Trade Using One Commodity as a Means of Payment

Lloyd Shapley

Rand Corporation

Martin Shubik

Yale University

A general model of noncooperative trading equilibrium is described in which prices depend in a natural way on the buying and selling decisions of the traders, avoiding the classical assumption that individuals must regard prices as fixed. The key to the approach is the use of a single, specified commodity as "cash," which may or may not have intrinsic value. The model, in several variants, is treated as a noncooperative game, in the spirit of Nash and Cournot. The *rules* of the game, including the price-forming mechanism, are independent of behavioral or equilibrium assumptions, which enter, instead, through the *solutions* of the game.

I. Introduction

In the development of a theory of money and financial institutions, there are three kinds of questions which should be clearly differentiated: (1) Why does money come into use, and how? (2) What keeps it in use? (3) What strategic limitations does its use impose on trade, what additional possibilities does it open up, and what are the institutional implications?

The first would involve both historical research and a study of transactions costs and markets. It would be interesting to explore such questions

We are grateful for suggestions and constructive criticisms from Pradeep Dubey, Robert Slighton, Rodney Smith, Ross Starr, and an anonymous referee as well as many others with whom we have discussed this model during the past 2 or 3 years. We are also grateful to the National Science Foundation (grant GS-31253) and the Office of Naval Research (contract N00014-76-C-0085) for their support.

[*Journal of Political Economy*, 1977, vol. 85, no. 5]

as the smallest size or level of trading activity at which a developing economic community should be expected to adopt a commodity money system, a fiat money system, or a banking system, etc. At least since the time of Cournot (1838), it has been observed that, if m commodities are to be bartered on a one-to-one basis with no use of a monetary medium, as many as $m(m - 1)/2$ pairwise trading markets might be needed, while the use of a generally accepted medium of exchange enables the system to function efficiently with only m or $m - 1$ trading posts. Similarly, a money can enable an economic system to function efficiently through purely bilateral transactions, without requiring tortuous sequences of middlemen who pass the goods along to their ultimate consumers (Ostroy and Starr 1974). In other words, a medium of exchange can "de-couple" cross-interactions both between traders and between commodities; and, if an economic society grows large, the social advantages of this soon outweigh any social disadvantages or costs that may be involved. This is of course not the whole story: information costs, search costs, transportation costs, the desire for safety or anonymity, and myriad other details of transactions technology may be invoked as further explanations of why money and financial institutions come into being (see Jones 1976).

Questions concerning the beginnings of money and finance are basically different, however, from the question of why money stays in use. The fish gets on the hook one way and off it in a radically different manner. Once enough people accept a money and a body of law and custom concerning its use, it is difficult, short of social breakdown, hyperinflation, or revolution, for an individual or small group to "opt out" of the system, even if the conditions attending the system's beginnings have changed. The acceptance of the system has a flavor of a noncooperative equilibrium like the "Prisoner's Dilemma": although a person may view, say, fiat money as being of dubious value as a store of wealth, he knows that most others will continue to use it for trade, and he may be in no position to do otherwise himself.

No modern society is a 100 percent monetized society. Segments with barter trade always exist, and the volume of such trade may vary with many factors such as technology, taxes, faith in financial institutions, constraining laws and regulations, and so forth. Nevertheless, if we accept the facts that money is the major means of exchange in a modern economy and that much of the exchange of money for goods, services, and financial instruments takes place via some form of organized market, be it a stock-market or a supermarket, then we can ask yet another kind of question. What are the restrictions imposed upon economic behavior if we *require* that trade be monetized? How do these restrictions influence the decisions of rational individuals, and how do they limit the range of feasible outcomes? The body of this paper is concerned largely with questions of this type.

Our exposition is based on a multicommodity model of trade as a game in strategic form, to which the noncooperative solution concept of Nash (1951) may be applied. The model is strategically closed, in that, unlike the classical equilibrium models, prices are determined by the actions of the traders, and the system as a whole responds meaningfully to the traders' decisions, even away from equilibrium.[1] The mathematical proof of the existence of a Nash equilibrium and its relationship to the competitive equilibrium of Walras will be presented elsewhere; here we concentrate on an elementary but, we hope, informative exposition, making much use of "Edgeworth" box diagrams. These will enable us to visualize the qualitative effects of monetized trade on such things as price formation, feasibility, Pareto optimality, and equilibrium.

It is worth emphasizing that we are presenting here not a single model with a fixed viewpoint on the nature of money but, rather, a general modeling approach, or framework, within which many different properties and users of money and financial institutions can be analyzed and compared. Our "money" can be a valuable, consumable commodity, or it can be merely paper, of no intrinsic worth. Even in a one-period model, various forms of credit can be introduced, and the trade-inhibiting effects of a limited money supply can be represented. In multiperiod models (not treated here, but see Sec. VIA below), one can go further and introduce lending institutions and derive interest rates. While we make no pretense that we are able to capture all of the complex factors associated with the use of money and financial institutions, it seems reasonable to hope that our approach, extended or elaborated in one direction or another, will prove useful as a basis for more extensive investigations of many facets of this subject.

II. Some Preliminaries on Economics and Modeling

Before proceeding to the description of a specific trading game, it may be useful to discuss some of the highlights of our approach in general terms. In particular, we would like to comment on (*a*) the differences between commodities, commodity money, and fiat money; (*b*) the institutional implications of cash and credit; and (*c*) the distinction between the (legal) rules of the game and the (behavioral) rules of play and the

[1] In his important early paper, Debreu (1952) represents the Walras exchange model as a game in strategic form for the technical purpose of applying a general existence theorem. But as a *descriptive* model his game shares the defect of the Walrasian model of being ill defined, or unrealistically defined, away from equilibrium. Indeed, if only one agent departs from equilibrium, he is presumed to be able to buy and sell at the stated prices, announced by an added fictive player whose objective is to minimize excess demand. But there is no explanation of how the excess demand thereby created is to be satisfied—unless it is out of the bottomless warehouses of the fictive player. See also Arrow and Debreu (1954) and Arrow and Hahn (1971, chap. 5).

reasons for our selection of the Nash noncooperative equilibrium for the analysis of the latter. In Section VI some other general points will be touched upon, in connection with possible variations and extensions of our model.

A. *Money and Commodities*

In this paper we are frankly concentrating on the "means of payment" role of money. We cannot contend that this is money's only or even its most important reason for being. But we do argue that this is its most conspicuous function in everyday economic life and that the payment *process* has hardly received a fair share of attention in modern mathematical economics. We shall therefore treat money as an element of strategy (in the game-theory sense) and not merely regard it as either an insubstantial price-reporting or book-balancing abstraction, on the one hand, or just another commodity to be traded and consumed, on the other.

Several authors, in the study of *cooperative*-game solutions to economic models, have had recourse to utility functions of the following special form:

$$U^i(x) = u^i(x^i_1, \ldots, x^i_m) + \lambda^i x^i_{m+1},$$

where the $(m + 1)$th commodity may be regarded as a kind of transferable utility, or "u-money" (see, e.g., Shapley and Shubik 1966, pp. 807–8; 1969*a*, 1972, 1976; Telser 1972, pp. 4–11; and Aumann and Shapley 1974, p. 180). Although the vague phrase "market game with money" would seem to cover both this approach and our present *noncooperative* model, the connection is only superficial. In cooperative solutions such as the core and the value, the underlying strategic form of the game is irrelevant, and all attention is directed to the actions and potential actions of coalitions. The "u-money," if available, serves only as a vehicle for side payments that adjust the final distribution of utility among traders in coalition. There has been little interest in formulating *cooperative* trading games in which money plays a distinguished role in the trading technology; in fact, core theory (with good reason) has traditionally stood aloof from the processes of trade and price formation.[2]

In our present work, we do not assume that the utility of the payment commodity is additively separable, as above. But it is possible that such an assumption might simplify some of our results or proofs or ensure uniqueness or other good behavior on the part of the noncooperative

[2] Telser (1972, chap. 3), however, has analyzed a Cournot-type oligopoly model in terms of the core; see also Shapley and Shubik (1972) for a model whose core has a direct interpretation in terms of prices.

equilibria, just as it does for the classical competitive equilibria.[3] At least the question merits further study.

