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On Buildings and their Applications

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The buildings considered in this talk[1] are some particular simplicial complexes naturally associated to algebraic simple groups. The "real estate" terminology, due to N. Bourbaki [8], originated in the fact that the maximal simplices of these complexes are called "chambers" (in French, "chambre", that is, "room"), because of their close connection with the "Weyl chamber" in the theory of root systems.

1. A construction procedure. Let us first describe in rough terms a trivial but fruitful procedure to build up complicated geometrical objects from simpler ones. Take an object C, for instance a space of some kind or a simplicial complex, and a group G. To each "component" x of C (point of the space, simplex of the complex), attach a subgroup G_x of G. Then, there exists a unique minimal object extending C, on which G acts in such a way that no two components of G are equivalent under G and that G_x is the stability group of x in G, namely the quotient of the product $G \times C$ by the equivalence relation $(g, x) \sim (g', x') \Leftrightarrow x = x'$ and $g^{-1}g' \in G_x$. To make this description precise, one has of course to specify in which category, say, the product and the quotient are taken. In the sequel, C will most of the time be just a simplex, to each face σ of which a subgroup G_σ of G is attached; furthermore, the relation $G_{\sigma \vee \sigma'} = G_\sigma \cap G_{\sigma'}$ will always hold. Then, it is clear how $G \times C/\sim$, "defined" as above, is endowed with a structure of simplicial complex. Notice that, in view of the above equality, all G_σ are known as soon as the groups attached to the vertices of C are given.

[1]When preparing this report, I made much use of information received form A. Borel and J. B. Wagoner, and, most of all, from J.-P. Serre, who kindly took the trouble of reviewing for me the main applications of the theory of buildings known to him. To these acknowledgements, I wish to associate the Deutsche Forschungsgemeinschaft whose generous support made it possible for me to attend the Congress.

EXAMPLES. (1) Let the vertices of the simplex C be numbered from 1 to n. To each edge (ij), associate an integer $m_{ij} \geqq 2$ or the symbol ∞. Set

$$G = \langle r_1, \cdots, r_n \,|\, r_h^2 = (r_i r_j)^{m_{ij}} = 1 \text{ for } m_{ij} \neq \infty \rangle$$

and attach to the face σ of C the group $G_\sigma = \langle r_i \,|\, i \notin \bar{\sigma} \rangle$. The resulting complex Δ is called a *Coxeter complex* (cf. [**43**, §2]). For instance, if C is a one-simplex ($n = 2$), Δ is a closed chain of length $2m_{12}$ or a doubly infinite chain according as $m_{12} \neq$ or $= \infty$; if C is a triangle and if the three m_{ij}'s are ${}_3{}^2{}_5$ (resp. ${}_3{}^3{}_3$; resp. ${}_3{}^2{}_6$), Δ is the barycentric subdivision of an icosahedron (resp. the paving of the plane by equilateral triangles; resp. the barycentric subdivision of the paving of the plane by hexagonal honeycombs). When G is finite, Δ can be realized as a simplicial decomposition of a Euclidean $(n - 1)$-sphere on which G acts as a group of isometries: Δ is then called *spherical*. It is called *Euclidean* if it can be realized as a simplicial decomposition of a Euclidean space on which G operates by Euclidean isometries. The matrices $((m_{ij}))$ giving rise to spherical and Euclidean Coxeter complexes have been determined by H.S.M. Coxeter [**17**] and E. Witt [**48**].

(2) Let $G = \mathrm{SL}_3(\boldsymbol{F}_2)$ and let G_1 (resp. G_2) be the subgroup of all matrices $((g_{ij}))$ with $g_{21} = g_{31} = 0$ (resp. $g_{31} = g_{32} = 0$). If C is a one-simplex to the vertices of which we attach G_1 and G_2, the resulting complex is the graph of Figure 1, which is also obtained as follows: Its vertices are the points and lines of the projective plane over $\boldsymbol{F}_2$ and its edges join the pairs forming a flag (point + line through it).

