfrak{a}+\mathfrak{b}}(M) &\longrightarrow H^n_{\mathfrak{a}}(M) \oplus H^n_{\mathfrak{b}}(M)\,,\\
z &\longmapsto (\theta^n_{\mathfrak{a}+\mathfrak{b},\mathfrak{a}}(z), \theta^n_{\mathfrak{a}+\mathfrak{b},\mathfrak{b}}(z)), \quad \text{and}\\
\pi_{\mathfrak{a},\mathfrak{b}}(M)\colon H^n_{\mathfrak{a}}(M) \oplus H^n_{\mathfrak{b}}(M) &\longrightarrow H^n_{\mathfrak{a}\cap\mathfrak{b}}(M)\,,\\
(x,y) &\longmapsto \theta^n_{\mathfrak{a},\mathfrak{a}\cap\mathfrak{b}}(x) - \theta^n_{\mathfrak{b},\mathfrak{a}\cap\mathfrak{b}}(y)\,.
\end{aligned}$$

Clearly the homomorphisms $\iota^n_{\mathfrak{a},\mathfrak{b}}(M)$ and $\pi^n_{\mathfrak{a},\mathfrak{b}}(M)$ are functorial in M.

Theorem 15.1 (Mayer-Vietoris sequence)**.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in a Noetherian ring R. For each R-module M, there exists a sequence of R-modules*

$$\begin{aligned}
0 \longrightarrow H^0_{\mathfrak{a}+\mathfrak{b}}(M) &\xrightarrow{\iota^0_{\mathfrak{a},\mathfrak{b}}(M)} H^0_{\mathfrak{a}}(M) \oplus H^0_{\mathfrak{b}}(M) \xrightarrow{\pi^0_{\mathfrak{a},\mathfrak{b}}(M)} H^0_{\mathfrak{a}\cap\mathfrak{b}}(M)\\
\longrightarrow H^1_{\mathfrak{a}+\mathfrak{b}}(M) &\xrightarrow{\iota^1_{\mathfrak{a},\mathfrak{b}}(M)} H^1_{\mathfrak{a}}(M) \oplus H^1_{\mathfrak{b}}(M) \xrightarrow{\pi^1_{\mathfrak{a},\mathfrak{b}}(M)} H^1_{\mathfrak{a}\cap\mathfrak{b}}(M) \longrightarrow \cdots\,,
\end{aligned}$$

which is exact, and functorial in M.

Proof. It is an elementary exercise to verify that one has an exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}+\mathfrak{b}}(M) \xrightarrow{\iota^0_{\mathfrak{a},\mathfrak{b}}(M)} \Gamma_{\mathfrak{a}}(M) \oplus \Gamma_{\mathfrak{b}}(M) \xrightarrow{\pi^0_{\mathfrak{a},\mathfrak{b}}(M)} \Gamma_{\mathfrak{a}\cap\mathfrak{b}}(M)\,.$$

We claim that $\pi^0_{\mathfrak{a},\mathfrak{b}}(M)$ is surjective whenever M is an injective R-module. Indeed, since $\Gamma_{\mathfrak{a}'}(-)$ commutes with direct sums for any ideal $\mathfrak{a}'$, it suffices to consider the case where $M = E_R(R/\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ of R. Then the asserted surjectivity is immediate from Example 7.6.

Let $I^\bullet$ be an injective resolution of M. In view of the preceding discussion, we have an exact sequence of complexes,

$$0 \longrightarrow \Gamma_{\mathfrak{a}+\mathfrak{b}}(I^\bullet) \longrightarrow \Gamma_{\mathfrak{a}}(I^\bullet) \oplus \Gamma_{\mathfrak{b}}(I^\bullet) \longrightarrow \Gamma_{\mathfrak{a}\cap\mathfrak{b}}(I^\bullet) \longrightarrow 0\,.$$

The homology exact sequence arising from this is the one we seek. The functoriality is a consequence of the functoriality of $\iota^n_{\mathfrak{a},\mathfrak{b}}(-)$ and $\pi^n_{\mathfrak{a},\mathfrak{b}}(-)$ and of the connecting homomorphisms in cohomology long exact sequences. □

Recall that a topological space is *connected* if it cannot be written as a disjoint union of two proper closed subsets. The Mayer-Vietoris sequence has applications to connectedness properties of algebraic varieties.

Exercise 15.2. Let R be a Noetherian ring. Prove that the support of a finitely generated indecomposable R-module is connected.

2. Punctured spectra

We focus now on punctured spectra of local rings.

Definition 15.3. The *punctured spectrum* of a local ring $(R,\mathfrak{m})$ is the set

$$\operatorname{Spec}^\circ R = \operatorname{Spec} R \smallsetminus \{\mathfrak{m}\}\,,$$

with topology induced by the Zariski topology on $\operatorname{Spec} R$. Similarly, if R is graded with homogeneous maximal ideal $\mathfrak{m}$, then its punctured spectrum refers to the topological space $\operatorname{Spec} R \smallsetminus \{\mathfrak{m}\}$.

Connectedness of the punctured spectrum can be interpreted entirely in the language of ideals.

Remark 15.4. Let $(R,\mathfrak{m})$ be a local ring and $\mathfrak{a}$ an ideal. The punctured spectrum $\operatorname{Spec}^\circ(R/\mathfrak{a})$ of the local ring $R/\mathfrak{a}$ is connected if and only if the following property holds: given ideals $\mathfrak{a}'$ and $\mathfrak{a}''$ in R with

$$\operatorname{rad}(\mathfrak{a}'\cap\mathfrak{a}'') = \operatorname{rad}\mathfrak{a} \quad \text{and} \quad \operatorname{rad}(\mathfrak{a}'+\mathfrak{a}'') = \mathfrak{m}\,,$$

either $\operatorname{rad}\mathfrak{a}'$ or $\operatorname{rad}\mathfrak{a}''$ equals $\mathfrak{m}$; equivalently, $\operatorname{rad}\mathfrak{a}''$ or $\operatorname{rad}\mathfrak{a}'$ equals $\operatorname{rad}\mathfrak{a}$.

Indeed, this is a direct translation of the definition of connectedness, keeping in mind that

$$V(\mathfrak{a}')\cup V(\mathfrak{a}'') = V(\mathfrak{a}'\cap\mathfrak{a}'') \quad \text{and} \quad V(\mathfrak{a}')\cap V(\mathfrak{a}'') = V(\mathfrak{a}'+\mathfrak{a}'')\,.$$

Exercise 15.5. Prove that if R is a local domain, then $\operatorname{Spec}^\circ R$ is connected.

Exercise 15.6. Let $R = \mathbb{R}[x, y, ix, iy]$ where $i^2 = -1$. Note that

$$R \cong \mathbb{R}[x, y, u, v]/(u^2 + x^2, v^2 + y^2, xy + uv, xv - uy)\,.$$

Show that $\operatorname{Spec}^\circ R$ is connected, but $\operatorname{Spec}^\circ(R \otimes_{\mathbb{R}} \mathbb{C})$ is not.

The next few results identify conditions under which the punctured spectrum is connected. The first is a straightforward application of the Mayer-Vietoris sequence, Theorem 15.1.

Proposition 15.7. *If R is local with* $\operatorname{depth} R \geqslant 2$*, then* $\operatorname{Spec}^\circ R$ *is connected.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R, and let $\mathfrak{a}'$ and $\mathfrak{a}''$ be ideals with $\operatorname{rad}(\mathfrak{a}' \cap \mathfrak{a}'') = \operatorname{rad}(0)$ and $\operatorname{rad}(\mathfrak{a}' + \mathfrak{a}'') = \mathfrak{m}$. Proposition 7.3(2) and the depth sensitivity of local cohomology, Theorem 9.1, imply that

$$H^0_{\mathfrak{a}' \cap \mathfrak{a}''}(R) = R \quad \text{ and } \quad H^0_{\mathfrak{a}' + \mathfrak{a}''}(R) = 0 = H^1_{\mathfrak{a}' + \mathfrak{a}''}(R)\,.$$

The Mayer-Vietoris sequence now yields an isomorphism of R-modules

$$H^0_{\mathfrak{a}'}(R) \oplus H^0_{\mathfrak{a}''}(R) \cong R\,.$$

But R is indecomposable as a module over itself by Exercise 15.8, so, without loss of generality, we may assume that $H^0_{\mathfrak{a}'}(R) = R$ and $H^0_{\mathfrak{a}''}(R) = 0$. This implies that $\operatorname{rad} \mathfrak{a}' = \operatorname{rad}(0)$, as desired. □

Exercise 15.8. Show that a local ring is indecomposable as a module.

The hypothesis on depth in Proposition 15.7 is optimal in view of Exercise 15.6 as well as the following example:

Example 15.9. The ring $R = \mathbb{K}[[x, y]]/(xy)$ is local with $\operatorname{depth} R = 1$. Moreover, one has that

$$V(x) \cup V(y) = \operatorname{Spec} R \quad \text{ and } \quad V(x) \cap V(y) = \{(x, y)\}\,.$$

Thus, $\operatorname{Spec}^\circ R$ is not connected.

Here is an amusing application of Proposition 15.7:

Example 15.10. Let $R = \mathbb{K}[[x, y, u, v]]/(x, y) \cap (u, v)$. It is not hard to check that $\dim R = 2$. On the other hand, $\operatorname{Spec}^\circ R$ is disconnected so $\operatorname{depth} R \leqslant 1$ by Proposition 15.7. In particular, R is not Cohen-Macaulay.

More sophisticated results on connectedness of punctured spectra can be derived from the Hartshorne-Lichtenbaum vanishing theorem. The one below was proved in the equicharacteristic case by Faltings [**36, 37**] using a different method; the argument presented here is due to Brodmann and Rung [**18**]. Recall that $\operatorname{ara} \mathfrak{a}$, the arithmetic rank of $\mathfrak{a}$, is the least number of generators of an ideal with the same radical as $\mathfrak{a}$.

Theorem 15.11 (Faltings' connectedness theorem)**.** *Let R be a complete local domain. If $\mathfrak{a}$ is an ideal of R with $\operatorname{ara}\mathfrak{a} \leqslant \dim R - 2$, then $\operatorname{Spec}^\circ(R/\mathfrak{a})$, the punctured spectrum of $R/\mathfrak{a}$, is connected.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R and $\mathfrak{a}', \mathfrak{a}''$ ideals with $\operatorname{rad}(\mathfrak{a}' \cap \mathfrak{a}'') = \operatorname{rad}\mathfrak{a}$ and $\operatorname{rad}(\mathfrak{a}' + \mathfrak{a}'') = \mathfrak{m}$. We prove that one of $\operatorname{rad}\mathfrak{a}'$ or $\operatorname{rad}\mathfrak{a}''$ equals $\mathfrak{m}$.

Set $d = \dim R$. Since $\operatorname{ara}\mathfrak{a} \leqslant d - 2$, we have

$$H^n_{\mathfrak{a}' \cap \mathfrak{a}''}(R) = H^n_{\mathfrak{a}}(R) = 0 \qquad \text{for } n = d-1, d\,,$$

where the first equality holds by Proposition 7.3(2) and the second by Proposition 9.12. Keeping in mind that $H^n_{\mathfrak{a}'+\mathfrak{a}''}(R) = H^n_{\mathfrak{m}}(R)$ for each n, again by Proposition 7.3(2), the Mayer-Vietoris sequence gives us

$$0 = H^{d-1}_{\mathfrak{a}' \cap \mathfrak{a}''}(R) \longrightarrow H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{a}'}(R) \oplus H^d_{\mathfrak{a}''}(R) \longrightarrow H^d_{\mathfrak{a}' \cap \mathfrak{a}''}(R) = 0\,.$$

Theorem 9.3 yields $H^d_{\mathfrak{m}}(R) \neq 0$, so the exact sequence above implies that one of $H^d_{\mathfrak{a}'}(R)$ or $H^d_{\mathfrak{a}''}(R)$ must be nonzero; say $H^d_{\mathfrak{a}'}(R) \neq 0$. This implies $\operatorname{cd}_R(\mathfrak{a}') \geqslant d$, and hence $\operatorname{rad}\mathfrak{a}' = \mathfrak{m}$ by Theorem 14.1. $\square$

Hochster and Huneke have obtained generalizations of Faltings' connectedness theorem. One such is [**74**, Theorem 3.3]:

Theorem 15.12. *Let $(R, \mathfrak{m})$ be a complete equidimensional ring of dimension d such that $H^d_{\mathfrak{m}}(R)$ is indecomposable as an R-module; equivalently, the canonical module ω_R is indecomposable.*

If $\mathfrak{a}$ is an ideal of R with $\operatorname{ara}\mathfrak{a} \leqslant d-2$, then $\operatorname{Spec}^\circ(R/\mathfrak{a})$ is connected. $\square$

Faltings' theorem leads to another connectedness result, originally due to Fulton and Hansen [**43**]. An interesting feature of the proof is the use of 'reduction to the diagonal' encountered in the proof of Theorem 1.33.

Theorem 15.13 (Fulton-Hansen theorem)**.** *Let $\mathbb{K}$ be an algebraically closed field, and let X and Y be irreducible sets in $\mathbb{P}^n_{\mathbb{K}}$.*

If $\dim X + \dim Y \geqslant n+1$, then $X \cap Y$ is connected.

Proof. Let $\mathfrak{p}$ and $\mathfrak{q}$ be homogeneous prime ideals of $\mathbb{K}[x_0, \dots, x_n]$ such that

$$\mathbb{K}[x_0, \dots, x_n]/\mathfrak{p} \quad \text{and} \quad \mathbb{K}[x_0, \dots, x_n]/\mathfrak{q}$$

are homogeneous coordinate rings of X and Y, respectively. Then

$$\mathbb{K}[x_0, \dots, x_n]/(\mathfrak{p} + \mathfrak{q})$$

is a homogeneous coordinate ring for the intersection $X \cap Y$. If $X \cap Y$ is disconnected, then so are the punctured spectra of the graded rings

$$\frac{\mathbb{K}[x_0, \dots, x_n]}{(\mathfrak{p} + \mathfrak{q})} \cong \frac{\mathbb{K}[x_0, \dots, x_n, y_0, \dots, y_n]}{(\mathfrak{p} + \mathfrak{q}' + \Delta)}\,,$$

where $\mathfrak{q}'$ is the ideal generated by polynomials obtained by substituting y_i for x_i in a set of generators for $\mathfrak{q}$, and $\Delta = (x_0 - y_0, \ldots, x_n - y_n)$ is the ideal defining the diagonal in $\mathbb{P}^{2n+1}_{\mathbb{K}}$. The complete local ring

$$R = \mathbb{K}[[x_0, \ldots, x_n, y_0, \ldots, y_n]]/(\mathfrak{p} + \mathfrak{q}')$$

is a domain by Exercises 15.15(4) and 15.17, and

$$\dim R = \dim X + 1 + \dim Y + 1 \geqslant n + 3\,.$$

Evidently $\operatorname{ara} \Delta \leqslant n + 1 = (n + 3) - 2$. This, however, contradicts Faltings' connectedness theorem. □

Exercise 15.14. If $\mathbb{K}$ is a field which is *not* algebraically closed, construct a $\mathbb{K}$-algebra R such that R is an integral domain, but $R \otimes_{\mathbb{K}} R$ is not.

Exercise 15.15. Let $\mathbb{K} \subseteq \mathbb{L}$ be fields, where $\mathbb{K}$ is algebraically closed.

(1) If a family of polynomials $\{f_i\}$ in $\mathbb{K}[x_1, \ldots, x_n]$ has no common root in $\mathbb{K}^n$, prove that it has no common root in $\mathbb{L}^n$.

(2) If $g \in \mathbb{K}[x_1, \ldots, x_n]$ is irreducible, prove that it is also irreducible as an element of $\mathbb{L}[x_1, \ldots, x_n]$. Hint: Use (1).

(3) If R is a finitely generated $\mathbb{K}$-algebra which is an integral domain, prove that $R \otimes_{\mathbb{K}} \mathbb{L}$ is an integral domain.

(4) If R and S are finitely generated $\mathbb{K}$-algebras which are integral domains, prove that $R \otimes_{\mathbb{K}} S$ is an integral domain. Hint: Take $\mathbb{L}$ to be the fraction field of S and use (3).

Exercise 15.16. Give an example of a local domain $(R, \mathfrak{m})$ whose $\mathfrak{m}$-adic completion is not a domain.

Exercise 15.17. If $(R, \mathfrak{m})$ is an $\mathbb{N}$-graded domain finitely generated over a field R_0, prove that its $\mathfrak{m}$-adic completion is also a domain.

Polyhedral Applications

Local cohomology and the concepts surrounding it have a lot to say about commutative algebra in general settings, as seen in previous lectures. In this lecture, we begin to see interactions with combinatorics: focusing on specific classes of rings allows for deep applications to polyhedral geometry.

1. Polytopes and faces

To begin with, suppose V is a subset of the Euclidean space $\mathbb{R}^d$. The smallest convex set in $\mathbb{R}^d$ containing V is the *convex hull* of V, denoted $\mathrm{conv}(V)$.

Definition 16.1. A *polytope* is the convex hull of a finite set V in $\mathbb{R}^d$. The *dimension* of $\mathrm{conv}(V)$ is the dimension of its affine span.

Example 16.2. Consider the six unit vectors in $\mathbb{R}^3$ along the positive and negative axes. The convex hull of these points is a regular octahedron:

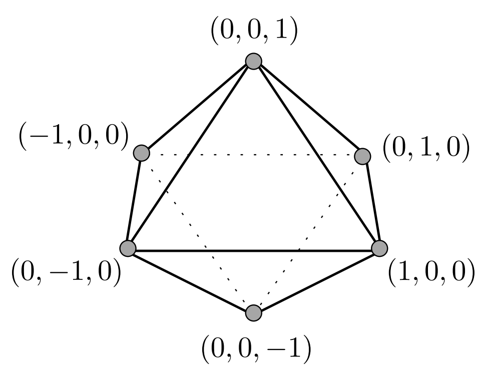


Figure 16.1. A regular octahedron

More generally, the convex hull of the $2d$ unit vectors along the positive and negative axes in $\mathbb{R}^d$ is the *regular cross-polytope* of dimension d.

A *closed halfspace* H^+ is the set of points weakly to one fixed side of an affine hyperplane H in $\mathbb{R}^d$. It is a fundamental and nontrivial—but intuitively obvious—theorem that a polytope can be expressed as an intersection of finitely many closed halfspaces, [**165**, Theorem 1.1]. This description makes it much easier to prove the following.

Exercise 16.3. The intersection of a polytope in $\mathbb{R}^d$ with any affine hyperplane is another polytope in $\mathbb{R}^d$.

In general, slicing a polytope P with affine hyperplanes yields infinitely many new polytopes, and their dimensions can vary from $\dim(P)$ down to 0. But if we only consider $P \cap H$ for *support hyperplanes* H, meaning that P lies on one side of H, say $P \subset H^+$, then only finitely many new polytopes occur. An intersection $F = P \cap H$ with a support hyperplane is called a *face* of P; that $P = \operatorname{conv}(V)$ has only finitely many faces follows because each face F is the convex hull of $F \cap V$.

Example 16.4. Consider the octahedron from Example 16.2. Any affine plane in $\mathbb{R}^3$ passing through $(0, 0, 1)$ and whose normal vector is sufficiently close to vertical is a support hyperplane; the corresponding face is the vertex $(0, 0, 1)$. The plane $x+y+z = 1$ is a support hyperplane; the corresponding face is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Definition 16.5. Let P be a polytope of dimension d. The *f-vector* of P is the vector $(f_{-1}, f_0, \ldots, f_{d-1}, f_d)$ in which f_i equals the number of dimension i faces of P. Here, $f_{-1} = 1$ counts the empty face of P, and $f_d = 1$ counts P itself which is also considered to be a face. The numbers f_0, f_1, and f_{d-1} are the number of *vertices*, *edges*, and *facets*, respectively.

The vertices of $P = \operatorname{conv}(V)$ are all elements of the finite set V, though V might contain points interior to P that are therefore not vertices of P.

Example 16.6. The octahedron has a total of 28 faces: 6 vertices, 12 edges, 8 facets, plus the whole octahedron itself and one empty face. Hence the f-vector of the octahedron is $(1, 6, 12, 8, 1)$.

Exercise 16.7. What is the f-vector of the cross-polytope in $\mathbb{R}^d$? You might find it easier to calculate the f-vector of the hypercube in $\mathbb{R}^d$ and reverse it; why does this work?

Counting faces raises the following basic issue, our main concern here:

Question 16.8. How many faces of each dimension could $P \subset \mathbb{R}^d$ possibly have, given that it has (say) a fixed number of vertices?

On the face of it, Question 16.8 has nothing at all to do with commutative algebra, let alone local cohomology; but we shall see that it does, given the appropriate class of rings.

2. Upper bound theorem

Fix a polytope $P = \mathrm{conv}(V)$ in $\mathbb{R}^d$ with n vertices, and assume that V equals the vertex set of P. Each face of P is determined by the set of vertices it contains, which is by definition the subset of V minimized by some (perhaps not uniquely determined) linear functional.

Now suppose that we wiggle the vertices V in $\mathbb{R}^d$ a tiny bit to get a set V', the vertex set of a new polytope $P' = \mathrm{conv}(V')$. Each face G of P' has a vertex set $V'_G \subseteq V'$. Unwiggling V'_G yields a subset $V_G \subseteq V$ whose affine span equals the affine span of some face F of P, although V_G might be a proper subset of the vertex set of F. In this way, to each face of P' is associated a well-defined face of P.

If our vertex-wiggling is done at random, what kind of polytope is P'?

Definition 16.9. A polytope of dimension i is a *simplex* if it has precisely $i + 1$ vertices. A polytope is *simplicial* if each proper face is a simplex.

To check that a polytope is simplicial, it is enough to check that its facets are simplices.

Exercise 16.10. A simplex of any dimension is a simplicial polytope. The octahedron and, more generally, any cross-polytope, is a simplicial polytope.

Exercise 16.11. Define a notion of *generic* for finite point sets in $\mathbb{R}^d$ such that (1) the convex hull of every generic set is a simplicial polytope; and (2) every finite set V can be made generic by moving each point in V less than any fixed positive distance.

Exercise 16.12. If P' is obtained by generically wiggling the vertices of P, prove that for every dimension i face of P there is a dimension i face of P' associated to it.

Hint: Let F be a given i-face of P, and call its vertex set V_F. Fix a support hyperplane H for F and a vector ν perpendicular to H. If F' is the orthogonal projection of $\mathrm{conv}(V'_F)$ to the affine span of F, then construct a *regular subdivision* [**165**, Definition 5.3] of F' using the orthogonal projection of V'_F to the affine span of $F + \nu$. Use a face of maximal volume in this subdivision to get the desired face of P'.

The maximal volume condition ensures that unwiggling the vertices of the chosen face of the subdivision of F' yields a subset of V_F whose convex hull has dimension $\dim(F)$. Regularity of the subdivision forces the corresponding subset of vertices of P' to be the vertex set of a face of P'; a support hyperplane is obtained by wiggling H.

Exercise 16.12 implies that the collection of subsets of V that are vertex sets of i-faces can only get bigger when V is made generic by wiggling. Consequently, among all polytopes with n vertices, there is a simplicial one that maximizes the number of i-faces. This reduces Question 16.8 to the simplicial case. In fact, there is a single simplicial polytope with n vertices that maximizes f_i for all i simultaneously. It is constructed as follows.

Definition 16.13. Consider the *rational normal curve* $(t, t^2, t^3, \ldots, t^d)$ in $\mathbb{R}^d$ parametrized by $t \in \mathbb{R}$. If $n \geqslant d+1$, then the convex hull of n distinct points on the rational normal curve is a *cyclic polytope* $C(n, d)$ of dimension d.

Exercise 16.14. Use the nonvanishing of Vandermonde determinants to show that a cyclic $C(n, d)$ polytope is indeed simplicial of dimension d.

It is nontrivial but elementary [**165**, Theorem 0.7] that the combinatorial type of $C(n, d)$ is independent of which n points on the rational normal curve are chosen as vertices. It is similarly elementary [**165**, Corollary 0.8] that for $i \leqslant d/2$, every set of i vertices of $C(n, d)$ is the vertex set of an $(i-1)$-face of $C(n, d)$; equivalently,

$$f_{i-1}(C(n, d)) = \binom{n}{i} \qquad \text{for } i \leqslant \frac{d}{2}\,.$$

Thus $C(n, d)$ is said to be *neighborly.*

Exercise 16.15. Find a simplicial non-neighborly polytope.

For obvious reasons, $f_{i-1}(P)$ is bounded above by $\binom{n}{i}$ for any polytope P with n vertices. Therefore, no polytope with n vertices has more $(i-1)$-faces than $C(n, d)$ when $i \leqslant d/2$. What about faces of higher dimension?

Theorem 16.16 (Upper bound theorem for polytopes)**.** *For any polytope $P \subset \mathbb{R}^d$ with n vertices, $f_i(P) \leqslant f_i(C(n, d))$ for each i.*

Exercise 16.17. Verify that the octahedron in Example 16.2 has the same f-vector as the cyclic polytope $C(6, 3)$; this is Theorem 16.16 for the octahedron. Now, using the result of Exercise 16.7, verify the statement of Theorem 16.16 for the 4-dimensional cross-polytope. This amounts to computing the f-vector of the appropriate cyclic polytope. (Which one is it?)

Theorem 16.16 was proved by McMullen [**117**]. Following Stanley [**150**], our present goal is to generalize the statement to simplicial spheres and see why it follows from certain rings being Cohen-Macaulay and Gorenstein.

3. The h-vector of a simplicial complex

We have assumed for these lectures that the reader has seen a little bit of algebraic topology, so simplicial complexes should be familiar, though perhaps not as combinatorial objects. Recall that a *simplicial complex* with vertices $1, \ldots, n$ is a collection Δ of subsets of $\{1, \ldots, n\}$ that is closed under taking subsets: if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then also $\tau \in \Delta$. For example, the boundary of a simplicial polytope is a simplicial complex.

We can record the numerical data associated to a general simplicial complex the same way we did for polytopes: let $f_i(\Delta)$ be the number of i-faces (in other words, simplices of dimension i) of Δ. Note, however, that if P is a simplicial polytope with boundary simplicial complex Δ, then we do not count the polytope itself as a face of Δ.

For simplicial complexes there is another, seemingly complicated but in fact often elegant, way to record the face numbers.

Definition 16.18. The *h-vector* $(h_0, h_1, \ldots, h_d)$ of a simplicial complex Δ of dimension $d-1$ is defined by

$$\sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i},$$

where $f_{i-1} = f_{i-1}(\Delta)$. The left-hand side is the *h-polynomial* of Δ. If P is a polytope, $h_i(P)$ is the i-th entry of the h-vector of the boundary of P.

The next section will tell us conceptual ways to compute h-polynomials; the octahedron example will appear later in Example 16.24. For now, we observe that there is a translation from the h-vector back to the f-vector.

Lemma 16.19. *For a simplicial complex Δ of dimension $d-1$, each f_i is a positive integer linear combination of $h_0, \ldots, h_d$. More precisely,*

$$f_{i-1} = \sum_{j=0}^{i} \binom{d-j}{i-j} h_j .$$

Proof. Dividing both sides of Definition 16.18 by $(1-t)^d$ yields

$$\sum_{i=0}^{d} h_i \frac{t^i}{(1-t)^i} \cdot \frac{1}{(1-t)^{d-i}} = \sum_{i=0}^{d} f_{i-1} \frac{t^i}{(1-t)^i} .$$

Setting $s = t/(1-t)$, we find that $1/(1-t) = s+1$, and hence

$$\sum_{i=0}^{d} h_i s^i (s+1)^{d-i} = \sum_{i=0}^{d} f_{i-1} s^i .$$

Now compare coefficients of s^i. $\square$

By Lemma 16.19, if we wish to bound the number of i-faces of a simplicial complex Δ by the number of i-faces of a cyclic polytope, then it is certainly enough to bound the h-vector of Δ entrywise by the h-vector of the cyclic polytope. This is what we shall do in Theorem 16.20.

It might seem from the discussion leading to Theorem 16.16 that its statement is geometric, in the sense that it says something fundamental about convexity. But, in fact, the statement turns out to be topological at heart: the same result holds for simplicial complexes that are homeomorphic to spheres, without any geometric hypotheses akin to convexity. It should be noted that in a precise sense, most simplicial spheres are not convex [**86**]. Here is the more general statement, proved later in Section 6.

Theorem 16.20 (Upper bound theorem for simplicial spheres)**.** *If Δ is a dimension $d-1$ simplicial sphere with n vertices, then $h_i(\Delta) \leqslant h_i(C(n,d))$ for each i, and consequently $f_i(\Delta) \leqslant f_i(C(n,d))$ for each i.*

4. Stanley-Reisner rings

The connection between the upper bound theorem and commutative algebra is via rings constructed from simplicial complexes. If $\sigma \subseteq \{1,\ldots,n\}$, then write $x^\sigma = \prod_{j\in\sigma} x_j$ for the corresponding squarefree monomial.

Definition 16.21. Let Δ be a simplicial complex with vertex set $\{1,\ldots,n\}$. The *Stanley-Reisner ring* of Δ is the quotient

$$\mathbb{K}[\Delta] = \mathbb{K}[x_1,\ldots,x_n]/(x^\sigma \mid \sigma \notin \Delta)$$

of the polynomial ring by the *Stanley-Reisner ideal* $I_\Delta = (x^\sigma \mid \sigma \notin \Delta)$.

Example 16.22. Let Δ be the boundary of the octahedron from Example 16.2. Let $R = \mathbb{K}[x_-, x_+, y_-, y_+, z_-, z_+]$, with the positively and negatively indexed x, y, and z-variables on the corresponding axes:

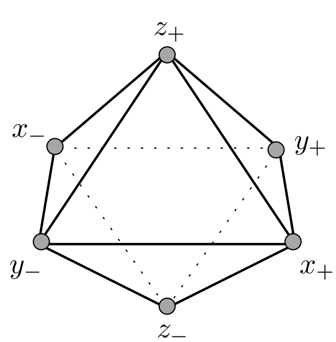


Figure 16.2. The octahedron revisited

The Stanley-Reisner ring of Δ is

$$\mathbb{K}[\Delta] = R/(x_- x_+, y_- y_+, z_- z_+)\,,$$

since every subset of the vertices not lying in Δ contains one of the main diagonals of the octahedron.

Geometrically, each set $\sigma \subseteq \{1, \ldots, n\}$ corresponds to a vector subspace of $\mathbb{K}^n$, namely the subspace spanned by the basis vectors $\{e_j \mid j \in \sigma\}$. Hence a simplicial complex Δ corresponds to a configuration of subspaces of $\mathbb{K}^n$, and $\mathbb{K}[\Delta]$ is simply the affine coordinate ring of this configuration.

The easiest way to understand the algebraic structure of $\mathbb{K}[\Delta]$ is to use the $\mathbb{N}^n$-*grading* of the polynomial ring:

$$\mathbb{K}[x_1, \ldots, x_n] = \bigoplus_{a \in \mathbb{N}^n} \mathbb{K} \cdot x^a\,,$$

where $x^a = x_1^{a_1} \cdots x_n^{a_n}$ for $a = (a_1, \ldots, a_n)$. The Stanley-Reisner ideal I_Δ is an $\mathbb{N}^n$-graded ideal because it is generated by monomials. Let us say that a monomial x^a has *support* $\sigma \subseteq \{1, \ldots, n\}$ if $a_j \neq 0$ precisely for $j \in \sigma$. By Definition 16.21, then, the monomials that remain nonzero in the quotient $\mathbb{K}[\Delta]$ are precisely those supported on faces of Δ.

Proposition 16.23. *The h-polynomial of a simplicial complex Δ of dimension $d-1$ equals the numerator of the Hilbert-Poincaré series of its Stanley-Reisner ring $\mathbb{K}[\Delta]$, that is to say*

$$P(\mathbb{K}[\Delta], t) = \frac{h_0(\Delta) + h_1(\Delta)t + h_2(\Delta)t^2 + \cdots + h_d(\Delta)t^d}{(1-t)^d}\,.$$

Proof. The $\mathbb{N}^n$-graded Hilbert-Poincaré series, which, by definition, is the sum of all nonzero monomials in $\mathbb{K}[\Delta]$, is

$$P(\mathbb{K}[\Delta], x) = \sum_{\sigma \in \Delta} x^\sigma \prod_{j \in \sigma} \frac{1}{1 - x_j}\,.$$

The summand indexed by σ is simply the sum of all monomials with support exactly σ. Now specialize $x_j = t$ for all j to get the Hilbert-Poincaré series

$$\begin{aligned} P(\mathbb{K}[\Delta], t) &= \sum_{\sigma \in \Delta} t^{|\sigma|} \cdot \frac{1}{(1-t)^{|\sigma|}} \\ &= \sum_{i=0}^{d} f_{i-1}(\Delta) \frac{t^i}{(1-t)^i} \\ &= \frac{1}{(1-t)^d} \sum_{i=0}^{d} f_{i-1}(\Delta) t^i (1-t)^{d-i}, \end{aligned}$$

whose numerator is the h-polynomial of Δ by definition. □

Example 16.24. It is now easy for us, as commutative algebraists, to compute the h-vector of the octahedron Δ: the Stanley-Reisner ideal I_Δ is a complete intersection generated by three quadrics, so the Hilbert-Poincaré series of $\mathbb{K}[\Delta]$ is

$$P(\mathbb{K}[\Delta], t) = \frac{(1-t^2)^3}{(1-t)^6} = \frac{(1+t)^3}{(1-t)^3}\,.$$

The h-polynomial of Δ is $1 + 3t + 3t^2 + t^3$, and its h-vector is $(1, 3, 3, 1)$.

Exercise 16.25. Let Δ be a simplicial complex with Stanley-Reisner ideal

$$I_\Delta = (de, abe, ace, abcd)\,.$$

Find the Betti numbers and Hilbert-Poincaré series of $\mathbb{K}[\Delta]$.

5. Local cohomology of Stanley-Reisner rings

The salient features of the h-vector in Example 16.24 are (1) every h_i is positive, and (2) the h-vector is symmetric. These combinatorial observations are attributable to the facts that (1) $\mathbb{K}[\Delta]$ is Cohen-Macaulay, and (2) $\mathbb{K}[\Delta]$ is, moreover, Gorenstein. In this section we explain how the proofs of these statements go, although a detail here and there is too nitpicky—and ubiquitous in the literature—to warrant including here. Our exposition in this section follows [**118**, §13.2], sometimes nearly verbatim.

The key point is a formula, usually attributed as an unpublished result of Hochster, for the local cohomology of Stanley-Reisner rings. More precisely, it is a formula for the $\mathbb{Z}^n$-graded Hilbert-Poincaré series of local cohomology. To state it, we need an elementary notion from simplicial topology.

Definition 16.26. The *link* of a face σ inside the simplicial complex Δ is

$$\mathrm{link}_\sigma(\Delta) = \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta \text{ and } \tau \cap \sigma = \varnothing\},$$

i.e., it is the set of faces disjoint from σ but whose unions with σ lie in Δ.

For the reader who has not seen links before, the precise definition is not so important at this stage; one can think of it simply as a way to associate a subcomplex of Δ to each face of Δ. The link records how Δ behaves near σ; see the proof of Theorem 16.28 for the relevant property of links.

Theorem 16.27 (Hochster's formula)**.** *Let $\mathbb{K}[\Delta] = \mathbb{K}[x_1, \ldots, x_n]/I_\Delta$ be a Stanley-Reisner ring with homogeneous maximal ideal $\mathfrak{m}$. Then the $\mathbb{Z}^n$-graded Hilbert-Poincaré series of the local cohomology $H^i_\mathfrak{m}(\mathbb{K}[\Delta])$ is*

$$P(H^i_\mathfrak{m}(\mathbb{K}[\Delta]), x) = \sum_{\sigma \in \Delta} \mathrm{rank}_\mathbb{K}\, \widetilde{H}^{i-|\sigma|-1}(\mathrm{link}_\sigma(\Delta), \mathbb{K}) \prod_{j \in \sigma} \frac{x_j^{-1}}{1 - x_j^{-1}}\,,$$

where $\widetilde{H}$ denotes reduced simplicial cohomology, and $|\sigma| = \dim(\sigma) + 1$. □

Let us parse the statement. The product over $j \in \sigma$ is the sum of all Laurent monomials whose exponent vectors are nonpositive and have support exactly σ. Therefore, the formula for the Hilbert-Poincaré series of $H^i_{\mathfrak{m}}(\mathbb{K}[\Delta])$ is just like the one for $\mathbb{K}[\Delta]$ itself in the first displayed equation in the proof of Proposition 16.23, except that here we consider monomials with negative exponents and additionally take into account the nonnegative coefficients $\operatorname{rank}_{\mathbb{K}} \widetilde{H}^{i-|\sigma|-1}(\operatorname{link}_\sigma(\Delta), \mathbb{K})$ depending on i and σ.

The proof of Theorem 16.27 is carried out one degree at a time: the Čech complex of $\mathbb{K}[\Delta]$ in each fixed $\mathbb{Z}^n$-graded degree is (essentially) the cochain complex for the desired link. Details of the proof of Hochster's formula can be found in [**20**, Chapter 5], [**118**, Chapter 13], or [**151**, Chapter II].

Theorem 16.28. *Let Δ be a simplicial sphere. Then $\mathbb{K}[\Delta]$ is a Gorenstein ring. In fact, $\mathbb{K}[\Delta]$ is its own canonical module, even under the $\mathbb{Z}^n$-grading.*

Proof. Set $d = \dim(\Delta) + 1$. First of all, we need that $\mathbb{K}[\Delta]$ is Cohen-Macaulay. By Theorems 10.36 and 16.27, the ring $\mathbb{K}[\Delta]$ is Cohen-Macaulay if and only if one has

$$\widetilde{H}^{i-|\sigma|-1}(\operatorname{link}_\sigma(\Delta), \mathbb{K}) = 0 \qquad \text{for } i \neq d\,.$$

As Δ is a sphere, the link of each face σ is a homology sphere of dimension $d - |\sigma| - 1$, [**121**, Theorem 62.3]. This means, by definition of homology sphere, that the homology of the link is isomorphic to that of a sphere of dimension $d - |\sigma| - 1$. Since the only nonvanishing reduced cohomology of a sphere is at the top index, the desired vanishing holds.

Since $\operatorname{link}_\sigma(\Delta)$ is a homology sphere, the sole nonvanishing reduced cohomology group is

$$\widetilde{H}^{d-|\sigma|-1}(\operatorname{link}_\sigma(\Delta), \mathbb{K}) \cong \mathbb{K}\,.$$

Hence the top local cohomology has $\mathbb{Z}^n$-graded Hilbert-Poincaré series

$$P(H^d_{\mathfrak{m}}(\mathbb{K}[\Delta]), x) = \sum_{\sigma \in \Delta} \prod_{j \in \sigma} \frac{x_j^{-1}}{1 - x_j^{-1}}\,.$$

Taking the Matlis dual of $H^d_{\mathfrak{m}}(\mathbb{K}[\Delta])$ results in the replacement $x_j^{-1} \rightsquigarrow x_j$ at the level of Hilbert-Poincaré series. Therefore $H^d_{\mathfrak{m}}(\mathbb{K}[\Delta])^\vee$, which is the canonical module $\omega_{\mathbb{K}[\Delta]}$, has the same $\mathbb{Z}^n$-graded Hilbert-Poincaré series as $\mathbb{K}[\Delta]$; see the proof of Proposition 16.23. In particular, $\omega_{\mathbb{K}[\Delta]}$ has a $\mathbb{Z}^n$-graded degree zero generator that is unique up to scaling. Since $\omega_{\mathbb{K}[\Delta]}$ is a faithful $\mathbb{K}[\Delta]$-module [**20**, Proposition 3.3.11], the $\mathbb{K}[\Delta]$-module map sending $1 \in \mathbb{K}[\Delta]$ to a $\mathbb{Z}^n$-graded degree zero generator of $\omega_{\mathbb{K}[\Delta]}$ is injective on each $\mathbb{Z}^n$-degree, and hence $\mathbb{K}[\Delta] \longrightarrow \omega_{\mathbb{K}[\Delta]}$ is an isomorphism. $\square$

6. Proof of the upper bound theorem

The key point about the Cohen-Macaulay condition in this combinatorial setting is that it implies positivity of the h-vector: each h_i counts the vector space dimension of a $\mathbb{Z}$-graded piece of a finite-dimensional $\mathbb{K}$-algebra.