Viewed as a commodity, real or fictitious, money is distinguished in practice by its near-universal acceptability in exchange for other commodities.[4] The reasons for its acceptance lie as much in the realms of the laws and customs of society as in pure individualistic economic reasoning. In an economic model that is not intended to encompass social and legal developments, it may be taken as axiomatic that the monetary good will be accepted at face value according to the existing conventions of the marketplace, regardless of its intrinsic worth or lack of worth. In our present model, we shall *require* that all exchanges be for money. An interesting consequence of this rule is that the set of attainable redistributions of goods will fail to include many redistributions that would be possible if arbitrary, transaction-cost-free barter were allowed.[5] It should be noted that this curtailment also occurs under the classical rules of Walrasian exchange at stated prices. This is not necessarily unrealistic. A similar curtailment of the feasible set occurs in practice, we believe, in most societies that rely on organized markets and relatively stable price systems for the redistribution of goods.

Because of its general acceptability a means of payment has obvious value to its holder, but this value need not show itself in his utility function.[6] Indeed a *fiat money* would enter the utility function only through the effect of truncation: unspent money at the game's end may be presumed to have buying power in the world to come. (You *can* take it with you!) A *commodity money*, on the other hand, like gold or silver, has utility in its own right, to which we may add any extra buying power it may be considered to possess—an alternative or incremental value conferred by society's acceptance of its special monetary role.[7]

It is obvious that an adequate investigation of the utility of money demands a dynamic treatment in a multiperiod model, with the possibility of durable as well as perishable goods. We forgo any such in-

[3] With an additively separable "u-money" in sufficient supply, the competitive allocations of the other goods are just those that maximize the sum $\sum u^t(x)/\lambda^t$; thus a "fixed-point" situation reduces to a simpler maximization problem.

[4] Clower (1967) goes so far as to assert, as a matter of definition, that *any* commodity universally acceptable in exchange is a "money."

[5] See the end of Sec. IIC.

[6] Many economic investigations include a "money" in the utility functions; see, e.g., Patinkin (1956), Telser (1972 and elsewhere), or the discussion in Samuelson (1947, pp. 119–21).

[7] Such an adjustment for buying power may be called for even in the case of pure barter. A person's utility for a large quantity of wheat, for example, evaluated at a particular point in time, may depend in part on his expectation of being able to trade it later for something more directly useful to him. It is difficult to treat this subject with any precision in a static, one-period model.

vestigation in this paper, adopting, instead, the expedient of reserving a place in the utility functions for the "means of payment" without insisting that it actually be consumable or have utility. Within this framework we can encompass the range between a commodity money of great intrinsic worth, at one extreme, and a perishable fiat money, at the other.

B. Cash and Credit

Our point of departure will be a basic model where "cash" payments are required in advance on all purchases. We then explore some ways of relaxing this condition, via a sort of shopkeepers' credit in the form of deferred payments secured by expected receipts. Since only a one-period model is considered, interest rates and the money market cannot be fairly represented. Nevertheless, some institution-modeling problems do appear, since beyond a narrow zone of "conservative" credit (see Sec. IVB below) the possibility of insolvency and default exists and must be faced if the game is to be well defined. It may seem overelaborate to burden the abstract model with the details of a bankruptcy proceeding in which the assets of the trader who cannot pay his bills are liquidated to satisfy the creditors. Yet important properties of the game and its equilibria may well depend on just how such details are handled, once we relax the "cash on the barrelhead" rule.[8] One simple expedient is to allow negative holdings of "cash" at the game's end while postulating a disutility (suitably concave and continuous) for being "in the red." Implicitly there are loan sharks, swimming outside the model, who pay off the creditors while making life uncomfortable for the debtors, but the burden of providing detailed rules for insolvency is avoided.

Even without default there are institutional overtones in the granting of credit. Suppose a trader pays for some goods with promissory notes rather than cash, expecting to redeem them with cash received from the sale of his own goods. What if some of his customers also use promissory notes? When and how does the redistribution of the actual "means of payment" commodity take place? It appears that a central clearinghouse must be created. This would represent a major change in the nature of the model, which—as will be seen—is otherwise quite clearly decentralized, with respect to both traders and commodities.

While we cannot enter into a detailed discussion here, it is important to note also that credit generally involves a *contract* between two parties, whereas cash generally does not (except perhaps in the weak sense that an

[8] The rule proposed by Postlewaite and Schmeidler (1975), which in effect confiscates and destroys all goods belonging to the bankrupt party, seems both drastic and unrealistic (particularly away from equilibrium), since avoiding insolvency in that model depends not only on personal prudence but on correctly estimating the actions of the other traders.

individual holder of a dollar bill may regard himself as a creditor of the government). We may for convenience describe a person's indebtedness as "negative cash," but it is basically a different instrument. In a society where all pay cash, default and the laws and procedures for dealing with it need not be considered. A society with even the simplest forms of credit is fundamentally more complex than one without credit, and it can adequately be represented only by a fundamentally more complex model.

C. Rules and Solutions

The game theorist is usually at great pains to assure himself that the rules of the game are completely defined before he turns to the problem of solving it. The reason for this caution stems from the fact that, while the *descriptive theory*—covering the moves and strategies and information and payoffs, etc.—can be "hard" and mathematically precise, the solution theory is often "soft" and indeterminate, since it expresses the actions of sophisticated, free-willed decision makers. In the multiperson non-zero-sum games common in economics, this indeterminacy shows itself both in the multitude of different solution *concepts* that game theory offers for consideration—like core, value, bargaining set, noncooperative equilibrium, etc.— and in the nonuniqueness of outcome that is so often exhibited by the actual solutions, under any one of these concepts.

We would like to stress, therefore, that we are presenting a well-defined game in the descriptive sense, formulated independently of any assumptions of equilibrium or of what might or might not be "rational" behavior. When the players have made their individual decisions, the market prices and the transfers of goods are *completely determined.* To this game a variety of solution concepts might be applied, but the rules of the game and its solution are two different things.

Having said this, we must admit that we have built our model in a particular way (in a "strategic" form, as opposed, say, to a "coalitional" form) in anticipation of applying a particular solution concept—the "Nash" noncooperative equilibrium. In much of our previous work in this general area we have, instead, used the coalitional form and (with many others) have considered static, essentially noninstitutional games of exchange and/or production, establishing some interesting links between the competitive equilibrium and the core or other *cooperative* game concepts.[9] However, this kind of coalitional approach fails to capture the dynamics of price formation and the essentially individualistic nature of much economic decision making, and so further investigations using *noncooperative* game concepts now seem to be in order.

[9] See, e.g., Shapley and Shubik (1966, 1969*a*, 1969*b*, 1976) or the books of Arrow and Hahn (1971), Scarf (1973), Aumann and Shapley (1974), and Hildenbrand (1974), where many additional references will be found.

The classical model is content to take prices as though given by an "invisible hand," insensitive to the actions of the traders at least in the short run. There is an implicit understanding, not reflected in any mathematical assumption of the classical model, that the traders are so numerous and their individual resources so small that this insensitivity is a good approximation to reality. Our model, on the other hand, has prices that depend in a reasonable way on the individual trading decisions. They are driven upward by increased buying and downward by increased selling. This puts us into a position to examine the validity of the classical assumption of unyielding prices when the traders are *not* individually insignificant and, indeed, to explore the transition zone between perfect competition à la Walras and oligopolistic competition à la Cournot. But to carry out this program, our game-theoretic solution concept must be strategy oriented and collusion free, and the Nash noncooperative equilibrium is ideally suited to this purpose.

It may be of interest to point out in conclusion that if we were to reduce our present model to its characteristic function,[10] we would get *a different cooperative game* from the unrestricted-barter or "Edgeworth" game that is usually considered in connection with the core. The reason is that our rules cause all trade to use a single set of prices. If goods, or goods and money, pass between Traders 1 and 2 in a certain ratio, then they cannot pass between 1 and 3 in a different ratio. From the coalitional standpoint, this restriction on trade can lead to a paradoxical result—a failure of superadditivity. In fact, if we make the natural assumption that subeconomies can form and establish their own price systems, then an economy that is fractured into opposing coalitions may be able to reach allocations on the Pareto surface that cannot be achieved by the same economy united.[11]

III. The Basic Model

In order to describe a well-defined game of exchange in which a specific commodity is used as a means of payment, we must spell out how the prices are formed. The classic general equilibrium model is content to establish the existence of prices (often not unique) *at equilibrium.* For a proper game model, however, we need rules that determine the prices for

[10] The "characteristic function" of a game characterizes the outcomes that can be achieved by any subset of players. See von Neumann (1928), Aumann and Peleg (1960), Scarf (1967), Shapley and Shubik (1973), etc.