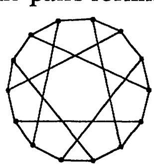

FIGURE 1

(3) More generally, let k be a division ring and $G = \mathrm{SL}_n(k)$. If we take for C an $(n - 1)$-simplex to the kth vertex of which we attach the group $\{((g_{ij})) \in G \,|\, g_{ij} = 0$ for $i > k \geqq j\}$, we get the "flag complex" of the $(n - 1)$-dimensional projective space Π over k, i.e., the complex whose vertices are the proper linear subspaces of Π and whose simplices are the flags of Π.

(4) Let k be a field with a discrete valuation whose residue field is $\boldsymbol{F}_2$, $\mathfrak{o}$ the ring of

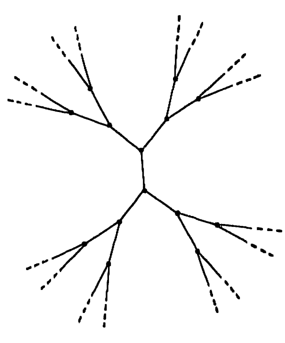

FIGURE 2

integers and π a uniformizing. Let $G = \mathrm{SL}_2(k)$. If we attach to the two vertices of a one-simplex C the subgroups $\mathrm{SL}_2(\mathfrak{o})$ and

$$G \cap \begin{pmatrix} \mathfrak{o} & \pi^{-1}\mathfrak{o} \\ \pi\mathfrak{o} & \mathfrak{o} \end{pmatrix}$$

we obtain a "homogeneous tree" whose vertices have order 3 (Figure 2).

(5) With k, $\mathfrak{o}$, π as above, let now $G = \mathrm{SL}_3(k)$ and let C be a two-simplex, to the vertices of which we attach the following subgroups:

$$\mathrm{SL}_3(\mathfrak{o}), \quad G \cap \begin{pmatrix} \mathfrak{o} & \pi^{-1}\mathfrak{o} & \pi^{-1}\mathfrak{o} \\ \pi\mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \pi\mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix}, \quad G \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \pi^{-1}\mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \pi^{-1}\mathfrak{o} \\ \pi\mathfrak{o} & \pi\mathfrak{o} & \mathfrak{o} \end{pmatrix}.$$

Then the resulting complex Δ is a kind of two-dimensional analogue of the tree of Figure 2: Every edge belongs to three two-simplices, the link of every vertex is the graph of Figure 1 and, in the same way as the tree contains "many" doubly infinite chains, so does Δ contain "many" subcomplexes isomorphic to the paving of a plane by equilateral triangles (cf. §3).

REMARKS. (a) In all examples given above, C was a simplex, but it may also be useful to start from other geometric objects. For instance, let G be the dihedral group of order 8, let us denote by U_2 its center, and by U_1, U_3 two other subgroups of order 2 such that $G = U_1U_2U_3$. Then, if one takes for C a hexagon to the vertices of which one attaches the groups G, U_1U_2, U_1, U_3, U_3U_2, G, the resulting complex is again the graph of Figure 1. Another instructive example is the following alternative construction of the tree of Figure 2: k, $\mathfrak{o}$ and π being defined as in Example (4), take for G the additive group of k, for C the "doubly infinite chain"

$$\begin{matrix} & -2 & & -1 & & 0 & & 1 & & 2 & \\ \cdots & | & \text{———} & | & \text{———} & | & \text{———} & | & \text{———} & | & \cdots \end{matrix}$$

and attach to the vertex i the subgroup $\pi^i\mathfrak{o}$ of G (this construction has an advantage over that of Example (4) in that it extends immediately to fields with nondiscrete valuations; cf. [**11**, §7]).

(b) In this article, we are essentially interested in buildings, but the general procedure described above can also be used to construct other interesting complexes, for instance graphs related to some sporadic simple groups.

2. Buildings. Let G be a semisimple algebraic group defined over some field k. (By a common abuse of language, we shall often make no distinction between an algebraic group and the "abstract" group of its rational points over some ground field; thus, G will also denote the group of k-rational points of the algebraic group G.) There are two types of buildings which one associates to such pairs (G, k) and which we want to describe:

for arbitrary k, the *spherical building* constructed by means of the k-parabolic subgroups of G;

when k is local (i.e., endowed with a complete discrete valuation, whose residue field we assume to be perfect, for safety) the *Euclidean building* constructed by means of the parahoric subgroups of G.