Proposition 16.29. *Let M be a finitely generated $\mathbb{Z}$-graded module over $\mathbb{K}[x_1, \ldots, x_n]$, where $\deg x_j = 1$. If M is Cohen-Macaulay and $\boldsymbol{y}$ is a linear system of parameters for M, then the numerator of $P(M, t)$ is $P(M/\boldsymbol{y}M, t)$.*

Proof. If $y \in \mathbb{K}[x_1, \ldots, x_n]$ is a linear nonzerodivisor on M, then the exact sequence arising from multiplication by y implies

$$P(M/yM, t) = P(M, t) - P(M(-1), t) = (1 - t)P(M, t)\,.$$

Use this inductively for $\boldsymbol{y}$. □

Remark 16.30. Stanley observed that the converse holds as well: if the numerator of $P(M, t)$ equals $P(M/\boldsymbol{y}M, t)$ for a linear system of parameters $\boldsymbol{y}$, then M is Cohen-Macaulay; see [**118**, Theorem 13.37.6].

Corollary 16.31. *Let R be a standard graded Gorenstein ring. If the numerator of $P(R, t)$ is $\sum_{i=0}^{r} h_i t^i$ with $h_r \neq 0$, then $h_i = h_{r-i}$ for each i.*

Proof. By Proposition 16.29, the numerator of $P(R, t)$ equals the Hilbert-Poincaré series of a finite-dimensional $\mathbb{K}$-algebra that is isomorphic to its own $\mathbb{K}$-vector space dual; see Exercises 18.4 and 18.5. □

Definition 16.32. When the ring in Corollary 16.31 is the Stanley-Reisner ring of a dimension $d - 1$ simplicial sphere, the relations $h_i = h_{d-i}$, which hold for each i by Theorem 16.28, are the *Dehn-Sommerville equations*.

The Dehn-Sommerville equations are extraordinarily simple when viewed in terms of h-vectors, but they represent quite subtle conditions on f-vectors. Note that if Δ has dimension $d-1$, then indeed $h_d(\Delta)$ is the highest nonzero entry of the h-vector, by Proposition 16.23.

Corollary 16.33. *Let Δ be a dimension $d - 1$ simplicial complex with n vertices such that $\mathbb{K}[\Delta]$ is Cohen-Macaulay. Then*

$$h_i(\Delta) \leqslant \binom{n - d - 1 + i}{i} \qquad \text{for each } i\,.$$

Proof. Fix a linear system of parameters $\boldsymbol{y}$ in the Stanley-Reisner ring $\mathbb{K}[\Delta]$. Then $h_i(\Delta)$ equals the vector space dimension of the degree i graded component of $\mathbb{K}[\Delta]/(\boldsymbol{y})$ by Proposition 16.23 and Proposition 16.29. Extending $\boldsymbol{y}$ to a basis of the linear polynomials $\mathbb{K}[x_1, \ldots, x_n]_1$, we see that $\mathbb{K}[\Delta]/(\boldsymbol{y})$ is isomorphic to a quotient of a polynomial ring in $n-d$ variables. Therefore the vector space dimension of the degree i graded component of $\mathbb{K}[\Delta]/(\boldsymbol{y})$ is at most $\binom{n-d-1+i}{i}$, the number of degree i monomials in $n-d$ variables. $\square$

Exercise 16.34. Consider any neighborly simplicial polytope P of dimension d with n vertices; for example, the cyclic polytope $C(n,d)$ satisfies this hypothesis. Using the argument of Corollary 16.33, prove that

$$h_i(C(n,d)) = \binom{n-d-1+i}{i} \qquad \text{for } i = 0, \ldots, \left\lfloor \frac{d}{2} \right\rfloor.$$

At this point, we are now equipped to complete the proof of the upper bound theorem for simplicial spheres.

Proof of Theorem 16.20. Assume that a given simplicial sphere Δ has dimension $d-1$ and n vertices. We need only show that $h_i(\Delta) \leqslant h_i(C(n,d))$ for $i = 0, \ldots, \lfloor \frac{d}{2} \rfloor$, because the result for the remaining values of i will then hold by the Dehn-Sommerville equations. Using the explicit computation of $h_i(C(n,d))$ in Exercise 16.34 for the desired values of i, the upper bound theorem follows from Corollary 16.33. $\square$

In retrospect, what has happened here?

(1) An explicit local cohomology computation allowed us to conclude that $\mathbb{K}[\Delta]$ is Gorenstein when Δ is a simplicial sphere.

(2) The Cohen-Macaulay property for $\mathbb{K}[\Delta]$ allowed us to write the h-polynomial as the Hilbert-Poincaré series of a finite-dimensional algebra, and to bound h_i by certain binomial coefficients.

(3) The Gorenstein property reduced the upper bound theorem to checking only half of the h-vector.

(4) Finally, an explicit computation for neighborly simplicial polytopes demonstrated that equality with the binomial coefficients indeed occurs for cyclic polytopes, at least on the first half of the h-vector.

Note that equality with the binomial coefficients does not occur—even for cyclic polytopes—on the second half of the h-vector: although $h_i = h_{d-i}$, the same symmetry does not occur with the binomial coefficients, which keep growing with i.

Lecture 17

D-modules

We introduce rings of differential operators and modules over these rings, i.e., D-modules. The focus is on Weyl algebras, with a view towards applications to local cohomology in Lecture 23.

1. Rings of differential operators

Let $\mathbb{K}$ be a field and R a commutative $\mathbb{K}$-algebra.

The composition of elements P, Q of $\mathrm{Hom}_{\mathbb{K}}(R,R)$ is denoted $P \cdot Q$. With this product, $\mathrm{Hom}_{\mathbb{K}}(R,R)$ is a ring; it is even a $\mathbb{K}$-algebra since each P is $\mathbb{K}$-linear. The *commutator* of P and Q is the element

$$[P,Q] = P \cdot Q - Q \cdot P\,.$$

Since R is commutative, the map $r \longmapsto (s \longmapsto rs)$ gives an embedding of $\mathbb{K}$-algebras: $R \subseteq \mathrm{Hom}_{\mathbb{K}}(R,R)$. Note that the natural left-module structure of R over $\mathrm{Hom}_{\mathbb{K}}(R,R)$ extends the one of R over itself.

Definition 17.1. Set $D_0(R;\mathbb{K}) = R$ viewed as a subring of $\mathrm{Hom}_{\mathbb{K}}(R,R)$, and for each $i \geqslant 0$, let

$$D_{i+1}(R;\mathbb{K}) = \{P \in \mathrm{Hom}_{\mathbb{K}}(R,R) \mid [P,r] \in D_i(R;\mathbb{K}) \text{ for each } r \in R\}\,.$$

It is easy to verify that if P is in $D_i(R;\mathbb{K})$ and Q is in $D_j(R;\mathbb{K})$, then $P \cdot Q$ is in $D_{i+j}(R;\mathbb{K})$. Thus, one obtains a $\mathbb{K}$-subalgebra of $\mathrm{Hom}_{\mathbb{K}}(R,R)$,

$$D(R;\mathbb{K}) = \bigcup_{i \geqslant 0} D_i(R;\mathbb{K})\,.$$

This is the ring of $\mathbb{K}$-linear *differential operators* on R. Elements of $D_i(R;\mathbb{K})$ are said to have *order* i. Note that $D_1(R;\mathbb{K})$ is the $\mathbb{K}$-span of R and the *derivations*, i.e., the maps δ with $\delta(rs) = \delta(r)s + r\delta(s)$ for $r, s \in R$.

The *tensor algebra* in $y_1, \ldots, y_m$ over $\mathbb{K}$ is denoted $\mathbb{K}\langle y_1, \ldots, y_m\rangle$.

Example 17.2. Let $\mathbb{K}$ be a field of characteristic zero. Set $R = \mathbb{K}[x]$ and $\partial = \partial/\partial x$, the derivative with respect to x; it is a derivation on R. The ring $D(R;\mathbb{K})$ is $\mathbb{K}\langle x, \partial\rangle$ modulo the two-sided ideal generated by $\partial \cdot x - x \cdot \partial - 1$.

As the example above shows, $D(R;\mathbb{K})$ need not be commutative.

Exercise 17.3. Find the rings of $\mathbb{C}$-linear differential operators on $\mathbb{C}[[x]]$ and on $\mathbb{C}[x,y]/(xy)$.

Definition 17.4. Set $R = \mathbb{K}[x_1, \ldots, x_n]$, a polynomial ring in $x_1, \ldots, x_n$. The n-th *Weyl algebra* over $\mathbb{K}$ is the ring

$$D_n(\mathbb{K}) = \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle/\mathfrak{a}\,,$$

where $\mathfrak{a}$ is the two-sided ideal generated by the elements

$$x_i \cdot x_j - x_j \cdot x_i, \qquad \partial_i \cdot x_j - x_j \cdot \partial_i - \delta_{i,j}, \qquad \partial_i \cdot \partial_j - \partial_j \cdot \partial_i, \tag{17.4.1}$$

with $\delta_{i,j}$ the Kronecker delta. Observe that R is a subring of $D_n(\mathbb{K})$.

Viewing the element $\partial_i \in D_n(\mathbb{K})$ as partial differentiation with respect to x_i, one can realize $D_n(\mathbb{K})$ as a subring of $D(R;\mathbb{K})$. The derivations are the R-linear combinations of $\partial_1, \ldots, \partial_n$. If $\mathbb{K}$ has characteristic zero, then $D_n(\mathbb{K})$ equals $D(R;\mathbb{K})$; see [**54**, Theorem 16.11.2] or [**25**, Theorem 2.3].

Exercise 17.5. When $\mathbb{K}$ is of positive characteristic, say p, show that $D(R;\mathbb{K})$ is bigger than the Weyl algebra, $D_n(\mathbb{K})$.

Hint: Consider *divided powers* $\dfrac{\partial^p}{p!\,\partial x_i^p}$.

Exercise 17.6. Prove that the only two-sided ideals of the Weyl algebra $D_n(\mathbb{K})$ are 0 and $D_n(\mathbb{K})$ itself; i.e., $D_n(\mathbb{K})$ is simple.

A theorem of Stafford [**149**] implies that every ideal of $D_n(\mathbb{K})$ can be generated by two elements; see also [**12**, §1.7]. Algorithms for obtaining the two generators may be found in [**66, 99**].

Remark 17.7. Let $\mathfrak{a}$ be an ideal in a commutative $\mathbb{K}$-algebra R, and set $S = R/\mathfrak{a}$. One can identify $D(S;\mathbb{K})$ with the subring of $D(R;\mathbb{K})$ consisting of operators that stabilize $\mathfrak{a}$, modulo the ideal generated by $\mathfrak{a}$.

For example, for $S = \mathbb{C}[x,y]/(xy)$, the ring $D(S;\mathbb{C})$ is the S-algebra generated by $x\partial_x^n$, $y\partial_y^n$ for $n \in \mathbb{N}$, modulo the ideal generated by xy.

Exercise 17.8. Let $S = \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$. Prove that $D(S;\mathbb{C}) = S\langle \partial_s, \partial_t\rangle$.

Let $R = \mathbb{C}[s, st, st^2]$, which is the semigroup ring for the semigroup generated by the columns of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

The ring $D(R;\mathbb{C})$ turns out to be the R-subalgebra of $D(S;\mathbb{C})$ generated by

$$I_{(p,q)}(\theta_s,\theta_t)\cdot s^p t^q, \qquad p,q\in\mathbb{Z},$$

where $\theta_s = s\cdot\partial_s$, $\theta_t = t\cdot\partial_t$ and $I_{(p,q)}(\alpha,\beta)$ is the defining ideal of the Zariski closure of the set of points $((p,q)+\mathbb{N}A)\smallsetminus\mathbb{N}A$ in $\operatorname{Spec}\mathbb{C}[\alpha,\beta]$. Check that these operators are indeed in $D(R;\mathbb{C})$.

For example, if $p=q=-1$, then

$$((p,q)+\mathbb{N}A)\smallsetminus\mathbb{N}A = \{(2k-1,k-1)\}_{k\geqslant 1}\cup\{(-1,k)\}_{k\geqslant -1}\,,$$

so that $I_{(-1,-1)}(\theta_s,\theta_t) = (\theta_t+1)(2\theta_2-\theta_t+1)$.

This exercise is a special case of a result about differential operators on affine semigroup rings; see [**137**, Theorem 2.1].

Exercise 17.9. Let f be an element in R. Use the order filtration on $D(R;\mathbb{K})$, as in Definition 17.1, to prove that each $P\in D(R;\mathbb{K})$ induces a differential operator on R_f. Thus, $R_f\otimes_R D(R;\mathbb{K})$ is a subset of $D(R_f;\mathbb{K})$.

Exercise 17.10. Check that $R_f\otimes_R D(R;\mathbb{K})$ is a ring. Prove that if R is a finitely generated $\mathbb{K}$-algebra, then $R_f\otimes_R D(R;\mathbb{K}) = D(R_f;\mathbb{K})$.

Hint: Each $P\in D(R_f;\mathbb{K})$ maps R to R_f. Since R is finitely generated, the image of P lies in $(1/f^k)R$ for some k. Use induction on the order of P.

Exercise 17.11. Show that $D(\mathbb{C}[x];\mathbb{C})$ has proper left ideals that are not principal. In contrast, every left ideal of $D(\mathbb{C}(x);\mathbb{C})$ is principal.

Remark 17.12. Let $\mathbb{K}$ be a field of characteristic zero, and R a domain finitely generated over $\mathbb{K}$. When R is regular, [**54**, Theorem 16.11.2] implies that $D(R;\mathbb{K})$ is the $\mathbb{K}$-algebra generated by R and the derivations. The converse is *Nakai's conjecture.*

2. The Weyl algebra

For the rest of this lecture, the field $\mathbb{K}$ is of characteristic zero, R is a polynomial ring over $\mathbb{K}$ in $\boldsymbol{x} = x_1,\dots,x_n$, and $D = D(R;\mathbb{K})$ is the Weyl algebra. We use $\bullet$ to denote the action of D on R; it is defined by

$$\partial_i\bullet f = \frac{\partial f}{\partial x_i}, \qquad x_i\bullet f = x_i f \qquad \text{for } f\in R\,.$$

For example, $\partial_i\bullet x_i = 1$ while $(\partial_i\cdot x_i)(f) = f + x_i\partial_i(f)$ for $f\in R$.

We study filtrations and associated graded rings on D.

Definition 17.13. The commutator relations (17.4.1) imply that each element of D can be written uniquely as

$$\sum_{\alpha,\beta\in\mathbb{N}^n} c_{\alpha,\beta}\boldsymbol{x}^\alpha\boldsymbol{\partial}^\beta,$$

where all but finitely many $c_{\alpha,\beta} \in \mathbb{K}$ are zero. The monomials $\boldsymbol{x}^\alpha \boldsymbol{\partial}^\beta$ form a $\mathbb{K}$-basis for D, which is the *Poincaré-Birkhoff-Witt basis* or *PBW basis.*

This basis can be used to solve Exercise 17.6. In what follows, we use it to define filtrations on D; the discussion involves notions from Lecture 5.

Definition 17.14. A *weight* on D is a pair of vectors ω_x and ω_∂ in $\mathbb{Z}^n$, denoting weights on $\boldsymbol{x}$ and $\boldsymbol{\partial}$, respectively, such that $\omega_x + \omega_\partial \geqslant 0$ componentwise. Given such an $\omega = (\omega_x, \omega_\partial)$, consider the subspaces of D

$$F_t = \mathbb{K} \cdot \{\boldsymbol{x}^\alpha \boldsymbol{\partial}^\beta \mid \langle \omega, (\alpha, \beta) \rangle \leqslant t\}\,.$$

The nonnegativity condition implies that no term in the expansion of $\boldsymbol{\partial}^\beta \boldsymbol{x}^\alpha$ in the PBW basis has weight larger than $\langle \omega, (\alpha, \beta)\rangle$. Check that $\bigcup_t F_t = D$ and that $F_s \cdot F_t \subseteq F_{s+t}$.

Strictly speaking, F is not a filtration in the sense of Definition 5.4, since F_t may be nonzero for $t < 0$, as in (3) below. This does not happen if the weight ω is nonnegative, as will be the case for most of this lecture.

(1) $\omega = (\mathbf{0}, \mathbf{1}) = (0, \dots, 0, 1, \dots, 1)$ yields the *order filtration*: operators in $F_t = \mathbb{K} \cdot \{\boldsymbol{x}^\alpha \boldsymbol{\partial}^\beta \mid \sum_{j=1}^n \beta_j \leqslant t\}$ are precisely those with order $\leqslant t$.

(2) $\omega = (\mathbf{1}, \mathbf{1}) = (1, \dots, 1, 1, \dots, 1)$ produces the *Bernstein filtration*, for which $F_t = \mathbb{K} \cdot \{\boldsymbol{x}^\alpha \boldsymbol{\partial}^\beta \mid \sum_{j=1}^n (\alpha_j + \beta_j) \leqslant t\}$.

(3) $\omega = (-1, 0, \dots, 0, 1, 0, \dots, 0)$ yields the *V-filtration along $x_1 = 0$*, where $F_t = \mathbb{K} \cdot \{\boldsymbol{x}^\alpha \boldsymbol{\partial}^\beta \mid \beta_1 - \alpha_1 \leqslant t\}$.

Remark 17.15. Let ω be a weight vector on D so that F, as in Definition 17.14, is a filtration. Recall that $D = T/\mathfrak{a}$, where T is the tensor algebra on $2n$ variables, and $\mathfrak{a}$ is the ideal generated by elements in (17.4.1).

The weight ω gives a filtration on T with the t-th level the linear combinations of all words that have ω-weight at most t. Then $\operatorname{gr}_\omega(D)$ is the quotient of $\operatorname{gr}_\omega(T)$ by $\operatorname{gr}_\omega(\mathfrak{a})$. Note that $\operatorname{gr}_\omega(\mathfrak{a})$ is generated by

$$\big\{[x_i, x_j]\big\}, \quad \big\{[x_i, \partial_j] \mid i \neq j\big\}, \quad \big\{[\partial_i, \partial_j]\big\},$$
$$\big\{[x_i, \partial_i] \mid \omega_{x_i} + \omega_{\partial_i} > 0\big\}, \quad \text{and} \quad \big\{[x_i, \partial_i] + 1 \mid \omega_{x_i} + \omega_{\partial_i} = 0\big\}.$$

It follows that the PBW basis on D gives one on $\operatorname{gr}_\omega(D)$.

We focus on the Bernstein filtration B.

Exercise 17.16. Prove that $\operatorname{gr}_B(D) = \mathbb{K}[\boldsymbol{x}, \boldsymbol{\partial}]$.

Proposition 17.17. *The Weyl algebra D is left and right Noetherian.*

Proof. Let $\mathfrak{b}$ be a proper left ideal of D. The Bernstein filtration B on D induces an increasing filtration on $\mathfrak{b}$ where

$$\mathfrak{b}_t = \mathfrak{b} \cap B_t \qquad \text{for } t \geqslant 0\,.$$

Note that $\mathfrak{b}_0 = 0$ and that $\bigcup_t \mathfrak{b}_t = \mathfrak{b}$. It is easily checked that $\mathrm{gr}(\mathfrak{b})$ is an ideal of $\mathrm{gr}(D)$. Since $\mathrm{gr}(D)$ is Noetherian by Exercise 17.16, there exist finitely many elements b_i in $\mathfrak{b}$ such that their images generate the ideal $\mathrm{gr}(\mathfrak{b})$. We claim that $\mathfrak{b} = \sum_i Db_i$, and hence that $\mathfrak{b}$ is finitely generated.

Indeed, if not, then let t be the least integer for which there exists an element b in $\mathfrak{b}_t \smallsetminus \sum_i Db_i$. By the choice of b_i, there exist elements d_i in D such that $b = \sum_i d_i b_i \bmod \mathfrak{b}_{t-1}$. But then

$$b \;=\; (b - \sum_i d_i b_i) + \sum_i d_i b_i \;\in\; \mathfrak{b}_{t-1} + \sum_i Db_i \;\subseteq\; \sum_i Db_i\,,$$

where the inclusion holds by choice of t. This is a contradiction.

At this point we know that D is left Noetherian. The proof that it is right Noetherian is similar. □

Definition 17.18. Let M be a finitely generated D-module. The Bernstein filtration B on D induces a filtration, say G, on M as in Exercise 5.15, and the $\mathrm{gr}_B(D)$-module $\mathrm{gr}_G(M)$ is Noetherian.

We write $\dim_B M$ and $e_B(M)$ for the *dimension* and *multiplicity* of M with respect to B; see Definition 5.19.

Example 17.19. One has $\dim_B D = 2n$ and $e_B(D) = 1$ since $\mathrm{gr}_B(D)$ is a polynomial ring in $2n$ variables.

One might expect the dimension of D-modules to take all values between 0 and $2n$. The next result, due to Bernstein, thus comes as a surprise.

Theorem 17.20. *If M is a finitely generated nonzero D-module, then one has $n \leqslant \dim_B M \leqslant 2n$.*

Proof. Let B be the Bernstein filtration on D, and G an induced filtration on M. We claim that the map

$$B_t \longrightarrow \mathrm{Hom}_{\mathbb{K}}(G_t, G_{2t}) \qquad \text{where } P \longmapsto (u \longmapsto Pu)$$

is injective. Indeed, this is trivially true for $t \leqslant 0$. Suppose $P \in B_t$ is nonzero and the claim holds for smaller t. If x_i, respectively, ∂_i, occurs in a term of P, then it is not hard to verify using the PBW basis that $[P, \partial_i]$, respectively, $[P, x_i]$, is in B_{t-1} and nonzero. Since the corresponding term does not annihilate G_{t-1}, one deduces that P cannot annihilate G_t. This settles the claim. It then follows that

$$\mathrm{rank}_{\mathbb{K}}(B_t) \leqslant \mathrm{rank}_{\mathbb{K}} \mathrm{Hom}_{\mathbb{K}}(G_t, G_{2t}) = (\mathrm{rank}_{\mathbb{K}} G_t)(\mathrm{rank}_{\mathbb{K}} G_{2t})\,.$$

Therefore $\dim_B D \leqslant 2 \dim_B M$; now recall Example 17.19. □

Exercise 17.21. Let B be the Bernstein filtration on D, and P an element in B_t. Prove the following statements.

(1) $[P, x_i]$ and $[P, \partial_i]$ are in B_{t-1}.
(2) $[P, x_i] \neq 0$, respectively, $[P, \partial_i] \neq 0$, if and only if ∂_i, respectively, x_i, occurs in the representation of P in terms of the PBW basis.

Extend this result to a general commutator $[P, Q]$.

We record some statements about general filtrations.

Remark 17.22. Let ω be a nonnegative weight on D. A spectral sequence argument as in [**13**, A:IV] shows that

$$\dim_\omega M = 2n - g(M) \qquad \text{for } g(M) = \min\{i \mid \operatorname{Ext}^i_D(M, D) \neq 0\}\,.$$

Thus, the dimension is independent of the nonnegative filtration. For example, the B-dimension and the dimension induced by the order filtration coincide. For the order filtration the analogue of Theorem 17.20 is the *weak fundamental theorem of algebraic analysis*; see [**11**] or [**25**].

Let ω be a weight with $\omega_x + \omega_\partial > 0$ componentwise. In this case, $\operatorname{gr}_\omega(D) = \mathbb{K}[\boldsymbol{x}, \boldsymbol{\partial}]$. Let M be a finitely generated D-module. Theorem 17.20 remains true, with the caveat that there may be nonzero modules M with $\operatorname{gr}_\omega(M) = 0$; consider $n = 1$, $M = D/D(x-1)$ and $\omega = (-1, 2)$. See [**148**].

The annihilator of the $\operatorname{gr}_\omega(D)$-module $\operatorname{gr}_\omega(M)$ is the *characteristic ideal* of M, with respect to ω. The associated subvariety of $\mathbb{A}^{2n}_{\mathbb{K}}$ is the *characteristic variety* of M. The weak fundamental theorem says that the characteristic variety with respect to the order filtration has dimension at least n. A stronger result is proved in [**135**]:

Theorem 17.23. *Let M be a nonzero finitely generated D-module. Each component V of its characteristic variety with respect to the order filtration satisfies $n \leqslant \dim V \leqslant 2n$.* □

3. Holonomic modules

Recall that R is a polynomial ring in n variables over a field of characteristic zero, and that D is the corresponding Weyl algebra.

We single out a class of D-modules which enjoy special properties:

Definition 17.24. A finitely generated D-module M is *holonomic* when $\dim_B M \leqslant n$; equivalently, $\dim_B M = n$ or $M = 0$.

It follows from Remark 17.22 that a finitely generated D-module M is holonomic if and only if $\operatorname{Ext}^i_D(M, D) = 0$ for $i < n$.

Exercise 17.25. Set $n = 1$. Prove that D/I is holonomic when $I \neq 0$.

Exercise 17.26. Prove that the D-module R is holonomic.

Exercise 17.27. Let $\mathfrak{m} = (\boldsymbol{x})$ in R. Verify that the image of $1/(x_1 \cdots x_n)$ generates $H^n_{\mathfrak{m}}(R)$ as a D-module. Prove that $H^n_{\mathfrak{m}}(R)$ is holonomic, and that it is isomorphic to $D/D\boldsymbol{x}$.

Remark 17.28. Consider an exact sequence of D-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0\,.$$

Since D is Noetherian, M is finitely generated if and only if M' and M'' are finitely generated. When this holds, it follows from Exercise 5.20 and Theorem 17.20 that M is holonomic if and only if M' and M'' are holonomic, and also that, in this case, one has an equality

$$e_B(M) = e_B(M') + e_B(M'')\,.$$

The definition of a holonomic module does not prepare us for the following remarkable result. Here B is the Bernstein filtration.

Theorem 17.29. *If M is a holonomic D-module, $\ell_D(M) \leqslant e_B(M) < \infty$.*

Proof. Let $0 \subset M_1 \subset \cdots \subset M_{l-1} \subset M_l = M$ be a strictly increasing filtration by D-submodules. It follows from Remark 17.28 that each M_i is holonomic. Moreover, the exact sequences of D-modules

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0$$

imply that $e_B(M_1) < e_B(M_2) < \cdots < e_B(M)$, since $M_i/M_{i-1} \neq 0$. Therefore one obtains $l \leqslant e_B(M)$. This implies the desired result. $\square$

Another noteworthy property of holonomic modules is that they are cyclic; see [**12**, Theorem 1.8.18].

Exercise 17.30. Suppose $n = 1$. The D-modules $R \oplus R$ and $R \oplus D/Dx$ are holonomic. Verify that they are cyclic.

4. Gröbner bases

Gröbner basis techniques can be transplanted to Weyl algebras. To begin with, fix a weight filtration on D. A Gröbner basis for an ideal $\mathfrak{a}$ is a system of generators for $\mathfrak{a}$ whose images in $\operatorname{gr}(D)$ generate $\operatorname{gr}(\mathfrak{a})$.

Exercise 17.31. Let ω be a weight vector that is componentwise positive, and let $\leqslant$ be the induced weight order. Show that $\leqslant$ can be refined to a term order $\leqslant_{\text{lex}}$ using a lexicographic order on the variables.

Show that if G is a $\leqslant_{\text{lex}}$-Gröbner basis for an ideal $\mathfrak{a}$, then it is also a Gröbner basis with respect to $\leqslant$.

For $\leqslant$ a term order, Buchberger's algorithm applied to a generating system for $\mathfrak{a}$ yields a Gröbner basis. If ω does not satisfy $\omega \geqslant 0$ componentwise, as in the case of the V-filtration, it cannot be refined to a term order. An approach to constructing Gröbner bases for such orders is outlined below.

Definition 17.32. The *homogenized Weyl algebra*, denoted $D^{(h)}$, is the ring $\mathbb{K}[h]\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ modulo the two-sided ideal generated by

$$x_i \cdot x_j - x_j \cdot x_i\,, \qquad \partial_i \cdot x_j - x_j \cdot \partial_i - \delta_{i,j} h^2\,, \qquad \partial_i \cdot \partial_j - \partial_j \cdot \partial_i\,.$$

The *dehomogenization map* $D^{(h)} \longrightarrow D$ is the surjective homomorphism induced by $h \longmapsto 1$.

If a term order $\prec$ on monomials of $D^{(h)}$ is such that $h^2 \prec x_i\partial_i$ for all i, then Algorithm 5.46 for the normal form terminates provided that G consists of homogeneous elements in $D^{(h)}$. Thus, Buchberger's algorithm terminates on homogeneous inputs in $D^{(h)}$; see [**136**, Proposition 1.2.2]. It remains to reduce Gröbner computations in D to homogeneous computations in $D^{(h)}$.

Exercise 17.33. Starting with the weight ω for D with $\omega_x + \omega_\partial \geqslant 0$ componentwise, let $t \geqslant 0$ be such that $2t \leqslant \omega_{x_i} + \omega_{\partial_i}$ for each i. Show that the weight (t, ω) for $D^{(h)}$ defines an order which, when refined with any term order for which $h < x_i$ and $h < \partial_i$ for each i, is an order $\prec$ as discussed in the previous paragraph.

Let $\mathfrak{a} = (P_1, \ldots, P_k)$ be an ideal in D. Show that the dehomogenization of a $\prec$-Gröbner basis for $P_1^{(h)}, \ldots, P_k^{(h)}$ gives an ω-Gröbner basis for $\mathfrak{a}$.

Macaulay 2 has a package [**98**] devoted to D-modules.

Local Duality Revisited

The aim of this lecture is to explain the 'duality' in Grothendieck's duality theorem by describing its connection to the classical Poincaré duality theorem for manifolds. We also offer another perspective on the local duality theorem which clarifies its relationship with Serre duality for sheaves on projective spaces. The last item is a discussion of global canonical modules, which are required to state and prove Serre duality.

1. Poincaré duality

Definition 18.1. Let $\mathbb{K}$ be a field and $R = \bigoplus_{n=0}^{d} R_n$ a graded $\mathbb{K}$-algebra with $\operatorname{rank}_{\mathbb{K}} R$ finite, $R_0 = \mathbb{K}$ and $R_d \neq 0$. The product on R provides, for each $0 \leqslant n \leqslant d$, a bilinear pairing

$$R_n \times R_{d-n} \longrightarrow R_d\,. \tag{18.1.1}$$

We say that R has *Poincaré duality* if this pairing is perfect for each n, i.e., if the induced map $R_n \longrightarrow \operatorname{Hom}_{\mathbb{K}}(R_{d-n}, R_d)$ is bijective.

The name originates from the prototypical example of such an algebra: the cohomology algebra of a compact connected orientable manifold; that the pairing (18.1.1) is perfect is the Poincaré duality theorem. Perhaps the following consequence of Poincaré duality is more familiar:

Exercise 18.2. Let R be a $\mathbb{K}$-algebra as above. Prove that when R has Poincaré duality, $\operatorname{rank}_{\mathbb{K}}(R_n) = \operatorname{rank}_{\mathbb{K}}(R_{d-n})$ for each $0 \leqslant n \leqslant d$. Give examples that show that the converse does not hold.

Exercise 18.3. Let $\langle\ ,\ \rangle$ be a bilinear form on a finite-rank $\mathbb{K}$-vector space V, and R the graded $\mathbb{K}$-algebra with $R_0 = \mathbb{K} = R_2$, $R_1 = V$, and $x \cdot y = \langle x, y\rangle$ for $x, y \in R_1$. Then R has Poincaré duality if and only if $\langle\ ,\ \rangle$ is perfect.

Here is why the notion of Poincaré duality is relevant to us:

Exercise 18.4. Let R be a graded $\mathbb{K}$-algebra as in Definition 18.1; assume furthermore that it is commutative. Prove that R is Gorenstein if and only if it has Poincaré duality.

We claim that Grothendieck duality, encountered in Theorem 11.26, is an extension of Poincaré duality to higher-dimensional Gorenstein rings. In order to justify the claim, we reinterpret Poincaré duality as follows: the pairing (18.1.1) yields, for each $0 \leqslant n \leqslant d$, a $\mathbb{K}$-linear map

$$R_n \longrightarrow \operatorname{Hom}_{\mathbb{K}}(R_{d-n}, R_d)\,.$$

Taking a direct sum yields a map of graded $\mathbb{K}$-vector spaces

$$R = \bigoplus_{n=0}^{d} R_n \longrightarrow \bigoplus_{n=0}^{d} \operatorname{Hom}_{\mathbb{K}}(R_{d-n}, R_d) = {}^*\operatorname{Hom}_{\mathbb{K}}(R, R_d)\,.$$

As a graded $\mathbb{K}$-vector space, ${}^*\operatorname{Hom}_{\mathbb{K}}(R, R_d)$ is concentrated in degrees $[-d, 0]$. The map above has degree $-d$ as it maps R_n to ${}^*\operatorname{Hom}_{\mathbb{K}}(R, R_d)_{n-d}$. We get a map of degree zero by shifting:

$$\chi\colon R \longrightarrow {}^*\operatorname{Hom}_{\mathbb{K}}(R, R_d)(-d)$$

is a morphism of graded $\mathbb{K}$-vector spaces. The natural R-module structure on R passes to an R-module structure on ${}^*\operatorname{Hom}_{\mathbb{K}}(R, R_d)$.

Exercise 18.5. Prove that χ is a morphism of R-modules, and that the following conditions are equivalent:

(1) The $\mathbb{K}$-algebra R has Poincaré duality.

(2) The homomorphism χ is bijective.

(3) As R-modules, $R \cong {}^*\operatorname{Hom}_{\mathbb{K}}(R, \mathbb{K})(-a)$ for some integer a.

2. Grothendieck duality

In this section $\mathbb{K}$ is a field and $R = \bigoplus_{n\geqslant 0} R_n$ is a finitely generated graded algebra over $R_0 = \mathbb{K}$. Its homogeneous maximal ideal is denoted $\mathfrak{m}$.

Exercise 18.6. Prove that the graded $\mathbb{K}$-dual

$${}^*\operatorname{Hom}_{\mathbb{K}}(R, \mathbb{K}) = \bigoplus_{n\geqslant 0} \operatorname{Hom}_{\mathbb{K}}(R_n, \mathbb{K})$$

is the graded injective hull of $\mathbb{K}$.

The following result is the graded version of Theorem 11.26.

Theorem 18.7. *Let R be a d-dimensional graded $\mathbb{K}$-algebra. When R is Gorenstein, $H^n_{\mathfrak{m}}(R) = 0$ for $n \neq d$, and there exists an integer a such that*

$$H^d_{\mathfrak{m}}(R) \cong {}^*\mathrm{Hom}_{\mathbb{K}}(R, \mathbb{K})(-a)\,. \qquad \square$$

The integer a that appears in the result is called, well, the *a-invariant* of R. You will find calculations of the a-invariant in Section 21.2.

When $\dim R = 0$, one has $H^0_{\mathfrak{m}}(R) = R$ so the theorem yields:

Corollary 18.8. *If R is Gorenstein and $\dim R = 0$, then there exists an integer a such that $R \cong {}^*\mathrm{Hom}_{\mathbb{K}}(R, \mathbb{K})(-a)$ as R-modules.* $\square$

Compare this result with Exercises 18.4, 18.5, and 18.6. In this way Grothendieck duality extends Poincaré duality.

Remark 18.9. Recall that the cohomology algebra of a compact connected orientable manifold satisfies Poincaré duality. Such manifolds may thus be seen as topological analogues of zero-dimensional Gorenstein rings. This analogy is strengthened when we note that the cohomology algebras of spaces are commutative, albeit in the graded sense: $a \cdot b = (-1)^{|a||b|} b \cdot a$.

One may then ask if higher-dimensional Gorenstein rings have counterparts in topology, and indeed they do: inspired by commutative algebra, Félix, Halperin, and Thomas [**38**] introduced and developed a theory of 'Gorenstein spaces.' Dwyer, Greenlees, and Iyengar [**31**] have proved a version of Grothendieck duality for such spaces.

3. Local duality

The basic context of this discussion is fairly general: let R be a ring, and let M and N be R-modules. Let $I^\bullet$ and $J^\bullet$ be injective resolutions of M and N respectively. Consider the complex of homomorphisms $\mathrm{Hom}_R(I^\bullet, J^\bullet)$, introduced in Definition 3.9, whose cohomology in degree n is

$$H^n\big(\mathrm{Hom}_R(I^\bullet, J^\bullet)\big) = \mathrm{Ext}^n_R(M, N)\,.$$

Remark 18.10. Let F be an additive covariant functor on the category of R-modules. For the moment, think of $\mathrm{Hom}_R(L, -)$, for some fixed R-module L; eventually, we would like to think of local cohomology functors, but not yet. Each cycle f in $\mathrm{Hom}_R(I^\bullet, J^\bullet)$ induces a homomorphism

$$\mathrm{F}(f)\colon\ \mathrm{F}(I^\bullet) \longrightarrow \mathrm{F}(J^\bullet)$$

of complexes which, in cohomology, provides for each n, d in $\mathbb{Z}$, a pairing

$$\mathrm{Ext}^{d-n}_R(M, N) \times R^n\,\mathrm{F}(M) \longrightarrow R^d\,\mathrm{F}(N)\,.$$

Here $R^i\,\mathrm{F}(-)$ is the i-th right derived functor of F. A natural question is whether there is a module N and an integer d for which the induced map

$$R^n\,\mathrm{F}(M) \longrightarrow \mathrm{Hom}_R(\mathrm{Ext}_R^{d-n}(M,N), R^d\,\mathrm{F}(N))$$

is bijective for each M. In practice, we may want to restrict M to a certain class of modules, for example, the finitely generated ones.

Remark 18.11. Let $(R,\mathfrak{m})$ be a local ring of dimension d. The preceding discussion, with $\mathrm{F} = \Gamma_{\mathfrak{m}}(-)$, yields for each integer n a pairing

$$\mathrm{Ext}_R^{d-n}(M,N) \times H_{\mathfrak{m}}^n(M) \longrightarrow H_{\mathfrak{m}}^d(N)\,.$$

This gives rise to an induced homomorphism of R-modules

$$H_{\mathfrak{m}}^n(M) \longrightarrow \mathrm{Hom}_R(\mathrm{Ext}_R^{d-n}(M,N), H_{\mathfrak{m}}^d(N))\,.$$

Given this discussion, the statement of the result below should come as no surprise. It is a refinement of Theorem 11.44.

Theorem 18.12. *Let $(R,\mathfrak{m},\mathbb{K})$ be a Cohen-Macaulay local ring with a canonical module ω, and M a finitely generated R-module. Set $d = \dim R$. Then $H_{\mathfrak{m}}^d(\omega) = E_R(\mathbb{K})$, and for each integer n the pairing*

$$\mathrm{Ext}_R^{d-n}(M,\omega) \times H_{\mathfrak{m}}^n(M) \longrightarrow H_{\mathfrak{m}}^d(\omega)$$

induces a bijection of R-modules

$$\chi^n(M)\colon H_{\mathfrak{m}}^n(M) \longrightarrow \mathrm{Hom}_R\left(\mathrm{Ext}_R^{d-n}(M,\omega), H_{\mathfrak{m}}^d(\omega)\right).$$

If, in addition, R is complete, then the pairing above is perfect.

Sketch of proof. The minimal injective resolution of ω, say $I^\bullet$, has the property that $I^n = 0$ for $n \notin [0,d]$ and that $E_R(\mathbb{K})$ occurs exactly once and in I^d; see Definition 11.31. Consequently, $H_{\mathfrak{m}}^n(\omega)$ is zero for $n \neq d$ and equal to $E_R(\mathbb{K})$, for $n = d$. This settles the first part of the theorem.

The bijectivity of $\chi^n(R)$ can be proved along the lines of the argument in Theorem 11.29. The essential points are that $I^\bullet$ is a bounded complex of injectives, and that homothety $R \longrightarrow \mathrm{Hom}_R(\omega,\omega)$ is an isomorphism; see Theorem 11.39. Then, from the fact that $\chi^d(R)$ is an isomorphism, it follows that so is $\chi^d(M)$, since the functors $H_{\mathfrak{m}}^d(-)$ and $\mathrm{Hom}_R(\mathrm{Hom}_R(-,\omega), H_{\mathfrak{m}}^d(\omega))$ are right-exact and additive. Finally, when $\chi^n(-)$ is bijective on finitely generated modules for some $n \leqslant d$, the long exact sequences associated with $\Gamma_{\mathfrak{m}}(-)$ and $\mathrm{Hom}_R(-,\omega)$ imply that $\chi^{n-1}(-)$ is bijective as well.