[11] It is not generally realized that there is a distinctive "Walras" cooperative game, with a more restrictive characteristic function than the Edgeworth game. Despite the possible failure of superadditivity, and hence of balancedness, the existence of the core in the Walras game is not threatened, since the competitive equilibrium must still have the core property. But the Walras core and Walras-Pareto set will in general be different from the Edgeworth core and Edgeworth-Pareto set. Superadditivity can be restored to the Walras game by the device of taking the "superadditive cover" (see Shapley and Shubik 1969*a*), but the result is still not the Edgeworth game.

positions of disequilibrium as well. With every player free to make an independent decision, the model must yield a well-defined output for every set of inputs.

Several types of price-forming mechanisms might be considered, each placing different restrictions on the strategic possibilities. In particular the traders might control only the quantities offered or demanded, or they might name reservation prices or price ranges or even complete demand curves for their individual transactions. Multistage bargaining might be introduced, or a centralized procedure that converts a set of unilateral price declarations into a unique, market-wide price for each good.[12] Here, as we are interested in general, anonymous exchange, with market-wide prices but with a minimum of ad hoc institutional detail, we adopt what is essentially a generalization of Cournot's original approach. The strategic variables will be *quantities*, not prices, but they will include quantities of the special good that serves as "cash" or "trading money." Indeed, in our simplest version, *only* that good will be subject to strategic choice.

Formally, in our prototype model we shall assume that there are n traders trading in $m + 1$ goods, where the $(m + 1)$th good has a special operational role in addition to its possible utility in consumption. We attribute to each trader an initial bundle of goods,[13]

$$a^i = (a^i_1, \ldots, a^i_m, a^i_{m+1}),$$

and a concave utility function,

$$u^i(x^i_1, \ldots, x^i_m, x^i_{m+1}).$$

We emphasize that u^i need not actually depend on x^i_{m+1}; the possibility of a fiat money is not excluded.

The general procedure will be for the traders to put up quantities of the first m goods to be sold and simultaneously to put up quantities of the $(m + 1)$th good to buy them, all at prices determined by the market-wide supply and demand for each good. For expository purposes, our prototype model will require the traders to offer for sale *all* of their holdings of the first m goods, though they need not spend all of their $(m + 1)$th good. A trader may well buy some of his original goods back, but they must go through the market. In other words, in this version of the model the trader does not own his initial bundle outright; he merely owns a *claim on the proceeds* when the bundle is sold.[14]

[12] Levitan and Shubik (1971, 1972) have given examples of different strategy spaces and Pareto surfaces that are possible in oligopolistic models.

[13] Superscripts will be used consistently to denote traders. Summation over traders will be denoted by a horizontal bar; thus $\bar{a}_j$ means

$$\sum_{i=1}^{n} a^i_j.$$

[14] This simplifying condition is not as unreasonable as it might appear at first glance. In a multiperiod context, it would amount to requiring that all "paper" profits or losses be realized every trading period. Some other modeling possibilities will be discussed in Sec. VI below.

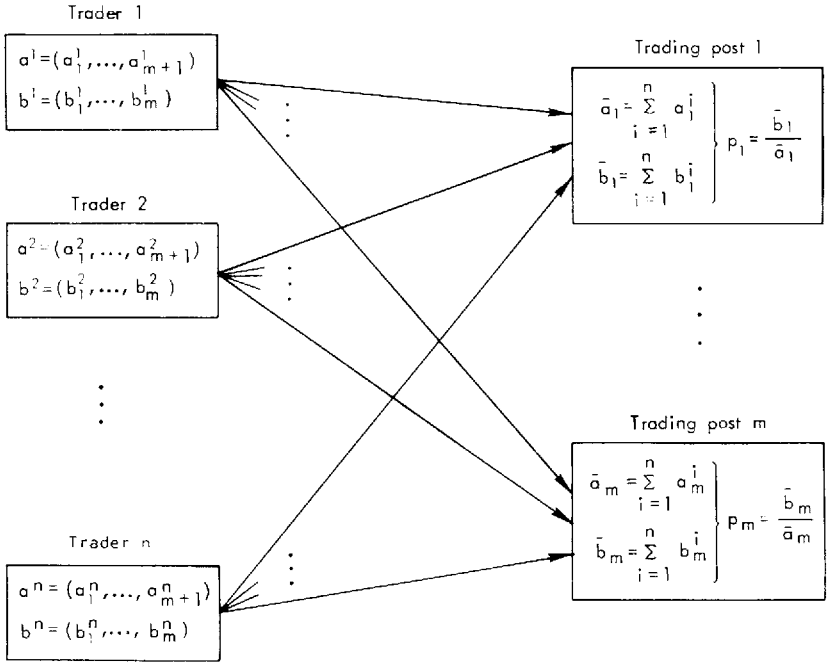

FIG. 1.—Decentralized trade

Let us imagine m separate *trading posts*, one for each of the first m commodities, where the total supplies $(\bar{a}_1, \ldots, \bar{a}_m)$, assumed to be positive, have been deposited for sale "on consignment." Figure 1 illustrates this. Each trader i makes bids by allocating amounts b_j^i of his $(m + 1)$th commodity among the m trading posts, $j = 1, \ldots, m$. We shall denote his *strategy*, in the game-theoretic sense, by the vector $b^i = (b_1^i, \ldots, b_m^i)$. There are a number of possible rules governing the permitted range of bids. In the simplest case, with no credit of any kind, the limits on b^i are given by

$$\sum_{j=1}^{m} b_j^i \leq a_{m+1}^i,$$

and

$$b_j^i \geq 0, \qquad j = 1, \ldots, m.$$

The interpretation of this spending limit is that the traders are required to pay *cash in advance*. More generally, we might allow them to defer payment, either in anticipation of receipts or under some other credit arrangement that would have to be made explicit in the model.[15]

[15] See Sec. IVB below. The simplest way to model credit is to remove the spending limit entirely, while extending the domain of the utility function to include negative

A. Price Formation

The prices now emerge in a natural way as a result of the simultaneous bids of all buyers; we define

$$p_j = \bar{b}_j/\bar{a}_j, \qquad j = 1, \ldots, m.$$

Thus bids precede prices. Traders allocate their budgets *fiscally*, committing quantities of their means of payment to the purchase of each good without definite knowledge of what the per-unit price will be. At an equilibrium this will not matter, as prices will be what the traders expect them to be. In a multiperiod multitrader context, moreover, the traders will know the previous prices and may expect that fluctuations in individual behavior in a mass market will not change prices by much. But any deviation from expectations will result in changing the quantity of goods received rather than the quantity of cash spent. In practice, if one allocates a portion of one's budget for purchase of a certain good in a mass market, this will be different—but not too different—from a decision to buy a specific amount at an unspecified price. It is a matter of letting one's stomach rather than one's purse absorb the fluctuations.

The prices in our model are so determined that they will exactly balance the books at each trading post. Note that there is no need for a central clearinghouse and there are no interpost or intertrader transactions. The amount of the jth good that the ith trader receives in return for his bid b_j^i is

$$x_j^i = \begin{cases} b_j^i/p_j & \text{if } p_j > 0, \\ 0 & \text{if}^{16}\ p_j = 0. \end{cases}$$

His final amount of the $(m + 1)$th good, taking account of his sales as well as his purchases, is

$$x_{m+1}^i = a_{m+1}^i - \sum_{j=1}^{m} b_j^i + \sum_{j=1}^{m} a_j^i p_j.$$

His *payoff*, in the game-theory sense, must be expressed as a function of all the traders' strategies; accordingly, we write

$$\Pi^i(b^1, \ldots, b^i, \ldots, b^n) = u^i(x_1^i, \ldots, x_{m+1}^i).$$

---

amounts of the $(m + 1)$th commodity so as to provide a suitable *disutility* for being caught short of cash when the sales and purchases are all added up. To close the model, an outside source of cash would have to be postulated, to cover the payments due the other traders when one trader defaults. In a dynamic, multiperiod context, we can conceive of durable goods, other than the $(m + 1)$th "payment" commodity, that are carried forward and used as security for the granting of credit for commercial loans (Shubik 1977).

[16] Note that $p_j = 0$ implies $b_j^i = 0$. Thus a trader receives nothing if and only if he bids nothing.