To avoid technical complications, we shall assume that G is absolutely almost simple[2] (i.e., has over no field extension of k a proper normal subgroup of strictly positive dimension) and, when talking about the local case, that G is simply connected (this is a technical condition, satisfied for instance by the groups SL_n, Spin_n, Sp_n and their "twisted forms").

We recall that the *parabolic subgroups* of G are the algebraic subgroups P such that G/P is a complete (in fact, projective) variety (cf., e.g., [2], [5]). There is no such simple characterization of the *parahoric subgroups*, a notion introduced by N. Iwahori and H. Matsumoto [23] in the case of Chevalley groups and successively extended by H. Hijikata [21] and by F. Bruhat and the author [10]; for a general definition, we refer the reader to [10] and [11]. Examples of parabolic and parahoric subgroups will be given in a moment, but we must first state a property of those subgroups which is essential for our purpose: There is a natural number l, called the *relative rank* of G, such that the following assertion holds:

(*) All minimal k-parabolic (resp. parahoric) subgroups of G are conjugate; if B is one of them and if $P_1, \cdots, P_r$ denote the maximal proper subgroups of G containing B, one has $r = l$ (resp. $r = l + 1$), the 2^r subgroups $P_{i_1} \cap \cdots \cap P_{i_s}$ are all distinct, they are the only proper subgroups of G containing B and they form a complete system of representatives of the conjugacy classes of proper k-parabolic (resp. of parahoric) subgroups of G.

Thus, if we want to describe the parabolic or the parahoric subgroups of G, it suffices to exhibit *one* minimal such subgroup B. We start with some examples of parabolic subgroups:

If $G = \mathrm{SL}_n(k)$, one can take for B the group of all upper triangular matrices.

If k is algebraically closed, B is any *Borel subgroup*, that is, any maximal connected (for the Zariski topology) solvable subgroup of G.

If k is perfect and if G is thought of as a group of matrices, one calls "unipotent subgroup" of G a subgroup consisting only of matrices all of whose eigenvalues are 1, and B is then the normalizer of any maximal unipotent subgroup (for instance, if char $k = p \neq 0$, B is the normalizer of any maximal p-subgroup of G: the "Sylow theorem" holds for such subgroups).

We now go over to the local case and denote by B a minimal parahoric subgroup of G. When k (and hence G) is locally compact, there is a characterization of B (essentially due to H. Matsumoto) similar to the characterization of minimal parabolic subgroups over perfect fields given above: B is the normalizer of any maximal pro-p-subgroup (projective limit of finite p-groups) of G, where p is the characteristic of the residue field. As a further example, let $G = \mathrm{SL}_n(k)$ over any local field k whose ring of integers we again denote by $\mathfrak{o}$; then, one can take for B the group of all elements of $\mathrm{SL}_n(\mathfrak{o})$ whose reduction modulo the prime ideal is upper triangular.

[2]In the sequel, the word "almost" will often be omitted when no confusion can arise.

The property (∗) makes the parabolic and parahoric subgroups well suited for applying the construction described in §1; one takes a simplex of dimension $r - 1 = l - 1$ (resp. l) whose vertices are numbered from 1 to r and one attaches to the face $(i_1, \cdots, i_q)$ the parabolic (resp. parahoric) subgroup $P_{i_1} \cap \cdots \cap P_{i_q}$ of G. The resulting complex is called the *spherical* (resp. *Euclidean*, or *affine*) *building* associated to G and k; simple examples are Examples (2) and (3) (resp. (4) and (5)) of §1. The first virtue of the geometric object thus attached to such pairs (G, k) is expressed by the

THEOREM. *If $l \geqq 2$, the building associated to (G, k) determines "canonically" the algebraic group G up to isogeny, the field k and, in the local case, the valuation of k.*

(For a more precise statement in the spherical case, cf. [**43**, 5.8].) In view of Example (3) of §1, that theorem can be regarded as a generalization of the "fundamental theorem of projective geometry"; it also includes the theorem of W. L. Chow and J. Dieudonné [**18**, III, §3] on the permutations of linear subspaces of quadrics which preserve the adjacency (at least for division rings which are finite dimensional over their center, but this restriction is not essential; cf. [**43**, §8]).