When R is complete, the Matlis dual of $\chi^n(M)$ is the homomorphism

$$\mathrm{Ext}_R^{d-n}(M,\omega) \longrightarrow \mathrm{Hom}_R\left(H_{\mathfrak{m}}^n(M), H_{\mathfrak{m}}^d(\omega)\right)$$

which is bijective. The pairing is therefore perfect. $\square$

Remark 18.13. The crucial point in the proof of Theorem 18.12 is the existence of *natural* maps $\chi^n(M)$. The argument works with ω replaced by a bounded complex $D^\bullet$ of injectives, with $H^n(D^\bullet)$ finitely generated for each n, and $H^d_{\mathfrak{m}}(D^\bullet) = E_R(\mathbb{K})$. Such an object exists when R is a quotient of a Gorenstein ring—for example, when R is complete—and is called a *dualizing complex*. With this at hand, one has a local duality statement for rings which may not be Cohen-Macaulay; see Hartshorne [**56**], Lipman [**101**], and P. Roberts [**132**].

While there are more efficient proofs of local duality, the argument given here has the merit that it immediately adapts to schemes.

Definition 18.14. Let X be a scheme of dimension d over a field $\mathbb{K}$. The discussion in Remarks 18.10 and 18.11 carries over to the context of schemes: for sheaves $\mathcal{F}$ and $\mathcal{G}$, one has, for each integer n, pairings of $\mathbb{K}$-vector spaces

$$\operatorname{Ext}^{d-n}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \times H^n(X,\mathcal{F}) \longrightarrow H^d(X,\mathcal{G})\,.$$

Serre's duality theorem for projective spaces then reads as follows; see [**140**]. Differential forms are defined later in this lecture.

Theorem 18.15. *Let $\mathbb{K}$ be a field, $X = \mathbb{P}^d_{\mathbb{K}}$, and $\mathcal{F}$ a coherent sheaf on X. Let Ω denote the sheaf of differential forms on X, and set $\omega = \wedge^d\Omega$. Then $H^d(X,\omega) \cong \mathbb{K}$, and for each integer n the pairing below is perfect:*

$$\operatorname{Ext}^{d-n}_{\mathcal{O}_X}(\mathcal{F},\omega) \times H^n(X,\mathcal{F}) \longrightarrow H^d(X,\omega) \cong \mathbb{K}\,. \qquad \square$$

Compare this result to Theorem 18.12, whose proof is modelled on Serre's proof of the theorem above. There is more to Serre's theorem than stated above: there is a canonical isomorphism

$$H^d(X,\omega) \longrightarrow \mathbb{K}\,,$$

called the *residue map*, and this is crucial for the proof.

4. Global canonical modules

Let R be a Cohen-Macaulay ring of finite Krull dimension; this last assumption is in line with the notion of Gorenstein rings adopted in Lecture 11.

Definition 18.16. A *canonical module* for R is a finitely generated R-module ω with the property that $\omega_{\mathfrak{m}}$ is a canonical module for $R_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R. It follows from Theorem 11.47 that if ω is a canonical module for R, then $\omega_{\mathfrak{p}}$ is a canonical module for $R_{\mathfrak{p}}$ for each $\mathfrak{p}$ in $\operatorname{Spec} R$.

When R is Gorenstein, any rank-one projective module, for example R itself, is a canonical module. As in the local case, it turns out that a Cohen-Macaulay ring R has a canonical module if and only if it is a quotient of a Gorenstein ring; see Proposition 18.21. The following exercises lead to this.

Exercise 18.17. When R is a Noetherian ring, prove that there exists an isomorphism $R \cong R_1 \times \cdots \times R_n$, where each $\operatorname{Spec} R_i$ is connected.

Exercise 18.18. Let $R_1, \ldots, R_n$ be rings, and set $R = R_1 \times \cdots \times R_n$. Prove the following statements:

(1) An R-module M is canonically isomorphic to $M_1 \oplus \cdots \oplus M_n$, where M_i is an R_i-module for each i.

(2) The ring R is Cohen-Macaulay if and only if each R_i is as well.

(3) Suppose that R is Cohen-Macaulay. Let ω be an R-module and write $\omega = \omega_1 \oplus \cdots \oplus \omega_n$, as in (1). Then ω is a canonical module for R if and only if for each i, the R_i-module ω_i is canonical.

Exercise 18.19. Let R be a Noetherian ring. Then $\operatorname{Spec} R$ is connected if and only if given minimal primes $\mathfrak{p}$ and $\mathfrak{p}'$ in R, there is a sequence of ideals

$$\mathfrak{p} = \mathfrak{p}_0,\ \mathfrak{m}_0,\ \mathfrak{p}_1,\ \mathfrak{m}_1,\ \ldots,\ \mathfrak{p}_n,\ \mathfrak{m}_n,\ \mathfrak{p}_{n+1} = \mathfrak{p}',$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are minimal primes, $\mathfrak{p}_i, \mathfrak{p}_{i+1} \subseteq \mathfrak{m}_i$ for $0 \leqslant i \leqslant n$, and the ideals $\mathfrak{m}_i$ are proper; evidently, they may be chosen to be maximal.

The next result extends Theorem 11.34.

Proposition 18.20. *Let $Q \longrightarrow R$ be a surjective homomorphism of Cohen-Macaulay rings, where $\operatorname{Spec} R$ is connected. If ω_Q is a canonical module for Q, then $\operatorname{Ext}^h_Q(R, \omega_Q)$, with $h = \dim Q - \dim R$, is a canonical module for R.*

Proof. Let $R = Q/\mathfrak{a}$. We claim that height $\mathfrak{a}_\mathfrak{m} = h$ for each maximal ideal $\mathfrak{m}$ of Q containing $\mathfrak{a}$. Indeed, let $\mathfrak{p}$ and $\mathfrak{p}'$ be minimal primes of $\mathfrak{a}$ contained in $\mathfrak{m}$. Then $\dim(Q_\mathfrak{m}/\mathfrak{p}Q_\mathfrak{m}) = \dim(Q_\mathfrak{m}/\mathfrak{p}'Q_\mathfrak{m})$ because $R_\mathfrak{m}$ is Cohen-Macaulay, hence equidimensional. Moreover $Q_\mathfrak{m}$ is Cohen-Macaulay, hence catenary, so height $\mathfrak{p}$ = height $\mathfrak{p}'$. Since $\operatorname{Spec} R$ is connected, Exercise 18.19 implies that height $\mathfrak{p}$ = height $\mathfrak{p}'$ for each pair of minimal primes $\mathfrak{p}, \mathfrak{p}'$ of $\mathfrak{a}$.

Set $\omega = \operatorname{Ext}^h_Q(R, \omega_Q)$; this is a finitely generated R-module. For each maximal ideal $\mathfrak{m}$ of Q containing $\mathfrak{a}$, since $Q_\mathfrak{m}$ is Cohen-Macaulay with canonical module $(\omega_Q)_\mathfrak{m}$ and h = height $\mathfrak{a}_\mathfrak{m}$, Theorem 11.34 implies that $\omega_\mathfrak{m}$ is a canonical module for $R_\mathfrak{m}$. Thus, ω is a canonical module for R. □

The following result is a global version of Corollary 11.43.

Proposition 18.21. *Let R be a Cohen-Macaulay ring. Then R has a canonical module if and only if it is a homomorphic image of a Gorenstein ring.*

Proof. Let ω be a canonical module for R and set $Q = R \ltimes \omega$. For each $\mathfrak{p}$ in $\operatorname{Spec} R$, the local ring $Q_\mathfrak{p} = R_\mathfrak{p} \ltimes \omega_\mathfrak{p}$ is Gorenstein because $\omega_\mathfrak{p}$ is a canonical module for $R_\mathfrak{p}$; see Theorem 11.42. Therefore, Q is Gorenstein, and the canonical projection $Q \longrightarrow R$ exhibits R as a homomorphic image of Q.

Let $Q \longrightarrow R$ be a surjective homomorphism of rings with Q Gorenstein. Set $h = \dim Q - \dim R$. As Q is a canonical module for itself, when $\operatorname{Spec} R$ is connected Proposition 18.20 implies that $\operatorname{Ext}_Q^h(R, Q)$ is a canonical module for R. In general, by Exercise 18.17, one has an isomorphism $R \cong R_1 \times \cdots \times R_d$ where each $\operatorname{Spec} R_i$ is connected. Since R is a quotient of a Gorenstein ring, so is each R_i. Hence R_i has a canonical module ω_i. Now $\omega_1 \oplus \cdots \oplus \omega_d$ is a canonical module for R by Exercise 18.18. □

The preceding result settles the question of the existence of canonical modules. The one below deals with uniqueness. One could not have expected a stronger result since over a Gorenstein ring any rank-one projective module is a canonical module. The proof is a straightforward localization argument using Lemma 11.38 and Theorem 11.39, and is left to the reader.

Proposition 18.22. *Let R be a Cohen-Macaulay ring and ω a canonical R-module. The following statements hold.*

(1) *If P is a rank-one projective module, $P \otimes_R \omega$ is a canonical module.*
(2) *If ω' is a canonical module, then $\operatorname{Hom}_R(\omega', \omega)$ is a rank-one projective module, and the evaluation homomorphism below is bijective:*

$$\operatorname{Hom}_R(\omega', \omega) \otimes_R \omega' \longrightarrow \omega\,.$$ □

Proposition 18.22 is the best one can do for general rings, in that there is no canonical choice of a canonical module; this is a problem when one wants to work with schemes. Fortunately, this problem has a solution for schemes that arise in (classical!) algebraic geometry. This leads us to the notion of smoothness.

Definition 18.23. Let R be a ring essentially of finite type over a field $\mathbb{K}$, i.e., a localization of a finitely generated $\mathbb{K}$-algebra. The $\mathbb{K}$-algebra R is *smooth* if for each field $\mathbb{L} \supseteq \mathbb{K}$, the ring $\mathbb{L} \otimes_{\mathbb{K}} R$ is regular.

As the $\mathbb{K}$-algebra R is essentially of finite type, the $\mathbb{L}$-algebra $\mathbb{L} \otimes_{\mathbb{K}} R$ is essentially of finite type, and hence Noetherian. Moreover, the following conditions are equivalent:

(1) R is a smooth $\mathbb{K}$-algebra;
(2) $\mathbb{L} \otimes_{\mathbb{K}} R$ is regular for each finite purely inseparable extension $\mathbb{L}$ of $\mathbb{K}$;
(3) $\mathbb{L} \otimes_{\mathbb{K}} R$ is regular for the algebraic closure $\mathbb{L}$ of $\mathbb{K}$.

In particular, when $\mathbb{K}$ has characteristic zero or is a perfect field, the $\mathbb{K}$-algebra R is smooth if and only if the ring R is regular.

Example 18.24. Polynomial rings $\mathbb{K}[x_1, \ldots, x_n]$ are smooth over $\mathbb{K}$, as are localizations of polynomial rings.

Exercise 18.25. Let $\mathbb{K}$ be a field and set

$$R = \mathbb{K}[x, y]/\left(y^3 - x(x-1)(x-2)\right).$$

Prove that R is smooth over $\mathbb{K}$ if and only if the characteristic of $\mathbb{K}$ is not 2.

Evidently, when R is smooth, it is regular. The converse does not hold:

Example 18.26. Let $\mathbb{K}$ be a field of characteristic p. Assume that $\mathbb{K}$ is not perfect, so that there exists an element $a \in \mathbb{K} \smallsetminus \mathbb{K}^p$. Set $R = \mathbb{K}[x]/(x^p - a)$. The ring R is a field, and, in particular, it is regular. However, for $\mathbb{L} = R$, the ring $\mathbb{L} \otimes_{\mathbb{K}} R$ is not reduced. Thus, R is not smooth.

More generally, for a finite field extension $\mathbb{K} \subseteq \mathbb{L}$, the $\mathbb{K}$-algebra $\mathbb{L}$ is smooth if and only if $\mathbb{L}$ is separable over $\mathbb{K}$; see [**114**, Theorem 62].

Our objective is to construct canonical modules over smooth algebras. This calls for more definitions.

Definition 18.27. Let R be a commutative $\mathbb{K}$-algebra; for now, we do not assume that $\mathbb{K}$ is a field. A $\mathbb{K}$-*derivation* of R with coefficients in an R-module M, is a $\mathbb{K}$-linear map $\delta\colon R \longrightarrow M$ satisfying the Leibniz rule:

$$\delta(rs) = \delta(r)s + r\delta(s) \quad \text{for all } r, s \in R.$$

The $\mathbb{K}$-linearity of δ implies that $\delta(1) = 0$.

Definition 18.28. A *module of Kähler differentials* of the $\mathbb{K}$-algebra R is an R-module Ω equipped with a $\mathbb{K}$-derivation $d\colon R \longrightarrow \Omega$ with the following universal property: each $\mathbb{K}$-derivation $\delta\colon R \longrightarrow M$ extends to a unique homomorphism of R-modules $\widetilde{\delta}\colon \Omega \longrightarrow M$, that is to say, the diagram

$$\begin{tikzcd} R \arrow[r, "d"] \arrow[d, "\delta"'] & \Omega \arrow[dl, dashed, "\widetilde{\delta}"] \\ M & \end{tikzcd}$$

commutes. Standard arguments show that a module of Kähler differentials, if it exists, is unique up to a unique isomorphism. We denote it $\Omega_{R|\mathbb{K}}$; the associated derivation $d\colon R \longrightarrow \Omega_{R|\mathbb{K}}$ is the *universal derivation* of R over $\mathbb{K}$.

The module of Kähler differentials exists, and can be constructed as follows. It illustrates yet another use of the diagonal homomorphism.

Construction 18.29. Let R be a $\mathbb{K}$-algebra. The tensor product $R \otimes_{\mathbb{K}} R$ is again a commutative $\mathbb{K}$-algebra, with product given by

$$(r \otimes s) \cdot (r' \otimes s') = (rr' \otimes ss').$$

In the following paragraphs, we view $R \otimes_{\mathbb{K}} R$ as an R-module, with product induced by the left-hand factor: $r' \cdot (r \otimes s) = r'r \otimes s$. The map

$$\mu\colon R \otimes_{\mathbb{K}} R \longrightarrow R \quad \text{where } \mu(r \otimes s) = rs$$

is a homomorphism of rings, since R is commutative. Set $\mathfrak{a} = \ker \mu$; it inherits an R-module structure from $R \otimes_{\mathbb{K}} R$. As an R-module, $\mathfrak{a}$ is spanned by $\{s \otimes 1 - 1 \otimes s \mid s \in R\}$. Moreover, if the $\mathbb{K}$-algebra R is generated by $s_1, \ldots, s_n$, then the ideal $\mathfrak{a}$ is generated by the elements

$$s_i \otimes 1 - 1 \otimes s_i \qquad \text{for } 1 \leqslant i \leqslant n\,.$$

Since $\mathfrak{a} = \ker \mu$, the action of $R \otimes_{\mathbb{K}} R$ on $\mathfrak{a}/\mathfrak{a}^2$ factors through R, and hence $\mathfrak{a}/\mathfrak{a}^2$ has a canonical R-module structure. The action of $r \in R$ on the residue class of $s \otimes 1 - 1 \otimes s$ in $\mathfrak{a}/\mathfrak{a}^2$ is given by

$$r \cdot (s \otimes 1 - 1 \otimes s) = rs \otimes 1 - r \otimes s\,.$$

Thus, this action coincides with the R-module structure induced by the action of R on $\mathfrak{a}$. It is easy to check that this is also the action induced by R acting on $R \otimes_{\mathbb{K}} R$ from the right.

Theorem 18.30. *The R-module $\mathfrak{a}/\mathfrak{a}^2$ is the module of Kähler differentials of R over $\mathbb{K}$, with universal derivation $d\colon R \longrightarrow \mathfrak{a}/\mathfrak{a}^2$ where $d(s) = s \otimes 1 - 1 \otimes s$.*

Proof. First we verify that d is a $\mathbb{K}$-derivation. Consider the $\mathbb{K}$-linear maps

$$\begin{aligned} \iota_1\colon R &\longrightarrow R \otimes_{\mathbb{K}} R \qquad \text{where } s \longmapsto s \otimes 1\,,\\ \iota_2\colon R &\longrightarrow R \otimes_{\mathbb{K}} R \qquad \text{where } s \longmapsto -1 \otimes s\,. \end{aligned}$$

One has then a diagram of $\mathbb{K}$-modules

$$R \xrightarrow{\ \Delta\ } R \oplus R \xrightarrow{\iota_1 \oplus \iota_2} (R \otimes_{\mathbb{K}} R) \oplus (R \otimes_{\mathbb{K}} R) \xrightarrow{\ \pi\ } R \otimes_{\mathbb{K}} R\,,$$

where $\Delta(s) = (s, s)$ and $\pi(y \otimes z, y' \otimes z') = y \otimes z + y' \otimes z'$. Let $\widetilde{d}$ denote the composed map; this is again $\mathbb{K}$-linear and a direct check shows that $\mu\widetilde{d} = 0$, so $\widetilde{d}(R) \subseteq \mathfrak{a}$. The map d is the composition

$$R \xrightarrow{\ \widetilde{d}\ } \mathfrak{a} \longrightarrow \mathfrak{a}/\mathfrak{a}^2\,,$$

and hence a well-defined homomorphism of $\mathbb{K}$-modules. For elements r, s of R, one has, in $\mathfrak{a}/\mathfrak{a}^2$, the equation

$$\begin{aligned} 0 &= (r \otimes 1 - 1 \otimes r)(s \otimes 1 - 1 \otimes s)\\ &= rs \otimes 1 - s \otimes r - r \otimes s + 1 \otimes rs\\ &= s(r \otimes 1 - 1 \otimes r) + r(s \otimes 1 - 1 \otimes s) - (rs \otimes 1 - 1 \otimes rs)\,. \end{aligned}$$

Thus, $d(rs) = d(r)s + rd(s)$, that is to say, d is a derivation.

It remains to prove that $d\colon R \longrightarrow \mathfrak{a}/\mathfrak{a}^2$ has the required universal property. Let $\delta\colon R \longrightarrow M$ be a $\mathbb{K}$-derivation. If there exists an R-linear homomorphism $\widetilde{\delta}\colon \mathfrak{a}/\mathfrak{a}^2 \longrightarrow M$ with $\widetilde{\delta} \circ d = \delta$, then it must be unique; this is because $d\colon R \longrightarrow \mathfrak{a}/\mathfrak{a}^2$ maps onto the R-module generators of $\mathfrak{a}/\mathfrak{a}^2$.

As to the existence of $\widetilde{\delta}$, the map δ induces an R-module homomorphism

$$R \otimes_{\mathbb{K}} \delta\colon R \otimes_{\mathbb{K}} R \longrightarrow M, \qquad \text{where } r \otimes s \longmapsto r\delta(s)\,.$$

Restriction yields an R-linear map $\delta'\colon \mathfrak{a} \longrightarrow M$. For r, s in R, one has

$$\begin{aligned} \delta'\big((r\otimes 1 - 1 \otimes r)(s \otimes 1 - 1 \otimes s)\big) &= \delta'\big(rs \otimes 1 - s \otimes r - r \otimes s + 1 \otimes rs\big) \\ &= -s\delta(r) - r\delta(s) + \delta(rs) \\ &= 0\,, \end{aligned}$$

where the second equality holds because $\delta(1) = 0$, and the third equality holds because δ is a derivation. Thus, $\delta'(\mathfrak{a}^2) = 0$, so δ' induces a homomorphism of R-modules $\widetilde{\delta}\colon \mathfrak{a}/\mathfrak{a}^2 \longrightarrow M$. It is clear that $\widetilde{\delta} \circ d = \delta$, as desired. This completes the proof. □

Example 18.31. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring with variables $x_1, \ldots, x_n$. The module of Kähler differentials is free on a basis dx_i:

$$\Omega_{R|\mathbb{K}} \cong \bigoplus_{i=1}^{n} R\, dx_i\,,$$

and the universal derivation is given by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \qquad \text{for } f \in R\,.$$

Thus, d is the usual gradient.

The next example follows from Example 18.31 and basic properties of modules of differentials; see [**114**, Theorem 58].

Example 18.32. Let $R = \mathbb{K}[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. The module of Kähler differentials is given by a presentation

$$R^c \xrightarrow{(\partial f_i/\partial x_j)} R^n \longrightarrow \Omega_{R|\mathbb{K}} \longrightarrow 0\,.$$

The matrix $(\partial f_i/\partial x_j)$ is called the *Jacobian matrix* of R.

The proof of the following lemma is straightforward: one checks that the module $U^{-1}\Omega_{Q|\mathbb{K}}$ satisfies the desired universal properties. Combining this result with the previous example, one can write down the module of differentials of any $\mathbb{K}$-algebra essentially of finite type.

Lemma 18.33. *If $R = U^{-1}Q$, where Q is a $\mathbb{K}$-algebra and U a multiplicatively closed subset of Q, then there is a canonical isomorphism of R-modules*

$$U^{-1}\Omega_{Q|\mathbb{K}} \cong \Omega_{R|\mathbb{K}}\,. \qquad \square$$

The next result addresses the question raised after Proposition 18.22. For a proof, see [**115**, §29].

Theorem 18.34. *Let $\mathbb{K}$ be a field and R a $\mathbb{K}$-algebra essentially of finite type. If R is smooth, then $\Omega_{R|\mathbb{K}}$ is a finitely generated projective R-module.*

When, in addition, R is a domain, for $d = \operatorname{rank}_R \Omega_{R|\mathbb{K}}$, the R-module $\wedge^d\Omega_{R|\mathbb{K}}$ is a rank-one projective, and hence a canonical module. $\square$

If R is a domain finitely generated over $\mathbb{K}$, then $\operatorname{rank}_R \Omega_{R|\mathbb{K}} = \dim R$.

The *Jacobian criterion* gives a converse to Theorem 18.34: if $\mathbb{K}$ has characteristic zero and $\Omega_{R|\mathbb{K}}$ is projective, then R is a smooth $\mathbb{K}$-algebra. There is a version of the criterion in positive characteristic, but the statement is more delicate; see [**32**, Theorem 16.19]

The following result is taken from [**71**].

Theorem 18.35. *Let $\mathbb{K}$ be a field, and R a $\mathbb{K}$-algebra essentially of finite type. If R is a d-dimensional Cohen-Macaulay domain and the non-smooth locus of R has codimension at least two, then*

$$\operatorname{Hom}_R(\operatorname{Hom}_R(\wedge^d\Omega_{R|\mathbb{K}}, R), R)$$

is a canonical module for R. $\square$

Example 18.36. Let $X = (x_{ij})$ be an $n \times (n+1)$ matrix of indeterminates, where $n \geqslant 2$. Set $R = \mathbb{K}[X]/I_n(X)$, where $I_n(X)$ is the ideal generated by the size n minors of X. Then ω_R is isomorphic to the ideal of size $n-1$ minors of any fixed $n-1$ columns of the matrix X; see [**20**, Theorem 7.3.6].

De Rham Cohomology

In this lecture we use ideas from calculus to link local cohomology of complex varieties in $\mathbb{C}^n$ to de Rham and singular cohomology of their complements; this is the content of the last section. In the first three sections we discuss de Rham cohomology in the real, complex, and algebraic cases, respectively. The proofs of the results are only sketched.

Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x_1, \dots, x_n]$ a polynomial ring. The module $\Omega_{R|\mathbb{K}}$ of $\mathbb{K}$-linear Kähler differentials on R is $\bigoplus_{i=1}^n R\,dx_i$ and the universal derivation $d\colon R \longrightarrow \Omega_{R|\mathbb{K}}$ is the *gradient map*; see Example 18.31.

The modules $\Omega^t_{R|\mathbb{K}} = \wedge^t \Omega_{R|\mathbb{K}}$ for $t \geqslant 0$ are the *higher order differentials*; note that $\Omega^0_{R|\mathbb{K}} = R$ and $\Omega^1_{R|\mathbb{K}} = \Omega_{R|\mathbb{K}}$. The $\mathbb{K}$-linear maps

$$\begin{aligned} d^t\colon \Omega^t_{R|\mathbb{K}} &\longrightarrow \Omega^{t+1}_{R|\mathbb{K}} \qquad \text{where} \\ (f\,dx_{i_1} \wedge \cdots \wedge dx_{i_t}) &\longmapsto df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_t}\,, \end{aligned}$$

satisfy $d^t \circ d^{t-1} = 0$. One thus obtains the *de Rham complex* $\Omega^\bullet_{R|\mathbb{K}}$ of R:

$$0 \longrightarrow R \longrightarrow \Omega_{R|\mathbb{K}} \longrightarrow \Omega^2_{R|\mathbb{K}} \longrightarrow \Omega^3_{R|\mathbb{K}} \longrightarrow \cdots\,.$$

Such a construction can be carried out for any $\mathbb{K}$-algebra R [**54**, §16.6]. Using Lemma 18.33, one can show that for each $f \in R$ there is a natural isomorphism of complexes $\Omega^\bullet_{R_f|\mathbb{K}} \cong R_f \otimes_R \Omega^\bullet_{R|\mathbb{K}}$. For a variety X over $\mathbb{K}$, this procedure can be carried out on affine open subsets to obtain a complex $\Omega^\bullet_{X|\mathbb{K}}$ of sheaves called the *algebraic de Rham complex* of X

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega_{X|\mathbb{K}} \longrightarrow \Omega^2_{X|\mathbb{K}} \longrightarrow \Omega^3_{X|\mathbb{K}} \longrightarrow \cdots\,.$$

Going a step further, this construction applies to ringed spaces over $\mathbb{K}$.

1. The real case: de Rham's theorem

In this section, all differentials are taken relative to the real numbers and we omit the subscript $\mathbb{R}$. For a smooth manifold M, we write $\Omega_M^\bullet$ for the complex of sheaves of smooth differential forms. The complex of its global sections, $\Omega^\bullet(M)$, is the *de Rham complex* of M.

We start with a discussion of Green's theorem.

Theorem 19.1. *Let $f, g\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be smooth functions with $f_y = g_x$. There then exists a smooth function $h\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ with $h_x = f$ and $h_y = g$.*

Sketch of proof. Set $h(x_0, y_0) = \int_{(0,0)}^{(x_0,y_0)} (f\,dx + g\,dy)$ where the integral is taken along an arbitrary smooth path; for each piecewise-smooth loop λ, one needs to verify that $\int_\lambda f\,dx + g\,dy = 0$. Such a loop can be uniformly approximated by piecewise-linear paths parallel to the coordinate axes, so it suffices to treat the case where λ is the boundary of a rectangle with corners $(0,0)$ and $(\varepsilon, \varepsilon')$. In this case

$$\begin{aligned}\int_\lambda f\,dx + g\,dy &= \int_0^\varepsilon (f(x,0) - f(x,\varepsilon'))\,dx + \int_0^{\varepsilon'} (g(\varepsilon,y) - g(0,y))\,dy \\ &= \int_0^\varepsilon \int_0^{\varepsilon'} -f_y(x,y)\,dy\,dx + \int_0^{\varepsilon'} \int_0^\varepsilon g_x(x,y)\,dx\,dy\,,\end{aligned}$$

which is zero since $f_y(x,y) = g_x(x,y)$. Note that $dh = f\,dx + g\,dy$. □

The hypothesis $f_y = g_x$ is necessary in the theorem since $h_{xy} = h_{yx}$. Suppose the domain of definition of $f_y = g_x$ is an open set $U \subset \mathbb{R}^2$. The proof shows that the path integral along a loop depends only on the number of times it winds around the holes of U. The integral vanishes along all loops in U if and only if $\int f\,dx + g\,dy$ is path-independent.

Example 19.2. On $\mathbb{R}^2 \smallsetminus \{0\}$, consider the functions

$$f(x,y) = -\frac{y}{x^2+y^2} \qquad \text{and} \qquad g(x,y) = \frac{x}{x^2+y^2}\,.$$

Verify that $f_y = g_x$ and that $\int_\lambda f\,dx = \pi = \int_\lambda g\,dy$ for any circle λ centered at the origin. Thus there can be no function h as in Green's theorem.

Let p be a point in an open set $U \subset \mathbb{R}^2$ and $\pi_1(U,p)$ the fundamental group of U based at p. The space of smooth differentials on U is

$$\Omega^1(U) = \{f\,dx + g\,dy \mid f, g \in C^\infty(U)\}\,.$$

Since U is open, each element of $\pi_1(U,p)$ has a smooth representative. Integration along a smooth path gives a linear map from $\Omega^1(U)$ to $\mathbb{R}$, which we restrict to $\Omega_0^1(U) = \{f\,dx + g\,dy \mid f_y = g_x\}$. For fixed $\omega \in \Omega_0^1(U)$, integrals on smooth contractible loops are zero, so one obtains a pairing

$\pi_1(U,p) \times \Omega_0^1(U) \longrightarrow \mathbb{R}$. Moreover, the induced map $\pi_1(U,p) \longrightarrow \mathbb{R}$ is a homomorphism of groups, hence it factors through the singular homology group $H_1(U,\mathbb{Z})$. Extending coefficients to $\mathbb{R}$, one obtains a pairing

$$\int \; : \; H_1(U,\mathbb{R}) \times \Omega_0^1(U) \longrightarrow \mathbb{R}\,.$$

The gradient of a differential form $f\,dx + g\,dy$ is $(g_x - f_y)\,dx \wedge dy$, so the form is in $\Omega_0^1(U)$ if and only if the gradient is zero, i.e., if it is *closed*. The form is *exact* if it arises as a gradient $dh = h_x\,dx + h_y\,dy$ for some $h \in \Omega^0(U)$. Since an exact differential gives a zero integral on all loops, the pairing above descends to an $\mathbb{R}$-linear pairing

$$\int \; : \; H_1(U,\mathbb{R}) \times H^1(\Omega^\bullet(U)) \longrightarrow \mathbb{R}\,.$$

If the pairing vanishes identically on a closed form ω, then, arguing as in the proof of Theorem 19.1, one sees that ω is exact. Thus, the pairing is faithful in the second argument. One can also show, using Example 19.2, that it is faithful in the first argument. Therefore, $H^1(\Omega^\bullet(U))$ is dual to $H_1(U,\mathbb{R})$ and hence isomorphic to singular cohomology, $H^1(U,\mathbb{R})$. Remarkably, there is a vast generalization of this observation; see Theorem 19.4.

Definition 19.3. Let M be a smooth manifold. The *de Rham cohomology* groups of M are the cohomology groups

$$H^t_{\mathrm{dR}}(M) = H^t(\Omega^\bullet(M)) \qquad \text{for } t \in \mathbb{Z}\,.$$

The cycles in $\Omega^\bullet(M)$ are the *closed* forms; the boundaries the *exact* forms.

Let M be a smooth manifold, and $S_t(M)$ the $\mathbb{R}$-vector space spanned by smooth maps from the standard t-simplex to M. Consider the pairing

$$\int \; : \; S_t(M) \times \Omega^t(M) \longrightarrow \mathbb{R} \qquad \text{with } (\sigma,\omega) \longmapsto \int_\sigma \omega\,,$$

where $\sigma\colon \Delta_t \longrightarrow M$ is a smooth t-simplex and $\int_\sigma \omega = \int_{\Delta_t} \sigma^*(\omega)$ for $\sigma^*(\omega)$ the pullback of ω along σ.

Theorem 19.4 (de Rham's theorem)**.** *Let M be a smooth manifold. The pairing above induces a perfect $\mathbb{R}$-linear pairing*

$$H_t(M,\mathbb{R}) \times H^t_{\mathrm{dR}}(M) \longrightarrow \mathbb{R}\,.$$

This yields isomorphisms $H^t_{\mathrm{dR}}(M) \cong \mathrm{Hom}_{\mathbb{R}}(H_t(M,\mathbb{R}),\mathbb{R}) \cong H^t(M,\mathcal{R})$, where $H^t(M,\mathcal{R})$ is the sheaf cohomology of the constant sheaf $\mathcal{R}$ of $\mathbb{R}$.

The pairing returns *periods* $\int_\sigma \omega$ which, typically, are transcendental.

Example 19.5. One has $H^0(\mathbb{S}^1,\mathbb{R}) = \mathbb{R} = H^1(\mathbb{S}^1,\mathbb{R})$. De Rham's theorem rediscovers two facts from Lecture 2. First, $H^0_{\mathrm{dR}}(\mathbb{S}^1)$ consists of constant functions, since these are the closed functions. The 0-form 1 is dual to any singular 0-chain in $\mathbb{S}^1$ that sends the 0-simplex (a point) to a point in $\mathbb{S}^1$. Second, $H^1_{\mathrm{dR}}(\mathbb{S}^1)$ is generated by the 1-chain that traverses $\mathbb{S}^1$ precisely once. Its dual 1-form is $\omega = dt/2\pi$, since 1-forms are closed and $\int_{\mathbb{S}^1}\omega = 1$.

Example 19.6. On $M = \mathbb{R}^3$ one has $H^0(M,\mathbb{R}) = \mathbb{R}$ and $H^{>0}(M,\mathbb{R}) = 0$. By Theorem 19.4, all closed forms of order $t \geqslant 1$ are exact on M. For $t = 1$ this is Stokes' theorem: if the curl of a global smooth vector field is zero, then the vector field is a gradient. When $t = 2$, this is Gauss' theorem: if a global smooth vector field has zero divergence, then it is equal to the curl of a vector field. Finally, every 3-form $w = f\,dx \wedge dy \wedge dz$ is a divergence.

As in the plane, the theorems of Stokes and Gauss may fail for forms not defined on all of $\mathbb{R}^3$. For example, the 1-form $(x^2+y^2)^{-1}(x\,dx - y\,dy)$ is closed, but not exact. The reason is the same as in Example 19.2: the domain of definition is not simply-connected. Similarly, the 2-form

$$\omega = \frac{1}{(x^2+y^2+z^2)^{3/2}}\,(x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

is closed, and the integral of ω over the 2-sphere is 2π, so ω cannot be a gradient. The point is that ω has a singularity at the origin.

Sketch of proof of Theorem 19.4. For $t \geqslant 1$, each global t-form on $\mathbb{R}^n$ is closed precisely when it is exact; this is proved by an explicit calculation akin to the one in the proof of Green's theorem. It follows that the complex of sheaves $\Omega^\bullet_M$ is a resolution of $\mathcal{R}$ as in Example 2.18.

The sheaves Ω^t_M are locally free, and hence allow for partitions of unity; compare Example 2.16. Using this, one shows that $H^{>0}(M,\Omega^t_M) = 0$ for each t, so $\Omega^\bullet_M$ is an acyclic resolution of $\mathcal{R}$; see Example 2.18. It follows from Theorem 2.26 that $H^t_{\mathrm{dR}}(M)$ agrees with $H^t(M,\mathcal{R})$.

The sheaf cohomology groups $H^\bullet(-,\mathcal{R})$ satisfy the *Eilenberg-Steenrod axioms* which characterize singular cohomology; see [**22, 83**]. Therefore de Rham cohomology, cohomology of the sheaf $\mathcal{R}$, and singular cohomology with $\mathbb{R}$-coefficients, all agree for the manifold M. It follows that $H^t(M,\mathcal{R})$ is the dual of singular homology $H_t(M,\mathbb{R})$.

A calculation similar to the one in the proof of Green's theorem shows that integrating a closed form over a boundary gives zero, so $\int_{\partial\sigma}\omega = \int_\sigma d\omega$. It follows that integration gives a pairing between $H^t_{\mathrm{dR}}(M)$ and $H_t(M,\mathbb{R})$. For $f\colon X \longrightarrow Y$ smooth, one has

$$\int_\sigma f^*(\omega) = \int_{f\circ\sigma}\omega = \int_{f_*(\sigma)}\omega\,,$$

so the pairing is functorial. Manifolds can be covered by copies of $\mathbb{R}^n$ and integration gives the right pairing on these copies. To verify that this is the pairing on all of M, one uses the Mayer-Vietoris sequence for de Rham cohomology and induction on open sets [**110**]. $\square$

2. Complex manifolds

In this section, M is a complex analytic manifold, and the complex $\Omega^\bullet_M$ consists of sheaves of holomorphic differential forms. The complex of its global sections, $\Omega^\bullet(M)$, is the *holomorphic de Rham complex* of M. Theorem 19.4 does not carry over directly to complex manifolds: $\mathbb{P}^1_{\mathbb{C}}$ has no second order complex differential forms, yet $H^2(\mathbb{P}^1_{\mathbb{C}}, \mathbb{C}) \neq 0$.

A crucial difference between the real and the complex case is that holomorphic functions can be locally represented by convergent power series. Hence there are no partitions of unity over $\mathbb{C}$, and sheaves of holomorphic functions, and of higher holomorphic differentials, may not be acyclic. We provide some examples below; these make use of a useful principle:

Remark 19.7. Let X be a smooth subvariety of $\mathbb{P}^n_{\mathbb{C}}$, defined by the vanishing of homogeneous polynomials $f_1, \dots, f_c$. These same polynomials also define a complex analytic submanifold X^{an} of $\mathbb{P}^n_{\mathbb{C}}$, with the usual Hausdorff topology. A theorem of Serre [**141**] implies that, for each t, the cohomology of the sheaf Ω^t_X can be identified with the cohomology of $\Omega^t_{X^{an}}$.

Example 19.8. Let $E \subset \mathbb{P}^2$ be an elliptic curve, defined by an irreducible homogeneous cubic polynomial $f \in R = \mathbb{C}[x, y, z]$. The exact sequence

$$0 \longrightarrow R(-3) \xrightarrow{f} R \longrightarrow R/(f) \longrightarrow 0$$

of R-modules induces an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_E \longrightarrow 0\,.$$

Taking cohomology yields an exact sequence

$$H^1(E, \mathcal{O}_E) \longrightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) \xrightarrow{f} H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})\,.$$

Theorem 13.25 implies that $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3))$ is the rank-one vector space

$$H^3_{(x,y,z)}(R)_{-3} = \mathbb{C} \cdot [1/(xyz)]$$

which, evidently, maps to zero in the exact sequence. Hence $H^1(E, \mathcal{O}_E) \neq 0$. By Remark 19.7, the sheaf of holomorphic functions on E^{an} is not acyclic.

Even when the structure sheaf is acyclic, the sheaf of higher order holomorphic differential forms may not be; this is the point of the next example.

Example 19.9. Let X be the projective line, $\operatorname{Proj}\mathbb{C}[x_0,x_1]$; then X^{an} is the analytic projective complex line. Remark 19.7 and Theorem 13.25 imply that $\mathcal{O}_{X^{an}}$ is acyclic; we verify that $H^1(X^{an},\Omega^1_{X^{an}}) = H^1(X,\Omega^1_X)$ is nonzero.

We use a Čech complex on the cover $\operatorname{Spec}\mathbb{C}[x_1/x_0]$ and $\operatorname{Spec}\mathbb{C}[x_0/x_1]$ to compute the cohomology of Ω^1_X. Setting $y = x_1/x_0$, this complex is

$$0 \longrightarrow \mathbb{C}[y]\,dy \;\oplus\; \mathbb{C}[y^{-1}]\,d(y^{-1}) \xrightarrow{d^0} \mathbb{C}[y,y^{-1}]\,dy \longrightarrow 0\,.$$

As $d^0(y^{-1}) = -y^{-2}dy$, one gets $H^0(X,\Omega^1_X) = 0$ and $H^1(X,\Omega^1_X) = \mathbb{C}y^{-1}dy$.

Remark 19.10. Suppose M is biholomorphic to $\mathbb{C}^n$. The holomorphic de Rham complex $\Omega^\bullet(M)$ then has a unique nonzero cohomology group, $H^0(\Omega^\bullet(M)) = \mathbb{C}$. This globalizes to the corresponding statement on sheaves: $\Omega^\bullet_M$ is a resolution of $\mathcal{C}$, the constant sheaf corresponding to $\mathbb{C}$. However, the sheaves of differential forms need not be cohomologically trivial, as seen in the examples above. In particular, Theorem 2.26 does not apply.

The concept below clarifies the relationship between $H^\bullet(M,\mathbb{C})$ and the cohomology of $\Omega^\bullet_M$, for a general complex analytic manifold M.

Definition 19.11. An analytic manifold S is a *Stein manifold* if each coherent analytic sheaf $\mathcal{F}$ on S is acyclic, i.e., $H^t(S,\mathcal{F}) = 0$ for $t \geqslant 1$.

A smooth complex affine variety is Stein; see [**59**, Proposition VI.3.1]. In the literature, a Stein space is sometimes defined as the zero set of a collection of analytic functions on $\mathbb{C}^n$. Our definition is more inclusive: a Stein space in our sense may not be embeddable as a whole in any $\mathbb{C}^n$, but for any compact subset $K \subset S$ one can find open neighborhoods of K that embed in $\mathbb{C}^n$, for some n. The distinction has no impact on our story as we are interested in algebraic varieties, where the two concepts agree.