Here the x's depend on the b's according to the three preceding displayed equations. It is noteworthy that the function Π^i is *concave* in b^i for each i; this is important for the existence proof.

Because of the mechanism for price formation and the anonymous allocation of sales, all traders pay the same price for the same good. However, the operation of the market system gives rise to what have been called *pecuniary externalities*, in the sense that the prices paid by one trader are dependent on the monetary actions of the others (Viner 1931; Shubik 1971). In the classic barter market this is not so; trading pairs or groups can form and exchange goods unaffected by the actions of others.

B. The Noncooperative Solution

A noncooperative or "best-response" equilibrium has been defined in game theory as a set of strategy choices by the players with the property that no player, given the choices of the others as fixed, can gain by changing his own choice.[17] Specialized to the present application, this solution concept will be recognized by economists as a close relative of the classic Cournot oligopoly solution (cf. Maňas 1972). It will consist of an n-tuple of strategies:

$$\hat{b} = (\hat{b}^1, \hat{b}^2, \ldots, \hat{b}^n)$$

such that for each i the function $\Pi^i(\hat{b}^1, \ldots, b^i, \ldots, \hat{b}^n)$, considered as a function of b^i alone, is maximized at $b^i = \hat{b}^i$.

The following existence theorem for the noncooperative equilibrium (NE) may be proved by an application of the Kakutani fixed-point theorem to the "best-response" adjustment process or transformation: $b \to b'$, with each b'^i being a best response by player i to the set of choices $\{b^j : j \neq i\}$. The details of the proof, which are rather lengthy, will be given elsewhere.

> *Theorem 1.* For each trader, $i = 1, \ldots, n$, let u^i be continuous, concave, and nondecreasing. For each good, $j = 1, \ldots, m$, let there be at least two traders with positive initial endowments of good $m + 1$ whose utility for good j is strictly increasing. Then a noncooperative equilibrium exists.

Note that there is no assumption that good $m + 1$ has intrinsic value to anyone. It must merely be available to enough people so that nontrivial markets for the other goods can be formed.

[17] The definition of Nash (1951), who, however, includes the possibility of mixed strategies. These have no plausible interpretation in our present model, so we shall be searching only for "Nash equilibria in pure strategies." The term "best-response equilibrium" is due to Robert Wilson.

Our reasons for concentrating on the noncooperative equilibrium rather than on one of the cooperative solutions of game theory have already been discussed in Section IIC. We would like merely to reiterate here that at least one purpose of our work is to formulate and study the *descriptive* game model, without reference to any particular solution concept. Indeed much of the discussion that follows concerns what are sometimes called "pre-solution" concepts like feasibility, Pareto optimality, or individual rationality, which do not depend on equilibrium notions at all.

IV. The Edgeworth Box

The case $m = 1, n = 2$ lends itself to simple two-dimensional descriptive analysis based on the familiar Edgeworth box. To avoid the confusion of too many lines and curves in one place, we shall make a sequence of diagrams, using the same labels as far as possible. As we shall see, much of this geometry will apply also to the general case, with many goods and traders, because of the way in which the operation of the system "decouples" both the traders and the trading posts.

In figure 2, the first trader's holdings are measured up and to the right of the point O^1, while the second's are measured down and to the left of O^2. The dimensions of the box are $\bar{a}_2$ high by $\bar{a}_1$ wide. The point R represents a typical *initial allocation*; thus we have $a_1^1 = |M^1R|$, $a_2^1 = |G^1R|$, etc.[18] The points S^1 and S^2 represent typical *strategy choices*; thus we have $b_1^1 = |M^1S^1|$ and $b_1^2 = |M^2S^2|$. They are restricted to lie along the edges M^1O^1 and M^2O^2, respectively, since we are requiring all of good 1 to be sent to market. (If we did not make this restriction, the strategies would lie arbitrarily in the rectangles $RM^1O^1G^1$ and $RM^2O^2G^2$, respectively; see Sec. VI below.)

Now consider the line joining S^1 and S^2. Its slope is $(b_1^1 + b_1^2)/(a_1^1 + a_1^2)$, which is just the price p_1. Moreover, it divides the line M^1M^2 into two segments which are equal in length to the amounts x_1^1 and x_1^2 of good 1 purchased by Traders 1 and 2, respectively. Similarly, it divides the line G^1G^2 into segments equal to their final holdings x_2^1, x_2^2 of the monetary commodity, or "cash." We see therefore that the *final allocation* is represented by the point F. The vector $R\bar{F}$ represents the actual transaction that takes place; its slope, naturally, is equal (in absolute value) to the price p_1.

It is essential that the reader understand the nomogram demonstrated in figure 2, as it is the key to all the diagrams that follow.

Figure 3 shows the effect of holding S^2 fixed while varying S^1. As S^1 moves over the interval from M^1 to O^1, the point F traces out the curve A^1RB^1. This curve is a portion of a hyperbola whose asymptotes are the

[18] The notation $|PQ|$ indicates the length of segment PQ.

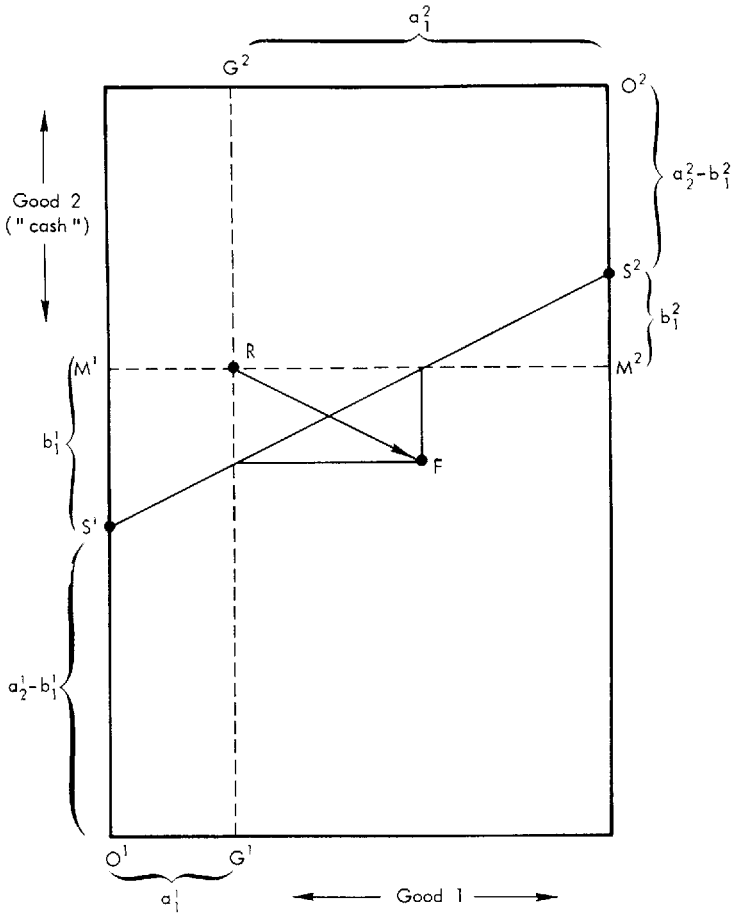

FIG. 2.—Price formation

horizontal and vertical lines through S^2. If we reverse the process and move S^2, holding S^1 fixed, we trace out the curve A^2B^2, which is part of a similar rectangular hyperbola centered at S^1. Endpoints A^1, A^2 correspond to zero bids, while endpoints B^1, B^2 reflect the upper limit on the amount a trader can bid. It happens in this case that we did not allow Trader 2 enough cash to be able to buy back his original holding when Trader 1 plays S^1, so the curve A^2B^2 stops short of R.