If $l = 0$ ("anisotropic group") the theory of buildings is of course empty (although, in the local case, buildings can also be used in the study of anisotropic groups; cf., e.g., [**10**, Proposition 6]). When $l = 1$, the Euclidean buildings are trees; they are quite useful (cf. for instance [**22**], [**32**], [**35**], [**36**], [**37**]) but do not have enough structure to give back G and k. The above theorem also suggests the following comment on Examples (4) and (5) of §1: If k and k' are two nonisomorphic totally ramified extensions of the field of dyadic numbers, the Euclidean buildings of $SL_2(k)$ and $SL_2(k')$ are isomorphic whereas those of $SL_3(k)$ and $SL_3(k')$ are not, though they look much alike "locally".

3. Apartments. The axiomatic approach. An important property of the buildings is that they contain "many" Coxeter subcomplexes. Indeed, every building Δ has a system $\mathscr{A}$ of Coxeter subcomplexes, called the *apartments* of Δ, such that:

(i) Every two simplices of Δ belong to an apartment.

(ii) If $\Sigma, \Sigma' \in \mathscr{A}$, there exists an isomorphism of Σ onto Σ' which fixes $\Sigma \cap \Sigma'$ (elementwise).

More precisely, Δ being associated to a group G (cf. §2):

(ii′) If $\Sigma, \Sigma' \in \mathscr{A}$, there exists an element of G which maps Σ onto Σ' and fixes $\Sigma \cap \Sigma'$.

For instance, in Examples (2), (3), (4), (5) of §1, the apartments are respectively hexagons (i.e., barycentric subdivisions of triangles), barycentric subdivisions of $(n - 1)$-simplices (the "coordinate frames" of the projective space in question), doubly infinite chains, and complexes isomorphic to the paving of a Euclidean plane by equilateral triangles. The terminology "spherical" and "Euclidean" introduced in §2 can now be motivated; the apartments of the buildings constructed by means of parabolic (resp. parahoric) subgroups are spherical (resp. Euclidean) Coxeter complexes.

Properties (i) and (ii) are responsible for many useful properties of the buildings. This suggests an axiomatic approach to the theory, in which these properties are taken as axioms. To avoid degeneracies, it is convenient to add the condition:

(iii) Every nonmaximal simplex of Δ is a face of at least three distinct simplices of Δ.

Thus, let us call "abstract building" a simplicial complex satisfying (iii) and having a system $\mathscr{A}$ of Coxeter subcomplexes such that (i) and (ii) hold. It can be shown that, if we require $\mathscr{A}$ to be maximal with these properties, it is unique. The question naturally arises to know how much more general this "abstract" notion is, compared to the "concrete" one introduced in §2. If the apartments are spherical of dimension $\geqq 2$ and "irreducible" (a Coxeter complex is irreducible if it is not the join of two nonempty Coxeter subcomplexes), or Euclidean of dimension $\geqq 3$, the answer is given by a classification theorem (for the spherical case, cf. [**43**]) which shows that the construction of §2 provides all such buildings if one extends the class of groups G considered so as to include the "classical groups" over arbitrary division rings and also some further "more exotic" groups. Thus, the notion of abstract building provides an elementary, "combinatorial", simultaneous approach to the algebraic semisimple groups and the classical groups of relative rank $\geqq 3$. For spherical abstract buildings of dimension one, a complete classification is out of the question but it is conjectured that a certain quite simple additional condition, the "Moufang property" (cf. [**43**, p. 274], and [**44**]), is sufficient to characterize among them the buildings associated to the classical groups, the algebraic simple groups and, again, some related "exotic" groups (e.g., the Ree groups of type ${}^2F_4$) of relative rank 2. Let us add here that the study of abstract buildings whose apartments are neither spherical nor Euclidean may be promising, as is suggested by the work of R. Moody and K. L. Teo [**29**] and R. Marcuson [**27**].