Let S be a Stein manifold. The sheaves in $\Omega^\bullet_S$ are coherent, and it is an acyclic resolution of the constant sheaf $\mathcal{C}$ on S. Using Theorem 2.26 one obtains the following result; see [**138, 139**].

Theorem 19.12. *On a Stein manifold S, singular and sheaf cohomology with coefficients in $\mathbb{C}$ are determined by the holomorphic de Rham complex:*

$$H^t(S,\mathbb{C}) = H^t(S,\mathcal{C}) = H^t(\Omega^\bullet(S))\,. \qquad \square$$

Remark 19.13. Suppose S has complex dimension n. The theorem implies that the singular cohomology of S vanishes beyond n because $\Omega^t_S = 0$ for $t > n$. Since the real dimension of S is $2n$, it thus has only half as much cohomology as one might expect. While every non-compact manifold fails Poincaré duality,[1] Stein manifolds fail in particularly grand style.

[1]Unless one uses compactly supported cochains.

Using Morse theory, one can prove that S is homotopy equivalent to a CW-complex of real dimension at most n; see [**120**].

Let $f = 0$ be a hypersurface on the analytic manifold $M = (\mathbb{C}^n)^{an}$. Its complement U_f can be identified with the submanifold of $M \times \mathbb{C}$ defined by the vanishing of $1 - tf$, where t is the coordinate function on $\mathbb{C}$, so it is Stein. Hence the complex $\Omega^\bullet(M)_f$ computes the singular cohomology of U_f. When f is a polynomial, this has been used to give algorithms for computing cohomology; see [**124, 125, 158**].

Let M be a complex analytic manifold. We want to relate singular cohomology to de Rham cohomology. Let $\mathcal{F}$ be a sheaf on M. If there is a cover $\mathfrak{V}$ of M as in Theorem 2.26, then $H^\bullet(M, \mathcal{F})$ can be computed using the Čech complex on any open cover that is acyclic for $\mathcal{F}$. This extends to a complex of sheaves; we outline the idea in an example. First a definition:

A *Stein cover* of M is a cover by Stein spaces $\{S_i\}$ for which finite intersections $S_I = \bigcap_{i\in I} S_i$ are Stein as well.

Example 19.14. Let M be as in Example 19.9, and set $U_1 = \operatorname{Spec} \mathbb{C}[x_1/x_0]$ and $U_2 = \operatorname{Spec} \mathbb{C}[x_0/x_1]$. The cover $M = U_1 \cup U_2$ is Stein, as each U_i is Stein, and so is their intersection $U_{1,2}$. By Theorem 19.12, one has:

$$\begin{aligned}
H^t(U_1, \mathbb{C}) &= H^t\big(\mathbb{C}\{y\} \longrightarrow \mathbb{C}\{y\}\, dy\big)\,,\\
H^t(U_2, \mathbb{C}) &= H^t\big(\mathbb{C}\{y^{-1}\} \longrightarrow \mathbb{C}\{y^{-1}\}\, d(y^{-1})\big)\,,\\
H^t(U_{1,2}, \mathbb{C}) &= H^t\big(\,\mathbb{C}\{y, y^{-1}\} \longrightarrow \mathbb{C}\{y, y^{-1}\}\, dy\,\big)\,,
\end{aligned}$$

where $\mathbb{C}\{y\}$, $\mathbb{C}\{y^{-1}\}$, and $\mathbb{C}\{y, y^{-1}\}$ are the global holomorphic functions on $\mathbb{P}^1 \smallsetminus \{\infty\}$, $\mathbb{P}^1 \smallsetminus \{0\}$, and $\mathbb{P}^1 \smallsetminus \{0, \infty\}$ respectively. One has a diagram

$$\begin{array}{ccc}
\mathbb{C}\{y\}\, dy \oplus \mathbb{C}\{y^{-1}\}\, d(y^{-1}) & \longrightarrow & \mathbb{C}\{y, y^{-1}\}\, dy \\
{\scriptstyle d}\big\uparrow & & \big\uparrow{\scriptstyle d} \\
\mathbb{C}\{y\} \oplus \mathbb{C}\{y^{-1}\} & \longrightarrow & \mathbb{C}\{y, y^{-1}\}
\end{array}$$

where the lower row is a Čech complex computing the cohomology of $\mathcal{O}_M$; the upper is the one for Ω^1_M. The column on the left is $\Omega^\bullet(U_1) \oplus \Omega^\bullet(U_2)$ and the one on the right is $\Omega^\bullet(U_{1,2})$. With a suitable choice of signs, one gets the holomorphic *Čech-de Rham complex* of M relative to the chosen cover.

The general construction is the following.

Definition 19.15. The holomorphic *Čech-de Rham complex* $\Omega^\bullet_M(\mathfrak{S})$ of an open cover $\mathfrak{S} = \{S_i\}_{i\in I}$ of M is a double complex where the t-th row is the Čech complex of Ω^t_M, with differential multiplied by $(-1)^t$, and the k-th column is a product of holomorphic de Rham complexes:

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\uparrow & & \uparrow & & \\
\prod_{i\in I}\Omega^1(S_i) & \longrightarrow & \prod_{i<j}\Omega^1(S_i\cap S_j) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \\
\prod_{i\in I}\mathcal{O}(S_i) & \longrightarrow & \prod_{i<j}\mathcal{O}(S_i\cap S_j) & \longrightarrow & \cdots
\end{array}$$

The following is the complex analytic version of de Rham's theorem.

Theorem 19.16. *Let M be a complex manifold, and $\mathfrak{S}$ a Stein cover. There are natural isomorphisms:*

$$H^t(M,\mathbb{C}) \cong H^t(\Omega_M^\bullet(\mathfrak{S}))\,,$$

for $\Omega_M^\bullet(\mathfrak{S})$ the total complex of the holomorphic Čech-de Rham complex.

Sketch of proof. Replace each of the groups $\prod_J \Omega^t(S_J)$ by the corresponding sheaves $\prod_J \Omega^t_{S_J}$. One gets a double complex of sheaves whose global sections form the holomorphic Čech-de Rham complex. Each column is a resolution of the corresponding constant sheaf $\prod_J \mathcal{C}_{S_J}$. Therefore the total complex is a resolution of $\mathcal{C}$. It is an acyclic resolution since each S_J is Stein. Hence, computing global sections gives the sheaf cohomology of $\mathcal{C}$ on M, which agrees with singular cohomology. $\square$

The total complex of the Čech-de Rham complex in Example 19.14 is

$$\mathbb{C}\{y\}\oplus\mathbb{C}\{y^{-1}\} \longrightarrow \mathbb{C}\{y\}dy\oplus\mathbb{C}\{y^{-1}\}d(y^{-1})\oplus\mathbb{C}\{y,y^{-1}\} \longrightarrow \mathbb{C}\{y,y^{-1}\}dy.$$

Evidently, the cohomology in degree zero is spanned by $(1,1)$, and there is no cohomology in degree one. In degree two, the cohomology is spanned by $y^{-1}dy$. These computations agree with the preconceptions one has about the singular cohomology of $\mathbb{P}^1_{\mathbb{C}} = \mathbb{S}^2$.

We close this section with an example that shows that not all Zariski-open subsets of $\mathbb{C}^n$ are Stein: complements of divisors are special.

Example 19.17. The space $M = \mathbb{C}^2 \smallsetminus \{0\}$ is homotopy equivalent to $\mathbb{S}^3$, so $H^3(M,\mathbb{C}) = \mathbb{C}$. As M is a 2-dimensional complex manifold, it does not have nonzero differentials of order 3. Hence M is not Stein; see Theorem 19.12.

3. The algebraic case

We investigate the extent to which results on complex analytic manifolds remain true for complex smooth varieties with the Zariski topology.

Exercise 19.18. Consider the algebraic de Rham complex of $R = \mathbb{C}[x, y]$:

$$0 \longrightarrow R \longrightarrow R\,dx \oplus R\,dy \longrightarrow R\,dx \wedge dy \longrightarrow 0\,.$$

Prove that $H^0(\Omega_R^\bullet) = \mathbb{C}$ and $H^{>0}(\Omega_R^\bullet) = 0$. Thus, $H^t(\mathbb{C}^2, \mathbb{C}) = H^t(\Omega_R^\bullet)$.

Let X be a smooth algebraic variety over $\mathbb{C}$. There is an associated complex analytic manifold X^{an}, as in Remark 19.7, and a natural continuous map $X^{an} \longrightarrow X$. Using the pullback functor, we view $\Omega_X^\bullet$ as a complex on X^{an}; there is an induced embedding $\Omega_X^\bullet \hookrightarrow \Omega_{X^{an}}^\bullet$. A natural question is whether this map is a quasi-isomorphism. A positive answer is of interest for $\Omega_X^\bullet$ is smaller than $\Omega_{X^{an}}^\bullet$.

We first look at the case where X is affine, for then X^{an} is Stein.

Example 19.19. Set $X = \operatorname{Spec} \mathbb{C}[x, x^{-1}]$. The form dx/x is not a global derivative since the branches of the local integral $\log x$ do not patch to a global function. The same problem carries over to the algebraic case. In the analytic situation, X has an open cover by simply connected sets on which $\log x$ has a well-defined branch. Proper open sets in the Zariski topology are complements of finitely many points, and are far from simply connected. It follows that the algebraic de Rham complex has a nontrivial first cohomology group on each Zariski open subset of X.

At this point we record a miracle: singular cohomology can be expressed in terms of purely algebraic data; see [**52, 60**]. It is also surprising that while the complex of shaves $\Omega_X^\bullet$ is typically not a resolution of the constant sheaf, its cohomology coincides with that of the constant sheaf.

Theorem 19.20 (Grothendieck's comparison theorem)**.** *Let $X = \operatorname{Spec} R$ be a smooth variety over $\mathbb{C}$. The singular cohomology of X^{an} is naturally isomorphic to the cohomology of the algebraic de Rham complex of R:*

$$H^t(X^{an}, \mathbb{C}) = H^t(\Omega_R^\bullet). \qquad \square$$

For example, for $X = \operatorname{Spec} \mathbb{C}[x, x^{-1}]$ the algebraic de Rham complex is

$$0 \longrightarrow \mathbb{C}[x, x^{-1}] \longrightarrow \mathbb{C}[x, x^{-1}]\,dx \longrightarrow 0\,.$$

Its cohomology is precisely the singular cohomology of X^{an}.

Exercise 19.21. Compute the singular cohomology of the affine open sets:

(1) $\mathbb{C}^1 \smallsetminus \{0, 1\}$.
(2) $\mathbb{C}^2$ with coordinate axes removed.
(3) The complement of $\operatorname{Var}(xy(x-1))$ in $\mathbb{C}^2$.
(4) The complement of $\operatorname{Var}(xy(x-y))$ in $\mathbb{C}^2$.
(5) The complement of $\operatorname{Var}(x(x-1),\, xy)$ in $\mathbb{C}^2$.

If you found this exercise hard, see what *Macaulay 2* can do for you.

Exercise 19.22. Show that Grothendieck's theorem does not extend to varieties over $\mathbb{R}$. Hint: Take the hyperbola $xy = 1$.

Remark 19.23. Grothendieck's comparison theorem extends to schemes. This requires *hypercohomology*, which is analogous to derived functors for complexes of modules.

Let $\mathcal{G}^\bullet$ be a finite complex of sheaves. One can construct a quasi-isomorphism $\mathcal{G}^\bullet \longrightarrow \mathcal{F}^\bullet$ of complexes of sheaves where $\mathcal{F}^\bullet$ consists of flasque sheaves; see [**22, 47**]. The t-th *hypercohomology* of the complex $\mathcal{G}^\bullet$, denoted $\mathbb{H}^t(X, \mathcal{G}^\bullet)$, is $H^t(\mathcal{F}^\bullet(X))$. For a sheaf $\mathcal{G}$, one has $\mathbb{H}^t(X, \mathcal{G}) = H^t(X, \mathcal{G})$.

The following result, which is an algebraic analogue of Theorem 19.16, extends Theorem 19.20.

Theorem 19.24. *Let X be a smooth scheme over $\mathbb{C}$ and $\mathfrak{U}$ an open affine cover of X. There are natural isomorphisms:*

$$H^t(X^{an}, \mathbb{C}) \cong H^t(\Omega^\bullet_X(\mathfrak{U})) \cong \mathbb{H}^t(X, \Omega^\bullet_X)\,,$$

for $\Omega^\bullet_X(\mathfrak{U})$ the total complex of the holomorphic Čech-de Rham complex. □

This result mirrors the theme indicated before: one may either replace the given sheaf (or complex of sheaves) by acyclic ones and compute cohomology, or use Čech cohomology. See [**60**] for the case of singular varieties, and [**124, 158, 159**] for algorithmic aspects.

4. Local and de Rham cohomology

In this section we use the Čech-de Rham complex to obtain an upper bound on the index of the top singular cohomology group of a Zariski open set $\mathbb{C}^n \smallsetminus X$ in terms of the local cohomological dimension of X. We then apply this bound in a familiar example.

Theorem 19.25. *Let $R = \mathbb{C}[x_1, \dots, x_n]$ and let $\mathfrak{a}$ be an ideal generated by elements $g_1, \dots, g_r$. Set $U = \mathbb{C}^n \smallsetminus \operatorname{Var}(\mathfrak{a})$.*

The singular cohomology groups $H^i(U^{an}, \mathbb{C})$ vanish for $i \geqslant \operatorname{cd}_R(\mathfrak{a}) + n$.

Sketch of a proof. The most transparent proof is via a spectral sequence arising from double complexes, [**22**], and it goes as follows.

The open cover $\{\mathbb{C}^n \smallsetminus \operatorname{Var}(g_i)\}$ is a Stein cover of U. For $t > \operatorname{cd}_R(\mathfrak{a})$ we have $H^t_{\mathfrak{a}}(M) = 0$ for all R-modules M, so the rows of the corresponding holomorphic Čech-de Rham complex are exact beyond the $\operatorname{cd}_R(\mathfrak{a}) - 1$ column. Since the double complex is zero beyond row n, it follows that the cohomology of the associated total complex is zero beyond cohomological degree $n + \operatorname{cd}_R(\mathfrak{a}) - 1$. Now use Theorem 19.24. □

We revisit Example 1.34.

Example 19.26. Let R be the ring associated to a generic 2×3 matrix over a field $\mathbb{K}$, and $\mathfrak{a}$ the ideal of 2×2 minors. Then $\mathfrak{a}$ has height 2 and is 3-generated, so $H^t_{\mathfrak{a}}(R) = 0$ for $t \neq 2, 3$. When $\mathbb{K}$ is of positive characteristic, $H^3_{\mathfrak{a}}(R) = 0$; see Example 21.31.

Suppose $\mathbb{K}$ has characteristic zero; we prove that $H^3_{\mathfrak{a}}(R) \neq 0$. By a base change argument, we may assume that $\mathbb{K} = \mathbb{C}$. By Theorem 19.25, it suffices to prove that $H^8_{\mathrm{dR}}(U, \mathbb{C}) \neq 0$, where $U = \mathbb{C}^6 \smallsetminus \mathrm{Var}(\mathfrak{a})$.

We claim that, as a topological space, U is homotopy equivalent to the space V of 2×3 matrices with orthogonal rows of unit length. Indeed, each matrix in U has rank 2. Scale the first row, varying continuously, until it is of unit length. Next, subtract a multiple of the first row from the second, continuously varying the multiplier, until the rows are orthogonal. Lastly, scale the second row such that it has unit length.

Since U and V are homotopy equivalent, one has $H^\bullet(U, \mathbb{C}) = H^\bullet(V, \mathbb{C})$. Consider the map from V to $\mathbb{C}^3 = \mathbb{R}^6$ sending a matrix to its first row. Its image is evidently $\mathbb{S}^5$, and the fibers of the map $V \longrightarrow \mathbb{S}^5$ are vectors of unit length in $\mathbb{C}^2 = \mathbb{R}^4$, and hence are identified with $\mathbb{S}^3$. Thus, V is an 8-dimensional real compact manifold.

As $\mathbb{S}^3$ and $\mathbb{S}^5$ are orientable, and the fibration is locally trivial, one can combine the two orientations to a local orientation of V. This gives a well-defined orientation on V: for any loop in V along which the orientation is locally constant, orientations at the start and finish coincide, because $\mathbb{S}^5$ is simply-connected. Since V is an 8-dimensional compact orientable manifold, one has $H^8(V, \mathbb{C}) = \mathbb{C}$, and so $H^8_{\mathrm{dR}}(U, \mathbb{C}) = \mathbb{C}$.

Remark 19.27. For any algebraically closed field $\mathbb{K}$, and a prime number ℓ different from the characteristic of $\mathbb{K}$, the *étale cohomology* groups $H^\bullet_{et}(-, \mathbb{Z}/\ell\mathbb{Z})$ satisfy many of the properties one is accustomed to, such as functoriality, Poincaré duality, Mayer-Vietoris sequences, and a Künneth formula; see [**42, 119**]. Using these, one can show that $H^8_{et}(U, \mathbb{Z}/\ell\mathbb{Z}) \neq 0$ where $U = \mathbb{A}^6 \smallsetminus \mathrm{Var}(\mathfrak{a})$ for $\mathfrak{a}$ as in the previous example. It follows that in all characteristics the arithmetic rank of $\mathfrak{a}$ is 3.

Exercise 19.28. Consider the algebraic set in $\mathbb{C}^{n \times (n+1)}$ consisting of complex $n \times (n+1)$-matrices of rank less than n. Find the minimum number of polynomials needed to define this set.

Local Cohomology over Semigroup Rings

Semigroup rings are rings generated over a field by monomials. Geometrically, they give rise to toric varieties. Their combinatorial polyhedral nature makes semigroup rings perhaps the easiest reasonably broad class of algebras over which to compute local cohomology explicitly. On the other hand, the singularities of semigroup rings are sufficiently general for their local cohomology to exhibit a wide range of interesting—and sometimes surprising—phenomena. The purpose of this lecture is to introduce some of the $\mathbb{Z}^d$-graded techniques used to do homological algebra over semigroup rings, including applications to quintessential examples. The key idea is to resolve modules by polyhedral subsets of $\mathbb{Z}^d$.

1. Semigroup rings

Definition 20.1. An *affine semigroup ring* is a subring of a Laurent polynomial ring generated by finitely many monomials.

Most of the affine semigroup rings in these lecture are *pointed* (see Definition 20.16), which is equivalent to their being subrings of honest polynomial rings instead of Laurent polynomial rings. The simplest example is a polynomial ring itself, but we have seen many other semigroup rings:

Example 20.2. The example from Section 1.4 is an affine semigroup ring. It appears also in Example 10.22 as a ring of invariants. This is not a coincidence; see [**118**, Chapter 10]. In Example 19.26, local cohomology is taken with support on its defining ideal.

Example 20.3. The ring $\mathbb{K}[s^4, s^3t, st^3, t^4]$ is an affine semigroup ring by definition. It has appeared in Examples 10.7, 10.12, and 10.19, which illustrate different ways to see the failure of the Cohen-Macaulay property; see Example 20.30 for yet another.

Example 20.4. The ring $\mathbb{K}[w, x, y, z]/(wx-yz)$, whose localization appears in Exercise 1.20, is isomorphic to the affine semigroup ring $\mathbb{K}[r, rst, rs, rt] \subset \mathbb{K}[r, s, t]$. Certain local cohomology modules of this ring behave quite badly; in Section 5, we will compute one explicitly.

By virtue of being generated by monomials, semigroup rings carry a lot of extra structure. Let us start by writing the monomials in a semigroup ring R using variables $t = t_1, \ldots, t_d$. Thus t^b for $b \in \mathbb{Z}^d$ is shorthand for the Laurent monomial $t_1^{b_1} \cdots t_d^{b_d}$. The set

$$Q = \{b \in \mathbb{Z}^d \mid t^b \in R\}$$

is a subset of $\mathbb{Z}^d$ that is closed under addition and contains $0 \in \mathbb{Z}^d$; hence Q is, by definition, a commutative *semigroup*.[1] Given the extra condition that Q is generated (under addition) by finitely many vectors—namely, the exponents on the generators of R—the semigroup Q is an *affine semigroup*.

Exercise 20.5. Find a subsemigroup of $\mathbb{N}^2$ containing 0 that is not finitely generated. Find uncountably many examples.

An affine semigroup $Q \subseteq \mathbb{Z}^d$ generates a subgroup $\langle Q \rangle$ under addition and subtraction. In general, $\langle Q \rangle$ might be a proper subgroup of $\mathbb{Z}^d$, and there are many reasons for wanting to allow $\langle Q \rangle \neq \mathbb{Z}^d$. Often in natural situations, the rank of $\langle Q \rangle$ can even be less than d; see [**151**, Chapter 1], for example, where the connection with solving linear diophantine equations is detailed (in terms of local cohomology, using techniques based on those in this lecture!). That being said, we adopt the following convention:

Notation 20.6. For the purposes of studying the intrinsic properties of a semigroup ring R, we can and do assume for simplicity that $\langle Q \rangle = \mathbb{Z}^d$.

Basic properties of R can be read directly off the semigroup Q. The reason is that

$$R = \mathbb{K}[Q] = \bigoplus_{b \in Q} \mathbb{K} \cdot t^b$$

is a $\mathbb{K}$-vector space, with basis consisting of the monomials in R.

[1] The correct term here is *monoid*, meaning "semigroup with unit element." An affine semigroup is defined to be a monoid; thus, when we say "Q is generated by a set A," we mean that every element in Q is a sum—perhaps with repeated terms and perhaps empty—of elements of A. Allowing the empty sum, which equals the identity element, requires us to generate Q as a monoid, not just as a semigroup.

Lemma 20.7. *If $\langle Q \rangle = \mathbb{Z}^d$ then the semigroup ring $\mathbb{K}[Q]$ has dimension d.*

Proof. The dimension can be computed as the transcendence degree of the fraction field of $\mathbb{K}[Q]$ over $\mathbb{K}$. Inverting the monomials that generate R, we reduce to the Laurent polynomial ring $\mathbb{K}[\mathbb{Z}^d]$ which, by a similar argument, has the same dimension as the polynomial ring $\mathbb{K}[\mathbb{N}^d]$. $\square$

The extra structure induced by the decomposition of R into rank-one vector spaces is a *fine grading*, as opposed to the *coarse grading* by $\mathbb{Z}$.

Exercise 20.8. Fix a semigroup ring $R = \mathbb{K}[t^{a_1}, \ldots, t^{a_n}]$ and take new variables $x = x_1, \ldots, x_n$. Prove that the kernel of the map $\mathbb{K}[x] \twoheadrightarrow R$ sending $x_i \longmapsto t^{a_i}$ is the ideal

$$I_A = (x^u - x^v \mid Au = Av)\,,$$

where A is the $d \times n$ matrix with columns $a_1, \ldots, a_n$ and $u, v \in \mathbb{Z}^n$ are column vectors of size n. The ideal I_A is called the *toric ideal* for A; find all examples of this notion in these twenty-four lectures.

Hint: Use the $\mathbb{Z}^d$-grading and mimic Exercises 1.35 and 1.36.

2. Cones from semigroups

Suppose that Q is an affine semigroup with $\langle Q \rangle = \mathbb{Z}^d$. Taking nonnegative real combinations of elements of Q instead of nonnegative integer combinations yields a *rational polyhedral cone* $C_Q = \mathbb{R}_{\geqslant 0} Q$. The adjective 'rational' means that C_Q is generated as a cone by integer vectors, while 'polyhedral' means that C_Q equals the intersection of finitely many closed halfspaces.

Example 20.9. Consider the semigroup $\mathbb{K}[s^4, s^3t, st^3, t^4]$ of Example 20.3. By our convention from Notation 20.6, the lattice $\langle Q \rangle = \mathbb{Z}^2$ is *not* the standard lattice in $\mathbb{R}^2$; instead, $\langle Q \rangle$ is generated as an Abelian group by, for instance, $(4,0)$ and $(-1,1)$. Under the isomorphism of $\langle Q \rangle$ with $\mathbb{Z}^2$ sending these two generators to the two basis vectors, Q is isomorphic to the semigroup generated by

$$\{(1,0),\ (1,1),\ (1,3),\ (1,4)\}\,.$$

The real cone C_Q in this latter representation consists of all (real) points above the horizontal axis and below the line of slope 4 through the origin.

Example 20.10. The ring $\mathbb{K}[r, rst, rs, rt]$ from Example 20.4 is $\mathbb{K}[Q]$ for the affine semigroup Q generated by

$$\{(1,0,0),\ (1,1,1),\ (1,1,0),\ (1,0,1)\}\,.$$

These four vectors are the vertices of a unit square in $\mathbb{R}^3$, and C_Q is the real cone over this square from the origin; see Figure 20.1. The lattice points in C_Q constitute Q itself.

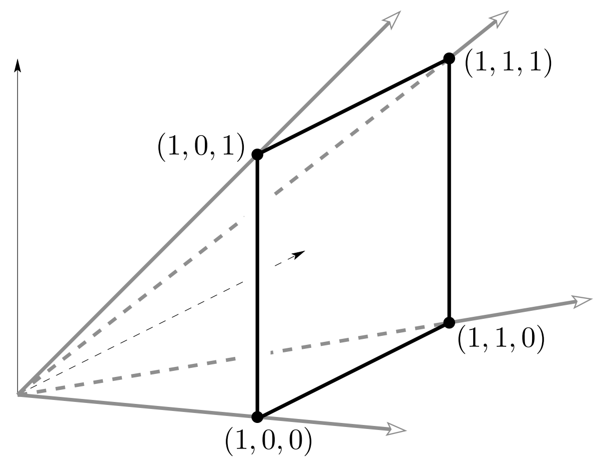


Figure 20.1. The cone over a square in Example 20.10

Theorem 20.11. *The ring* $\mathbb{K}[Q]$ *is normal if and only if* $Q = C_Q \cap \mathbb{Z}^d$.

Proof. Suppose first that $Q = C_Q \cap \mathbb{Z}^d$, and write $C_Q = \bigcap H_i^+$ as an intersection of closed halfspaces. Then $Q = \bigcap \big(H_i^+ \cap \mathbb{Z}^d\big)$ is an intersection of semigroups, each of the form $H^+ \cap \mathbb{Z}^d$. Hence $\mathbb{K}[Q]$ is an intersection of semigroup rings $\mathbb{K}[H^+ \cap \mathbb{Z}^d]$ inside the Laurent polynomial ring $\mathbb{K}[\mathbb{Z}^d]$. Each of these subrings is isomorphic to $\mathbb{K}[t_1, \ldots, t_d, t_2^{-1}, \ldots, t_d^{-1}]$; note that t_1 has not been inverted. This localization of a polynomial ring is normal, and therefore so is $\mathbb{K}[Q]$.

Next let us assume that $Q \subsetneq C_Q \cap \mathbb{Z}^d$. Exercise 20.12 implies that there is a monomial t^a in $\mathbb{K}[\mathbb{Z}^d]$ such that $(t^a)^m \in \mathbb{K}[Q]$ but $t^a \notin \mathbb{K}[Q]$. This monomial is a root of the monic polynomial $y^m - t^{am}$. □

Exercise 20.12. Show that if $a \in C_Q \cap \mathbb{Z}^d$, then $m \cdot a \in Q$ for all sufficiently large integers m.

Example 20.13. The semigroup ring from Examples 20.3 and 20.9 is not normal, since $s^2t^2 \notin \mathbb{K}[s^4, s^3t, st^3, t^4]$ but

$$s^2t^2 = \frac{s^4 \cdot st^3}{s^3t} \quad \text{and} \quad (s^2t^2)^2 \in \mathbb{K}[s^4, s^3t, st^3, t^4]\,.$$

Example 20.14. The semigroup ring from Example 20.4 is normal by the last sentence of Example 20.10.

Any rational polyhedral cone C has a unique smallest face (the definition of face is unchanged from the notion for polytopes as in Lecture 16). This smallest face clearly contains the origin, but it also contains any vector $v \in C$ such that $-v$ also lies in C. The cone C is called *pointed* if 0 is the only vector in C whose negative also lies in C. Geometrically, there is a hyperplane H such that C lies on one side of H and intersects H only at 0. Thus C "comes to a point" at the origin.

Thinking of Q instead of C_Q, any vector $a \in Q$ such that $-a \in Q$ corresponds to a monomial $t^a \in \mathbb{K}[Q]$ whose inverse also lies in $\mathbb{K}[Q]$; that is, t^a is a unit in $\mathbb{K}[Q]$. Such a monomial can't lie in any proper ideal of $\mathbb{K}[Q]$. On the other hand, the ideal generated by all nonunit monomials is a proper ideal, the *maximal monomial ideal*. It is the largest ideal of $\mathbb{K}[Q]$ that is $\mathbb{Z}^d$-graded, but it need not be a maximal ideal.

Exercise 20.15. For an affine semigroup Q, the following are equivalent.

(1) The maximal monomial ideal of $\mathbb{K}[Q]$ is a maximal ideal.

(2) The real cone C_Q is pointed.

(3) Q has no nonzero units; that is, $a \in Q$ and $-a \in Q$ implies $a = 0$.

Definition 20.16. If the conditions of Exercise 20.15 hold, Q is *pointed*.

A cone C is pointed when there is a hyperplane H intersecting it in exactly one point. Intersecting a (positively translated) parallel hyperplane with C yields a *transverse section* of C, which is a polytope P. The geometry of P can depend on the support hyperplane H, but the combinatorics of P is intrinsic to Q: the poset of faces of P is the same as the poset of faces of Q.

Example 20.17. The square in Example 20.10, namely the convex hull of the four generators of Q, is a transverse section of the cone C_Q.

In the coming sections, we will see how the homological properties of a semigroup ring $\mathbb{K}[Q]$ are governed by the combinatorics of C_Q.

3. Maximal support: the Ishida complex

Let Q be a pointed affine semigroup with associated real cone C_Q, and P_Q a transverse section. The polytope P_Q is a cell complex, so, choosing relative orientations for its faces, it has algebraic chain and cochain complexes.

Example 20.18. Let Q be as in Example 20.17, and let P_Q be the square there. The reduced cochain complex of P_Q with coefficients in $\mathbb{K}$ is

$$0 \longrightarrow \underset{\varnothing}{\mathbb{K}} \xrightarrow{\varphi^0} \underset{\text{vertices}}{\mathbb{K}^4} \xrightarrow{\varphi^1} \underset{\text{edges}}{\mathbb{K}^4} \xrightarrow{\varphi^2} \underset{\text{all of } P_Q}{\mathbb{K}} \longrightarrow 0\,.$$

The map φ^0 takes the $\varnothing$-basis vector to the sum of the basis vectors in $\mathbb{K}^4$. The map φ^1 takes the basis vector corresponding to a vertex v to the signed sum of all edges with v as an endpoint; the signs are determined by an (arbitrary) orientation: positive if the edge ends at v, negative if the edge begins at v. Since P_Q is convex—and hence contractible—the cohomology of the above complex is identically zero.

In the same way that the cochain complex of a simplex gives rise to the Čech complex over the polynomial ring, the cochain complex of P_Q gives rise to a complex of localizations of any pointed affine semigroup ring $\mathbb{K}[Q]$. Describing this complex precisely and presenting its role in local cohomology is the goal of this section. Let us first describe the localizations.

Recall that a face of the real cone C_Q is by definition the intersection of C_Q with a support hyperplane. Since C_Q is finitely generated as a cone, it has finitely many faces, just as the transverse section polytope P_Q does, although most of the faces of C_Q are unbounded, being themselves cones.

Definition 20.19. The intersection of Q with a face of C_Q is a *face* of Q.

Lemma 20.20. *Let $F \subseteq Q$ be a face. The set of monomials $\{t^b \mid b \notin F\}$ is a prime ideal $\mathfrak{p}_F$ of $\mathbb{K}[Q]$.*

Proof. To check that $\mathfrak{p}_F$ is an ideal it is enough, in view of the ambient $\mathbb{Z}^d$-grading, to check that it is closed under multiplication by monomials from $\mathbb{K}[Q]$. Let ν be a normal vector to a support hyperplane for F such that $\nu(Q) \geqslant 0$. Thus $\nu(f) = 0$ for some $f \in Q$ if and only if $f \in F$. Assume that $t^b \in \mathfrak{p}_F$ and $t^a \in \mathbb{K}[Q]$. Then $\nu(a+b) \geqslant \nu(b) > 0$, whence the product $t^a t^b = t^{a+b}$ lies in $\mathfrak{p}_F$.

The ideal $\mathfrak{p}_F$ is prime because the quotient $\mathbb{K}[Q]/\mathfrak{p}_F$ is isomorphic to the affine semigroup ring $\mathbb{K}[F]$, which is an integral domain. $\square$

Notation 20.21. Let F be a face of Q. Write $\mathbb{K}[Q]_F$ for the localization of $\mathbb{K}[Q]$ by the set of monomials t^f for $f \in F$.

Exercise 20.22. The localization $\mathbb{K}[Q]_F$ is the semigroup ring $\mathbb{K}[Q-F]$ for the affine semigroup

$$Q - F = \{q - f \mid q \in Q \text{ and } f \in F\}.$$

Note that $Q - F$ is non-pointed as long as F is nonempty.

Here now is the main definition of this lecture.

Definition 20.23. The *Ishida complex* $\mho_Q^\bullet$ of the semigroup Q, or of the semigroup ring $\mathbb{K}[Q]$, is the complex

$$0 \to \mathbb{K}[Q] \to \bigoplus_{\text{rays } F} \mathbb{K}[Q]_F \to \cdots \bigoplus_{i\text{-faces } F} \mathbb{K}[Q]_F \xrightarrow{\delta^i} \cdots \bigoplus_{\text{facets } F} \mathbb{K}[Q]_F \to \mathbb{K}[\mathbb{Z}^d] \to 0\,,$$

where an i-face is a face F of Q such that $\dim \mathbb{K}[F] = i$; a *ray* is a 1-face and a *facet* is a $(d-1)$-face. The differential δ is composed of natural localization maps $\mathbb{K}[Q]_F \longrightarrow \mathbb{K}[Q]_G$ with signs as in the algebraic cochain complex of the transverse section P_Q. The terms $\mathbb{K}[Q]$ and $\mathbb{K}[\mathbb{Z}^d]$ sit in cohomological degrees 0 and d respectively.

Exercise 20.24. Write down explicitly the Ishida complex for the cone over the square—the semigroup ring from Example 20.10—using Example 20.18.

Note that when $Q = \mathbb{N}^d$, the Ishida complex is precisely the Čech complex on the variables $t_1, \ldots, t_d$. In general, we still have the following.

Theorem 20.25. *Let $\mathbb{K}[Q]$ be a pointed affine semigroup ring with maximal monomial ideal $\mathfrak{m}$. The local cohomology of any $\mathbb{K}[Q]$-module M supported at $\mathfrak{m}$ is the cohomology of the Ishida complex tensored with M, i.e.,*

$$H^i_{\mathfrak{m}}(M) \cong H^i(M \otimes \mho^\bullet_Q)\,. \qquad \square$$

The proof of Theorem 20.25 would take an extra lecture (well, probably less than that); it is mostly straightforward homological algebra. To give you an idea, it begins with the following.

Exercise 20.26. Check that $H^0(M \otimes \mho^\bullet_Q) \cong H^0_{\mathfrak{m}}(M)$.

What's left is to check that $H^i(M \otimes \mho^\bullet_Q)$ is zero when M is injective and $i > 0$, for then $H^i(- \otimes \mho^\bullet_Q)$ agrees with the derived functors of $\Gamma_{\mathfrak{m}}$. The polyhedral nature of $\mho^\bullet_Q$, in particular the contractibility of certain subcomplexes of P_Q, enters into the proof of higher vanishing for injectives; see [**20**, Theorem 6.2.5] and its proof.

The natural maps between localizations in the Ishida complex are $\mathbb{Z}^d$-graded of degree zero, so the local cohomology of a $\mathbb{Z}^d$-graded module is naturally $\mathbb{Z}^d$-graded (we could have seen this much from the Čech complex). Sometimes it is the $\mathbb{Z}^d$-graded degrees of the nonzero local cohomology that are interesting, rather than the cohomological degrees or the module structure; see Lecture 24. In any case, the finely graded structure makes local cohomology computations over semigroup rings much more tractable, since they can be done degree-by-degree.

We could have computed the local cohomology in Theorem 20.25 using a Čech complex, but there is no *natural* choice of elements on which to build one. In contrast, the Ishida complex is based entirely on the polyhedral nature of Q. Combining this with the $\mathbb{Z}^d$-grading provides the truly polyhedral description of the maximal support local cohomology in Corollary 20.29.

Notation 20.27. Write $(\mho^\bullet_Q)_b$ for the complex of $\mathbb{K}$-vector spaces constituting the $\mathbb{Z}^d$-graded degree b piece of $\mho^\bullet_Q$. In addition, let $P_Q(b)$ be the set of faces of P_Q corresponding to faces F of Q with $(\mathbb{K}[Q]_F)_b = 0$, or equivalently, $b \notin Q - F$.

Exercise 20.28. Let Q be a pointed affine semigroup.

(1) Prove that $P_Q(b)$ is a cellular subcomplex of the cell complex P_Q.

(2) Show that $(\mho_Q^\bullet)_b$ is the relative cochain complex for $P_Q(b) \subset P_Q$, up to shifting the cohomological degrees by 1.

Corollary 20.29. *Let Q be a pointed affine semigroup. The degree b part of the local cohomology of $\mathbb{K}[Q]$ is isomorphic to the relative cohomology of the pair $P_Q(b) \subset P_Q$ with coefficients in $\mathbb{K}$, that is,*

$$H^i_{\mathfrak{m}}(\mathbb{K}[Q])_b \cong H^{i-1}(P_Q, P_Q(b); \mathbb{K}) . \qquad \square$$

Example 20.30. We know that the ring $\mathbb{K}[s^4, s^3t, st^3, t^4]$ is not Cohen-Macaulay. Let us see it yet again, this time by way of its local cohomology. Set $b = (2,2)$. Then $\mathbb{K}[Q]_b = 0$, so $\varnothing$ is a face of $P_Q(b)$, but every other localization of $\mathbb{K}[Q]$ appearing in the Ishida complex $\mho_Q^\bullet$ is nonzero in degree b. Thus the complex $(\mho_Q^\bullet)_b$ of $\mathbb{K}$-vector spaces is the complex

$$C^\bullet(P_Q, \varnothing; \mathbb{K}) : \qquad 0 \longrightarrow \mathbb{K} \oplus \mathbb{K} \longrightarrow \mathbb{K} .$$

The cohomology is $\mathbb{K}$ in cohomological degree 1 and zero elsewhere. Hence $H^1_{\mathfrak{m}}(\mathbb{K}[Q])$ is nonzero, so $\mathbb{K}[Q]$ is not Cohen-Macaulay.

The Ishida complex in Example 20.30 turns out to be a Čech complex, as is typical of two-dimensional pointed affine semigroup rings. For higher-dimensional examples, Čech complexes are almost always bigger and less natural than Ishida complexes.

Exercise 20.31. In which $\mathbb{Z}^3$-graded degrees and cohomological degrees is the local cohomology with maximal support of the pointed affine semigroup ring $\mathbb{K}[Q]$ nonzero, if Q is generated by the columns of

$$\begin{bmatrix} 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} ?$$

Hint: These five vectors have equal last coordinates; plot the first two coordinates of each in the plane. Try also drawing the intersection of Q with the coordinate plane in $\mathbb{R}^3$ spanned by $(1,0,0)$ and $(0,0,1)$. Which lattice points are “missing”?

The local cohomology of normal affine semigroup rings behaves so uniformly that we can treat them all at once. The following exercise outlines a proof that normal affine semigroup rings are Cohen-Macaulay. This was proved by Hochster [**67**].

Exercise 20.32 (Hochster’s theorem)**.** Let $\mathbb{K}[Q]$ be a normal affine semigroup ring.

(1) Show that if $P_Q(b)$ equals the boundary of P_Q, consisting of all proper faces of P_Q, then the complex $(\mho^\bullet_Q)_b$ has $\mathbb{K}$ in cohomological degree d and zeros elsewhere.

(2) Prove that if $P_Q(b)$ is properly contained in the boundary of P_Q, then $P_Q(b)$ is contractible. Hint: It has a convex homeomorphic projection.

(3) Deduce that the complex $\mho^\bullet_Q$ has nonzero cohomology only in cohomological degree d.

(4) Conclude that $\mathbb{K}[Q]$ is Cohen-Macaulay and find its canonical module.