These traces are comparable to the price rays or "budget sets" that confront a trader in the classical Walrasian model with its fixed prices. The difference is that in the present case the price is not constant but reacts to variations in a trader's own decisions, so we get a curve instead of a line. The curve is concave, as one would expect, so that if a trader increases his purchase he drives the price up, while if he bids less the price

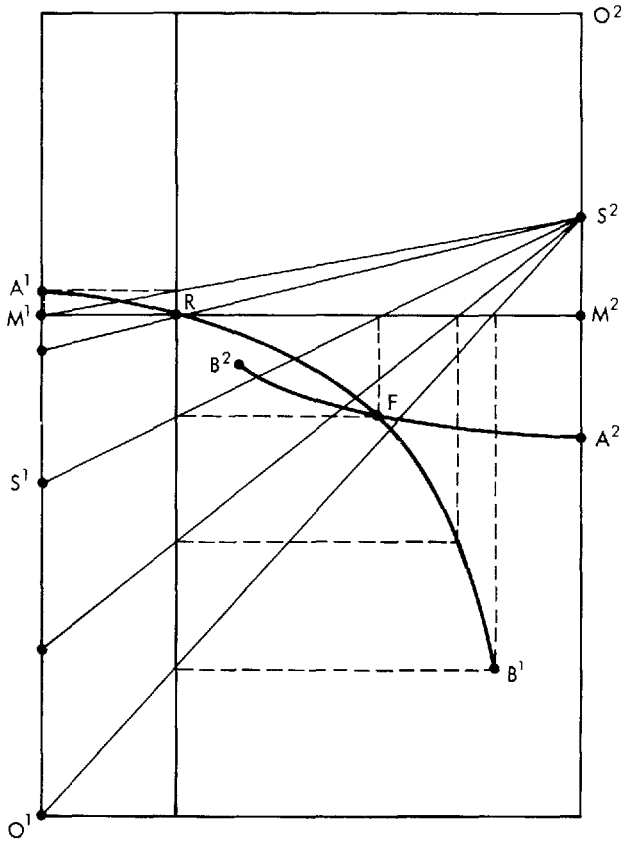

FIG. 3.—"Budget sets"

falls. The connection between the two approaches may be illustrated in figure 3 by supposing the second trader's holdings and bid to be very large, pushing points O^2 and S^2 far off the page. The first trader's hyperbolic "budget set" would then approximate a straight line through R and F, reflecting the fact that he now has little influence on the price.

A. Two Kinds of Equilibrium

So far we have discussed only the mechanics of the rules of exchange. We are now ready to add the traders' preferences to the picture, by superimposing the contours of the utility functions u^1 and u^2. As shown in figure 4, S^1 happens to be the *best response* to S^2, since A^1B^1 is tangent at F to the contour of u^1. (Note that the curvature of A^1B^1 is such that there is always a unique point of tangency.) Similarly, S^2 is the best response to S^1, since A^2B^2 is tangent at F to one of the contours of u^2. Thus figure 4

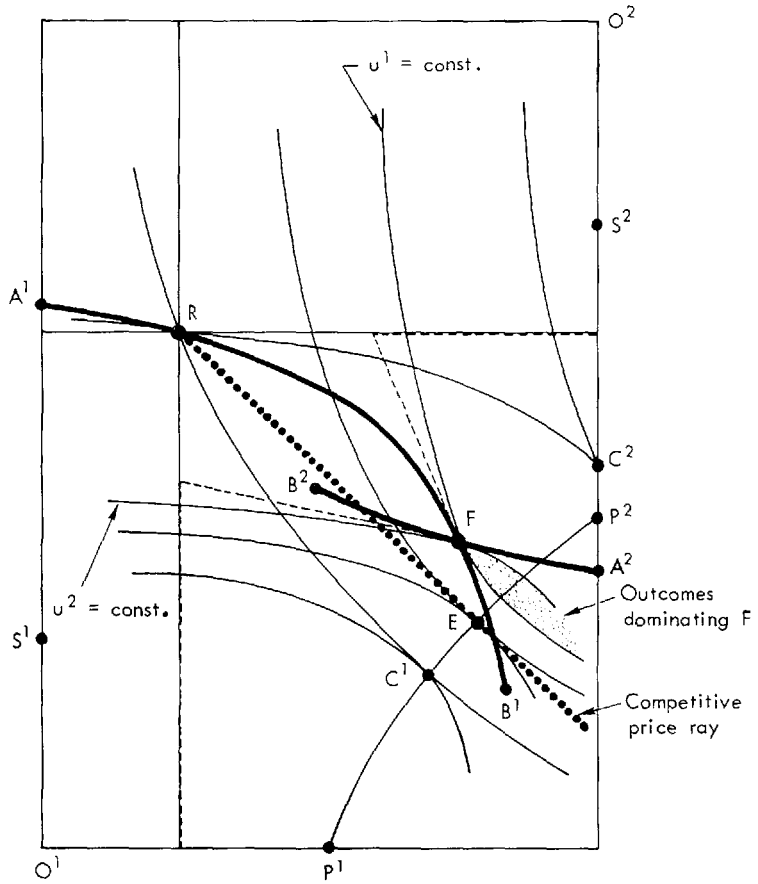

FIG. 4.—Noncooperative (F) and competitive (E) equilibrium

illustrates a noncooperative or "Nash" equilibrium (NE) for the market. Neither trader, knowing the strategy of the other, would wish to change.

A striking feature of this kind of equilibrium is its nonoptimality. Since curves A^1B^1 and A^2B^2 are not generally tangent to each other, the point F cannot be expected to be Pareto optimal or "efficient."[19] In effect, the traders are working with unequal *marginal* prices, represented by the unequal slopes of A^1B^1 and A^2B^2 at F. Any outcome in the shaded region would be preferred by both traders to the NE allocation at F. In particular, they would both profit from increased trade at the *average* price p_1, represented by the slope of RF.

The reader familiar with the Edgeworth diagram will recognize the

[19] Indeed Pareto optimality in a NE can only occur at corners of the indifference curves or in special cases like $F = R$ or $F = B^2$.

contract curve $C^1EP^2C^2$ in figure 4, which is a subset of the more extensive *Pareto set* $O^1P^1C^1EP^2C^2O^2$. The competitive equilibrium (CE) is represented by the competitive price ray RE, which is tangent to both indifference curves at the competitive allocation E. The situation illustrated seems to be typical: there is less volume of trade at the NE than at the CE. But the reverse is also possible. In fact, by a somewhat contorted but perfectly legitimate arrangement of the indifference contours, we could make any given point outside the dotted lines in figure 4 the location of a unique CE, while keeping a unique NE at F.

Although we have not illustrated it, it is also not difficult to construct utility functions for which several distinct strategy pairs (S_1, S_2) are in noncooperative equilibrium. This may or may not be accompanied by a corresponding multiplicity of CE; the two kinds of solution are not directly interlinked. But their general mathematical properties are quite similar. For example, we would expect that even for general m and n, if the utility functions are smooth and the initial allocation is chosen according to a nonatomic probability distribution, then with probability 1 there will be a finite, odd number of NE (cf. Debreu 1970).

B. The Feasible Set and Credit

The set of feasible outcomes can be determined by holding one trader's bid at its upper extreme O^i or lower extreme M^i and sweeping through the strategies of the other trader. The feasible set is shown in figure 5 (solid shading); for example, outcome L results if both traders bid their upper limit.[20] This feasible set has nothing to do with rationality or motivation; it merely describes those outcomes that the mechanism *can* be made to produce. It necessarily contains all NE allocations, but, as for the CE allocations, we know a priori only that they lie in the quadrants of the Edgeworth box "northwest" or "southeast" of the point R. It may well be that some or all of the CE allocations fall outside of the feasible set, under our present trading rules.

By introducing *credit* we can enlarge the feasible set, permitting the traders to bid more of the "means of payment" commodity than they actually possess as cash on hand. In diagrammatic terms, the first trader, say, could be allowed the segment C^1M^1 instead of O^1M^1 (fig. 5). The

[20] Most boundary points are attainable, but those on the line M^1M^2 are not, except for R itself. There is, however, an exceptional "null" outcome, not in the Edgeworth box, in which Trader 1 bids and gets M^1 and Trader 2 bids and gets M^2. Our rule is that if a trader bids nothing he gets nothing, even if the price is zero. So if both traders bid nothing, the entire stock of good 1 is lost. (We may imagine that it goes to an otherwise unnoticed "scavenger," who sends infinitesimal bids to each trading post in the hope of making a killing.) This exceptional outcome is manifestly undesirable and unstable, and it has no real effect on the solutions of the game, though it causes technical difficulties in the existence proof.