4. Metric. Topology. So far, we have only been interested in the "combinatorial" structure of the simplicial complexes we have considered. Now, it will be necessary to imagine the simplices "concretely" realized as spherical or Euclidean simplices. If Δ is the spherical (resp. Euclidean) building associated to a group G (cf. §2), its apartments are Euclidean spheres (resp. Euclidean spaces) endowed with a natural metric, well defined up to a scalar multiplication. It is easily seen that the distance functions in the various apartments can be chosen in such a way that for every $g \in G$ and every apartment Σ, g induces an isometry of Σ onto $g\Sigma$. Then, by property (ii′) of §2, the metrics on any two apartments agree on their intersection. Since, by (i), any two points belong to an apartment, Δ itself is endowed with a distance function d which can be shown to satisfy the triangular inequality. Thus, Δ is a metric space on which G acts as a group of isometries (*N.B.*: the metrics of Figures 1 and 2 are not induced by the natural metric of the underlying sheet of paper!). If Δ is spherical, its diameter is the common diameter of its apartments. We say that two points p, q of Δ are *opposite* if Δ is spherical, of diameter $d(p, q)$.

Let p, q be two nonopposite points of the building Δ. In any apartment Σ containing them, which is a Euclidean space or a sphere, they can be joined by a unique shortest geodesic, which turns out to be independent of Σ (cf. [**11**, 2.5.4] for the Euclidean case). From this, one deduces in the usual way that:

(i) *Euclidean buildings are contractible*;

(ii) *a spherical building minus the set of all points opposite to a given point is contractible.*

This last property readily implies that

(ii′) *a spherical building has the homotopy type of a bouquet of spheres.*

Furthermore, the number N of these spheres is easily determined; for instance, if the ground field k is finite of characteristic p, N is the p-contribution to the order of G.

The above properties are useful facts, as was first recognized by L. Solomon [**40**] who observed that, since G acts on Δ, it also operates on $\tilde{H}_{l-1}(\Delta) = \boldsymbol{Z}^N$ (l being, as before, the relative rank of G). One thus obtains a special G-module whose rank—in the finite case—is the order of the p-Sylow subgroups of G; as one may expect, this is nothing else but the *Steinberg module* of G. A similar idea was used by A. Borel and J.-P. Serre (unpublished, cf. however [**6**]) to define the "Steinberg module" of an algebraic simple group over a $\mathfrak{p}$-adic field: Here, one lets G operate on H^l_c (cohomology with compact support) of the Euclidean building of G, which is shown to be isomorphic with the Čech cohomology group H^{l-1} of the spherical building of G endowed with a nonstandard topology.

Further applications of the spherical buildings to the representation theory of finite simple groups "of Lie type" have been made by T. A. Springer (unpublished, except for some indications in [**41**]) and by G. Lusztig ([**25**], [**42**]) who considers moreover other complexes (e.g., the complex of "affine flags away from 0" in affine spaces) closely related to the buildings.

Properties (i) and (ii) are also used by D. Quillen in his proofs of various finiteness theorems in algebraic K-theory (cf. [**9**], [**34**] and other, unpublished results concerning the function field case). For further applications of buildings or "building-type constructions" to algebraic K-theory, we refer to [**1**], [**45**], [**46**] [**47**].

5. Euclidean buildings and symmetric spaces. In many respects, *the Euclidean buildings are the "ultrametric analogues" of the Riemannian symmetric spaces.* In other words, they play, in the study of $\mathfrak{p}$-adic simple groups, a role similar to that of the symmetric spaces in the theory of simple Lie groups. We shall illustrate this assertion by a few examples.

E. Cartan has shown that, in an irreducible, noncompact, simply connected symmetric space, every compact group of isometries has a fixed point (cf. [**12**, p. 19]). The same is true of a compact (and even a bounded) group of isometries of a Euclidean building [**11**, 3.2]. In fact, G. Prasad has observed that Cartan's proof itself can be carried over to Euclidean buildings: One just has to prove for the latter a certain metric inequality [**33**, 5.12] which, in the case of Riemannian spaces, characterizes the spaces with negative curvature. That the Euclidean buildings behave like spaces with negative curvature is further illustrated by other inequalities (e.g., [**11**, 3.2.1]) and by the unicity of the geodesic joining two points (cf. §4).