You will need to use normality, of course: by Theorem 20.11, checking whether a $\mathbb{Z}^d$-graded degree b lies in Q (or in $Q - F$ for some face F) amounts to checking that b satisfies a collection of linear inequalities coming from the facets of the real cone C_Q. A detailed solution can be found in [**118**, §12.2], although the arguments there have to be Matlis-dualized to agree precisely with the situation here.

Exercise 20.33. Given an affine semigroup ring R, exhibit a finitely generated maximal Cohen-Macaulay R-module.

The intrinsic polyhedral nature of the Ishida complex makes the line of reasoning in the above proof of Hochster's theorem transparent. It would be more difficult to carry out using a Čech complex.

4. Monomial support: $\mathbb{Z}^d$-graded injectives

In the category of $\mathbb{Z}^d$-graded modules over an affine semigroup ring, the injective objects are particularly uncomplicated. In this section we exploit the polyhedral nature of $\mathbb{Z}^d$-graded injectives to calculate local cohomology supported on monomial ideals. As the motivating example, we will discover the polyhedral nature of Hartshorne's famous local cohomology module whose socle is not finitely generated.

Notation 20.34. Given a subset $S \subset \mathbb{Z}^d$, write $\mathbb{K}\{S\}$ for the $\mathbb{Z}^d$-graded vector space with basis S, and let $-S = \{-b \mid b \in S\}$.

Example 20.35. Let Q be a pointed affine semigroup.

(1) As a graded vector space, $\mathbb{K}[Q]$ itself is expressed as $\mathbb{K}\{Q\}$.

(2) The injective hull $E_{\mathbb{K}[Q]}$ of $\mathbb{K}$ as a $\mathbb{K}[Q]$-module is $\mathbb{K}\{-Q\}$.

(3) By Exercise 20.22, the localization of $\mathbb{K}[Q]$ along a face F is $\mathbb{K}\{Q-F\}$.

(4) The vector space $\mathbb{K}\{F - Q\}$ is the $\mathbb{Z}^d$-*graded injective hull* of $\mathbb{K}[F]$.

By $F - Q$ we mean $-(Q - F)$. The justification for the statement in (2) and the nomenclature in (4) are essentially Matlis duality in the $\mathbb{Z}^d$-graded category; see [**118**, §11.2].

Exercise 20.36. Show that $\mathbb{K}\{F-Q\}$ has a natural $\mathbb{K}[Q]$-module structure in which multiplication by the monomial t^f is bijective for all $f \in F$.

The module structure you just found makes $\mathbb{K}\{F-Q\}$ injective in the category of $\mathbb{Z}^d$-graded modules. This statement is not difficult [**118**, Proposition 11.24]; it is more or less equivalent to the statement that the localization $\mathbb{K}[Q-F]$ is flat. As a consequence of injectivity, derived functors of left-exact functors on $\mathbb{Z}^d$-graded modules can be computed using resolutions by such modules. Let us be more precise.

Definition 20.37. Let Q be an affine semigroup. An *indecomposable $\mathbb{Z}^d$-graded injective* is a $\mathbb{Z}^d$-graded translate of $\mathbb{K}\{F-Q\}$ for a face F of Q. A *$\mathbb{Z}^d$-graded injective resolution* of a $\mathbb{Z}^d$-graded $\mathbb{K}[Q]$-module M is a complex

$$0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

of $\mathbb{Z}^d$-graded modules and homomorphisms such that each I^j is a $\mathbb{Z}^d$-graded injective, $H^0(I^\bullet) \cong M$, and $H^j(I^\bullet) = 0$ for $j \geqslant 1$.

The right derived functors that interest us are, of course, local cohomology. In order to return a $\mathbb{Z}^d$-graded module, the support must be $\mathbb{Z}^d$-graded.

Theorem 20.38. *Let $\mathfrak{a} \subset \mathbb{K}[Q]$ be a monomial ideal. The local cohomology of a $\mathbb{Z}^d$-graded module M supported at $\mathfrak{a}$ can be calculated as*

$$H^i_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(I^\bullet)\,,$$

where $I^\bullet$ is any $\mathbb{Z}^d$-graded injective resolution of M. □

This theorem is useful because of the polyhedral nature of $\mathbb{Z}^d$-graded injectives, in analogy with the polyhedral nature of the Ishida complex, combined with the following characterization of $\Gamma_{\mathfrak{a}}$ on $\mathbb{Z}^d$-graded injectives.

Exercise 20.39. The module $\Gamma_{\mathfrak{a}}\mathbb{K}\{F-Q\}$ equals $\mathbb{K}\{F-Q\}$ if the prime ideal $\mathfrak{p}_F$ (in the notation of Lemma 20.20) contains $\mathfrak{a}$, and is zero otherwise.

Next we consider the *$\mathbb{Z}^d$-graded Matlis dual* of the Ishida complex.

Definition 20.40. Let Q be a pointed affine semigroup. The *dualizing complex* $\Omega^\bullet_Q$ of the semigroup ring $\mathbb{K}[Q]$ is

$$0 \to \mathbb{K}[\mathbb{Z}^d] \to \bigoplus_{\text{facets } F} \mathbb{K}[F-Q] \to \cdots \to \bigoplus_{i\text{-faces } F} \mathbb{K}[F-Q] \xrightarrow{\omega^{d-i}} \cdots \to \mathbb{K}[-Q] \to 0\,,$$

where ω is composed of natural surjections $\mathbb{K}[F-Q] \longrightarrow \mathbb{K}[G-Q]$ with signs as in the algebraic chain complex of the transverse section P_Q. The terms $\mathbb{K}[\mathbb{Z}^d]$ and $\mathbb{K}[-Q]$ sit in cohomological degrees 0 and d respectively.

The reader seeing the $\mathbb{Z}^d$-graded point of view for the first time should make sure to understand the following exercise before continuing.

Exercise 20.41. The complex of $\mathbb{K}$-vector spaces in the $\mathbb{Z}^d$-graded degree b piece of the dualizing complex is the $\mathbb{K}$-dual of the complex in $\mathbb{Z}^d$-graded degree $-b$ of the Ishida complex, up to a cohomological degree shift by d:

$$(\Omega_Q^\bullet)_b[d] \cong ((\mho_Q^\bullet)_{-b})^* \qquad \text{for all } b \in \mathbb{Z}^d\,.$$

Remark 20.42. For readers familiar with dualizing complexes as in [**56**], Ishida proved that the complex in Definition 20.40 really is one [**81, 82**]. The "normalized" dualizing complex would place $\mathbb{K}[\mathbb{Z}^d]$ in cohomological degree $-d$ and $\mathbb{K}[-Q]$ in cohomological degree 0.

Exercise 20.41 combined with Hochster's theorem implies the following.

Corollary 20.43. *If the affine semigroup ring $\mathbb{K}[Q]$ is Cohen-Macaulay, then its dualizing complex $\Omega_Q^\bullet$ is a $\mathbb{Z}^d$-graded injective resolution of a module ω_Q. By local duality, it follows that ω_Q is the canonical module $\omega_{\mathbb{K}[Q]}$.*

As a first application, one can compute the Hilbert series of the local cohomology of the canonical module when $\mathbb{K}[Q]$ is normal. The next exercise proves a result of Yanagawa; see [**163**] or [**118**, Theorem 13.14]. Compare with Theorem 16.27, where the support is maximal but the module is the quotient by an arbitrary monomial ideal.

Exercise 20.44. Let $\mathbb{K}[Q]$ be a normal affine semigroup ring, and $\mathfrak{a} \subset \mathbb{K}[Q]$ a monomial ideal.

(1) Associate a polyhedral subcomplex $\Delta \subseteq P_Q$ to (the radical of) $\mathfrak{a}$.

(2) For $b \in \mathbb{Z}^d$, write a polyhedral homological expression in terms of Δ and P_Q for the vector space dimension of the graded piece $H^i_{\mathfrak{a}}(\omega_{\mathbb{K}[Q]})_b$.

Exercise 20.45. Let Δ be the simplicial complex consisting of the isolated point z and the line segment (x, y). Find I_Δ, $H^\bullet_{I_\Delta}(\mathbb{K}[x, y, z])$, and $H^\bullet_{\mathfrak{m}}(\mathbb{K}[\Delta])$.

5. Hartshorne's example

Our final example is the main example in [**58**], although the methods here are different, since they rely on the $\mathbb{Z}^d$-grading. Prior to Hartshorne's example, Grothendieck had conjectured that the socle of a local cohomology module should always have finite dimension as a vector space over $\mathbb{K}$.

Let Q be the cone-over-the-square semigroup in Examples 20.4 and 20.10. Retain the notation from those two examples.

The ideal $\mathfrak{a} = (rst, rt)$ is the prime ideal $\mathfrak{p}_F$ for the 2-dimensional facet F of Q generated by $(1,0,0)$ and $(1,1,0)$ and hence lying flat in the horizontal plane. Let us compute the local cohomology modules $H^i_{\mathfrak{a}}(\omega_Q)$ of the canonical module using the dualizing complex.

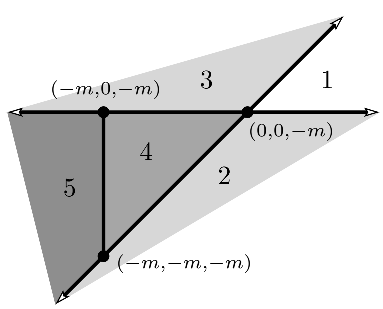
Exercise 20.46. Prove that, ignoring the grading, $\mathbb{K}[Q] \cong \omega_Q$, so the results below really hold for the local cohomology modules $H^i_{\mathfrak{a}}(\mathbb{K}[Q])$ of the semigroup ring itself.

Let A and B be the rays of Q forming the boundary of F, with A along the axis spanned by $(1,0,0)$ and B cutting diagonally through the horizontal plane. The only monomial prime ideals containing $\mathfrak{a}$ are $\mathfrak{a}$ itself, the ideals $\mathfrak{p}_A$ and $\mathfrak{p}_B$, and the maximal monomial ideal $\mathfrak{m}$. By Exercise 20.39, applying $\Gamma_{\mathfrak{p}}$ to the dualizing complex therefore yields the complex $\Gamma_{\mathfrak{a}} I^\bullet$ below:

$$0 \longrightarrow \mathbb{K}\{F - Q\} \longrightarrow \mathbb{K}\{A - Q\} \oplus \mathbb{K}\{B - Q\} \longrightarrow \mathbb{K}\{-Q\} \longrightarrow 0\,.$$

For lack of a better term, we call each of the four indecomposable $\mathbb{Z}^3$-graded injectives a *summand*.

Consider the nonzero contributions of the four summands to a $\mathbb{Z}^3$-graded degree $b = (\alpha, \beta, \gamma)$. If $\gamma > 0$, then none of the four summands contribute, because then $\mathbb{K}\{G - Q\}_b = 0$ whenever G is one of the faces F, A, B, or $\{0\}$ of Q. However, the halfspace beneath the horizontal plane, consisting of vectors b with $\gamma \leqslant 0$, is partitioned into five *sectors*. For degrees b in a single sector, the subset of the four summands contributing a nonzero vector space to degree b remains constant. The summands contributing to each sector are listed in Figure 20.2, which depicts the intersections of the sectors with the plane $\gamma = -m$ as the five regions.



$1 : \mathbb{K}\{F{-}Q\}$
$2 : \mathbb{K}\{F{-}Q\}, \mathbb{K}\{A{-}Q\}$
$3 : \mathbb{K}\{F{-}Q\}, \qquad \mathbb{K}\{B{-}Q\}$
$4 : \mathbb{K}\{F{-}Q\}, \mathbb{K}\{A{-}Q\}, \mathbb{K}\{B{-}Q\}$
$5 : \mathbb{K}\{F{-}Q\}, \mathbb{K}\{A{-}Q\}, \mathbb{K}\{B{-}Q\}, \mathbb{K}\{-Q\}$

Figure 20.2. Intersections of sectors with a horizontal plane

Only in sectors 1 and 4 does $\Gamma_{\mathfrak{a}} I^\bullet$ have any cohomology. The cone of integer points in sector 1 and the cohomology of $\Gamma_{\mathfrak{a}} I^\bullet$ there are as follows:

$$\text{sector } 1: \qquad \gamma \leqslant 0 \text{ and } \alpha > \beta > 0 \qquad \Longleftrightarrow \qquad H^1_{\mathfrak{a}}(\omega_Q)_b = \mathbb{K}\,.$$

For sector 4, we get the cone of integer points and cohomology as follows:

$$\text{sector 4}: \qquad 0 \geqslant \beta \geqslant \alpha > \gamma \quad \iff \quad H^2_{\mathfrak{a}}(\omega_Q)_b = \mathbb{K}\,.$$

We claim that sector 4 has infinitely many degrees with socle elements of $H^2_{\mathfrak{a}}(\omega_Q)$: they occupy all degrees $(0, 0, -m)$ for $m > 0$. This conclusion is forced by the polyhedral geometry. To see why, keep in mind that sector 4 is not just the triangle depicted in Figure 20.2 (which sits in a horizontal plane below the origin), but the cone from the origin over that triangle. Consider any element $h \in H^2_{\mathfrak{a}}(\omega_Q)$ of degree $(0, 0, -m)$. Multiplication by any nonunit monomial of $\mathbb{K}[Q]$ takes h to an element whose $\mathbb{Z}^3$-graded degree lies outside of sector 4 (this is the polyhedral geometry at work!). Since $H^2_{\mathfrak{a}}(\omega_Q)$ is zero in degrees outside of sector 4, we conclude that h must be annihilated by every nonunit monomial of $\mathbb{K}[Q]$.

Exercise 20.47. What is the annihilator of $H^1_{\mathfrak{a}}(\omega_Q)$? What elements of this module have annihilator equal to a prime ideal of $\mathbb{K}[Q]$? Is $H^1_{\mathfrak{a}}(\omega_Q)$ finitely generated? In what $\mathbb{Z}^3$-graded degrees do its generators lie?

Hartshorne's example raises the following open problem.

Problem 20.48. Characterize the normal affine semigroup rings $\mathbb{K}[Q]$, monomial ideals $\mathfrak{a} \subset \mathbb{K}[Q]$, and cohomological degrees i such that $H^i_{\mathfrak{a}}(\mathbb{K}[Q])$ has infinite-dimensional socle.

All we know at present is that $\mathbb{K}[Q]$ has a monomial ideal $\mathfrak{a}$ and a finitely generated $\mathbb{Z}^d$-graded module M with $\mathrm{soc}(H^i_{\mathfrak{a}}(M))$ of infinite dimension for some i, if and only if P_Q is not a simplex [**64**].

The Frobenius Endomorphism

In this lecture, p is a prime number and R a commutative Noetherian ring. We describe various methods by which the Frobenius homomorphism has been used to study local cohomology for rings of characteristic p.

Definition 21.1. When R has characteristic p, the *Frobenius endomorphism* of R is the map

$$\varphi\colon R \longrightarrow R \qquad \text{where } \varphi(r) = r^p\,.$$

Since R has characteristic p, the binomial formula yields $(r+s)^p = r^p + s^p$ for elements r, s in R. It follows that φ is a homomorphism of rings, and hence so are its iterates φ^e, for each integer $e \geqslant 0$. Evidently, $\varphi^e(r) = r^{p^e}$.

1. Homological properties

A homomorphism of rings $\varphi\colon R \longrightarrow S$ is said to be *flat* if S is flat when viewed as an R-module via φ. The following theorem of Kunz [**94**] exemplifies the general principle that properties of the Frobenius endomorphism of a ring entail properties of the ring itself. Recent work of Avramov, Iyengar, C. Miller, Sather-Wagstaff and others reveals that this principle applies to a much larger class of endomorphisms; see [**7, 85**].

Theorem 21.2. *Let R be a Noetherian ring of characteristic p, and φ the Frobenius endomorphism of R. The following conditions are equivalent:*

1. *The ring R is regular;*
2. *The homomorphism φ^e is flat for each integer $e \geqslant 1$;*
3. *The homomorphism φ^e is flat for some integer $e \geqslant 1$.*

Observe also the similarity between this result and Theorem 8.22. We prove (1) $\Longrightarrow$ (2) for only this is used in the sequel. Our proof is based on the result below due to André [**4**, Lemma II.58]. The reader may refer to [**94**] for the original argument, and for other implications; see Theorem 21.10 and [**7**] for different perspectives.

Theorem 21.3. *Let* $(R, \mathfrak{m}, \mathbb{K}) \longrightarrow S$ *be a local homomorphism of local rings, and* N *a finitely generated* S*-module. If* $\operatorname{Tor}_i^R(\mathbb{K}, N) = 0$ *for each* $i \geqslant 1$*, then* N *is a flat* R*-module.*

Proof. It suffices to prove that $\operatorname{Tor}_i^R(M, N) = 0$ for each finitely generated R-module M, and $i \geqslant 1$. This we achieve by an induction on $\dim M$.

When $\dim M = 0$, we further induce on the length of M. If $\ell_R(M) = 1$, then $M \cong \mathbb{K}$, so the desired result is the hypothesis. When $\ell_R(M) \geqslant 2$, one can get an exact sequence of R-modules $0 \longrightarrow \mathbb{K} \longrightarrow M \longrightarrow M' \longrightarrow 0$. Applying $- \otimes_R N$ yields an exact sequence

$$\operatorname{Tor}_i^R(\mathbb{K}, N) \longrightarrow \operatorname{Tor}_i^R(M, N) \longrightarrow \operatorname{Tor}_i^R(M', N)\,.$$

Since $\ell_R(M') = \ell_R(M) - 1$, the induction hypothesis yields the vanishing.

Let $d \geqslant 1$ be an integer such that for $i \geqslant 1$ the functor $\operatorname{Tor}_i^R(-, N)$ vanishes on finitely generated R-modules of dimension up to $d-1$. Let M be a finitely generated R-module of dimension d. Consider the exact sequence of R-modules

$$0 \longrightarrow \Gamma_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow M' \longrightarrow 0\,,$$

and the induced exact sequence on $\operatorname{Tor}_i^R(-, N)$. Since $\ell_R(\Gamma_{\mathfrak{m}}(M))$ is finite, it suffices to verify the vanishing for M'. Thus, replacing M by M', one may assume $\operatorname{depth}_R M \geqslant 1$. Let x in R be an M-regular element; then $\dim(M/xM) = \dim M - 1$. In view of the induction hypothesis, the exact sequence $0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$ induces an exact sequence

$$\operatorname{Tor}_i^R(M, N) \xrightarrow{\ \ x\ \ } \operatorname{Tor}_i^R(M, N) \xrightarrow{\qquad} \operatorname{Tor}_i^R(M/xM, N) = 0$$

for $i \geqslant 1$. As an S-module, $\operatorname{Tor}_i^R(M, N)$ is finitely generated: compute it using a resolution of M by finitely generated free R-modules. Since xS is in the maximal ideal of S, the exact sequence above implies $\operatorname{Tor}_i^R(M, N) = 0$ by Nakayama's lemma. This completes the induction step. $\square$

It should be clear from the proof that there are versions of the preceding result for any half-decent half-exact functor; this is André's original context. The exercise below leads to a global version of André's theorem.

Exercise 21.4. Let $\varphi \colon R \longrightarrow S$ be a homomorphism of Noetherian rings. Prove that φ is flat if and only if for each prime ideal $\mathfrak{q}$ in S, the local homomorphism $\varphi_{\mathfrak{q}} \colon R_{\mathfrak{p}} \longrightarrow S_{\mathfrak{q}}$, where $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$, is flat.

Definition 21.5. Let R be a ring of characteristic p, and M an R-module. We write ${}^{\varphi^e}M$ for the Abelian group underlying M with R-module structure defined via $\varphi^e\colon R \longrightarrow R$. Thus, for $r \in R$ and $m \in {}^{\varphi^e}M$, one has

$$r \cdot m = \varphi^e(r)m = r^{p^e}m\,.$$

In particular, ${}^{\varphi^e}R$ denotes R viewed as an R-module via φ^e.

Proof of (1) $\Longrightarrow$ (2) in Theorem 21.2. It suffices to prove that φ is flat, since a composition of flat maps is flat. For each prime ideal $\mathfrak{q}$ in R, it is easy to check that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{q}$ and that the induced map $\varphi_{\mathfrak{q}}\colon R_{\mathfrak{q}} \longrightarrow R_{\mathfrak{q}}$ is again the Frobenius endomorphism. By Exercise 21.4, it is thus enough to consider the case where $(R, \mathfrak{m}, \mathbb{K})$ is a local ring.

Let $x_1, \dots, x_d$ be a minimal generating set for the ideal $\mathfrak{m}$. The ring R is regular, hence the sequence $\boldsymbol{x}$ is regular. Therefore the Koszul complex $K^\bullet(\boldsymbol{x}; R)$ is a free resolution of $R/(\boldsymbol{x}) = \mathbb{K}$, so one has isomorphisms

$$\operatorname{Tor}_i^R(\mathbb{K}, {}^{\varphi}R) = H^{-i}(\boldsymbol{x}; {}^{\varphi}R) \qquad \text{for each } i\,.$$

Observe that the sequence $\boldsymbol{x}$ is regular on ${}^{\varphi}R$ as $\varphi(x) = x^p$. Corollary 6.22 thus yields $\operatorname{Tor}_i^R(\mathbb{K}, {}^{\varphi}R) = 0$ for $i \neq 0$. Now apply Theorem 21.3. $\square$

Exercise 21.6. Verify that the Frobenius endomorphism is not flat for the ring $\mathbb{F}_2[x^2, xy, y^2]$.

Exercise 21.7. Find a ring R of characteristic p with the property that its Frobenius homomorphism $\varphi\colon R \longrightarrow R$ is not flat, but R is flat over $\varphi(R)$.

Given a homomorphism of rings $R \longrightarrow S$, restriction of scalars is a functor from S-modules to R-modules, while base change is a functor from R-modules to S-modules. For the Frobenius endomorphism the first functor is described explicitly in Definition 21.5. We now discuss the second functor.

Definition 21.8. Let R be a commutative ring containing a field of characteristic p. The assignment

$$M \longmapsto F(M) = {}^{\varphi}R \otimes_R M$$

is a functor on the category of R-modules, called the *Frobenius functor*. For each integer $e \geqslant 1$, one has $F^e(M) = {}^{\varphi^e}R \otimes_R M$. Given a complex $P_\bullet$, the complex obtained by applying F^e is denoted $F^e(P_\bullet)$.

Evidently, $F^e(R) = {}^{\varphi^e}R \otimes_R R \cong R$ as R-modules. If $\theta\colon R^s \longrightarrow R^t$ is given by a matrix (a_{ij}) with respect to some choice of bases for R^s and R^t, then $F^e(\theta)\colon R^s \longrightarrow R^t$ is given by the matrix $(a_{ij}^{p^e})$. This implies that if M is finitely generated, then so is $F^e(M)$.

Next we prove the Lemme d'acyclicité of Peskine and Szpiro [**128**, I.1.8].

Lemma 21.9. *Let R be a local ring and $0 \longrightarrow C_n \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0$ a complex of R-modules with nonzero homology. Set $s = \max\{i \mid H_i(C_\bullet) \neq 0\}$.*

If $\operatorname{depth}_R C_i \geqslant i$ for each $i \geqslant 1$, then either $s = 0$ or $\operatorname{depth}_R H_s(C_\bullet) \geqslant 1$.

Proof. Assume $s \geqslant 1$ and $\operatorname{depth}_R H_s(C_\bullet) = 0$. Set $Z_s = \ker(C_s \longrightarrow C_{s-1})$ and $B_i = \operatorname{image}(C_{i+1} \longrightarrow C_i)$ for $i \geqslant s$. One has exact sequences

$$0 \longrightarrow B_s \longrightarrow Z_s \longrightarrow H_s(C_\bullet) \longrightarrow 0\,,$$
$$0 \longrightarrow B_{i+1} \longrightarrow C_{i+1} \longrightarrow B_i \longrightarrow 0 \qquad \text{for } i \geqslant s\,.$$

Since $Z_s \subseteq C_s$ and $\operatorname{depth}_R C_s \geqslant s \geqslant 1$, one has $\operatorname{depth}_R Z_s \geqslant 1$. Repeated use of Lemma 8.7 on the exact sequences above yields that for $s \leqslant i \leqslant n-1$ one has $\operatorname{depth}_R B_i = i - s + 1$. It remains to note that $B_{n-1} = C_n$, so that $\operatorname{depth}_R C_n = n - s \leqslant n - 1$, which contradicts the hypotheses. □

Peskine and Szpiro [**128**] used the acyclicity lemma to prove the result below. It generalizes the implication (1) $\Longrightarrow$ (2) in Theorem 21.2.

Theorem 21.10. *Let R be a local ring of characteristic p and M a finitely generated R-module of finite projective dimension. If $P_\bullet$ is a minimal free resolution of M, then $F^e(P_\bullet)$ is a minimal free resolution of $F^e(M)$.*

Proof. Set $s = \max\{i \mid \operatorname{Tor}_i^R({}^{\varphi^e}\!R, M) \neq 0\}$. It suffices to prove that $s = 0$, for then $F^e(P_\bullet)$ is a minimal free resolution of $H_0(F^e(P_\bullet)) = F^e(M)$.

Suppose $s \geqslant 1$. Pick a prime $\mathfrak{p}$ associated to $\operatorname{Tor}_s^R({}^{\varphi^e}\!R, M)$. Let $P_\bullet$ be a minimal free resolution of $M_\mathfrak{p}$ over $R_\mathfrak{p}$ and set $C_\bullet = F^e(P_\bullet)$. Observe that

$$H_i(C_\bullet) \cong \operatorname{Tor}_i^R({}^{\varphi^e}\!R, M)_\mathfrak{p} \qquad \text{for each } i \geqslant 0\,.$$

The choice of $\mathfrak{p}$ ensures that $\operatorname{depth}_{R_\mathfrak{p}} H_s(C_\bullet) = 0$, while $H_i(C_\bullet) = 0$ for $i \geqslant s+1$. Since $P_\bullet$ is a minimal resolution of $M_\mathfrak{p}$, the Auslander-Buchsbaum formula implies $P_i = 0$ for $i > \operatorname{depth} R_\mathfrak{p}$. It is now easy to check that the complex $C_\bullet$ of $R_\mathfrak{p}$-modules contradicts Lemma 21.9. □

Here is one consequence of the preceding theorem:

Corollary 21.11. *Let R be a Noetherian ring of characteristic p and M a finitely generated R-module of finite projective dimension. For each integer $e \geqslant 0$, one has that $\operatorname{pd}_R F^e(M) = \operatorname{pd}_R M$ and $\operatorname{Ass} F^e(M) = \operatorname{Ass} M$.*

Proof. We may assume that the ring is local, and then the result on projective dimensions is contained in Theorem 21.10. The claim about associated primes reduces to proving that over a local ring R, the maximal ideal $\mathfrak{m}$ is associated to M if and only if it is associated to $F^e(M)$. Since M has finite projective dimension, the Auslander-Buchsbaum formula implies that $\mathfrak{m}$ is in $\operatorname{Ass} M$ if and only if $\operatorname{pd}_R M = \operatorname{depth} R$; likewise for $F^e(M)$. The result now follows from the result on projective dimension. □

Finiteness of projective dimension is necessary in Corollary 21.11.

Example 21.12. Let $R = \mathbb{F}_2[x, y]/(x^2, xy)$. The R-module $M = R/Rx$ satisfies $F(M) = R$. It is not hard to check that

$$\begin{aligned} &\operatorname{pd}_R M = \infty \qquad &&\text{and} \qquad &&\operatorname{depth}_R M = 1\,, \\ &\operatorname{pd}_R F(M) = 0 \qquad &&\text{and} \qquad &&\operatorname{depth}_R F(M) = 0\,. \end{aligned}$$

Theorem 21.10, which is reformulated as (1) $\Longrightarrow$ (2) in the statement below, has a converse due to Herzog [**65**]:

Theorem 21.13. *Let R be a Noetherian ring of characteristic p and M a finitely generated R-module. The following conditions are equivalent:*

(1) $\operatorname{pd}_R M < \infty$*;*

(2) $\operatorname{Tor}_i^R(M, {}^{\varphi^e}R) = 0$ *for each* $i \geqslant 1$ *and each* $e \geqslant 1$*;*

(3) $\operatorname{Tor}_i^R(M, {}^{\varphi^e}R) = 0$ *for each* $i \geqslant 1$ *and infinitely many* $e \geqslant 1$. □

The implication (3) $\Longrightarrow$ (1) of this result implies the corresponding implication in Theorem 21.2: if ${}^{\varphi^e}R$ is flat for some integer $e \geqslant 1$, it follows that ${}^{\varphi^e}R$ is flat for infinitely many $e \geqslant 1$, because a composition of flat maps is flat. For all such e, one has that $\operatorname{Tor}_i^R(\mathbb{K}, {}^{\varphi^e}R) = 0$ for $i \geqslant 1$, and hence $\operatorname{pd}_R \mathbb{K} < \infty$ by the result above. Theorem 8.22 implies that R is regular.

2. Frobenius action on local cohomology modules

Next we bring local cohomology into the picture.

Definition 21.14. Let $\mathfrak{a} = (x_1, \dots, x_n)$ be an ideal of R and $\varphi\colon R \longrightarrow S$ a homomorphism of rings. Recall that the local cohomology modules $H^i_{\mathfrak{a}}(R)$ can be computed as the cohomology modules of the Čech complex $\check{C}^\bullet(\boldsymbol{x}; R)$. Tensoring with S provides a morphism of complexes

$$\check{C}^\bullet(\boldsymbol{x}; R) \longrightarrow S \otimes_R \check{C}^\bullet(\boldsymbol{x}; R) = \check{C}^\bullet(\varphi(\boldsymbol{x}); S)\,.$$

Taking homology gives, for each integer i, a homomorphism of R-modules $H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\varphi(\mathfrak{a})S}(S)$, where R acts on the target via φ.

Now specialize to the case where R is a ring of characteristic p and φ is the Frobenius endomorphism of R. One has then an induced map

$$f\colon H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\varphi(\mathfrak{a})R}(R) = H^i_{\mathfrak{a}}(R)\,,$$

which is the *Frobenius action* on $H^i_{\mathfrak{a}}(R)$; note that $\varphi(\mathfrak{a})R = (x_1^p, \dots, x_n^p)$ so $\operatorname{rad}\varphi(\mathfrak{a})R = \operatorname{rad}\mathfrak{a}$. For $r \in R$ and $\eta \in H^i_{\mathfrak{a}}(R)$ one has

$$f(r \cdot \eta) = \varphi(r) f(\eta) = r^p f(\eta)\,.$$

In particular, f is not an endomorphism of R-modules.

Example 21.15. Let $\mathbb{K}$ be a field of characteristic p. Let $R = \mathbb{K}[x_1, \dots, x_d]$ be a polynomial ring and set $\mathfrak{a} = (x_1, \dots, x_d)$. As noted in Example 7.16, one has $H^i_{\mathfrak{a}}(R) = 0$ for $i \neq d$, and $H^d_{\mathfrak{a}}(R)$ has a $\mathbb{K}$-basis consisting of elements

$$\left[\frac{1}{x_1^{n_1} \cdots x_d^{n_d}}\right] \qquad \text{where each } n_i \geqslant 1\,.$$

The Frobenius action f on $H^d_{\mathfrak{a}}(R)$ preserves addition and is given by

$$\left[\frac{\lambda}{x_1^{n_1} \cdots x_d^{n_d}}\right] \longmapsto \left[\frac{\lambda^p}{x_1^{pn_1} \cdots x_d^{pn_d}}\right]$$

for λ in $\mathbb{K}$. Evidently f is injective, and it is $\mathbb{K}$-linear only when $\mathbb{K} = \mathbb{F}_p$; it is not R-linear. Setting $\deg x_i = 1$ imposes a $\mathbb{Z}$-grading on $H^d_{\mathfrak{a}}(R)$, and

$$f\big(H^d_{\mathfrak{a}}(R)_j\big) \subseteq H^d_{\mathfrak{a}}(R)_{pj}\,,$$

where $H^d_{\mathfrak{a}}(R)_j$ is the degree j graded component of $H^d_{\mathfrak{a}}(R)$. Observe that the module $H^d_{\mathfrak{a}}(R)$ is supported in degrees $j \leqslant -d$.

Example 21.16. Let $\mathbb{K}$ be a field of characteristic p. Let $R = \mathbb{K}[x_0, \dots, x_d]$, with $\deg x_i = 1$, and let h be a homogeneous polynomial of degree $n \geqslant d+2$. Set $S = R/Rh$; thus $\dim S = d$. We prove that the Frobenius action on $H^d_{\mathfrak{a}}(S)$, where $\mathfrak{a} = (x_0, \dots, x_d)$, is not injective.

The exact sequence of graded R-modules

$$0 \longrightarrow R(-n) \xrightarrow{\;h\;} R \longrightarrow S \longrightarrow 0$$

induces an exact sequence of graded modules:

$$0 \longrightarrow H^d_{\mathfrak{a}}(S) \longrightarrow H^{d+1}_{\mathfrak{a}}(R)(-n) \xrightarrow{\;h\;} H^{d+1}_{\mathfrak{a}}(R) \longrightarrow 0\,.$$

Examining the graded pieces of this exact sequence, one obtains that

$$H^d_{\mathfrak{a}}(S)_{n-d-1} \neq 0 \quad \text{and} \quad H^d_{\mathfrak{a}}(S)_{\geqslant n-d} = 0\,.$$

Since $n \geqslant d+2$, one has $p(n-d-1) \geqslant (n-d)$, and it follows that

$$f\colon H^d_{\mathfrak{a}}(S)_{n-d-1} \longrightarrow H^d_{\mathfrak{a}}(S)_{p(n-d-1)} = 0$$

must be the zero map.

Exercise 21.17. Let $\mathbb{K}$ be a field, $S = \mathbb{K}[x,y]/(xy)$, and set $\mathfrak{a} = (x,y)$. Determine a $\mathbb{K}$-vector space basis for $H^1_{\mathfrak{a}}(S)$. When $\mathbb{K}$ has characteristic p, describe the Frobenius action on $H^1_{\mathfrak{a}}(S)$.

Exercise 21.18. Let $\mathbb{K}$ be a field of characteristic $p \geqslant 3$, and set

$$R = \mathbb{K}[x,y]/(x^2+y^2) \qquad \text{and} \qquad S = \mathbb{K}[x,y]/(x^3+y^3)\,.$$

Prove that the Frobenius acts injectively on $H^1_{(x,y)}(R)$, but not on $H^1_{(x,y)}(S)$.

The proof of the next result illustrates a typical use of the Frobenius action on local cohomology modules.

Theorem 21.19. *Let R be a local ring of characteristic p. If $x_1, \dots, x_d$ is a system of parameters for R, then for each $t \geqslant 0$ one has that*

$$(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \dots, x_d^{t+1})R\,.$$

Proof. Suppose there exist elements $r_i \in R$ and an integer $t \geqslant 0$ such that

$$(x_1 \cdots x_d)^t = r_1 x_1^{t+1} + \cdots + r_d x_d^{t+1}\,.$$

Set $\mathfrak{a} = (\boldsymbol{x})$ and compute $H^d_{\mathfrak{a}}(R)$ as the cohomology of the Čech complex on $\boldsymbol{x}$. The equation above implies that

$$\eta = \left[\frac{1}{x_1 \cdots x_d}\right] = \left[\sum_{i=1}^{d} \frac{r_i}{(x_1 \cdots \widehat{x}_i \cdots x_d)^{t+1}}\right] = 0$$

in $H^d_{\mathfrak{a}}(R)$, so $\varphi^e(\eta) = 0$ for $e \geqslant 1$. Each element of $H^d_{\mathfrak{a}}(R)$ can be written as

$$\left[\frac{a}{(x_1 \cdots x_d)^{p^e}}\right] = a\varphi^e(\eta)$$

for a in R and $e \geqslant 1$. Thus $H^d_{\mathfrak{a}}(R) = 0$, which contradicts Theorem 9.3. □

Remark 21.20. Let R be a local ring, and $x_1, \dots, x_d$ a system of parameters. Hochster's *monomial conjecture* states that for each $t \geqslant 0$, one has

$$(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \dots, x_d^{t+1})R\,.$$

Theorem 21.19 proves it when R is of characteristic p; Hochster [**70**] proved it for rings containing a field. Work of Heitmann [**63**] settles it in the affirmative for rings of dimension at most three. The general case remains open; it is equivalent to the direct summand conjecture, the canonical element conjecture, and the improved new intersection conjecture; see [**30, 70**].

Let R be a local ring. A *big Cohen-Macaulay module* is an R-module on which a system of parameters for R is regular; the module need not be finitely generated. Hochster [**68**] proved that they exist when R contains a field. In [**73**] Hochster and Huneke proved that each excellent local ring containing a field has a big Cohen-Macaulay algebra; see also the work of Huneke and Lyubeznik [**79**].

Exercise 21.21. If $x_1, \dots, x_d$ is a regular sequence on an R-module, prove that $(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \dots, x_d^{t+1})$ for each $t \geqslant 0$.

Let R be a local ring, or an $\mathbb{N}$-graded ring with R_0 a field. Hartshorne asked whether there exist maximal Cohen-Macaulay modules, that is to say, finitely generated modules with depth equal to $\dim R$; they should be graded when R is graded.

Example 21.22. When R is a one-dimensional local ring, the R-module $R/\operatorname{rad}(0)$ is maximal Cohen-Macaulay. Compare with Exercise 20.33.

Suppose R is an excellent local ring, for example, an algebra essentially of finite type over a field. Let $\mathfrak{p}$ be a prime ideal with $\dim R/\mathfrak{p} = \dim R$. The integral closure R' of $R/\mathfrak{p}$ is a finitely generated R-module. Since $\dim R' = \dim R$ and $\operatorname{depth} R' \geqslant \min\{\dim R, 2\}$, when R has dimension two the R-module R' is maximal Cohen-Macaulay.

Little is known in higher dimensions. The next result was discovered independently by Hartshorne, Hochster, and Peskine and Szpiro.

Theorem 21.23. *Let $\mathbb{K}$ be a perfect field of characteristic p, and let R be an $\mathbb{N}$-graded ring finitely generated over $R_0 = \mathbb{K}$, with $\dim R \geqslant 3$.*

Then there exists a finitely generated R-module M with $\operatorname{depth} M \geqslant 3$. When $\dim R = 3$, the module M is maximal Cohen-Macaulay.

We first establish an intermediate step.

Proposition 21.24. *Let $(R, \mathfrak{m})$ be a local domain which is a homomorphic image of a Gorenstein ring, and let M be a finitely generated R-module.*

If, for each $\mathfrak{p}$ in $\operatorname{Spec} R \smallsetminus \{\mathfrak{m}\}$, the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is a maximal Cohen-Macaulay, then the R-module $H^i_\mathfrak{m}(M)$ has finite length for $i \leqslant \dim R - 1$.

Proof. Let Q be a Gorenstein ring mapping onto R. The local duality Theorem 11.30 provides an isomorphism

$$H^i_\mathfrak{m}(M) \cong \operatorname{Ext}_Q^{\dim Q - i}(M, Q)^\vee .$$

It thus suffices to prove that the R-module $\operatorname{Ext}_Q^{\dim Q - i}(M, Q)$ has finite length for relevant i. Since the module is finitely generated, this is equivalent to proving that for each $\mathfrak{p}$ in the punctured spectrum, the $R_\mathfrak{p}$-module

$$\operatorname{Ext}_Q^{\dim Q - i}(M, Q)_\mathfrak{p} \cong \operatorname{Ext}_{Q_\mathfrak{q}}^{\dim Q - i}(M_\mathfrak{p}, Q_\mathfrak{q})$$

is zero; here $\mathfrak{q}$ is the preimage of $\mathfrak{p}$ in Q. Another application of local duality, now over $Q_\mathfrak{q}$, yields an isomorphism

$$\operatorname{Ext}_{Q_\mathfrak{q}}^{\dim Q - i}(M_\mathfrak{p}, Q_\mathfrak{q})^\vee \cong H_{\mathfrak{q}Q_\mathfrak{q}}^{\dim Q_\mathfrak{q} - \dim Q + i}(M_\mathfrak{p}) .$$

The vanishing of the local cohomology module follows from the fact that

$$\dim Q_\mathfrak{q} - \dim Q + i = \dim R_\mathfrak{p} - \dim R + i < \dim R_\mathfrak{p} = \dim M_\mathfrak{p} ,$$

where the first equality holds because Q is catenary and R is a domain, and the second holds because $M_\mathfrak{p}$ is maximal Cohen-Macaulay. $\square$

Proof of Theorem 21.23. Replacing R by the integral closure of $R/\mathfrak{p}$ for $\mathfrak{p}$ a minimal prime with $\dim R/\mathfrak{p} = \dim R$, we assume that R is a normal domain. Then $H^0_{\mathfrak{m}}(R) = 0 = H^1_{\mathfrak{m}}(R)$ where $\mathfrak{m}$ is the homogeneous maximal ideal of R. By Proposition 21.24, the length of $H^2_{\mathfrak{m}}(R)$ is finite.