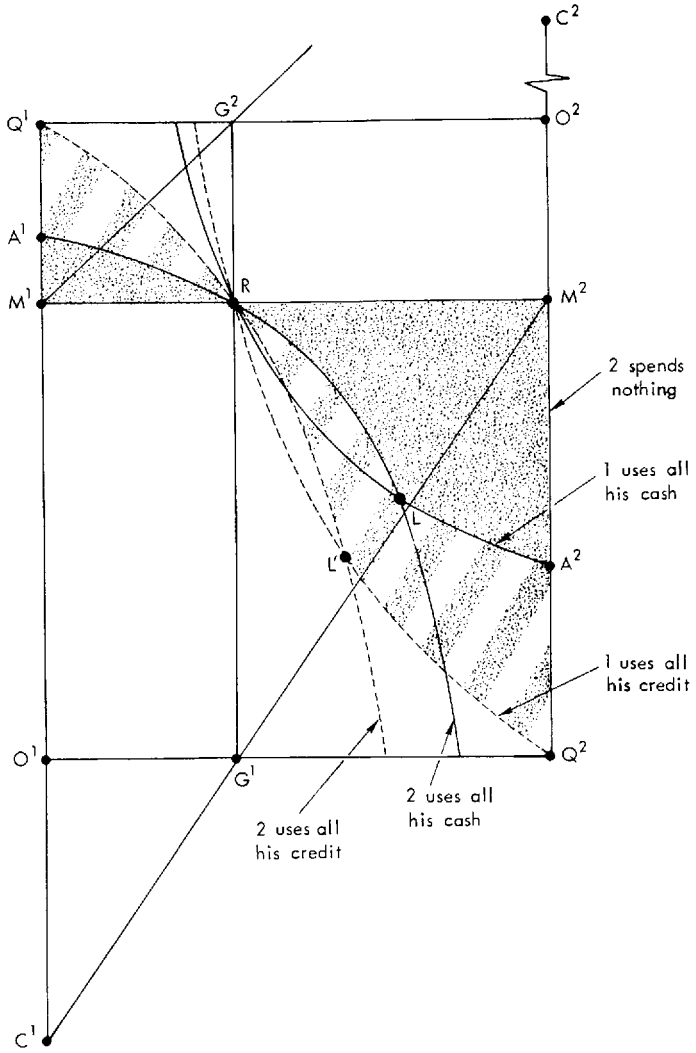

FIG. 5.—The feasible set with and without credit

length $|O^1C^1|$ has a simple interpretation in terms of banking: it is the largest amount of good 2 that could be loaned to Trader 1 with the certainty that he will be able to repay. To see this, consider that the worst case is $S^1 = C^1, S^2 = M^2$, with Trader 1 bidding the limit and Trader 2 bidding nothing. This leads to outcome Q^2, at which Trader 1 is just barely solvent. A similar "conservative" credit limit C^2 for Trader 2 can be determined in the same way.[21]

[21] It will become clear (Secs. IVD and VA) that when there are many traders the amount of "conservative" credit is likely to be very small, since it depends on the amount

If we alter the rules of the game to enable both traders to use this conservative banking credit, the feasible set is extended as indicated by the stripes in figure 5. For example, outcome L' results if both traders bid up to their new limits. The new feasible set still does not fill up the two basic quadrants, however, and so we still cannot be sure that any competitive outcomes will be attainable.

With more liberal credit, the feasible set would cover a larger part of the two basic rectangles and eventually all of them, but it would also include areas above and below the Edgeworth box, representing situations in which one or the other trader is "caught short." If we regard credit as the issue of a financial instrument, the interpretation of a point outside the box is simple and familiar—it amounts to stating that after trade an individual ends up with none of the monetary commodity in hand and with outstanding claims that he cannot meet. In order to complete the model we must either describe the utility to any player at such an outcome or add further moves to the game, corresponding to bankruptcy proceedings (see Sec. IIB).

C. Individual Rationality

In the present version of our model we are requiring that all of the initial endowments, except for the "payment" commodity, be put up for sale.[22] Consequently the concept of ownership is somewhat different from that usually assumed in microeconomics and is closer to that in law. All goods are monetized, and trade is virtually anonymous. The economy is an accountant's dream, but by forcing goods to pass through the market ownership rights are weakened. In particular, it may be impossible for a person to recover the bundle he originally started with or obtain an equivalent bundle. On the other hand, he may be able unilaterally to guarantee an outcome superior to his initial holding. (This contrasts with the classical Edgeworth and modern "core" models, where an individual trader can always defend his initial holdings if he so desires, but no more.)

The "defensive" possibilities are easily illustrated. In figure 6 the first trader is fortunate, since he has relatively little of good 1 that must go to market and relatively much of good 2 that he can use at his discretion. He cannot, of course, protect his exact initial holding, since this would mean being able to enforce outcome R regardless of the other's choice of S^2. But he can force the outcome to *have as much utility* for him as R. In fact, by playing S^1 as shown, he restricts the possible outcomes to the set A^2B^2, a set which lies entirely on the "high" side of the indifference

---

of a trader's own money that will surely come back to him from his own bids. In this respect our two-trader exposition may be misleading. We may also remark that "conservative" credit does not require the aid of a central clearinghouse (see Sec. IIB), as any promissory notes can be redeemed at each trading post independently.

[22] But see Sec. VI, where this requirement is removed.

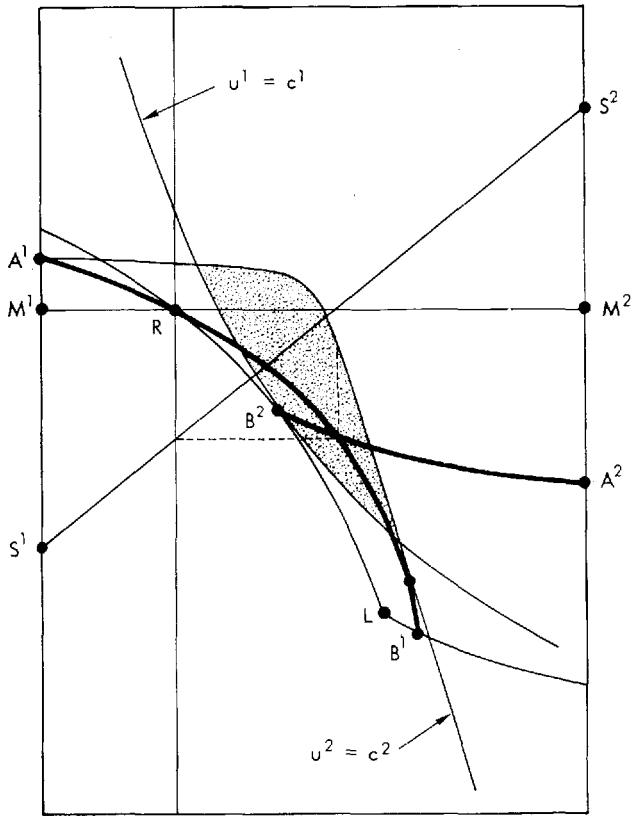

FIG. 6.—"Max-min" strategies and the individual rationality zone

curve $u^1 = c^1$, as drawn. This is in fact the best he can do defensively: S^1 is what would be called his "max-min" strategy, and c^1 is his "max-min" payoff. We see that moving S^1 either up or down would move B^2 into a region of lower payoff, either toward R or toward L.

The second trader, on the other hand, has a relatively poor defensive position. The highest u^2 indifference curve that completely contains one of the A^1B^1 sets is the curve $u^2 = c^2$, which is distinctly inferior to R for him. There are two critical points: the endpoint A^1 and the tangency point near B^1. As S^2 moves, the A^1B^1 curve rotates on R. Raising S^2 would make A^1 worse, while lowering S^2 would make the critical point near B^1 worse, so S^2 is the "max-min" strategy.

The shaded area in figure 6 represents the so-called individually rational allocations, bounded by the curves $u^1 = c^1$ and $u^2 = c^2$. The *feasible* individually rational allocations are just those that lie beneath the horizontal line RM^2, and it is a theorem that every NE solution will lie in that region. In contrast, the Walras-Edgeworth individually rational

zone, which contains every CE solution, would be defined by the two indifference curves through R (not shown). It is easy to see that this region could be entirely disjoint from the other. This is not too surprising, as the two regions arise from games with the same economic data but different rules of trade.

D. Many Traders

The same diagrams can be used, one-sidedly, when there are more than two traders. This is because our model has the *aggregation* property, with both strategies and outcomes being additive over traders. Viewed by the other traders, a set or coalition of traders acting together is hardly distinguishable from a single, larger trader. In the diagrams, we may regard the point S^2 not as a single bid but as the vector sum of bids by Traders 2, 3, . . . , n, using R as the "origin" from which the vectors are defined. The first trader may notice a quantitative but not a qualitative difference from the two-person case: since the other traders combined may have far more resources than he does alone, the distance of S^2 from R and S^1 may make the slope of S^1S^2 (i.e., the price) almost insensitive to his own choice.