The fixed-point theorem mentioned above was used by Cartan to show the conjugacy of all maximal compact subgroups of a real simple Lie group. Its analogue for buildings enabled F. Bruhat and the author [**11**, §3] to show that, in a $\mathfrak{p}$-adic simple group (assumed to be simply connected, as previously agreed), the maximal compact subgroups are the maximal parahoric subgroups, and thus form $l + 1$ conjugacy classes (l = the relative rank). The fixed-point theorem is also an essential tool in the process of extending the theory of Iwahori and Matsumoto to arbitrary $\mathfrak{p}$-adic simple groups (cf. [**10**, §6]): This is done by "Galois descent", and the compact group to which the theorem is applied is the Galois group of a "splitting field" of the $\mathfrak{p}$-adic group in question.

Another domain where Euclidean buildings are used as substitutes for the symmetric spaces is the cohomology of discrete subgroups. Let G be a real noncompact simple Lie group and Γ a discrete subgroup which, for simplicity, we shall assume without torsion. Then, Γ operates freely on the symmetric space X of G and, since X is contractible, $H^i(\Gamma) = H^i(X/\Gamma)$ for arbitrary coefficients. In particular, cd Γ is $\leqq$ dim X. Furthermore, using some differential operators on X related to the Riemannian curvature, Y. Matsushima was able to obtain more precise information on the groups $H^i(\Gamma,\boldsymbol{R})$; his results show, for instance, that for cocompact Γ and i "sufficiently small" $H^i(\Gamma, \boldsymbol{R})$ depends only on G and not on Γ. As J.-P. Serre pointed out, the Euclidean building X of a $\mathfrak{p}$-adic simple group G can be used similarly to investigate the cohomology of discrete subgroups Γ of G: The most obvious observation is that, since X is contractible (cf. §4), the above argument shows that if Γ is torsion-free, cd $\Gamma \leqq$ dim $X = l$ (relative rank of G); in [**38**] similar but more elaborate techniques are used to estimate—among other things—the cohomological dimension of S-arithmetic groups. (This dimension is *determined* in [**6**].) As for the result of Matsushima mentioned above, it can be compared with a conjecture of Serre proved by H. Garland [**19**] for "sufficiently large residue fields" (a restriction lifted by W. Casselman later on; cf. [**15**], [**20**]): If Γ is a torsion-free cocompact discrete subgroup of a $\mathfrak{p}$-adic simple group, then $H^i(\Gamma, \boldsymbol{R}) = 0$ for $0 < i < l$. The method of Garland bears striking formal similarities with that of Matsushima; the differential operators considered by the latter are here replaced by some "local combinatorial operators", regarded by Garland as the "$\mathfrak{p}$-adic curvature" of the building X (cf. also [**3**]).

Mentioning those operators naturally leads us to another formal analogy between symmetric spaces and Euclidean buildings, namely the possibility of doing "harmonic analysis" on the latter as well as on the former. The simplest case is that of a locally finite tree T (remember Figure 2). If f is a complex-valued function on the set of vertices of T and if, for every vertex s, we denote by $\Lambda(f)(s)$ the average of the values of f in the vertices neighbouring s, it is well known that the operator $\Delta = \Lambda - 1$ is the "analogue" for T of the Laplace-Beltrami operator on a Riemannian manifold. The harmonic analysis on trees has been extensively studied by P. Cartier ([**13**], [**14**]). Instead of considering functions on vertices, i.e., 0-cochains, one may consider 1-cochains or, more generally, l-cochains on a locally finite Euclidean building X of dimension l, that is, functions defined on the set of all l-simplices. Such a function f is called *harmonic* if for every simplex σ of dimension $l - 1$, the sum of the values of f on all maximal simplices whose closure contains σ is zero. Taking for X the building of a $\mathfrak{p}$-adic simple group G and letting G operate on the Hilbert space of L^2 harmonic l-cochains on X, one obtains the so-called *special representation* of G which contains the Steinberg module (cf. §4) tensorized with $\boldsymbol{C}$ as a dense submodule, and which plays an important role in the theory of unitary representations of G. This representation was introduced by H. Matsumoto [**28**] and J. A. Shalika [**39**] (by I. M. Gelfand and M. I. Graev for GL_2); its interpretation as a representation on L^2 forms is due to A. Borel who also showed that the space of admissible vectors is the Steinberg module [**4**] and who constructed other, similar representations, using the Euclidean building [**4**]. (For related questions, cf. also [**26**] and its bibliography.)