Since $\mathbb{K}$ is perfect, for each nonnegative integer e the R-module ${}^{\varphi^e}R$ is finitely generated. Moreover, the length of the R-module

$$ {}^{\varphi^e}H^i_{\mathfrak{m}}(R) \cong H^i_{\mathfrak{m}}({}^{\varphi^e}R) $$

does not depend on e. Note that for m in $[{}^{\varphi^e}R]_n$ and r in R_j, one has

$$ r \cdot m = r^{p^e} m \in \left[{}^{\varphi^e}R\right]_{jp^e+n} . $$

Hence, for each e, the R-module ${}^{\varphi^e}R$ is a direct sum of the p^e modules

$$ M_{e,i} = R_i + R_{p^e+i} + R_{2p^e+i} + R_{3p^e+i} + \cdots , $$

where $0 \leqslant i \leqslant p^e - 1$. Consequently, one has

$$ H^2_{\mathfrak{m}}({}^{\varphi^e}R) = H^2_{\mathfrak{m}}(M_{e,0}) \oplus H^2_{\mathfrak{m}}(M_{e,1}) \oplus \cdots \oplus H^2_{\mathfrak{m}}(M_{e,p^e-1}) . $$

The length of this R-module is independent of e, so for $e \gg 0$ there exists $M_{e,i} \neq 0$ with $H^2_{\mathfrak{m}}(M_{e,i}) = 0$. Therefore, $\operatorname{depth}_R M_{e,i} \geqslant 3$. $\square$

Exercise 21.25. Let R be an $\mathbb{N}$-graded domain, finitely generated over a perfect field $R_0 = \mathbb{K}$ of characteristic p. Let M be a finitely generated graded R-module such that the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is maximal Cohen-Macaulay for each $\mathfrak{p} \in \operatorname{Spec}^{\circ} R$. Prove that R has a maximal Cohen-Macaulay module.

3. A vanishing theorem

In this section we present yet another result of Peskine and Szpiro, this one on the vanishing of certain local cohomology modules.

Definition 21.26. Let $\mathfrak{a} = (x_1, \dots, x_n)$ be an ideal of a ring R of characteristic p. For each nonnegative integer e, set

$$ \mathfrak{a}^{[p^e]} = (x_1^{p^e}, \dots, x_n^{p^e})R . $$

These ideals are called the *Frobenius powers* of $\mathfrak{a}$.

Exercise 21.27. Check that the ideals $\mathfrak{a}^{[p^e]}$ do not depend on the choice of generators for the ideal $\mathfrak{a}$.

Remark 21.28. If $\mathfrak{a}$ is generated by n elements, then $\mathfrak{a}^{np^e} \subseteq \mathfrak{a}^{[p^e]} \subseteq \mathfrak{a}^{p^e}$. It follows from Remark 7.9 that for each i one has isomorphisms:

$$ H^i_{\mathfrak{a}}(M) \cong \varinjlim_{e} \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, M) . $$

The Frobenius powers of $\mathfrak{a}$ can be expressed in terms of the Frobenius functor from Definition 21.8: if $\mathfrak{a} = (x_1, \dots, x_n)$, then $R/\mathfrak{a}$ has a presentation

$$R^n \xrightarrow{[x_1 \; \dots \; x_n]} R \longrightarrow R/\mathfrak{a} \longrightarrow 0\,.$$

Applying $F^e(-)$, the right-exactness of tensor gives us the exact sequence

$$R^n \xrightarrow{\left[x_1^{p^e} \; \dots \; x_n^{p^e}\right]} R \longrightarrow F^e(R/\mathfrak{a}) \longrightarrow 0\,,$$

which shows that $F^e(R/\mathfrak{a}) \cong R/\mathfrak{a}^{[p^e]}$.

The next result is due to Peskine and Szpiro [**128**, Proposition III.4.1]. See Theorem 22.1 for an interesting extension due to Lyubeznik.

Theorem 21.29. *Let R be a regular domain of characteristic p. If $\mathfrak{a} \subset R$ is an ideal such that $R/\mathfrak{a}$ is Cohen-Macaulay, then*

$$H^i_{\mathfrak{a}}(R) = 0 \qquad \textit{for } i \neq \operatorname{height} \mathfrak{a}\,.$$

Proof. Since R is regular, $\operatorname{height} \mathfrak{a} = \operatorname{depth}_R(\mathfrak{a}, R)$; see Remark 10.2. Hence Theorem 9.1 yields $H^i_{\mathfrak{a}}(R) = 0$ for $i < \operatorname{height} \mathfrak{a}$. For the rest of the proof we may assume, with no loss of generality, that $(R, \mathfrak{m})$ is a regular local ring. Recall that $R/\mathfrak{a}^{[p^e]} \cong F^e(R/\mathfrak{a})$ from Remark 21.28. Using Corollary 21.11, the Auslander-Buchsbaum formula, and the assumption that $R/\mathfrak{a}$ is Cohen-Macaulay, one obtains the equalities

$$\begin{aligned}
\operatorname{pd}_R(R/\mathfrak{a}^{[p^e]}) &= \operatorname{pd}_R R/\mathfrak{a} \\
&= \dim R - \operatorname{depth} R/\mathfrak{a} \\
&= \dim R - \dim R/\mathfrak{a} \\
&= \operatorname{height} \mathfrak{a}\,.
\end{aligned}$$

Thus $\operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R) = 0$ for $i \geqslant \operatorname{height} \mathfrak{a} + 1$, and hence

$$H^i_{\mathfrak{a}}(R) = \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R) = 0\,. \qquad \square$$

Exercise 21.30. Let R be a Noetherian ring of characteristic p and $\mathfrak{a}$ an ideal such that $\operatorname{pd}(R/\mathfrak{a})$ is finite. Prove that

$$H^i_{\mathfrak{a}}(R) = 0 \qquad \text{for each } i > \operatorname{depth} R - \operatorname{depth} R/\mathfrak{a}\,.$$

The following example shows that the assertion of Theorem 21.29 does not hold over regular rings of characteristic zero. This was constructed independently by Hartshorne-Speiser [**62**, page 75] and Hochster.

Example 21.31. Let $R = \mathbb{K}[u, v, w, x, y, z]$ be a polynomial ring over a field $\mathbb{K}$, and $\mathfrak{a}$ the ideal generated by the 2×2 minors of the matrix

$$X = \begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}\,.$$

Hence $\mathfrak{a} = (\Delta_1, \Delta_2, \Delta_3)$ where

$$\Delta_1 = vz - wy, \quad \Delta_2 = wx - uz, \quad \Delta_3 = uy - vx\,.$$

Note that $\operatorname{depth}_R(\mathfrak{a}, R) = 2$, so $H^2_{\mathfrak{a}}(R) \neq 0$, and $R/\mathfrak{a}$ is Cohen-Macaulay. Thus, when $\mathbb{K}$ has positive characteristic, Theorem 21.29 implies $H^3_{\mathfrak{a}}(R) = 0$.

Suppose the characteristic of $\mathbb{K}$ is zero. In Example 19.26 we saw that $H^3_{\mathfrak{a}}(R) \neq 0$; here is an alternative argument. The group $G = SL_2(\mathbb{K})$ acts on the ring R as in Example 10.31. The ring of invariants for this action is $R^G = \mathbb{K}[\Delta_1, \Delta_2, \Delta_3]$, a polynomial ring of dimension three, since the minors are algebraically independent over $\mathbb{K}$. Thus, $H^3_{\mathfrak{n}}(R^G)$ is nonzero where $\mathfrak{n} = (\Delta_1, \Delta_2, \Delta_3)R^G$ is the homogeneous maximal ideal of R^G. As $\mathbb{K}$ has characteristic zero, G is linearly reductive, and hence R^G is a direct summand of R, that is to say, $R \cong R^G \oplus M$ for an R^G-module M. Therefore

$$H^3_{\mathfrak{a}}(R) = H^3_{\mathfrak{n}}(R) \cong H^3_{\mathfrak{n}}(R^G) \oplus H^3_{\mathfrak{n}}(M) \neq 0\,.$$

In terms of cohomological dimensions this discussion amounts to:

$$\operatorname{cd}_R(\mathfrak{a}) = \begin{cases} 3 & \text{if } \mathbb{K} \text{ has characteristic } 0, \\ 2 & \text{if } \mathbb{K} \text{ has characteristic } p > 0\,. \end{cases}$$

See Example 22.5 for a more striking example of this phenomenon.

Curious Examples

In Example 21.31 we saw that local cohomology may depend on the characteristic. In this lecture we present more striking examples of such behavior, and also discuss finiteness properties of local cohomology modules.

1. Dependence on characteristic

The following theorem of Lyubeznik [**108**, Theorem 1.1] may be viewed as a generalization of Theorem 21.29. See Definition 21.14 for the Frobenius action on local cohomology modules.

Theorem 22.1. *Let $(R, \mathfrak{m})$ be a regular local ring of positive characteristic and $\mathfrak{a}$ an ideal. Then $H^i_{\mathfrak{a}}(R) = 0$ if and only if the e-th Frobenius iteration*

$$f^e\colon H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a}) \longrightarrow H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a})$$

is zero for all $e \gg 0$, where $d = \dim R$.

Sketch of proof. Recall that for each i one has

$$H^i_{\mathfrak{a}}(R) = \varinjlim_e \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R) \cong \varinjlim_e F^e\big(\operatorname{Ext}^i_R(R/\mathfrak{a}, R)\big),$$

where the isomorphism holds because the Frobenius endomorphism is flat. Therefore $H^i_{\mathfrak{a}}(R) = 0$ if and only if there exists an integer e such that

$$\operatorname{Ext}^i_R(R/\mathfrak{a}, R) \longrightarrow F^e\big(\operatorname{Ext}^i_R(R/\mathfrak{a}, R)\big)$$

is the zero map. Taking Matlis duals, this is equivalent to the map

$$F^e\big(H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a})\big) \cong {}^{\varphi^e}R \otimes_R H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a}) \longrightarrow H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a})$$

being zero. The map above sends $r \otimes \eta$ to $r f^e(\eta)$, so it is zero precisely when $f^e(\eta) = 0$ for each $\eta \in H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a})$; see [**108**] for details. $\square$

Exercise 22.2. For a field $\mathbb{K}$ of characteristic p, take $R = \mathbb{K}[w, x, y, z]$ and
$$\mathfrak{a} = (x^3 - w^2 y, x^2 z - wy^2, xy - wz, y^3 - xz^2)\,.$$
Note that $R/\mathfrak{a} \cong \mathbb{K}[s^4, s^3 t, st^3, t^4]$ as in Example 10.19. Use Theorem 22.1 to prove that $H^3_{\mathfrak{a}}(R) = 0$.

We summarize some facts about Segre embeddings:

Definition 22.3. Let A and B be $\mathbb{N}$-graded rings over a field $A_0 = \mathbb{K} = B_0$. The *Segre product* of A and B is the ring
$$A\#B = \bigoplus_{n \geqslant 0} A_n \otimes_{\mathbb{K}} B_n$$
which is a subring—in fact a direct summand—of the tensor product $A \otimes_{\mathbb{K}} B$. The Segre product has a natural $\mathbb{N}$-grading with $[A\#B]_n = A_n \otimes_{\mathbb{K}} B_n$. If $U \subseteq \mathbb{P}^r$ and $V \subseteq \mathbb{P}^s$ are projective varieties with homogeneous coordinate rings A and B respectively, then $A\#B$ is a homogeneous coordinate ring for the Segre embedding of $U \times V$ in $\mathbb{P}^{rs+r+s}$.

Given $\mathbb{Z}$-graded modules M and N over A and B respectively, their Segre product is the $\mathbb{Z}$-graded $A\#B$-module
$$M\#N = \bigoplus_{n \in \mathbb{Z}} M_n \otimes_{\mathbb{K}} N_n \qquad \text{with } [M\#N]_n = M_n \otimes_{\mathbb{K}} N_n\,.$$

Remark 22.4. Let A and B be normal $\mathbb{N}$-graded rings over $\mathbb{K}$. For reflexive $\mathbb{Z}$-graded modules M and N over A and B respectively, there is a Künneth formula for local cohomology due to Goto-Watanabe [**48**, Theorem 4.1.5]:
$$\begin{aligned} H^k_{\mathfrak{m}_{A\#B}}(M\#N) \cong \big(M\#H^k_{\mathfrak{m}_B}(N)\big) \oplus \big(H^k_{\mathfrak{m}_A}(M)\#N\big) \\ \oplus \bigoplus_{i+j=k+1} \big(H^i_{\mathfrak{m}_A}(M)\#H^j_{\mathfrak{m}_B}(N)\big) \quad \text{for each } k \geqslant 0\,. \end{aligned}$$
If $\dim A = r \geqslant 1$ and $\dim B = s \geqslant 1$, then $\dim A\#B = r + s - 1$.

Example 22.5. This example is due to Hartshorne-Speiser [**62**, page 75], though we present a different argument, based on Theorem 22.1. Let $\mathbb{K}$ be a field of prime characteristic $p \neq 3$, and S the Segre product of
$$A = \mathbb{K}[a, b, c]/(a^3 + b^3 + c^3) \quad \text{and} \quad B = \mathbb{K}[r, s]\,.$$
Then S is the $\mathbb{K}$-subalgebra of $\mathbb{K}[a, b, c, r, s]/(a^3 + b^3 + c^3)$ generated by ar, br, cr, as, bs, cs; see also Example 10.28. Note that $\operatorname{Proj} S = E \times \mathbb{P}^1$ where $E = \operatorname{Proj} A$ is an elliptic curve.

Consider the polynomial ring $R = \mathbb{K}[u, v, w, x, y, z]$ mapping onto S with
$$\begin{aligned} &u \longmapsto ar\,, \quad v \longmapsto br\,, \quad w \longmapsto cr\,, \\ &x \longmapsto as\,, \quad y \longmapsto bs\,, \quad z \longmapsto cs\,. \end{aligned}$$

The kernel $\mathfrak{a}$ of this surjection is generated by

$$u^3 + v^3 + w^3,\ u^2x + v^2y + w^2z,\ ux^2 + vy^2 + wz^2,\ x^3 + y^3 + z^3,$$
$$vz - wy,\ wx - uz,\ uy - vx.$$

We compute $H^2_{\mathfrak{m}_S}(S)$ using the Künneth formula. The Čech complex

$$0 \longrightarrow A \longrightarrow A_a \oplus A_b \longrightarrow A_{ab} \longrightarrow 0$$

shows that $H^2_{\mathfrak{m}_A}(A)_0$ is a 1-dimensional $\mathbb{K}$-vector space spanned by

$$\left[\frac{c^2}{ab}\right] \in \frac{A_{ab}}{A_a + A_b}\,.$$

The Künneth formula shows that the only nonzero component of $H^2_{\mathfrak{m}_S}(S)$ is

$$H^2_{\mathfrak{m}_S}(S)_0 \cong H^2_{\mathfrak{m}_A}(A)_0 \# B_0\,,$$

which is the vector space spanned by $[c^2/ab] \otimes 1$. In particular, S is not Cohen-Macaulay; it is a normal domain since it is a direct summand of the normal domain $A \otimes_{\mathbb{K}} B$. Since $H^2_{\mathfrak{m}_S}(S)$ is a 1-dimensional vector space, an iteration f^e of the Frobenius map

$$f\colon H^2_{\mathfrak{m}_S}(S) \longrightarrow H^2_{\mathfrak{m}_S}(S)$$

is nonzero if and only if f is nonzero, and this is equivalent to the condition

$$f\left(\left[\frac{c^2}{ab}\right]\right) = \left[\frac{c^{2p}}{a^p b^p}\right] \neq 0 \qquad \text{in } \frac{A_{ab}}{A_a + A_b}\,.$$

If $c^{2p}/a^p b^p$ is in $A_a + A_b$, then there exist $\alpha, \beta \in A$ and $n \gg 0$ such that

$$\frac{c^{2p}}{a^p b^p} = \frac{\alpha}{a^n} + \frac{\beta}{b^n}\,,$$

so $c^{2p}(ab)^{n-p} \in (a^n, b^n)A$. Since A is Cohen-Macaulay, this is equivalent to

$$c^{2p} \in (a^p, b^p)A\,.$$

We determine the primes p for which the above holds. If $p = 3k + 2$, then

$$c^{2p} = c^{6k+4} = -c(a^3 + b^3)^{2k+1} \in (a^{3k+3}, b^{3k+3})A \subseteq (a^p, b^p)A\,.$$

On the other hand, if $p = 3k + 1$ then the binomial expansion of

$$c^{2p} = c^{6k+2} = c^2(a^3 + b^3)^{2k}$$

when considered modulo (a^p, b^p) has a nonzero term

$$\binom{2k}{k} c^2 a^{3k} b^{3k} = \binom{2k}{k} c^2 a^{p-1} b^{p-1}\,,$$

which shows that $c^{2p} \notin (a^p, b^p)A$. We conclude that $f\colon H^2_{\mathfrak{m}_S}(S) \longrightarrow H^2_{\mathfrak{m}_S}(S)$ is the zero map if $p \equiv 2 \bmod 3$, and is nonzero if $p \equiv 1 \bmod 3$. Using Theorem 22.1, it follows that

$$H^4_{\mathfrak{a}}(R) \neq 0 \quad \text{if } p \equiv 1 \bmod 3\,, \qquad\qquad H^4_{\mathfrak{a}}(R) = 0 \quad \text{if } p \equiv 2 \bmod 3\,.$$

Exercise 22.6. Let $\mathbb{K}$ be a field, and consider homogeneous polynomials $g \in \mathbb{K}[x_0, \ldots, x_m]$ and $h \in \mathbb{K}[y_0, \ldots, y_n]$ where $m, n \geqslant 1$. Let

$$A = \mathbb{K}[x_0, \ldots, x_m]/(g) \quad \text{and} \quad B = \mathbb{K}[y_0, \ldots, y_n]/(h)\,.$$

Prove that $A\#B$ is Cohen-Macaulay if and only if $\deg g \leqslant m$ and $\deg h \leqslant n$.

Remark 22.7. Let E be a smooth elliptic curve over a field $\mathbb{K}$ of characteristic $p > 0$. There is a Frobenius action

$$f\colon H^1(E, \mathcal{O}_E) \longrightarrow H^1(E, \mathcal{O}_E)$$

on the 1-dimensional cohomology group $H^1(E, \mathcal{O}_E)$. If $E = \operatorname{Proj} A$, then the map f above is precisely the action of the Frobenius on

$$H^1(E, \mathcal{O}_E) = H^2_{\mathfrak{m}}(A)_0\,.$$

The elliptic curve E is *supersingular* if f is zero, and is *ordinary* otherwise. For example, the cubic polynomial $a^3+b^3+c^3$ defines a smooth elliptic curve E in any characteristic $p \neq 3$. Our computation in Example 22.5 says that E is supersingular for primes $p \equiv 2 \bmod 3$, and is ordinary if $p \equiv 1 \bmod 3$.

Let $g \in \mathbb{Z}[a, b, c]$ be a cubic polynomial defining a smooth elliptic curve $E_{\mathbb{Q}} \subset \mathbb{P}^2_{\mathbb{Q}}$. Then the Jacobian ideal of g in $\mathbb{Q}[a, b, c]$ is primary to the maximal ideal (a, b, c). After inverting a nonzero integer in $\mathbb{Z}[a, b, c]$, the Jacobian ideal contains a power of (a, b, c). Consequently, for all but finitely many primes p, the polynomial $g \bmod p$ defines a smooth elliptic curve $E_p \subset \mathbb{P}^2_{\mathbb{F}_p}$. If the elliptic curve $E_{\mathbb{C}} \subset \mathbb{P}^2_{\mathbb{C}}$ has complex multiplication, then it is a classical result that the *density* of the supersingular prime integers p, i.e.,

$$\lim_{n \longrightarrow \infty} \frac{\big|\{p \text{ prime} \mid p \leqslant n \text{ and } E_p \text{ is supersingular}\}\big|}{\big|\{p \text{ prime} \mid p \leqslant n\}\big|}$$

is $1/2$, and that this density is 0 if $E_{\mathbb{C}}$ does not have complex multiplication. Even when $E_{\mathbb{C}}$ does not have complex multiplication, the set of supersingular primes is infinite [**33**]. It is conjectured that if $E_{\mathbb{C}}$ does not have complex multiplication, then the number of supersingular primes less than n grows asymptotically like $C\sqrt{n}/\log n$, where C is a positive constant.

Example 22.8. Let $E \subset \mathbb{P}^2_{\mathbb{Q}}$ be a smooth elliptic curve, and $\mathfrak{a}$ an ideal of $R = \mathbb{Z}[x_0, \ldots, x_{3n+2}]$ defining the Segre embedding $E \times \mathbb{P}^n \subset \mathbb{P}^{3n+2}$. Then

$\operatorname{cd}_R(\mathfrak{a}, R/pR) \geqslant \operatorname{height} \mathfrak{a} = 2n+1$. Imitating the methods in Example 22.5, we shall see that

$$\operatorname{cd}_R(\mathfrak{a}, R/pR) = \begin{cases} 2n+1 & \text{if } E_p \text{ is supersingular,} \\ 3n+1 & \text{if } E_p \text{ is ordinary.} \end{cases}$$

The ring $R/(\mathfrak{a}+pR)$ may be identified with $A\#B$ where

$$A = \mathbb{F}_p[a,b,c]/(g) \quad \text{and} \quad B = \mathbb{F}_p[y_0, \dots, y_n]\,.$$

Let p be a prime for which E_p is smooth. The Künneth formula shows that

$$H^i_{\mathfrak{m}}(R/(\mathfrak{a}+pR)) = \begin{cases} \mathbb{F}_p & \text{if } i = 2, \\ 0 & \text{if } 3 \leqslant i \leqslant n+1. \end{cases}$$

The Frobenius action on $H^2_{\mathfrak{m}}(R/(\mathfrak{a}+pR))$ can be identified with

$$H^1(E_p, \mathcal{O}_{E_p}) \xrightarrow{f} H^1(E_p, \mathcal{O}_{E_p})\,,$$

which is the zero map precisely when E_p is supersingular. Consequently every element of $H^2_{\mathfrak{m}}(R/(\mathfrak{a}+pR))$ is killed by the Frobenius action (equivalently, by an iteration of the action) if and only if E_p is supersingular. The assertion now follows from Theorem 22.1.

Exercise 22.9. Let $R = \mathbb{F}_p[x,y,z]/(x^3+xy^2+z^3)$ and

$$\eta = \left[\frac{z^2}{xy}\right] \in \frac{R_{xy}}{R_x + R_y} = H^2_{\mathfrak{m}}(R)\,.$$

For which primes p is $f(\eta) = 0$, i.e., for which p is the elliptic curve $\operatorname{Proj} R$ supersingular? Hint: Consider p mod 6.

2. Associated primes of local cohomology modules

While local cohomology modules may not be finitely generated, they sometimes possess useful finiteness properties. For example, for a local ring $(R, \mathfrak{m})$, the modules $H^i_{\mathfrak{m}}(R)$ are Artinian. This implies that the socle

$$\operatorname{Hom}_R\left(R/\mathfrak{m}, H^i_{\mathfrak{m}}(R)\right)$$

has finite length for each $i \geqslant 0$. Grothendieck [**55**, Exposé XIII, page 173] conjectured that for each $\mathfrak{a} \subset R$, the module

$$\operatorname{Hom}_R\left(R/\mathfrak{a}, H^i_{\mathfrak{a}}(R)\right)$$

is finitely generated; in particular, the socle of $H^i_{\mathfrak{a}}(R)$ has finite length. Hartshorne [**58**] gave a counterexample to this, as we saw in Section 20.5. A related question was raised by Huneke [**77**]:

Question 22.10. Is the number of associated prime ideals of a local cohomology module $H^i_{\mathfrak{a}}(R)$ finite?

This issue was discussed briefly in Lecture 9. The first general results were obtained by Huneke and Sharp [**80**, Corollary 2.3]:

Theorem 22.11. *Let R be a regular ring of positive characteristic, and $\mathfrak{a} \subset R$ an ideal. Then*

$$\operatorname{Ass} H^i_{\mathfrak{a}}(R) \subseteq \operatorname{Ass} \operatorname{Ext}^i_R(R/\mathfrak{a}, R)$$

for each $i \geqslant 0$. In particular, $\operatorname{Ass} H^i_{\mathfrak{a}}(R)$ is a finite set.

Proof. Let p denote the characteristic of R. Suppose $\mathfrak{p} \in \operatorname{Ass} H^i_{\mathfrak{a}}(R)$. After localizing at $\mathfrak{p}$, we can assume that R is local with maximal ideal $\mathfrak{p}$. Then $\mathfrak{p} \in \operatorname{Ass} H^i_{\mathfrak{a}}(R)$ implies that the socle of $H^i_{\mathfrak{a}}(R)$ is nonzero. By Remark 21.28

$$H^i_{\mathfrak{a}}(R) = \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R)$$

so, for some integer e, the module $\operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R)$ has a nonzero socle. But then $\mathfrak{p}$ is an associated prime of $\operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R)$. Since R is regular,

$$F^e\left(\operatorname{Ext}^i_R(R/\mathfrak{a}, R)\right) \cong \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R)\,.$$

Corollary 21.11 completes the proof since

$$\operatorname{Ass} \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R) = \operatorname{Ass} \operatorname{Ext}^i_R(R/\mathfrak{a}, R)\,. \qquad \square$$

Remark 22.12. The proof of the Huneke-Sharp theorem uses the flatness of the Frobenius endomorphism, Theorem 21.2, which characterizes regular rings of positive characteristic. The containment

$$\operatorname{Ass} H^i_{\mathfrak{a}}(R) \subseteq \operatorname{Ass} \operatorname{Ext}^i_R(R/\mathfrak{a}, R)$$

fails for regular rings of characteristic zero: consider $\mathfrak{a} \subset R$ as in Example 21.31, where R is a polynomial ring over a field of characteristic zero. Then $\operatorname{Ext}^3_R(R/\mathfrak{a}, R) = 0$ since $\operatorname{pd}_R R/\mathfrak{a} = 2$. However $H^3_{\mathfrak{a}}(R)$ is nonzero.

Question 22.10 has an affirmative answer for unramified regular local rings, obtained by combining Theorem 22.11 with the following results of Lyubeznik, [**104**, Corollary 3.6 (c)] and [**107**, Theorem 1].

Theorem 22.13. *Let $\mathfrak{a}$ be an ideal in a regular ring R containing a field of characteristic zero. For every maximal ideal $\mathfrak{m}$ of R, the set of associated primes of $H^i_{\mathfrak{a}}(R)$ contained in $\mathfrak{m}$ is finite. If R is finitely generated over a field of characteristic zero, then $\operatorname{Ass} H^i_{\mathfrak{a}}(R)$ is a finite set.* $\square$

The case where R is a polynomial ring is treated in Corollary 23.5.

Theorem 22.14. *Let $\mathfrak{a}$ be an ideal of an unramified regular local ring R of mixed characteristic. Then $\operatorname{Ass} H^i_{\mathfrak{a}}(R)$ is finite for each i.* $\square$

Lyubeznik [**104**, Remark 3.7] conjectured that for R regular and $\mathfrak{a}$ an ideal, $H^i_{\mathfrak{a}}(R)$ has finitely many associated primes. This conjecture is unresolved for ramified regular local rings of mixed characteristic, and also for regular rings such as $\mathbb{Z}[x_1, \ldots, x_d]$; see [**145**] for some observations.

For a finitely generated R-module M, the submodule $H^0_{\mathfrak{a}}(M)$ is finitely generated, and so $\operatorname{Ass} H^0_{\mathfrak{a}}(M)$ is finite. If i is the smallest integer for which $H^i_{\mathfrak{a}}(M)$ is not finitely generated, then $\operatorname{Ass} H^i_{\mathfrak{a}}(M)$ is also finite; see [**17, 90**] and Remark 9.2. Other positive answers to Question 22.10 include the following result of Marley [**112**, Corollary 2.7].

Theorem 22.15. *Let R be a local ring and M a finitely generated R-module of dimension at most three. Then $\operatorname{Ass} H^i_{\mathfrak{a}}(M)$ is finite for all ideals $\mathfrak{a}$.* □

The following example is taken from [**144**, §4].

Example 22.16. Consider the hypersurface

$$R = \mathbb{Z}[u, v, w, x, y, z]/(ux + vy + wz)$$

and ideal $\mathfrak{a} = (x, y, z)R$. We show that for every prime integer p, the local cohomology module $H^3_{\mathfrak{a}}(R)$ has a p-torsion element; it follows that $H^3_{\mathfrak{a}}(R)$ has infinitely many associated prime ideals.

Using the Čech complex on x, y, z to compute $H^i_{\mathfrak{a}}(R)$, we have

$$H^3_{\mathfrak{a}}(R) = \frac{R_{xyz}}{R_{yz} + R_{zx} + R_{xy}}\,.$$

For a prime integer p, the fraction

$$\lambda_p = \frac{(ux)^p + (vy)^p + (wz)^p}{p}$$

can be written as a polynomial with integer coefficients, and is therefore an element of R. We claim that

$$\eta_p = \left[\frac{\lambda_p}{(xyz)^p}\right] \in H^3_{\mathfrak{a}}(R)$$

is nonzero and p-torsion. Note that

$$p \cdot \eta_p = \left[\frac{p\lambda_p}{(xyz)^p}\right] = \left[\frac{u^p}{(yz)^p} + \frac{v^p}{(zx)^p} + \frac{w^p}{(xy)^p}\right] = 0\,,$$

so all that remains to be checked is that η_p is nonzero. If $\eta_p = 0$, then there exist $c_i \in R$ and an integer $n \gg 0$ such that

$$\frac{\lambda_p}{(xyz)^p} = \frac{c_1}{(yz)^n} + \frac{c_2}{(zx)^n} + \frac{c_3}{(xy)^n}\,.$$

Clearing denominators, this gives the equation

$$\lambda_p(xyz)^{n-p} = c_1 x^n + c_2 y^n + c_3 z^n\,.$$

We prescribe a $\mathbb{Z}^4$-grading on R as follows:

$$\begin{aligned} \deg x &= (1,0,0,0)\,, & \deg u &= (-1,0,0,1)\,,\\ \deg y &= (0,1,0,0)\,, & \deg v &= (0,-1,0,1)\,,\\ \deg z &= (0,0,1,0)\,, & \deg w &= (0,0,-1,1)\,. \end{aligned}$$

With this grading, λ_p is a homogeneous element of degree $(0,0,0,p)$, and there is no loss of generality in assuming that the c_i are homogeneous. Comparing degrees, we see that $\deg(c_1) = (-p, n-p, n-p, p)$, i.e., c_1 must be an integer multiple of the monomial $u^p y^{n-p} z^{n-p}$. Similarly c_2 is an integer multiple of $v^p z^{n-p} x^{n-p}$ and c_3 of $w^p x^{n-p} y^{n-p}$. Consequently

$$\lambda_p (xyz)^{n-p} \;\in\; (xyz)^{n-p}(u^p x^p, v^p y^p, w^p z^p)R\,,$$

so $\lambda_p \in (u^p x^p, v^p y^p, w^p z^p)R$. Specializing $u, v, w \longmapsto 1$, this implies

$$\frac{x^p + y^p + (-1)^p (x+y)^p}{p} \;\in\; (p, x^p, y^p)\mathbb{Z}[x,y]\,,$$

which is easily seen to be false.

Katzman [**89**] constructed the first example where $\operatorname{Ass} H^i_{\mathfrak{a}}(R)$ is infinite for R containing a field:

Example 22.17. Let $\mathbb{K}$ be an arbitrary field, and consider the hypersurface

$$R = \mathbb{K}[s,t,u,v,x,y]/\big(sv^2x^2 - (s+t)vxuy + tu^2y^2\big)\,.$$

We will see that $H^2_{(x,y)}(R)$ has infinitely many associated primes. For a local example, one may localize at the homogeneous maximal ideal (s,t,u,v,x,y).

For $n \in \mathbb{N}$, let $\tau_n = s^n + s^{n-1}t + \cdots + t^n$ and set

$$z_n = \left[\frac{sxy^n}{uv^n}\right] \in \frac{R_{uv}}{R_u + R_v} = H^2_{(u,v)}(R)\,.$$

Exercises 22.18–22.20 below show that $\operatorname{ann}(z_n) = (u,v,x,y,\tau_n)R$. It follows that $H^2_{(u,v)}(R)$ has an associated prime $\mathfrak{p}_n$ of height 4 with

$$(u,v,x,y,\tau_n) \subseteq \mathfrak{p}_n \subsetneq (s,t,u,v,x,y)\,.$$

The set $\{\mathfrak{p}_n\}_{n\in\mathbb{N}}$ is infinite by Exercise 22.21, completing the argument.

In the following four exercises, we use the notation of Example 22.17.

Exercise 22.18. Show that $(u,v,x,y) \subseteq \operatorname{ann}(z_n)$.

Exercise 22.19. Let $A \subset R$ be the $\mathbb{K}$-algebra generated by $a = vx$, $b = uy$, s, t. For $g \in \mathbb{K}[s,t]$, show that $gz_n = 0$ if and only if $gab^n \in (a^{n+1}, b^{n+1})A$.

Hint: You may find the following $\mathbb{N}^3$-grading useful.

$$\begin{aligned} \deg s &= (0,0,0), & \deg u &= (1,0,1), & \deg x &= (1,0,0),\\ \deg t &= (0,0,0), & \deg v &= (0,1,1), & \deg y &= (0,1,0). \end{aligned}$$

Exercise 22.20. Let $A = \mathbb{K}[s,t,a,b]/(sa^2 - (s+t)ab + tb^2)$. Show that

$$(a^{n+1}, b^{n+1})A :_{\mathbb{K}[s,t]} ab^n$$

is the ideal of $\mathbb{K}[s,t]$ generated by τ_n. This exercise completes the proof that

$$\operatorname{ann}(z_n) = (u, v, x, y, \tau_n)R\,.$$

Exercise 22.21. Let $\mathbb{K}[s,t]$ be a polynomial ring over $\mathbb{K}$ and

$$\tau_n = s^n + s^{n-1}t + \cdots + t^n \qquad \text{for } n \in \mathbb{N}\,.$$

For relatively prime integers m and n, show that

$$\operatorname{rad}(\tau_{m-1}, \tau_{n-1}) = (s,t)\,.$$

Remark 22.22. The defining equation of the hypersurface factors as

$$sv^2x^2 - (s+t)vxuy + tu^2y^2 = (svx - tuy)(vx - uy)\,,$$

so the ring in Example 22.17 is not an integral domain. In [**146**] there are extensions of Katzman's construction to obtain examples such as the following. Let $\mathbb{K}$ be an arbitrary field, and consider the hypersurface

$$T = \frac{\mathbb{K}[r,s,t,u,v,w,x,y,z]}{\big(su^2x^2 + sv^2y^2 + tuxvy + rw^2z^2\big)}\,.$$

Then T is a unique factorization domain for which $H^3_{(x,y,z)}(T)$ has infinitely many associated primes. This is preserved if we replace T by the localization at its homogeneous maximal ideal. The ring T has rational singularities if $\mathbb{K}$ has characteristic zero and is F-regular in the case of positive characteristic.

Algorithmic Aspects of Local Cohomology

In this lecture we present applications of D-modules to local cohomology. Local cohomology modules are hard to compute, as they are typically not finitely generated over the base ring. The situation is somewhat better for polynomial rings over fields, and we describe methods to detect their vanishing, and in characteristic zero also the module structure, over such rings. In characteristic zero this involves the Weyl algebra, while in positive characteristics the key ingredient is the Frobenius morphism.

Except in the last section, $\mathbb{K}$ is a field of characteristic zero and R the ring of polynomials in $x_1, \dots, x_n$ over $\mathbb{K}$. We write D for the corresponding Weyl algebra; it is the ring of $\mathbb{K}$-linear differential operators on R. As in Lecture 17, we write B for the Bernstein filtration on D. It induces an increasing, exhaustive filtration on each D-module; see Exercise 5.15.

1. Holonomicity of localization

Recall that R is a left D-module. For each element $f \in R$, the localization R_f has a natural D-module structure defined by

$$x_i \bullet \frac{g}{f^k} = \frac{x_i g}{f^k} \qquad \text{and} \qquad \partial_i \bullet \frac{g}{f^k} = \frac{1}{f^k}\frac{\partial g}{\partial x_i} - \frac{kg}{f^{k+1}}\frac{\partial f}{\partial x_i}\,.$$

We prove that R_f is a holonomic D-module, and that it is cyclic. This fact is at the basis of applications of D-module theory to local cohomology. We begin with a result from [**12**, Theorem 1.5.4]. Note that, a priori, we do not assume that M is finitely generated.

Lemma 23.1. *Let M be a D-module with an increasing exhaustive filtration G compatible with the Bernstein filtration on D.*

If there exists an integer c such that

$$\operatorname{rank}_{\mathbb{K}} G_t \leqslant ct^n + (\textit{lower order terms in } t)\,,$$

then the D-module M is finitely generated and holonomic.

Proof. Let M_0 be a finitely generated D-submodule of M, and consider a filtration F on M_0 induced by the Bernstein filtration on D. Theorem 5.16 gives an integer s such that $F_i \subseteq G_{i+s} \cap M_0$ for each i. It follows that $\dim_B(M_0) \leqslant n$ and that $e_B(M_0) \leqslant n!c$. Thus, any finitely generated submodule of M is holonomic of bounded multiplicity. Therefore Theorem 17.29 implies that any chain of finitely generated submodules of M stabilizes. Hence M itself is finitely generated and holonomic. $\square$

Theorem 23.2. *For each $f \in R$, the D-module R_f is holonomic.*

Proof. Consider the filtration on R_f with t-th level

$$\left\{ gf^{-t} \mid \deg g \leqslant t(1 + \deg f) \right\},$$

and verify that this satisfies the hypotheses of the lemma. $\square$

Let M be a D-module. The product rule endows a natural D-module structure on M_f extending the one on M.

Exercise 23.3. Prove that if M is holonomic, so is M_f. Hint: take the induced filtrations F and G on R_f and M, and filter M_f by the submodules

$$\sum_{i=0}^{t} F_i \otimes_{\mathbb{K}} G_{t-i}\,, \qquad \text{for } t \in \mathbb{Z}\,.$$

There are algorithms to compute presentations for R_f. In the following example, *Macaulay 2* computes a presentation for R_f where $f = x^2 + y^2$.

```
i1 : load "D-modules.m2";

i2 : D=QQ[x,y,dx,dy, WeylAlgebra=>{x=>dx, y=>dy}];

i3 : f=x^2+y^2;

i4 : L=Dlocalize(D^1/ideal(dx,dy), f)

o4 = cokernel | ydx-xdy xdx+ydy+4 x2dy+y2dy+4y |

                                    1
o4 : D-module, quotient of D

i5 : isHolonomic L

o5 = true
```

2. Local cohomology as a D-module

Let $\boldsymbol{f}$ be a finite set of generators for $\mathfrak{a}$, and M a D-module. The modules in the Čech complex $\check{C}^\bullet(\boldsymbol{f}; M)$ are direct sums of modules of the form M_f. It is clear that its differential is D-linear, so $\check{C}^\bullet(\boldsymbol{f}; M)$ is a complex of D-modules. Thus, its cohomology modules, $H^i_{\mathfrak{a}}(M)$, are also D-modules.

The following theorem of Lyubeznik [**104**] is at the heart of many applications of D-modules to local cohomology.