It might appear at first glance that the *feasible set*, when there are many traders, would continue to be a sizable fraction of the set of outcomes that are feasible under unrestricted, Edgeworthian barter. But this is not the case. Let us count dimensions. The unrestricted allocation space has dimension $(m + 1)(n - 1)$.[23] Even in the case $m = 1$, there are $2n - 2$ dimensions of possible outcomes if trade is not required to pass through the "trading post" mechanism. This compares with the n dimensions at most that can arise when n traders bid by each selecting a point on a line as his strategy. Thus, starting at $m = 1$, $n = 3$, the feasible set is only a lower-dimensional surface or manifold in the set of outcomes that would be possible under unrestricted trading.

E. Many Commodities

Since the trading posts operate essentially independently of each other, it is possible to continue to use these diagrams when there are more than two commodities (i.e., more than one trading post). But there are two ways in which the markets remain intercoupled: (1) through the spending limits, which apply to all bids combined, so that a trader's upper bound at each trading post will depend on what he spends at the others, and (2) through the utility functions, which in general reflect complementaries,

[23] The "+ 1" here is for the monetary good, while the "− 1" reflects the fact that the sum of the holdings is constant.

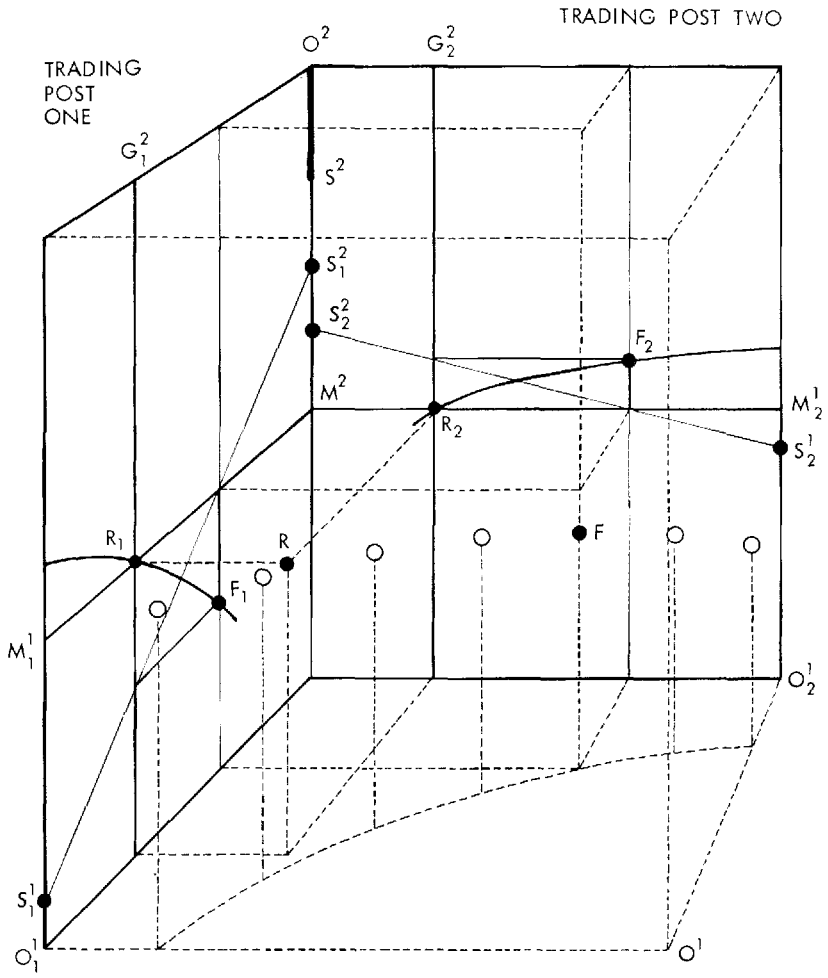

FIG. 7.—Two trading posts

substitution effects, etc., among the different goods, so that the utility maps we superimpose on each Edgeworth box will depend on what is happening in the others. The second effect (as in the classical model) can only be visualized with hyperdimensional eyesight, but the first is still within reach of our diagrams.

Figure 7 illustrates a three-commodity situation. The two "boxes" are erected at right angles to each other, in the back of the three-dimensional figure (solid lines). The initial point, R, projects to R_1 in the first box and R_2 in the second.[24] We may conveniently regard R as the

[24] The labels are generally consistent with the previous diagrams, but with subscripts added to distinguish the two trading posts.

vector sum of M^2R_1 and M^2R_2. Trader 2's pair of bids b_1^2 and b_2^2 are shown at S_1^2 and S_2^2 and their sum at S^2. Thus $|M^2S^2| = |M^2S_1^2| + |M^2S_2^2|$. The unspent balance is represented by the vertical bar O^2S^2.

Trader 1's bids are shown at S_1^1 and S_2^1. As S_1^1 varies over the permitted range $O_1^1M_1^1$, the outcome in the first marketplace sweeps out the curve through R_1 and F_1 in the usual way. Similarly, the curve through R_2 and F_2 describes his possibilities in the second marketplace. But his joint choice of S_1^1, S_2^1 is restricted by the spending limit

$$|M_1^1S_1^1| + |M_2^1S_2^1| \leq |M_1^1O_1^1| = |M_2^1O_2^1|.$$

The case actually illustrated has him spending most of his cash on good 1 and the rest on good 2. (Had he held some back, we would have shown it by a vertical bar at O^1.) The resulting final allocation is found by taking the vector sum of M^2F_1 and M^2F_2; this yields the point F.

Now suppose that Trader 1 changes his allocation of cash between goods 1 and 2, causing points S_1^1 and S_2^1 to move in opposite directions. The point F will then trace out a curve in space, which we have tried to suggest by the open dots perched above the curve's projection on the base. This projection on the base shows the tradeoff between goods 1 and 2 when a fixed amount of cash is bid by Trader 1; note that it is concave to his origin O^1. If Trader 1 bids less than the maximum allowed (strict inequality above), a curved triangular surface is generated, extending out from F and the open dots toward the viewer and rising to an apex directly above O^1. This locus (not shown) indicates ways in which Trader 1 can obtain more of good 3 at the expense of goods 1 and/or 2; it is concave to the origin O^1, and its projection is the large, roughly triangular region in the base of the diagram. The first trader's best response to the other trader's given strategy (S_1^2, S_2^2) is determined by the relationship of this surface to the indifference surfaces of u^1, in the three-dimensional commodity space with origin at O^1.

V. Replication

Inquiries into the behavior of economic models with large numbers of participants often make sweeping assumptions of symmetry, in the hope of keeping the models mathematically tractable and easy to visualize, while capturing at least some of the characteristic effects of large numbers. A favorite technique, involving a high but not total degree of symmetry, is called "replication." Imagine a basic economic system juxtaposed to a large number of identical replicas of itself; then take away all barriers, and form a "common market." Equivalently, assume that all traders in the full model are drawn from a small number of *types*, with an equal number of individuals of each type. Traders of the same type have identical endowments and identical tastes, but they are not constrained to act

alike. That is, they are not programmed robots or members of a bloc or cartel but independent decision makers.[25]

The number of members of a type—the "replication number"—provides the modeler with a simple size parameter that he can vary without calling for additional data. Of course we do not intend to treat replication as though it were some kind of actual economic event, like a homogeneous population increase or an accretion of similar countries to a common market. Replication should be considered only as a technical device of comparative statics. As such, however, it has repeatedly proved its worth in explorations of the size effect (see Edgeworth 1881; Shubik 1959, 1968; Debreu and Scarf 1963; Shapley and Shubik 1966, 1969*b*; Debreu 1975; Owen 1975; Shapley 1975; etc.).

A. A Simple Case

The effect of replication on our model is illustrated in figure 8. We have $m = 1$ and $n = 2k$, where k is the replication number. The point S^1 represents the *average* of the bids of the k traders of type 1; similarly S^2 for type 2.[26] The point F therefore indicates the average final bundles for the two types.