6. Spherical buildings and symmetric spaces. We shall again introduce this section with a metamathematical statement, which will however be considerably vaguer than that of §5. Let G be a real or a $\mathfrak{p}$-adic simple group and let X be its symmetric space or Euclidean building. When studying various questions, one is sometimes led to add to G or X "points at infinity"; it turns out that

the "most natural choice" for the "space at infinity" of G or X is "often" closely related to the spherical building of G.

Restriction of space and competence forces me to be very brief in commenting on that sentence. With some good will however, the reader will grant that it is illustrated by the results enumerated below, and whose interconnections have perhaps not yet been fully investigated.

In [**6**], A. Borel and J.-P. Serre compactify the Euclidean building of a $\mathfrak{p}$-adic simple group G by adding to it the spherical building of G suitably retopologized (cf. also [**11**, 5.1.33]). In [**7**], considering an algebraic semisimple group G defined over a field $k \subset \boldsymbol{R}$ they enlarge the symmetric space X of the real Lie group $G(\boldsymbol{R})$ in a "manifold with corners" $\bar{X}$ and, if k is countable, $\bar{X} - X$ has the homotopy type of the spherical building of G over k. Both papers are primarily aimed at the study of arithmetic and S-arithmetic groups and, in particular, of their cohomology.

Let now G be an algebraic simple group over any field k. In [**31**, Chapter 2, §2], for the purpose of studying the "stability" in G-spaces, D. Mumford interprets the points of a certain dense subset $X_{\boldsymbol{Q}}$ of the spherical building of G over k as the equivalence classes of "one-parameter subgroups" (one-dimensional split tori) of G for a suitable equivalence relation. Intuitively, that relation describes a certain "asymptotic" behavior of the one-parameter subgroups, so that $X_{\boldsymbol{Q}}$ can be regarded as "lying at infinity" of G. A similar viewpoint is developed further in [**24**, IV, §2] (and in forthcoming continuations), where G is effectively enlarged into a scheme $\bar{G}$ by adding "at infinity" a scheme related with the spherical building of G on k (roughly speaking, $\bar{G} - G$ has a stratification whose "k-rational nerve" is the building).

Finally, it is appropriate to mention under the same heading the work of G.D.

Mostow [30] and G. Prasad [33] on the strong rigidity of cocompact discrete subgroups of real and $\mathfrak{p}$-adic simple groups, and perhaps also some aspects of the spectacular result of G. A. Margulis on the arithmeticity of lattices, which became known during this Congress. To conclude the article at a somewhat more "concrete" level, I shall try to give in a few words an extremely oversimplified idea of Mostow's proof of the following special but significant case of his result:

Let G, G' be two absolutely simple noncompact Lie groups of relative rank $\geqq 2$ and let $\Gamma \subset G$, $\Gamma' \subset G'$ be torsion-free, cocompact discrete subgroups; then, every isomorphism $\alpha\colon \Gamma \to \Gamma'$ extends to an isomorphism of G onto G'.

Let X, X' be the symmetric spaces of G and G' and admit that the real spherical buildings Y and Y' of G and G' "lie at infinity" of X and X'. Because X, X' are topological cells, the manifolds X/Γ and X'/Γ' are $K(\Gamma, 1)$ and $K(\Gamma', 1)$, so that there exists a homotopy equivalence $X/\Gamma \to X'/\Gamma'$ which lifts to a mapping $\beta\colon X \to X'$ "compatible with α". Because X/Γ and X'/Γ' are compact, β "does not disturb much" the distance function in the large, from which one infers that it induces an isomorphism $\beta'\colon Y \to Y'$ of the buildings at infinity. Finally, it follows from the canonicity assertion of the Theorem of §2 that β' is induced by an isomorphism of G onto G'.

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