Theorem 23.4. *Let M be a holonomic D-module and $\mathfrak{a}$ an ideal of R. The following statements hold.*

(1) *The set of associated primes of M as an R-module is finite.*
(2) *Each $H^i_{\mathfrak{a}}(M)$ is a holonomic D-module, and hence finitely generated.*
(3) *For each integer i, the set of associated primes of $H^i_{\mathfrak{a}}(M)$ is finite.*

Proof. (1) The length of M as a D-module is finite by Theorem 17.29. We induce on length, the statement being clear for length 0. If $\mathfrak{p} \in \operatorname{Spec} R$ is an associated prime of M, then $\Gamma_{\mathfrak{p}}(M)$, the $\mathfrak{p}$-torsion submodule of M, is a nonzero D-submodule of M; verify this. Thus, $\Gamma_{\mathfrak{p}}(M)$ and $M/\Gamma_{\mathfrak{p}}(M)$ are holonomic and of length less than that of M. It remains to note that

$$\operatorname{Ass}(M) \subseteq \operatorname{Ass}(\Gamma_{\mathfrak{p}}(M)) \cup \operatorname{Ass}(M/\Gamma_{\mathfrak{p}}(M)).$$

Part (2) follows from Exercise 23.3 and Remark 17.28, and part (3) is a consequence of (2) and (1). □

The corollary below is the characteristic zero analogue of Theorem 22.11.

Corollary 23.5. *Let $\mathfrak{a}$ be an ideal of R. Then, for each integer i, the set of associated primes of $H^i_{\mathfrak{a}}(R)$ is finite.* □

Remark 23.6. Theorem 23.2 implies that the D-module R_f is cyclic, and hence generated by f^a for some integer a. To obtain explicit presentations for $H^i_{\mathfrak{a}}(R)$, it is critical to pin down the value of a. This is accomplished via Bernstein-Sato polynomials, which is the subject of the next section. As an added bonus, one gets an alternative proof that R_f is a cyclic D-module; see Corollary 23.9.

The D-module structure on local cohomology modules has indeed been exploited to obtain effective computational algorithms; see [**125, 157**]. Here is Example 19.26 revisited using *Macaulay 2*:

```
i1 : load "D-modules.m2"

i2 : D=QQ[x_1..x_6, dx_1..dx_6,
              WeylAlgebra => toList(1..6)/(i -> x_i => dx_i)];

i3 : a=minors(2,matrix{{x_1,x_2,x_3}, {x_4,x_5,x_6}});
```

```
o3 : Ideal of D

i4 : LC=localCohom(3,a,D^1/ideal{dx_1,dx_2,dx_3,dx_4,dx_5,dx_6},
              Strategy=>Walther,LocStrategy=>Oaku);

i5 : pruneLocalCohom LC

o5 = HashTable{3 => | x_4dx_4+x_5dx_5+x_6dx_6+6 x_1dx_4+...

o5 : HashTable
```

This reconfirms the fact that $H^3_{\mathfrak{a}}(R_6) \neq 0$, established in Example 19.26.

3. Bernstein-Sato polynomials

Let s be a new indeterminate and set $D[s] = \mathbb{K}[s] \otimes_{\mathbb{K}} D$; in this ring, s is a central element. The following theorem is due to Bernstein [**11**]. There are similar results for rings of convergent power series and for rings of formal power series; see [**12**].

Theorem 23.7. *For each polynomial f in R, there exist elements $b(s)$ in $\mathbb{K}[s]$ and $\Delta(s)$ in $D[s]$ such that for each integer a one has*

$$b(a)f^a = \Delta(a) \bullet f^{a+1} .$$

Sketch of proof. Let M be the free $\mathbb{K}[s,\boldsymbol{x}]$-module $\bigoplus_{j\in\mathbb{Z}} \mathbb{K}[s,\boldsymbol{x}]e_j$ modulo the submodule generated by elements $fe_j - e_{j+1}$, for $j \in \mathbb{Z}$. This has a structure of a $D[s]$-module with action defined by

$$\partial_i \bullet (ge_j) = (\partial_i \bullet g)e_j + g(s+j)(\partial_i \bullet f)e_{j-1}$$

for $g \in \mathbb{K}[s,\boldsymbol{x}]$. As a $\mathbb{K}[s]$-module, M is free. Set

$$M(s) = \mathbb{K}(s) \otimes_{\mathbb{K}[s]} M \qquad \text{and} \qquad D(s) = \mathbb{K}(s) \otimes_{\mathbb{K}} D .$$

The action of $D[s]$ on M extends to an action of $D(s)$ on $M(s)$; we use $\bullet$ to denote these actions. Note that $D(s)$ is a Weyl algebra over $\mathbb{K}(s)$.

The crucial point is that $M(s)$ is a holonomic $D(s)$-module. To see this, consider the filtration of $M(s)$ given by

$$G_t = \{ge_{-t} \mid \deg g \leqslant t(1+\deg f)\} \qquad \text{for } t \in \mathbb{N} .$$

Verify that this filtration is increasing, exhaustive, compatible with the Bernstein filtration on $D(s)$, and that it satisfies the equality

$$\operatorname{rank}_{\mathbb{K}(s)} G_t = \binom{n + t(1+\deg f)}{n} .$$

Lemma 23.1 implies the holonomicity of $M(s)$.

Observe that one has a sequence of $D(s)$-submodules of $M(s)$:

$$\cdots \supseteq D(s)e_{j-1} \supseteq D(s)e_j \supseteq D(s)e_{j+1} \supseteq \cdots .$$

This sequence stabilizes as $M(s)$ has finite length. Hence there exists an integer d and an element $\theta \in D(s)$ such that $e_d = \theta \bullet e_{d+1}$. Clearing denominators in θ yields $b'(s) \in \mathbb{K}[s]$ such that $\Delta'(s) = b'(s)\theta$ is in $D[s]$ and

$$b'(s)e_d = \Delta'(s) \bullet e_{d+1}$$

in $M(s)$, and hence in M, since M is a subset of $M(s)$.

Set $b(s) = b'(s-d)$ and $\Delta(s) = \Delta'(s-d)$. Fix $a \in \mathbb{Z}$ and consider the homomorphism of rings $D[s] \longrightarrow D$ induced by $s \longmapsto a-d$. The map $M \longrightarrow R_f$ induced by $e_j \longmapsto f^{a-d+j}$ is equivariant with respect to the ring homomorphism. Thus, the equality above translates to

$$b(a)f^a = \Delta(a) \bullet f^{a+1}$$

which is the desired result. □

It is customary to view the equality in Theorem 23.7 as defining a formal functional equation

$$b(s)f^s = \Delta \bullet f^{s+1}\,.$$

Definition 23.8. Fix $f \in R$. Polynomials $b(s)$ as in Theorem 23.7 form an ideal in $\mathbb{K}[s]$. The unique monic generator of this ideal is the *Bernstein-Sato polynomial*, or the *b-function*, of f and is denoted $b_f(s)$.

Corollary 23.9. *For each polynomial f in R there exists an integer a such that the D-module R_f is generated by f^a.*

Proof. We may assume that f is not a unit. Take $a \in \mathbb{Z}$ to be the smallest integer root of $b_f(s)$; there is one such by the exercise below. For any integer $i \leqslant a$, Theorem 23.7 shows that $f^{i-1} = \Delta \bullet f^i$ for some $\Delta \in D$. □

Exercise 23.10. When f is not a unit, prove that $(s+1)$ divides $b_f(s)$.

Let us look at a few examples.

Example 23.11. Consider $R = \mathbb{K}[x]$ and $f = x$. Since $(s+1)x^s = \partial_x \cdot x^{s+1}$, Exercise 23.10 implies that $b_x(s) = s+1$.

A polynomial $f \in R$ is *regular* if $f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$ generate the unit ideal.

Theorem 23.12. *A polynomial f is regular if and only if $b_f(s) = s+1$.*

Proof. If f is regular, there exist $c_i \in R$ such that

$$1 = c_0 f + c_1 \frac{\partial f}{\partial x_1} + \cdots + c_n \frac{\partial f}{\partial x_n}\,.$$

One can easily verify that the following equality holds:

$$(s+1)f^s = \bigl(c_0(s+1) + c_1\partial_1 + \cdots + c_n\partial_n\bigr) \bullet f^{s+1}\,.$$

The converse is less trivial; see [**15**, Proposition 2.6]. □

Example 23.13. Let $f = x_1^2 + \cdots + x_n^2 \in R$. Then

$$\begin{aligned}\frac{\partial f^{s+1}}{\partial x_i} &= 2(s+1)x_i f^s,\\ \frac{\partial^2 f^{s+1}}{\partial x_i^2} &= 4(s+1)s x_i^2 f^{s-1} + 2(s+1)f^s.\end{aligned}$$

Setting $\Delta = \partial_1^2 + \cdots + \partial_n^2$, one has

$$\Delta \bullet f^{s+1} = 4(s+1)sf^s + 2n(s+1)f^s = 4(s+1)(s+\frac{n}{2})f^s.$$

Since f is not regular, Theorem 23.12 implies that $b_f(s)$ is a proper multiple of $s+1$. It follows that $b_f(s) = (s+1)(s+n/2)$.

A number of interesting computations of Bernstein-Sato polynomials are carried out in [**164**]; the case of generic hyperplane arrangements is studied in [**160**]. Oaku [**123**] gave an algorithm for computing b-functions.

Let R be an $\mathbb{N}$-graded polynomial ring over $\mathbb{K}$, and f a homogeneous element in R. Set $a_i = \deg x_i / \deg f$. One then has

$$\sum_{i=1}^n a_i x_i \frac{\partial f}{\partial x_i} = f\,.$$

The following theorem is proved in [**164**].

Theorem 23.14. *Let R be an $\mathbb{N}$-graded polynomial ring, and f a homogeneous element such that R/fR is regular on the punctured spectrum.*

Set $v = \sum a_i x_i \partial_i$ where $a_i = \deg x_i / \deg f$ and let $\{\lambda_1, \dots, \lambda_t\}$ be the eigenvalues of v on $R/(\partial f/\partial x_1, \dots, \partial f/\partial x_n)$. Then

$$b_f(s) = (s+1)\ \operatorname{lcm}\big(s + \lambda_j + \sum_{i=1}^n a_i \mid 1 \leqslant j \leqslant t\big)\,. \qquad \square$$

Example 23.15. Let $f = x_1^{m_1} + \cdots + x_n^{m_n}$, where each $m_i \geqslant 2$. Then

$$\{\boldsymbol{x}^\alpha \mid 0 \leqslant \alpha_i \leqslant m_i - 2\}$$

is a basis of eigenvectors for $\sum (1/m_i)x_i\partial_i$, and the corresponding eigenvalues are $\sum \alpha_i/m_i$. The preceding theorem then yields

$$b_f(s) = (s+1)\ \operatorname{lcm}\big(s + \sum_{i=1}^n \frac{\beta_i}{m_i}\big)\,,$$

where the lcm runs over n-tuples $(\beta_1, \dots, \beta_n)$ with $1 \leqslant \beta_j \leqslant m_j - 1$.

For example, let $f = x_1^3 + x_2^4$. Then $v = \frac{1}{3}x_1\partial_1 + \frac{1}{4}x_2\partial_2$ and

$$\begin{aligned} b_f(s) &= (s+1)(s+\frac{1}{3}+\frac{1}{4})(s+\frac{1}{3}+\frac{2}{4})(s+\frac{1}{3}+\frac{3}{4}) \\ &\qquad \times (s+\frac{2}{3}+\frac{1}{4})(s+\frac{2}{3}+\frac{2}{4})(s+\frac{2}{3}+\frac{3}{4}) \\ &= (s+1)(s+\frac{7}{12})(s+\frac{5}{6})(s+\frac{11}{12})(s+\frac{13}{12})(s+\frac{7}{6})(s+\frac{17}{12})\,. \end{aligned}$$

Macaulay 2 confirms the last calculation:

```
i1 : load "D-modules.m2"

i2 : D=QQ[x_1,x_2,dx_1,dx_2, WeylAlgebra=>{x_1=>dx_1,x_2=>dx_2}];

i3 : factorBFunction globalBFunction(x_1^3+x_2^4)

             5                      17         13         7         7         11
o3 = ($s + -)($s + 1)($s + --)($s + --)($s + --)($s + -)($s + --)
             6                      12         12        12         6         12

o3 : Product
```

Remark 23.16. Differential operators localize, therefore the equation in Theorem 23.7 may be interpreted over the ring of $\mathbb{K}$-linear differential operators on $R_{\mathfrak{m}}$, for $\mathfrak{m}$ the maximal ideal corresponding to a point $p \in \mathbb{K}^n$. Once again, the ideal of polynomials that satisfy such an equation is nonzero and principal, and hence generated by a divisor $b_{f,p}(s)$ of $b_f(s)$, called the *local b-function* of f at p. The polynomial $b_f(s)$ is the least common multiple of the $b_{f,p}(s)$, almost all of which equal 1.

Exercise 23.17. Compute $b_f(s)$ for f equal to xy, x^3+y^3, and x^3+xy.

Exercise 23.18. Let $f = x_1^m + \cdots + x_n^m \in \mathbb{Q}[x_1,\ldots,x_n]$. Prove that

$$\liminf_{m\longrightarrow\infty}\{\alpha \mid b_f(\alpha)=0\} = -n\,.$$

Quite generally, if f is a polynomial in n variables, then $-n$ is a lower bound for roots of $b_f(s)$; see [**156**].

We record a striking property of Bernstein-Sato polynomials, proved by Malgrange [**111**] for isolated singularities, and by Kashiwara [**88**] in the general case. For other properties, see [**13**, Chapter VI] and [**92**, §10].

Theorem 23.19. *For each nonzero polynomial $f \in R$, the roots of $b_f(s)$ are negative rational numbers. In particular, $b_f(s)$ has rational coefficients.* $\square$

The following theorem is due to Lyubeznik [**106**].

Theorem 23.20. *For each $d \in \mathbb{N}$, the set $\{b_f \mid f \in R,\ \deg f \leqslant d\}$ is finite.*

Sketch of proof. Consider $g = \sum_{|\alpha|\leqslant d} a_\alpha \boldsymbol{x}^\alpha$, a generic polynomial of degree d, i.e., where the coefficients a_α are indeterminates. Viewed as an element of $\mathbb{K}(\boldsymbol{a})[\boldsymbol{x}]$, the polynomial g has a b-function $b_g(s)$ which satisfies

$$b_g(s)g^s = \Delta \bullet g^{s+1} \qquad \text{for some } \Delta \in D(\mathbb{K}(\boldsymbol{a}))[s]\,.$$

Let $E \in \mathbb{K}[\boldsymbol{a}]$ be the least common denominator of the coefficients of Δ. Consider $f = \sum_{|\alpha|\leqslant d} c_\alpha \boldsymbol{x}^\alpha$ with $c_\alpha \in \mathbb{K}$. If $E(\boldsymbol{c}) \neq 0$ then $b_f(s)$ divides $b_g(s)$, since the specialization $\boldsymbol{a} \longmapsto \boldsymbol{c}$ produces a functional equation for f. Since the number of the monic divisors of $b_g(s)$ is finite, it remains to show that the algebraic set $E(\boldsymbol{c}) = 0$ gives rise to a finite number of Bernstein-Sato polynomials. For this, we refer the reader to [**106**]. □

Consider specializations $\boldsymbol{a} \longmapsto \boldsymbol{c}$ in $\mathbb{A}^N$. As proved in [**97**], the subset of $\mathbb{A}^N$ corresponding to a fixed Bernstein-Sato polynomial is constructible.

4. Computing with the Frobenius morphism

Let R be a regular ring of characteristic $p > 0$, and $\mathfrak{a}$ an ideal. Lyubeznik [**105**, Remark 2.4] gave the following algorithm to determine whether $H^i_{\mathfrak{a}}(R)$ is zero: Remark 21.28 yields that

$$H^i_{\mathfrak{a}}(R) = \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R)\,,$$

where the maps on Ext are induced by the surjections $R/\mathfrak{a}^{[p^{e+1}]} \longrightarrow R/\mathfrak{a}^{[p^e]}$. Compositions of these maps give homomorphisms of R-modules

$$\beta_e\colon \operatorname{Ext}^i_R(R/\mathfrak{a}, R) \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R)\,.$$

Since R is Noetherian, the sequence $\ker\beta_1 \subseteq \ker\beta_2 \subseteq \ker\beta_3 \subseteq \cdots$ stabilizes. Let r be the least integer with $\ker\beta_r = \ker\beta_{r+1}$. Then $H^i_{\mathfrak{a}}(R)$ is zero if and only if $\ker\beta_r = \operatorname{Ext}^i_R(R/\mathfrak{a}, R)$; confer Theorem 22.1.

Remark 23.21. We end this lecture by mentioning some of the differences between D-modules in characteristic zero and characteristic $p > 0$. Let R be the polynomial ring in $x_1, \dots, x_n$ over a field $\mathbb{K}$.

When $\mathbb{K}$ has characteristic $p > 0$, the Weyl algebra does not equal the ring of $\mathbb{K}$-linear differential operators on R. Indeed, the differential operators

$$\frac{1}{(tp)!}\frac{\partial^{tp}}{\partial x_i^{tp}} \qquad \text{for } t \in \mathbb{N}$$

are not in the Weyl algebra. In fact, $D(R;\mathbb{K})$ is generated by the Weyl algebra and these operators [**54**, §16.11]; thus $D(R;\mathbb{K})$ is not Noetherian.

If $\mathbb{K}$ is of characteristic zero and c is the smallest integer root of $b_f(s)$, the D-module R_f is generated by f^c but not by f^{c+1}; see [**160**, Lemma 1.3]. In contrast, when $\mathbb{K}$ has positive characteristic, R_f is generated by $1/f$ as a $D(R;\mathbb{K})$-module; this is a theorem of Blickle, Lyubeznik, and Montaner [**3**].

Lecture 24

Holonomic Rank and Hypergeometric Systems

In previous lectures, we have seen applications of D-modules to the theory of local cohomology. In this lecture, we shall see an application of local cohomology to certain D-modules. Specifically, associated to any affine semigroup ring is a family of holonomic D-modules, and the $\mathbb{Z}^d$-graded local cohomology of the semigroup ring indicates how the ranks of these D-modules behave in the family. Substantial parts of this lecture are based on [**116**].

1. GKZ A-hypergeometric systems

> One of the main branches of Mathematics which interacts with D-module theory is doubtlessly the study of hypergeometric functions, both in its classical and generalized senses, in its algebraic, geometric and combinatorial aspects.
>
> —*Workshop on D-modules and hypergeometric functions, Lisbon, Portugal, 11–14 July, 2005.*

Generally speaking, hypergeometric functions are power series solutions to certain systems of differential equations. Classical univariate hypergeometric functions go back at least to Gauss, and by now there are various multivariate generalizations. One class was introduced in the late 1980s by Gelfand, Graev, and Zelevinskiĭ [**44**]. These systems, now called *GKZ systems* or *A-hypergeometric systems*, are closely related to affine semigroup rings and toric varieties. They are constructed as follows.

For the rest of this lecture, fix a $d \times n$ integer matrix $A = (a_{ij})$ of rank d. We do not assume that the columns $a_1, \ldots, a_n$ of A lie in an affine hyperplane, but we do assume that A is *pointed*, i.e., that the affine semigroup

$$Q_A = \Big\{ \sum_{i=1}^{n} \gamma_i a_i \mid \gamma_1, \ldots, \gamma_n \in \mathbb{N} \Big\}$$

generated by the column vectors $a_1, \ldots, a_n$ is pointed. To get the appropriate interaction with D-modules, we express the semigroup ring $\mathbb{C}[Q_A]$ as the quotient $\mathbb{C}[\partial_1, \ldots, \partial_n]/I_A$, where

$$I_A = (\partial^\mu - \partial^\nu \mid \mu, \nu \in \mathbb{Z}^n, A \cdot \mu = A \cdot \nu)$$

is the toric ideal of A as in Exercise 20.8. Note that $\mathbb{C}[Q_A]$ and $\mathbb{C}[\partial]$ are naturally $\mathbb{Z}^d$-graded by $\deg(\partial_j) = -a_j$, the negative of the j-th column of A.

Our choice of signs in the $\mathbb{Z}^d$-grading of $\mathbb{C}[Q_A]$ is compatible with a $\mathbb{Z}^d$-grading on the Weyl algebra D in which $\deg(x_j) = a_j$ and $\deg(\partial_j) = -a_j$. Under this $\mathbb{Z}^d$-grading, the i-th *Euler operator*

$$E_i = \sum_{j=1}^{n} a_{ij} x_j \partial_j \in D$$

is homogeneous of degree 0 for $i = 1, \ldots, d$. The terminology arises from the case where A has a row of 1's, in which case the corresponding Euler operator is $x_1\partial_1 + \cdots + x_n\partial_n$. When applied to a homogeneous polynomial $f(x_1, \ldots, x_n)$ of total degree λ, this operator returns $\lambda \cdot f(x_1, \ldots, x_n)$. Therefore, power series solutions f to

$$(E_1 - \beta_1) \cdot f = 0, \quad \ldots, \quad (E_d - \beta_d) \cdot f = 0$$

can be thought of as being homogeneous of multidegree $\beta \in \mathbb{C}^d$.

Definition 24.1. Given a parameter vector $\beta \in \mathbb{C}^d$, write $E - \beta$ for the sequence $E_1 - \beta_1, \ldots, E_d - \beta_d$. The *$A$-hypergeometric system with parameter β* is the left ideal of the Weyl algebra D given by

$$H_A(\beta) = D \cdot (I_A, E - \beta)\,.$$

The *A-hypergeometric D-module with parameter β* is

$$\mathcal{M}_\beta^A = D/H_A(\beta)\,.$$

Example 24.2. Letting $d = 2$ and $n = 4$, consider the 2×4 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}.$$

The semigroup ring associated to A is then $\mathbb{C}[Q_A] = \mathbb{C}[s, st, st^3, st^4]$, which is isomorphic to the semigroup ring $\mathbb{C}[s^4, s^3t, st^3, t^4]$ from Example 20.30 via the isomorphism in Example 20.9. As we have seen in Example 10.19, in different notation, the toric ideal for A is

$$I_A = (\partial_2\partial_3 - \partial_1\partial_4,\ \partial_1^2\partial_3 - \partial_2^3,\ \partial_2\partial_4^2 - \partial_3^3,\ \partial_1\partial_3^2 - \partial_2^2\partial_4)\,,$$

these generators corresponding to the equations

$$A\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = A\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\quad A\begin{bmatrix}2\\0\\1\\0\end{bmatrix} = A\begin{bmatrix}0\\3\\0\\0\end{bmatrix},\quad A\begin{bmatrix}0\\1\\0\\2\end{bmatrix} = A\begin{bmatrix}0\\0\\3\\0\end{bmatrix},\quad A\begin{bmatrix}1\\0\\2\\0\end{bmatrix} = A\begin{bmatrix}0\\2\\0\\1\end{bmatrix}.$$

Given $\beta = (\beta_1, \beta_2)$, the homogeneities from A are the classical Euler operator

$$x_1\partial_1 + x_2\partial_2 + \ \ x_3\partial_3 + \ \ x_4\partial_4 - \beta_1$$

and the "0134 equation"

$$x_2\partial_2 + 3x_3\partial_3 + 4x_4\partial_4 - \beta_2\,.$$

The left ideal $H_A(\beta)$ is generated by I_A and the above two homogeneities.

Definition 24.3. Let H be a left ideal of D with D/H holonomic. The rank of the $\mathbb{C}$-vector space of analytic solutions of the system H of differential equations in a neighborhood of a general point of $\mathbb{C}^n$ is independent of the point; see [**136**, Theorem 1.4.19]; this is the *holonomic rank* of D/H.

The first fundamental results about the systems $H_A(\beta)$, regarding holonomicity and rank, were proved by Gelfand, Graev, Kapranov, and Zelevinskiĭ (whence the 'K' in 'GKZ'). These results concern the case where $\mathbb{C}[Q_A]$ is Cohen-Macaulay and standard $\mathbb{Z}$-graded [**44, 45**]. Subsequently, the Cohen-Macaulay and standard $\mathbb{Z}$-graded assumptions were relaxed. Results of [**1, 45, 76, 136**] imply the following nontrivial statement.

Theorem 24.4. *The module $\mathcal{M}_\beta^A$ is holonomic of nonzero rank.* □

This result is what motivates the story here.

Question 24.5. For fixed A, what is the holonomic rank of the module $\mathcal{M}_\beta^A$ as a function of β?

As we shall see, when $\mathbb{C}[Q_A]$ is Cohen-Macaulay or when the parameter β is generic, this rank is an easily described constant. The qualitative change in rank as β varies is partially understood; however, we know little about the quantitative behavior.

Remark 24.6. The solutions of A-hypergeometric systems appear as toric residues [**24**], and special cases are mirror transforms of generating functions for intersection numbers on moduli spaces of curves [**27**]. In the latter case, the A-hypergeometric systems are Picard-Fuchs equations governing the variation of Hodge structures for Calabi-Yau toric hypersurfaces. In general, A-hypergeometric systems constitute an important class of D-modules, playing a role similar to that of toric varieties in algebraic geometry and semigroup rings in commutative algebra: they possess enough combinatorial underpinning to make calculations feasible, but enough diversity of behavior to make them interesting as a test class for conjectures and computations.

2. Rank vs. volume

In Question 24.5, one might wonder what role the integer matrix A plays in determining the rank. The answer is pleasantly combinatorial—well, polyhedral. It requires a simple lemma and definition.

Lemma 24.7. *Fix a lattice L of full rank d in $\mathbb{R}^d$. Among all simplices of dimension d having lattice points for vertices, there is one with minimum Euclidean volume.*

Proof. After translating, assume that the origin is a vertex. The volume of any d-simplex in $\mathbb{R}^d$ (with nonrational vertices allowed) is $1/d!$ times the absolute value of the determinant of the remaining d vertices. The minimal absolute value for the determinant is attained on any basis of L. $\square$

A polytope with its vertices at lattice points is called a *lattice polytope*; if the polytope is a simplex, then it is a *lattice simplex*.

Definition 24.8. Fix a lattice $L \subset \mathbb{R}^d$ of rank d, and let P be a polytope. The *normalized volume* of P is the ratio $\mathrm{vol}_L(P)$ between the Euclidean volume of P and the smallest volume of a lattice simplex.

Example 24.9. When $L = \mathbb{Z}^d$, the normalized volume $\mathrm{vol}_{\mathbb{Z}^d}(P)$ of a lattice polytope P is simply $d!$ times the usual volume of P, since the smallest Euclidean volume of a lattice simplex is $1/d!$.

Notation 24.10. We write $\mathrm{vol}(P) = \mathrm{vol}_{\mathbb{Z}^d}(P)$. Given a $d \times n$ integer matrix A, set $\mathrm{vol}(A)$ equal to the normalized volume of the convex hull of the columns of A and the origin $0 \in \mathbb{Z}^d$.

The following result is from [**44, 45**].

Theorem 24.11. *If the affine semigroup ring $\mathbb{C}[Q_A]$ is Cohen-Macaulay and admits a standard grading, then the A-hypergeometric D-module $\mathcal{M}_\beta^A$ has holonomic rank* $\mathrm{vol}(A)$. $\square$

The remarkable fact here is that the rank formula holds independent of the parameter β. What if $\mathbb{C}[Q_A]$ is not Cohen-Macaulay, or not $\mathbb{Z}$-graded? Adolphson [**1**] further proved that for generic β the characterization of rank through volume in Theorem 24.11 is still correct.

Theorem 24.12. *The rank of* $\mathcal{M}_\beta^A$ *equals* $\text{vol}(A)$ *as long as* β *lies outside of a certain closed locally finite arrangement of countably many "semiresonant" affine hyperplanes. If the ring* $\mathbb{C}[Q_A]$ *is Cohen-Macaulay then* $\text{rank}(\mathcal{M}_\beta^A) = \text{vol}(A)$ *for all* β. □

Adolphson made no claim concerning the parameters β lying in the semiresonant hyperplanes; he did not, in particular, produce a parameter β where the rank did not equal $\text{vol}(A)$. It seemed natural enough to conjecture that perhaps the rank is actually always constant. It came as quite a surprise when an example was given by Sturmfels and Takayama [**152**] showing that if $\mathbb{C}[Q_A]$ is not Cohen-Macaulay, then not all parameters β give the same rank [**152**]. Which example did they give? Why, 0134, of course!

Example 24.13. Let A be as in Example 24.2 and set $\beta = (1, 2)$. Then $\text{rank}(\mathcal{M}_\beta^A) = 5$, whereas $\text{vol}(A) = 4$, the latter because the convex hull of A and the origin is a triangle with base length 4 and height 1 in $\mathbb{R}^2$.

Sturmfels and Takayama produced five linearly independent series solutions when $\beta = (1, 2)$. They also showed that $\text{rank}(\mathcal{M}_\beta^A) = \text{vol}(A)$ for $\beta \neq (1, 2)$, so that $\beta = (1, 2)$ is the only *exceptional parameter* where the rank changes from its generic value. Around the same time, the case of projective toric curves, the $2 \times n$ case with first row $(1, \ldots, 1)$, was settled by Cattani, D'Andrea, and Dickenstein [**23**]: the set of exceptional parameters is finite in this case, and empty precisely when $\mathbb{C}[Q_A]$ is Cohen-Macaulay; moreover, at each exceptional parameter β, the rank exceeds the volume by 1. These observations led Sturmfels to the following reasonable surmise.

Conjecture 24.14. $\mathcal{M}_\beta^A$ has rank $\text{vol}(A)$ for all $\beta \in \mathbb{C}^d$ precisely when $\mathbb{C}[Q_A]$ is Cohen-Macaulay.

This turns out to be true; see Corollary 24.36.

3. Euler-Koszul homology

What is it about a parameter $\beta \in \mathbb{C}^d$ where the rank jumps that breaks the Cohen-Macaulay condition for $\mathbb{C}[Q_A]$? Stepping back from hypergeometric systems for a little while, what is it about vectors in $\mathbb{C}^d$ in general that witness the failure of the Cohen-Macaulay condition for $\mathbb{C}[Q_A]$?

These being lectures on local cohomology, you can guess where the answer lies. Recall from Lecture 20 that the local cohomology of $\mathbb{C}[Q_A]$ is $\mathbb{Z}^d$-graded, since the columns of A lie in $\mathbb{Z}^d$.

Definition 24.15. Let $L^i = \{a \in \mathbb{Z}^d \mid H^i_\mathfrak{m}(\mathbb{C}[Q_A])_a \neq 0\}$ be the set of $\mathbb{Z}^d$-graded degrees where the i-th local cohomology of $\mathbb{C}[Q_A]$ is nonzero. The *exceptional set* of A is

$$\mathcal{E}_A = \bigcup_{i=0}^{d-1} -L^i\,.$$

Write $\overline{\mathcal{E}}_A$ for the Zariski closure of $\mathcal{E}_A$.

Try not to be confused about the minus sign on L^i in the definition of $\mathcal{E}_A$. Keep in mind that the degree of ∂_j is the *negative* of the j-th column of A, and our convention in this lecture is that $\mathbb{C}[Q_A]$ is graded by $-Q_A$ rather than the usual Q_A. These sign conventions are set up so that the $\mathbb{Z}^d$-grading on the Weyl algebra looks right. The simplest way to think of the sign on the exceptional set is to *pretend* that $\mathbb{C}[Q_A]$ is graded by Q_A, not $-Q_A$; with this pretend convention, $a \in \mathcal{E}_A$ if and only if $H^i_{\mathfrak{m}}(\mathbb{C}[Q_A])_a \neq 0$ for some $i < d$. In particular, the $\mathbb{Z}^d$-graded degrees of nonvanishing local cohomology in Lecture 20 are exceptional degrees; no minus signs need to be introduced.

Example 24.16. For A as in Example 24.2, $L^0 = \varnothing$ and

$$-L^1 = \{(1,2)\} = \mathcal{E}_A = \overline{\mathcal{E}}_A\,.$$

This is the "hole" in the semigroup Q_A generated by the columns of A; see Example 20.30, where a different grading was used.

Any time $\mathcal{E}_A$ contains infinitely many lattice points along a line, the Zariski closure $\overline{\mathcal{E}}_A$ contains the entire (complex) line through them.

Exercise 24.17. Calculate the exceptional set $\mathcal{E}_A$ for the matrix A displayed in Exercise 20.31.

Hint: the Zariski closure of L^2 is a line; which line is it?

Exercise 24.18. What conditions on a set of points in $\mathcal{E}_A$ lying in a plane guarantee that the Zariski closure $\overline{\mathcal{E}}_A$ contains the entire (complexified) plane in which they lie?

Exercise 24.19. To get a feel for what the Zariski closure means in this context, prove that $\overline{\mathcal{E}}_A$ is a finite union of affine subspaces in $\mathbb{C}^d$, each of which is parallel to one of the faces of Q_A (or of C_{Q_A}).

Hint: Use local duality to show that the only associated primes of the Matlis dual $H^i_{\mathfrak{m}}(\mathbb{C}[Q_A])^\vee$ come from faces of Q_A.

Lemma 24.20. *Definition* 24.15 *associates to the matrix A a finite affine subspace arrangement $\overline{\mathcal{E}}_A$ in $\mathbb{C}^d$ that is empty if and only if the ring $\mathbb{C}[Q_A]$ is Cohen-Macaulay.*

Proof. Exercise 24.19 says that the Zariski closure $\overline{\mathcal{E}}_A$ is a finite subspace arrangement. It is empty if and only if $\mathcal{E}_A$ is itself empty, and this occurs precisely when $H^i_{\mathfrak{m}}(\mathbb{C}[Q_A]) = 0$ for $i < d$. By Theorem 9.3 and Theorem 10.36, this condition is equivalent to the Cohen-Macaulay property for $\mathbb{C}[Q_A]$. $\square$

We were after rank jumps, but took a detour to define an affine subspace arrangement from local cohomology. How does it relate to D-modules?

The Weyl algebra contains a commutative polynomial ring $\mathbb{C}[\Theta]$ where

$$\Theta = \theta_1, \ldots, \theta_n \quad \text{ and } \quad \theta_j = x_j \partial_j \, .$$

Each of the Euler operators E_i lies in $\mathbb{C}[\Theta]$, as do the constants. Therefore

$$E_i - \beta_i \in \mathbb{C}[\Theta] \qquad \text{for all } i \, .$$

Consequently, the Weyl algebra has a commutative subalgebra $\mathbb{C}[E-\beta] \subset D$. The linear independence of the rows of A (we assumed from the outset that the $d \times n$ matrix A has full rank d) implies that $\mathbb{C}[E - \beta]$ is isomorphic to a polynomial ring in d variables.

Recall the $\mathbb{Z}^d$-grading of D from Section 1. Suppose N is a $\mathbb{Z}^d$-graded left D-module. If $y \in N$ is a homogeneous element, let $\deg_i(y)$ be the i-th component in the degree of y, so

$$\deg(y) = (\deg_1(y), \ldots, \deg_d(y)) \in \mathbb{Z}^d \, .$$

The $\mathbb{Z}^d$-grading allows us to define a nonstandard action of $\mathbb{C}[E - \beta]$ on N.

Notation 24.21. Let N be a $\mathbb{Z}^d$-graded left D-module. For each nonzero homogeneous element $y \in N$, set

$$(E_i - \beta_i) \circ y = \big(E_i - \beta_i - \deg_i(y)\big) y \, ,$$

where the left-hand side uses the left D-module structure.

The funny $\circ$ action is defined on each $\mathbb{Z}^d$-graded piece of N, so N is really just a direct sum of $\mathbb{C}[E - \beta]$-modules, one for each graded piece of N.

Definition 24.22. Fix a $\mathbb{Z}^d$-graded $\mathbb{C}[\partial]$-module M. Then $D \otimes_{\mathbb{C}[\partial]} M$ is a $\mathbb{Z}^d$-graded left D-module. The *Euler-Koszul complex* is the Koszul complex

$$\mathcal{K}_\bullet(E - \beta; M) = K_\bullet(E - \beta; D \otimes_{\mathbb{C}[\partial]} M)$$

over $\mathbb{C}[E - \beta]$ using the sequence $E - \beta$ under the $\circ$ action on $D \otimes_{\mathbb{C}[\partial]} M$. The i-th *Euler-Koszul homology* of M is

$$\mathcal{H}_i(E - \beta; M) = H_i(\mathcal{K}_\bullet(E - \beta; M)) \, .$$

Why the curly $\mathcal{K}$ instead of the usual Koszul complex K? First of all, we have done more to M than simply placed it in a Koszul complex: we have tensored it with D first. But more importantly, we want to stress that the Euler-Koszul complex is not just a big direct sum (over $\mathbb{Z}^d$) of Koszul complexes in each degree.

Exercise 24.23. Prove that $\mathcal{K}_\bullet(E-\beta;-)$ constitutes a functor from $\mathbb{Z}^d$-graded $\mathbb{C}[\partial]$-modules to complexes of D-modules. In particular, prove that the maps in Definition 24.22 are homomorphisms of D-modules.

Hint: See [**116**, Lemma 4.3].

In a special form, Euler-Koszul homology was known to Gelfand, Kapranov, and Zelevinskiĭ, as well as to Adolphson, who exploited it in their proofs. In the remainder of this lecture, we shall see why Euler-Koszul homology has been so effective for dealing with ranks of hypergeometric systems. A first indication is the following:

Exercise 24.24. For the $\mathbb{Z}^d$-graded $\mathbb{C}[\partial]$-module $\mathbb{C}[\partial]/I_A$, prove that

$$\mathcal{H}_0(E-\beta;\mathbb{C}[Q_A]) = \mathcal{M}_\beta^A\,.$$

The next indication of the utility of Euler-Koszul homology is that it knows about local cohomology. This result explains why we bothered to take the Zariski closure of the exceptional degrees in Definition 24.15.

Theorem 24.25. *The Euler-Koszul homology $\mathcal{H}_i(E-\beta;\mathbb{C}[Q_A])$ is nonzero for some $i \geqslant 1$ if and only if $\beta \in \overline{\mathcal{E}}_A$.* □

A more general statement, in which $\mathbb{C}[Q_A]$ is replaced by an arbitrary finitely generated $\mathbb{Z}^d$-graded $\mathbb{C}[Q_A]$-module, appears in [**116**, Theorem 6.6]. The proof involves a spectral sequence combining holonomic duality and local duality; it relies on little—if anything—beyond what is covered in these twenty-four lectures.

Corollary 24.26. *Euler-Koszul homology detects Cohen-Macaulay rings: the ring $\mathbb{C}[Q_A]$ is Cohen-Macaulay if and only if*

$$\mathcal{H}_i(E-\beta;\mathbb{C}[Q_A]) = 0 \qquad \textit{for all } i \geqslant 1 \textit{ and all } \beta \in \mathbb{C}^d\,.$$

Proof. The vanishing of higher Euler-Koszul homology is equivalent to $\overline{\mathcal{E}}_A = \varnothing$ by Theorem 24.25. On the other hand, $\overline{\mathcal{E}}_A = \varnothing$ if and only if $\mathbb{C}[Q_A]$ is Cohen-Macaulay by Lemma 24.20. □

The content of this section is that we have a hypergeometric D-module criterion for the failure of the Cohen-Macaulay condition for semigroup rings. To complete the picture, we need to see what this criterion has to do with changes in the holonomic ranks of hypergeometric systems as β varies.

4. Holonomic families

Recall that D is the Weyl algebra, $\mathbb{C}\langle x_1,\dots,x_n,\partial_1,\dots,\partial_n\rangle$. For any D-module M, we set $M(x) = M \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$.

The connection between Euler-Koszul homology and rank defects of hypergeometric systems proceeds by characterizing rank defects in general families of holonomic modules. This, in turn, begins with Kashiwara's characterization of holonomic rank; see [**136**, Theorem 1.4.19 and Corollary 1.4.14].

Theorem 24.27. *If M is holonomic, then* $\operatorname{rank} M = \operatorname{rank}_{\mathbb{C}(x)} M(x)$. □

Now suppose that $\mathcal{M}$ is an *algebraic family* of D-modules over $\mathbb{C}^d$. By definition, this means that $\mathcal{M}$ is a finitely generated left $D[b]$-module, where $b = b_1, \ldots, b_d$ is a collection of commuting variables.

Definition 24.28. $\mathcal{M}$ is a *holonomic family* if

(1) the fiber $\mathcal{M}_\beta = \mathcal{M}/(b-\beta)\mathcal{M}$ is a holonomic D-module for all $\beta \in \mathbb{C}^d$, where $b - \beta$ is the sequence $b_1 - \beta_1, \ldots, b_d - \beta_d$ in $D[b]$; and

(2) $\mathcal{M}(x)$ is a finitely generated module over $\mathbb{C}[b](x)$.

Condition (2) here is a subtle coherence requirement. It is quite obvious that $\mathcal{M}(x)$ is finitely generated as a left module over $D[b](x)$, and it follows from condition (1) along with Theorem 24.27 that the fibers of $\mathcal{M}(x)$ over $\mathbb{C}^d$ are finite-dimensional $\mathbb{C}(x)$-vector spaces, but this does not guarantee that $\mathcal{M}(x)$ will be finitely generated over $\mathbb{C}[b](x)$.

Exercise 24.29. Take $\mathcal{M} = D[b]/(bx\partial - 1)$ where $D = \mathbb{C}\langle x, \partial\rangle$. When $\beta \neq 0$, the fiber $\mathcal{M}_\beta$ is the rank-one holonomic module corresponding to the solution $x^{1/\beta}$. But when $\beta = 0$, the fiber $\mathcal{M}_\beta$ is zero. Prove that $\mathcal{M}(x)$ is not finitely generated as a module over $\mathbb{C}[b](x)$.

The point of looking at $\mathcal{M}(x)$ is that its fiber over $\beta \in \mathbb{C}^d$ is a $\mathbb{C}(x)$-vector space of dimension $\operatorname{rank}(\mathcal{M}_\beta)$. Condition (2) in Definition 24.28 implies that constancy of holonomic rank in a neighborhood of β is detected by ordinary Koszul homology.

Proposition 24.30. *The rank function $\beta \longmapsto \operatorname{rank}(\mathcal{M}_\beta)$ for any holonomic family $\mathcal{M}$ is upper-semicontinuous on $\mathbb{C}^d$, and it is constant near $\beta \in \mathbb{C}^d$ if and only if the Koszul homology $H_i(b - \beta; \mathcal{M}(x))$ is zero for all $i \geqslant 1$.*

Proof. Upper-semicontinuity follows from the coherence condition; details are omitted. The fiber dimension of $\mathcal{M}(x)$ is constant near β if and only if (by the coherence condition again) $\mathcal{M}(x)$ is flat near $\beta \in \mathbb{C}^d$, and this occurs if and only if $H_i(b - \beta; \mathcal{M}(x)) = 0$ for all $i \geqslant 1$. □

Upper-semicontinuity means that the holonomic ranks of the fibers in a holonomic family can only jump up on closed sets. In particular, for any holonomic family, there is a well-defined generic rank taken on by the fibers over a Zariski open subset of $\mathbb{C}^d$.

Example 24.31. In Exercise 24.29, the holonomic rank of $\mathcal{M}_\beta$ equals 1 if $\beta \neq 0$, but the rank drops to zero when $\beta = 0$. This actually constitutes a solution to Exercise 24.29, if one is willing to accept Proposition 24.30!

Definition 24.32. Set $\mathcal{M}^A = D[b]/D[b](I_A, E - b)$.

The definition of $\mathcal{M}^A$ is obtained from that of $\mathcal{M}_\beta^A$ by replacing the constants $\beta_1, \ldots, \beta_d \in \mathbb{C}^d$ with the commuting variables $b_1, \ldots, b_d$.

Proposition 24.33. *$\mathcal{M}^A$ is a holonomic family whose fiber over each parameter vector $\beta \in \mathbb{C}^d$ is the A-hypergeometric D-module $\mathcal{M}_\beta^A$.* □

This result is proved in [**116**]. The proof of the first part uses some criteria for when an algebraic family of D-modules is a holonomic family. It is not particularly difficult, but we will not go into it here. We leave it as an easy exercise to check the second part: the fiber $(\mathcal{M}^A)_\beta$ is $\mathcal{M}_\beta^A$.

Example 24.34. Proposition 24.33 implies that the upper-semicontinuity dictated by Proposition 24.30 holds for the algebraic family $\mathcal{M}^A$ constructed from the matrix A in Example 24.2. We have seen this in Example 24.13.

We have a D-module homological theory, Euler-Koszul homology on $E - \beta$, for detecting the failure of the Cohen-Macaulay condition in hypergeometric systems, and a commutative algebraic homological theory, Koszul homology on $b - \beta$, for detecting jumps of holonomic ranks in hypergeometric families. The final point is that they coincide; see [**116**].

Theorem 24.35. $H_i(b - \beta; \mathcal{M}^A) \cong \mathcal{H}_i(E - \beta; \mathbb{C}[Q_A])$. □

As a corollary, we see that Conjecture 24.14 is true:

Corollary 24.36. *The equality* $\operatorname{rank}(\mathcal{M}_\beta^A) = \operatorname{vol}(A)$ *holds for all* $\beta \in \mathbb{C}^d$ *if and only if* $\mathbb{C}[Q_A]$ *is Cohen-Macaulay. In general,* $\operatorname{rank}(\mathcal{M}_\beta^A) \geqslant \operatorname{vol}(A)$*, and*

$$\operatorname{rank}(\mathcal{M}_\beta^A) > \operatorname{vol}(A) \quad \textit{if and only if} \quad \beta \in \overline{\mathcal{E}}_A\,.$$

Proof. The left side of Theorem 24.35 detects when β yields $\operatorname{rank}(\mathcal{M}_\beta^A) > \operatorname{vol}(A)$; the right side detects when β violates the Cohen-Macaulayness of $\mathbb{C}[Q_A]$, via the exceptional set (Theorem 24.25 and Corollary 24.26). □

Example 24.37. For A as in Example 24.2, the coincidence of the jump in rank of $\mathcal{M}_\beta^A$ at $\beta = (1, 2)$ in Example 24.13 and the inclusion of $\beta = (1, 2)$ in the exceptional set from Example 24.16 is a consequence of Corollary 24.36.

Exercise 24.38. Prove that the set of exceptional parameters for a GKZ hypergeometric system $\mathcal{M}_\beta^A$ has codimension at least 2 in $\mathbb{C}^d$.

Appendix

Injective Modules and Matlis Duality

In this appendix R is a commutative ring. We discuss the theory of injective modules; here is a summary:

(1) Each R-module M has an *injective hull*, denoted $E_R(M)$, which is an injective module containing M with the property that any injective containing M has $E_R(M)$ as a direct summand.

(2) An injective module over a Noetherian ring has a unique decomposition as a direct sum of indecomposable injectives.

(3) When R is Noetherian, the indecomposable injective R-modules are of the form $E_R(R/\mathfrak{p})$ for prime ideals $\mathfrak{p}$ of R.

(4) *Matlis duality.* Let R be a complete local ring and E the injective hull of the residue field. The functor $(-)^\vee = \operatorname{Hom}_R(-, E)$ has the following properties:
   (a) If M is Noetherian, then $M^\vee$ is Artinian.
   (b) If M is Artinian, then $M^\vee$ is Noetherian.
   (c) If M is Noetherian or Artinian, then $M^{\vee\vee} \cong M$.

Matlis duality was developed in [**113**]; see also [**20**, §3] and [**115**, §18].

1. Essential extensions

Definition A.1. An R-module E is *injective* if the functor $\operatorname{Hom}_R(-, E)$ is exact; equivalently, if it takes injective maps to surjective maps.

Theorem A.2 (Baer's criterion)**.** *An R-module E is injective if and only if each homomorphism $\mathfrak{a} \longrightarrow E$, where $\mathfrak{a}$ is an ideal, extends to $R \longrightarrow E$.*

Proof. One direction is obvious. For the other, if $M \subseteq N$ are R-modules and $\varphi\colon M \longrightarrow E$, we need to show that φ extends to a homomorphism $N \longrightarrow E$. By Zorn's lemma, there is a module N' with $M \subseteq N' \subseteq N$ maximal with respect to the property that φ extends to a homomorphism $\varphi'\colon N' \longrightarrow E$. Suppose there exists an element y in $N \smallsetminus N'$, and consider the ideal $\mathfrak{a} = (N' :_R y)$. By hypothesis, the composite homomorphism

$$\mathfrak{a} \xrightarrow{y} N' \xrightarrow{\varphi'} E$$

extends to a homomorphism $\psi\colon R \longrightarrow E$. Define $\varphi''\colon N' + Ry \longrightarrow E$ by $\varphi''(y' + ry) = \varphi'(y') + \psi(r)$. This homomorphism is well-defined and its existence contradicts the maximality of φ', so we must have $N' = N$. $\square$

Exercise A.3. Show that $E \oplus E'$ is injective if and only if E, E' are injective.

Exercise A.4. Let R be an integral domain. An R-module M is *divisible* if $rM = M$ for every nonzero element $r \in R$. Verify the following statements.

(1) An injective R-module is divisible.

(2) When R is a principal ideal domain, an R-module is divisible if and only if it is injective. Thus $\mathbb{Q}/d\mathbb{Z}$ is an injective $\mathbb{Z}$-module for $d \in \mathbb{Z}$.

(3) Every nonzero Abelian group has a nonzero homomorphism to $\mathbb{Q}/\mathbb{Z}$.

(4) Set $(-)^\vee = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$. For each $\mathbb{Z}$-module M, the natural map $M \longrightarrow M^{\vee\vee}$ is injective.

Exercise A.5. Let R be an A-algebra. Prove the following statements.

(1) If E is an injective A-module and F a flat R-module, then $\operatorname{Hom}_A(F, E)$ is an injective R-module. Hint: Use the adjointness of $\otimes$ and Hom.

(2) Every R-module embeds in an injective R-module. Hint: Given an R-module M, take an R-linear surjection $F \twoheadrightarrow \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ where F is free, and apply $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$.

The proof of the following proposition is straightforward.

Proposition A.6. *Let $\theta\colon M \longrightarrow N$ be an injective map of R-modules. The following are equivalent:*

(1) *For any homomorphism $\varepsilon\colon N \longrightarrow Q$, if $\varepsilon \circ \theta$ is injective, then so is ε.*

(2) *Every nonzero submodule of N has a nonzero intersection with $\theta(M)$.*

(3) *Every nonzero element of N has a nonzero multiple in $\theta(M)$.* $\square$

Definition A.7. If $\theta\colon M \hookrightarrow N$ satisfies the equivalent conditions of the previous proposition, we say that N is an *essential extension* of M.

Example A.8. If $U \subseteq R$ is a multiplicatively closed set of nonzerodivisors in R, then $U^{-1}R$ is an essential extension of R.

Example A.9. Let $(R, \mathfrak{m})$ be a local ring and N a module in which each element is killed by a power of $\mathfrak{m}$. The submodule $\operatorname{soc}(N) = (0 :_N \mathfrak{m})$ is the *socle* of N. The extension $\operatorname{soc}(N) \subseteq N$ is essential: if $y \in N$ is a nonzero element and t the smallest integer with $\mathfrak{m}^t y = 0$, then $\mathfrak{m}^{t-1} y \subseteq \operatorname{soc}(N)$.

Exercise A.10. Let I be an index set. Then $M_i \subseteq N_i$ is essential for each i in I if and only if $\bigoplus_i M_i \subseteq \bigoplus_i N_i$ is essential.

Exercise A.11. Let $\mathbb{K}$ be a field, $R = \mathbb{K}[[x]]$, and set $N = R_x/R$. Prove that $\operatorname{soc}(N) \subseteq N$ is an essential extension but $\prod_{\mathbb{N}} \operatorname{soc}(N) \subseteq \prod_{\mathbb{N}} N$ is not.

The proofs of (1) and (2) below are easy; for (3) use Zorn's lemma.

Proposition A.12. *Let $L \subseteq M \subseteq N$ be nonzero R-modules.*

(1) *The extension $L \subseteq N$ is essential if and only if the extensions $L \subseteq M$ and $M \subseteq N$ are essential.*

(2) *Suppose $M \subseteq N_i \subseteq N$ with $N = \bigcup_i N_i$. Then $M \subseteq N$ is essential if and only if each $M \subseteq N_i$ is essential.*

(3) *There exists a unique module N' with $M \subseteq N' \subseteq N$ maximal with respect to the property that $M \subseteq N'$ is an essential extension.* □

Definition A.13. The module N' in Proposition A.12(3) is the *maximal essential extension of M in N*. If $M \subseteq N$ is essential and N has no proper essential extensions, we say that N is a *maximal essential extension of M*.

Proposition A.14. *Let M be an R-module. The following are equivalent:*

(1) *M is injective;*

(2) *every injective homomorphism $M \longrightarrow N$ splits;*

(3) *M has no proper essential extensions.*

Proof. We only prove (3) $\Longrightarrow$ (1); the others are left as an exercise. Pick an embedding $M \hookrightarrow E$ where E is injective. By Zorn's lemma, there exists a submodule $N \subseteq E$ maximal with respect to the property that $N \cap M = 0$. So $M \hookrightarrow E/N$ is an essential extension, and hence an isomorphism. But then $E = M + N$ implies $E = M \oplus N$. Since M is a direct summand of an injective module, it must be injective. □

Proposition A.15. *Let M be an R-module. If $M \subseteq E$ with E injective, then the maximal essential extension of M in E is an injective module, hence a direct summand of E. Maximal essential extensions of M are isomorphic.*

Proof. Let E' be the maximal essential extension of M in E and let $E' \subseteq Q$ be an essential extension. Since E is injective, the inclusion $E' \longrightarrow E$ lifts to a homomorphism $Q \longrightarrow E$; this map is injective since Q is an essential extension of E'. One has $M \subseteq E' \subseteq Q \hookrightarrow E$, and the maximality of

E' implies $Q = E'$. Hence E' has no proper essential extensions, so it is injective by Proposition A.14.

Let $M \subseteq E'$ and $M \subseteq E''$ be maximal essential extensions. Then E'' is injective by Proposition A.14, so $M \subseteq E''$ extends to a homomorphism $\varphi\colon E' \longrightarrow E''$. The inclusion $M \subseteq E'$ is essential, so φ is injective. But then $E' \cong \varphi(E')$ is an injective module, and hence a direct summand of E''. Since $M \subseteq \varphi(E') \subseteq E''$ is an essential extension, $\varphi(E') = E''$. □

Definition A.16. The *injective hull* of an R-module M is the maximal essential extension of M, denoted $E_R(M)$.

Verify that M is indecomposable if and only if $E_R(M)$ is indecomposable.

Definition A.17. Let M be an R-module. An *injective resolution* of M is a complex of injective R-modules

$$0 \longrightarrow E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} E^2 \xrightarrow{\partial^2} \cdots$$

with $H^0(E^\bullet) = M$ and $H^i(E^\bullet) = 0$ for $i \geqslant 1$; it is *minimal* if $E^0 = E_R(M)$, $E^1 = E_R(E^0/M)$, and E^{i+1} is the injective hull of $\operatorname{coker} \partial^i$ for each $i \geqslant 1$.

2. Noetherian rings

The following characterization of Noetherian rings is due to Bass [**8**].

Proposition A.18. *A ring R is Noetherian if and only if every direct sum of injective R-modules is injective.*

Proof. Let R be a Noetherian ring and $\{E_i\}_{i\in I}$ injective R-modules. For $\mathfrak{a}$ an ideal, the natural map $\operatorname{Hom}_R(R, E_i) \longrightarrow \operatorname{Hom}_R(\mathfrak{a}, E_i)$ is surjective. This induces the surjection in the bottom row of the following diagram:

$$\begin{array}{ccc}
\operatorname{Hom}_R(R, \bigoplus_i E_i) & \longrightarrow & \operatorname{Hom}_R(\mathfrak{a}, \bigoplus_i E_i) \\
\cong\big\uparrow & & \big\uparrow\cong \\
\bigoplus_i \operatorname{Hom}_R(R, E_i) & \longrightarrow & \bigoplus_i \operatorname{Hom}_R(\mathfrak{a}, E_i)
\end{array}$$

The vertical maps are isomorphisms as $\mathfrak{a}$ is finitely generated. Thus, the top map is surjective. Baer's criterion implies that $\bigoplus_i E_i$ is injective.

For the converse, consider ideals $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ and set $\mathfrak{a} = \bigcup_i \mathfrak{a}_i$. The natural maps $\mathfrak{a} \hookrightarrow R \twoheadrightarrow R/\mathfrak{a}_i \hookrightarrow E_R(R/\mathfrak{a}_i)$ give a homomorphism

$$\mathfrak{a} \longrightarrow \prod_{i\in\mathbb{N}} E_R(R/\mathfrak{a}_i)\,.$$

Verify that this map factors through the direct sum to give a homomorphism $\mathfrak{a} \longrightarrow \bigoplus_i E_R(R/\mathfrak{a}_i)$. By hypothesis, this extends to a homomorphism

$$\varphi\colon R \longrightarrow \bigoplus_{i\in\mathbb{N}} E_R(R/\mathfrak{a}_i)\,.$$

In particular, $\varphi(1)_j = 0$ for $j \gg 0$ so the composite map $\mathfrak{a} \longrightarrow E_R(R/\mathfrak{a}_j)$ is zero. Thus $\mathfrak{a} = \mathfrak{a}_j$ for $j \gg 0$. $\square$

Definition A.19. Let $\mathfrak{a}$ be an ideal of R. An R-module M is $\mathfrak{a}$-*torsion* if every element of M is killed by a power of $\mathfrak{a}$.

Theorem A.20. *Let $\mathfrak{p}$ be a prime ideal of a Noetherian ring R, and set $\kappa = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$, the fraction field of $R/\mathfrak{p}$. Let $E = E_R(R/\mathfrak{p})$.*

(1) *If $x \in R \smallsetminus \mathfrak{p}$, then $E \xrightarrow{x} E$ is an isomorphism, hence $E = E_{\mathfrak{p}}$;*

(2) $(0 :_E \mathfrak{p}) = \kappa$*;*

(3) *$\kappa \subseteq E$ is an essential extension of $R_{\mathfrak{p}}$-modules and $E = E_{R_{\mathfrak{p}}}(\kappa)$;*

(4) *E is $\mathfrak{p}$-torsion and* $\operatorname{Ass}(E) = \{\mathfrak{p}\}$*;*

(5) $\operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa, E) \cong \kappa$ *and* $\operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa, E_R(R/\mathfrak{q})_{\mathfrak{p}}) = 0$ *for primes $\mathfrak{q} \neq \mathfrak{p}$.*

Proof. (1) By Example A.8 the extension $R/\mathfrak{p} \subseteq \kappa$ is essential. Hence E contains a copy of κ; we may assume that $R/\mathfrak{p} \subseteq \kappa \subseteq E$. Multiplication by $x \in R \smallsetminus \mathfrak{p}$ is injective on κ, and so also on its essential extension E. The submodule $xE \cong E$ is injective, thus a direct summand of E. However $\kappa \subseteq xE \subseteq E$ are essential extensions, so $xE = E$.

(2) Part (1) implies that $(0 :_E \mathfrak{p}) = (0 :_E \mathfrak{p}R_{\mathfrak{p}})$ is a κ-vector space, so the inclusion $\kappa \subseteq (0 :_E \mathfrak{p})$ splits. As $\kappa \subseteq E$ is essential, $(0 :_E \mathfrak{p}) = \kappa$.

(3) The containment $\kappa \subseteq E$ is an essential extension of R-modules, hence also of $R_{\mathfrak{p}}$-modules. Let $E \subseteq M$ be an essential extension of $R_{\mathfrak{p}}$-modules. Each $m \in M$ has a nonzero multiple $(r/s)m$ in E, where $s \in R \smallsetminus \mathfrak{p}$. But then rm is a nonzero multiple of m in E, so $E \subseteq M$ is an essential extension of R-modules and $M = E$.

(4) Let $\mathfrak{q} \in \operatorname{Ass}(E)$ and pick $e \in E$ with $(0 :_R e) = \mathfrak{q}$. Since $R/\mathfrak{p} \subseteq E$ is essential, e has a nonzero multiple re in $R/\mathfrak{p}$. In particular, $r \notin \mathfrak{q}$ and

$$\mathfrak{p} = (0 :_R re) = ((0 :_R e) :_R r) = (\mathfrak{q} :_R r) = \mathfrak{q}\,.$$

If $\mathfrak{a}$ is the annihilator of a nonzero element of E, then $R/\mathfrak{a} \hookrightarrow E$, and hence $\mathfrak{p}$ is the only associated prime of $R/\mathfrak{a}$, that is to say, $\operatorname{rad}\mathfrak{a} = \mathfrak{p}$.

(5) The first assertion follows from

$$\operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa, E) = \operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, E) = (0 :_E \mathfrak{p}R_{\mathfrak{p}}) = \kappa\,.$$

The R-module $E(R/\mathfrak{q})$ is $\mathfrak{q}$-torsion, so $E_R(R/\mathfrak{q})_{\mathfrak{p}} = 0$ if $\mathfrak{q} \not\subseteq \mathfrak{p}$. If $\mathfrak{q} \subseteq \mathfrak{p}$, then one has

$$\operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa, E_R(R/\mathfrak{q})_{\mathfrak{p}}) = \left(0 :_{E_R(R/\mathfrak{q})_{\mathfrak{p}}} \mathfrak{p}R_{\mathfrak{p}}\right) = \left(0 :_{E_R(R/\mathfrak{q})} \mathfrak{p}R_{\mathfrak{p}}\right)\,.$$

If this is nonzero, then there is a nonzero element of $E_R(R/\mathfrak{q})$ killed by $\mathfrak{p}$, which forces $\mathfrak{q} = \mathfrak{p}$ since $\operatorname{Ass} E_R(R/\mathfrak{q}) = \{\mathfrak{q}\}$. $\square$

We now prove a structure theorem for injective modules.

Theorem A.21. *Let E be an injective module over a Noetherian ring R. Then there exists a direct sum decomposition*

$$E \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} E_R(R/\mathfrak{p})^{\mu_{\mathfrak{p}}}\,,$$

and the numbers $\mu_{\mathfrak{p}}$ are independent of the decomposition.

Proof. By Zorn's lemma, there exists a maximal family $\{E_i\}_{i \in I}$ of injective submodules of E such that $E_i \cong E_R(R/\mathfrak{p}_i)$ and $\sum E_i$ is a direct sum, say E'. By Proposition A.18, the module E' is injective and hence $E = E' \oplus E''$. Suppose E'' is nonzero. Let $\mathfrak{p}$ be an associated prime of E'' and consider an embedding $R/\mathfrak{p} \hookrightarrow E''$. This gives a copy of $E_R(R/\mathfrak{p})$ contained in E'', which contradicts the maximality of $\{E_i\}$. This proves the existence of a decomposition. As to the uniqueness, Theorem A.20(5) implies that $\mu_{\mathfrak{p}}$ is the rank of the $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$-vector space

$$\operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, E_{\mathfrak{p}})\,. \qquad \square$$

The following proposition can be proved along the lines of Theorem A.20.

Proposition A.22. *Let $U \subset R$ be a multiplicative set.*

(1) *If E is an injective R-module, the $U^{-1}R$-module $U^{-1}E$ is injective.*

(2) *If $M \subseteq N$ is an essential extension (or a maximal essential extension) of R-modules, then the same is true for $U^{-1}M \subseteq U^{-1}N$ over $U^{-1}R$.*

(3) *The indecomposable injectives over $U^{-1}R$ are the modules $E_R(R/\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \cap U = \varnothing$.* $\square$

Definition A.23. Let M be an R-module, and $E^\bullet$ its minimal injective resolution. For each i, Theorem A.21 gives a decomposition

$$E^i = \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p},M)}\,.$$

The number $\mu_i(\mathfrak{p}, M)$ is the i-th *Bass number of M with respect to* $\mathfrak{p}$. The following theorem shows that it is well-defined.

Theorem A.24. *Let R be a Noetherian ring and M an R-module. Let $\mathfrak{p}$ be a prime ideal, and set $\kappa = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Then*

$$\mu_i(\mathfrak{p}, M) = \operatorname{rank}_\kappa \operatorname{Ext}^i_{R_{\mathfrak{p}}}(\kappa, M_{\mathfrak{p}})\,.$$

Proof. Let $E^\bullet$ be a minimal injective resolution of M. By Proposition A.22, $E^\bullet_{\mathfrak{p}}$ is a minimal injective resolution of $M_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$. Moreover, the number of copies of $E_R(R/\mathfrak{p})$ occurring in E^i is the same as the number of copies of

$E_R(R/\mathfrak{p})$ in $E^i_{\mathfrak{p}}$. By definition, $\operatorname{Ext}^i_{R_{\mathfrak{p}}}(\kappa, M_{\mathfrak{p}})$ is the i-th cohomology module of the complex

$$0 \longrightarrow \operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa, E^0_{\mathfrak{p}}) \longrightarrow \operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa, E^1_{\mathfrak{p}}) \longrightarrow \cdots .$$

We claim that the maps in this complex are zero; equivalently for φ in $\operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa, E^i_{\mathfrak{p}})$ the composition

$$\kappa \xrightarrow{\varphi} E^i_{\mathfrak{p}} \xrightarrow{\delta} E^{i+1}_{\mathfrak{p}}$$

is the zero map. If $\varphi(x) \neq 0$, then the minimality of $E^\bullet$ implies that $\varphi(x)$ has a nonzero multiple in $\operatorname{image}(E^{i-1}_{\mathfrak{p}} \longrightarrow E^i_{\mathfrak{p}})$. Since κ is a field,

$$\varphi(\kappa) \subseteq \operatorname{image}(E^{i-1}_{\mathfrak{p}} \longrightarrow E^i_{\mathfrak{p}}),$$

and hence $\delta \circ \varphi = 0$. This completes the proof that the complex above has zero differential. It remains to note that, by Theorem A.20(5), one has

$$\operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa, E^i_{\mathfrak{p}}) \cong \kappa^{\mu_i(\mathfrak{p}, M)}. \qquad \square$$

A homomorphism $\varphi\colon (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ is *local* if $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$.

Theorem A.25. *Let $\varphi\colon (R, \mathfrak{m}, \mathbb{K}) \longrightarrow (S, \mathfrak{n}, \mathbb{L})$ be a local homomorphism of local rings. If S is module-finite over R, then $\operatorname{Hom}_R(S, E_R(\mathbb{K})) = E_S(\mathbb{L})$.*

Proof. The S-module $\operatorname{Hom}_R(S, E_R(\mathbb{K}))$ is injective and $\mathfrak{n}$-torsion, and hence isomorphic to $E_S(\mathbb{L})^\mu$. To determine μ, consider the isomorphisms

$$\begin{aligned}\operatorname{Hom}_S(\mathbb{L}, \operatorname{Hom}_R(S, E_R(\mathbb{K}))) &\cong \operatorname{Hom}_R(\mathbb{L}, E_R(\mathbb{K})) \\ &\cong \operatorname{Hom}_{\mathbb{K}}(\mathbb{L}, \operatorname{Hom}_R(\mathbb{K}, E_R(\mathbb{K}))) \\ &\cong \operatorname{Hom}_{\mathbb{K}}(\mathbb{L}, \mathbb{K}).\end{aligned}$$

Therefore $\mathbb{L}^\mu \cong \operatorname{Hom}_{\mathbb{K}}(\mathbb{L}, \mathbb{K})$ and hence $\mu = 1$. $\square$

Remark A.26. Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring. For each ideal $\mathfrak{a}$ of R, the theorem above implies that the injective hull of $\mathbb{K}$ over $R/\mathfrak{a}$ is $(0 :_{E_R(\mathbb{K})} \mathfrak{a})$. In particular, since $E_R(\mathbb{K})$ is $\mathfrak{m}$-torsion, one has

$$E_R(\mathbb{K}) = \bigcup_{t \in \mathbb{N}} \left(0 :_{E_R(\mathbb{K})} \mathfrak{m}^t\right) = \bigcup_{t \in \mathbb{N}} E_{R/\mathfrak{m}^t}(\mathbb{K}).$$

This motivates the study of $E_R(\mathbb{K})$ for Artinian local rings.

3. Artinian rings

The *length* of a module M is the length of a maximal composition series for M and is denoted $\ell(M)$. Length is additive on exact sequences. Observe that over a local ring R, the residue field $\mathbb{K}$ is the only simple module, so the subquotients in any composition series are isomorphic to it.

Lemma A.27. *Let R be a local ring with residue field $\mathbb{K}$ and set $(-)^\vee = \operatorname{Hom}_R(-, E_R(\mathbb{K}))$. Let M be an R-module. The following statements hold.*

(1) *The natural map $M \longrightarrow M^{\vee\vee}$ is injective.*
(2) $\ell(M^\vee) = \ell(M)$.

In particular, $(-)^\vee$ is faithful, i.e., $M^\vee = 0$ if and only if $M = 0$.

Proof. For nonzero $x \in M$, set $L = Rx \subseteq M$, and consider the composed map $L \longrightarrow L/\mathfrak{m}L \cong \mathbb{K} \hookrightarrow E_R(\mathbb{K})$. This map extends to a homomorphism of R-modules $\varepsilon\colon M \longrightarrow E_R(\mathbb{K})$ with $\varepsilon(x) \neq 0$. This justifies (1).

For (2), we assume first that $\ell(M)$ is finite, and induce on length. If $\ell(M) = 1$, then $M \cong \mathbb{K}$, and the result follows since $\operatorname{Hom}_R(\mathbb{K}, E_R(\mathbb{K})) \cong \mathbb{K}$. When $\ell(M) \geqslant 2$, the composition series of M gives an exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow \mathbb{K} \longrightarrow 0\,.$$

Additivity of length on exact sequences yields $\ell(L) = \ell(M) - 1$. Since $E_R(\mathbb{K})$ is an injective module, applying $(-)^\vee$ to the exact sequence above, one obtains an exact sequence $0 \longrightarrow \mathbb{K} \longrightarrow M^\vee \longrightarrow L^\vee \longrightarrow 0$. Additivity and the induction hypothesis give

$$\ell(M^\vee) = \ell(L^\vee) + 1 = \ell(L) + 1 = \ell(M)\,.$$

When $\ell(M)$ is infinite, part (1) and the already established case of (2) imply that $\ell(M^\vee)$ is infinite as well. □

Applying the preceding result with $M = R$ yields:

Corollary A.28. *One has an equality $\ell(E_R(\mathbb{K})) = \ell(R)$. Hence the length of the R-module $E_R(\mathbb{K})$ is finite if and only if the local ring R is Artinian.* □

Theorem A.29. *Let R be a local ring. Then R is an injective as a module over itself if and only if it is Artinian and* $\operatorname{rank}_{\mathbb{K}} \operatorname{soc}(R) = 1$.

Proof. A local ring is indecomposable by Nakayama's lemma. Thus, when R is injective, it is isomorphic to $E_R(R/\mathfrak{p})$ for some $\mathfrak{p}$ in $\operatorname{Spec} R$. This implies that R is $\mathfrak{p}$-torsion and the elements of $R \smallsetminus \mathfrak{p}$ are units, so $\mathfrak{p}$ is the only prime ideal of R. In particular, R is Artinian, with maximal ideal $\mathfrak{p}$, and $\operatorname{soc}(R)$ is isomorphic to $\operatorname{soc}(E_R(\mathfrak{p}))$, which is a rank-one vector space over $R/\mathfrak{p}$.

For the converse, note that when R is Artinian, it is an essential extension of its socle. Thus, since $E_R(\mathbb{K})$ is injective, when $\operatorname{soc}(R) \cong \mathbb{K}$, there is an embedding $R \subseteq E_R(\mathbb{K})$, and then $\ell(E_R(\mathbb{K})) = \ell(R)$ implies $E_R(\mathbb{K}) = R$. □

4. Matlis duality

The following results about completions can be found in [**6**].

Remark A.30. Let $\mathfrak{a}$ be an ideal in a ring R, and let M be an R-module. Consider the natural surjections

$$\cdots \longrightarrow M/\mathfrak{a}^3M \longrightarrow M/\mathfrak{a}^2M \longrightarrow M/\mathfrak{a}M \longrightarrow 0\,.$$

The $\mathfrak{a}$-*adic completion* of M, denoted $\widehat{M}$, is the inverse limit of this system:

$$\varprojlim_i (M/\mathfrak{a}^iM) = \left\{ (\dots, \overline{m}_2, \overline{m}_1) \in \prod_i M/\mathfrak{a}^iM \mid m_{i+1} - m_i \in \mathfrak{a}^iM \right\}.$$

There is a canonical homomorphism of R-modules $M \longrightarrow \widehat{M}$. The salient properties of this construction are summarized below:

(1) $\ker(M \longrightarrow \widehat{M}) = \bigcap_{i\in\mathbb{N}} \mathfrak{a}^iM$;

(2) $\widehat{R}$ is a ring and $R \longrightarrow \widehat{R}$ is a homomorphism;

(3) $\widehat{M}$ is an $\widehat{R}$-module and $M \longrightarrow \widehat{M}$ is compatible with these structures.

When the ring R is Noetherian and M is an R-module, one has in addition:

(4) The ring $\widehat{R}$ is Noetherian;

(5) If $(R, \mathfrak{m})$ is local, then $\widehat{R}$ is local with maximal ideal $\mathfrak{m}\widehat{R}$;

(6) When the R-module M is finitely generated, one has $\widehat{M} = \widehat{R} \otimes_R M$, and hence $\widehat{M}$ is a finitely generated $\widehat{R}$-module;

(7) One has $\widehat{R}/\mathfrak{a}^i\widehat{R} \cong R/\mathfrak{a}^i$ for each i. If M is $\mathfrak{a}$-torsion, it has a natural $\widehat{R}$-module structure and the map $M \longrightarrow \widehat{R} \otimes_R M$ is an isomorphism;

(8) If M and N are $\mathfrak{a}$-torsion, then $\mathrm{Hom}_{\widehat{R}}(M, N) = \mathrm{Hom}_R(M, N)$.

Theorem A.31. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring, $\widehat{R}$ its $\mathfrak{m}$-adic completion, and set $E = E_R(\mathbb{K})$. One then has $E_{\widehat{R}}(\mathbb{K}) = E$, and the following map is an isomorphism of $\widehat{R}$-modules:*

$$\widehat{R} \longrightarrow \mathrm{Hom}_R(E, E), \qquad \textit{where } r \longmapsto (e \longmapsto re)\,.$$

Proof. The containment $\mathbb{K} \subseteq E$ is an essential extension of R-modules, hence also of $\widehat{R}$-modules. Let $E \subseteq M$ be an essential extension of $\widehat{R}$-modules; it is easy to check that M is $\mathfrak{m}$-torsion. Thus $Rm = \widehat{R}m$ for each $m \in M$, which implies that $E \subseteq M$ is also an essential extension of R-modules. Hence $M = E$, and we conclude that $E = E_{\widehat{R}}(\mathbb{K})$.

Since $\mathrm{Hom}_{\widehat{R}}(E, E) = \mathrm{Hom}_R(E, E)$, for the rest of the proof we assume that R is complete. We first check that $R \longrightarrow \mathrm{Hom}_R(E, E)$ is injective. If

$rE = 0$ for some r in R, applying $(-)^\vee = \operatorname{Hom}_R(-, E)$ to the sequence

$$R \xrightarrow{r} R \longrightarrow R/rR \longrightarrow 0$$

implies $(R/rR)^\vee \cong R^\vee$. This gives the isomorphism in the diagram below:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R/rR \arrow[d] \\ R^{\vee\vee} \arrow[r, "\cong"] & (R/rR)^{\vee\vee} \end{tikzcd}$$

The vertical maps are injective by Lemma A.27, so we deduce that $r = 0$. This justifies the injectivity.

When R is Artinian, the injectivity of the map $R \longrightarrow \operatorname{Hom}_R(E, E)$ implies that it is bijective since $\ell(R) = \ell(\operatorname{Hom}_R(E, E))$; see Lemma A.27. For each $i \geqslant 1$, set $R_i = R/\mathfrak{m}^i$. The R_i-module $E_i = (0 :_E \mathfrak{m}^i)$ is the injective hull of $\mathbb{K}$ by Remark A.26. For each homomorphism φ in $\operatorname{Hom}_R(E, E)$, one has $\varphi(E_i) \subseteq E_i$, so φ restricts to an element of $\operatorname{Hom}_{R_i}(E_i, E_i) = R_i$ where the equality holds as R_i is Artinian. Consequently φ restricted to E_i is multiplication by an element $\overline{r_i} \in R_i$ and $r_{i+1} - r_i \in \mathfrak{m}^i$. Thus φ is multiplication by the element $\varprojlim_i r_i$. $\square$

Corollary A.32. *For a local ring $(R, \mathfrak{m}, \mathbb{K})$, the module $E_R(\mathbb{K})$ is Artinian.*

Proof. Consider a chain of submodules $E_R(\mathbb{K}) \supseteq E_1 \supseteq E_2 \supseteq \cdots$. Applying the functor $(-)^\vee = \operatorname{Hom}_R(-, E_R(\mathbb{K}))$ yields surjections

$$\widehat{R} \twoheadrightarrow E_1^\vee \twoheadrightarrow E_2^\vee \twoheadrightarrow \cdots .$$

The ideals $\ker(\widehat{R} \twoheadrightarrow E_i^\vee)$ stabilize so $E_i^\vee \twoheadrightarrow E_{i+1}^\vee$ is an isomorphism for $i \gg 0$. Since $(-)^\vee$ is faithful, it follows that $E_i = E_{i+1}$ for $i \gg 0$. $\square$

Theorem A.33. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and M an R-module. The following conditions are equivalent:*

(1) *M is $\mathfrak{m}$-torsion and $\operatorname{rank}_\mathbb{K} \operatorname{soc}(M)$ is finite;*

(2) *M is an essential extension of a $\mathbb{K}$-vector space of finite rank;*

(3) *M can be embedded in a finite direct sum of copies of $E_R(\mathbb{K})$;*

(4) *M is Artinian.*

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are straightforward.

(4) $\Longrightarrow$ (1). For $x \in M$, the descending chain $Rx \supseteq \mathfrak{m}x \supseteq \mathfrak{m}^2 x \supseteq \cdots$ stabilizes, so $\mathfrak{m}^{t+1}x = \mathfrak{m}^t x$ for some t, and then Nakayama's lemma yields $\mathfrak{m}^t x = 0$. Therefore M is $\mathfrak{m}$-torsion. Finally, since $\operatorname{soc}(M)$ is Artinian and a $\mathbb{K}$-vector space, it must have finite rank. $\square$

Example A.34. Let $(R, \mathfrak{m}, \mathbb{K})$ be a DVR with $\mathfrak{m} = Rx$. For example, R may be $\mathbb{K}[[x]]$ or the p-adic integers. We claim that $E_R(\mathbb{K}) \cong R_x/R$.

Indeed, it is easy to check that the R-module R_x/R is divisible and hence injective, by Exercise A.4, and that the $\mathbb{K}$-vector space $\operatorname{soc}(R_x/R)$ is generated by the image of $1/x$.

The following theorem was proved by Matlis [**113**].

Theorem A.35 (Matlis duality)**.** *Let $(R, \mathfrak{m}, \mathbb{K})$ be a complete local ring, and set $(-)^\vee = \operatorname{Hom}_R(-, E_R(\mathbb{K}))$. Let M be an R-module.*

(1) *If M is Noetherian, respectively Artinian, then $M^\vee$ is Artinian, respectively Noetherian.*
(2) *If M is Artinian or Noetherian, then the map $M \longrightarrow M^{\vee\vee}$ is an isomorphism.*

Proof. Let $E = E_R(\mathbb{K})$. If M is Noetherian, consider a presentation

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0\,.$$

Applying $(-)^\vee$, we get an exact sequence $0 \longrightarrow M^\vee \longrightarrow (R^n)^\vee \longrightarrow (R^m)^\vee$. By Theorem A.31 the module $(R^n)^\vee \cong E^n$ is Artinian, hence so is the submodule $M^\vee$. Applying $(-)^\vee$ again, we get the commutative diagram

$$\begin{tikzcd}
R^m \arrow[r] \arrow[d, "\cong"'] & R^n \arrow[r] \arrow[d, "\cong"'] & M \arrow[r] \arrow[d] & 0 \\
(R^m)^{\vee\vee} \arrow[r] & (R^n)^{\vee\vee} \arrow[r] & M^{\vee\vee} \arrow[r] & 0
\end{tikzcd}$$

with exact rows, where the isomorphisms hold by Theorem A.31. It follows that $M \longrightarrow M^{\vee\vee}$ is an isomorphism as well.

Using Theorem A.33, a similar argument works when M is Artinian. $\square$

Remark A.36. Let M be a finitely generated module over a complete local ring $(R, \mathfrak{m}, \mathbb{K})$. One has isomorphisms

$$\begin{aligned}
\operatorname{Hom}_R(\mathbb{K}, M^\vee) &\cong \operatorname{Hom}_R(\mathbb{K} \otimes_R M, E_R(\mathbb{K})) \\
&\cong \operatorname{Hom}_R(M/\mathfrak{m}M, E_R(\mathbb{K})) \\
&\cong \operatorname{Hom}_{\mathbb{K}}(M/\mathfrak{m}M, \mathbb{K})\,.
\end{aligned}$$

Thus the number of generators of M as an R-module is $\operatorname{rank}_{\mathbb{K}} \operatorname{soc}(M^\vee)$; this number is the *type* of $M^\vee$.

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