Let us focus on a typical member of type 1. His own bid might be, say, at $\tilde{S}^1$. To discover his personal final bundle, $\tilde{F}$, we must draw the line through $\tilde{S}^1$ that is *parallel* to S^1S^2; we then locate $\tilde{F}$ in the usual way. Of course $\tilde{F}$ necessarily falls on the straight line through R and F, since all transactions take place at the same price.

Now suppose our trader changes his bid from $\tilde{S}^1$ to M^1. That is, he decides to buy nothing. This depresses the average bid for type 1, moving it from S^1 to T^1, where $|S^1T^1|/|\tilde{S}^1M^1| = 1/k$. The new price is the slope of T^1S^2, so our trader's new final bundle is $\tilde{A}^1$, determined by the line through M^1 parallel to T^1S^2. Similarly, at the other extreme, if he decides to bid the limit, the result is $\tilde{B}^1$, determined by the line through O^1 parallel to U^1S^2, where $|U^1S^1|/|\tilde{S}^1O^1| = 1/k$. We should not be surprised to see $\tilde{B}^1$ fall outside the box. This means merely that our trader happens to have enough "cash" to buy more than the combined initial bundles of one trader of each type.

The locus of possible final bundles for our variable trader is the curve $\tilde{A}^1R\tilde{F}\tilde{B}^1$, shown in figure 8 for the case $k = 4$. The flattening of the curve, as compared with the analogous curve A^1RFB^1 in figure 3 (which is the case $k = 1$), clearly shows the diminished influence of a single individual on price after replication. Indeed the price variation in the present case

[25] This distinction can be very important in the game-theory approach but is less important in a behavioristic theory. As we shall see, replication changes the NE but not the CE.

[26] More generally, if there were many types, then the analogue of S^2 would be not the average but $1/k$ times the sum of all bids from traders not of type 1.

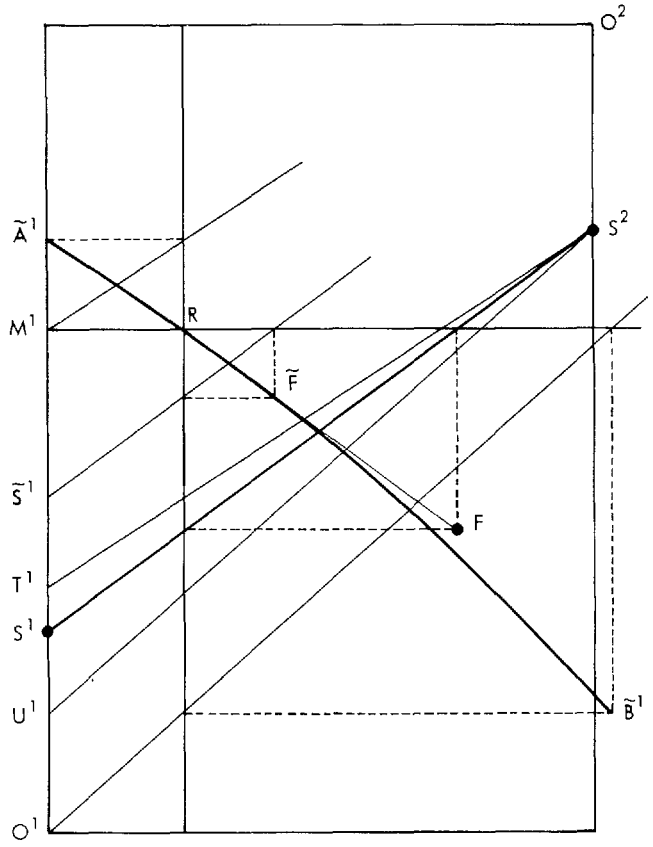

FIG. 8.—Price formation in a replicated market

is encompassed by the thin pencil of sloping lines within the angle $U^1S^2T^1$. In the limit, as $k \to \infty$, the curve becomes the fixed-price "budget set" of the classical CE model—but with one important difference: it is truncated at the $\tilde{B}^1$ endpoint instead of extending to the horizontal axis through O^1, in recognition of the limited quantity of the payment commodity available.[27]

B. The General Case

For the replicated form of the general model presented in Section II, we consider a market with T types of traders and k of each type; thus $n = kT$. The initial holding of good j by trader s of type t will be denoted

[27] It may be observed that the "conservative" line of credit for an individual shrinks to zero as $k \to \infty$. A simple calculation shows in fact that, in the present case, the amount of such credit is equal to $a_2^1 a_2^2/[k(a_2^1 + a_2^2) - a_2^1]$.

a_j^{ts}, which we can shorten to a_j^t since it does not depend on s. The total amount of good j (summed over *both* t and s) will be denoted $\bar{a}_j$, as before. A typical strategy will be denoted $b^{ts} = (b_1^{ts}, \ldots, b_m^{ts})$. The defining equations of Section II then become

$$p_j = \bar{b}_j/\bar{a}_j, \qquad j = 1, \ldots, m;$$

$$x_j^{ts} = b_j^{ts}/p_j, \qquad j = 1, \ldots, m \text{ and } p_j \neq 0;$$

$$x_j^{ts} = 0 \qquad j = 1, \ldots, m \text{ and } p_j = 0;$$

$$x_{m+1}^{ts} = a_{m+1}^{ts} - \sum_{j=1}^{m} b_j^{ts} + \sum_{j=1}^{m} a_j^{ts} p_j;$$

and

$$\Pi^{ts}(b^{11}, \ldots, b^{1k}, b^{21}, \ldots, \ldots, b^{nk}) = u^t(x_1^{ts}, \ldots, x_{m+1}^{ts}),$$

where $u^t(x) \equiv u^{ts}(x)$ is the common utility function for members of type t.

Suppose that a NE has been found in which all traders of the same type make the same bids:[28] $\hat{b}^{ts} = \hat{b}^t$, $s = 1, \ldots, k$. It is instructive to look at the equilibrium conditions for the "interior" case, that is, the case where all $\hat{b}_j^t$ are positive and $\sum_{j=1}^{m} \hat{b}_j^t < a_{m+1}^t$, $t = 1, \ldots, T$. Assuming that the concave functions $u^t(x)$ are differentiable, we let $u_j^t(x)$ denote their partial derivatives with respect to x_j. Then setting $du^t/db_j^{ts} = 0$ gives us

$$u_j^t(x^{ts}) \left\{ \frac{1}{p_j} - \frac{b_j^{ts}}{(p_j)^2 \bar{a}_j} \right\} = u_{m+1}^t(x^{ts}) \left\{ 1 - \frac{a_j^{ts}}{\bar{a}_j} \right\},$$

which must hold at $b = \hat{b}$. Since prices are in terms of good $m + 1$ as numeraire, we can introduce $p_{m+1} \equiv 1$ and rewrite the condition for a symmetric, interior NE as follows:

$$\frac{u_j^t(\hat{x}^t)}{\hat{p}_j} \left\{ 1 - \frac{\hat{b}_j^t}{\bar{\hat{b}}_j} \right\} = \frac{u_{m+1}^t(\hat{x}^t)}{\hat{p}_{m+1}} \left\{ 1 - \frac{a_j^t}{\bar{a}_j} \right\}, \qquad j = 1, \ldots, m,$$

where $\hat{p}$ and $\hat{x}$ are the prices and allocation corresponding to $\hat{b}$.

The expressions in curly brackets reveal the effect of oligopoly, but note that they lie between 1 and $1 - 1/k$. In the limit, therefore, these conditions for NE reduce to the conditions for CE, with the prices of the goods proportional to the marginal utilities of every trader. Thus we may state a convergence theorem, as follows:[29]

> *Theorem 2.* Assume that for infinitely many values of k the market has a symmetric, interior NE, and let $\hat{p}^{(k)}$ be the corresponding m-vector of prices. Let $\tilde{p}$ be any limit point of the $\hat{p}^{(k)}$,

[28] The conditions given in Theorem 1 in Sec. IIIB are sufficient to guarantee the existence of such a symmetric NE. It is quite possible also to have nonsymmetric NEs.

[29] A somewhat more general form of this theorem will be found in Shapley (1976).

and define $\tilde{p}_{m+1} = 1$. Then the $m + 1$ prices $\tilde{p}_1, \ldots, \tilde{p}_m, \tilde{p}_{m+1}$ will be competitive for the market (for any value of k); that is, an allocation x will exist for which $\bar{\tilde{x}} = \bar{a}$ and, for each s and t, $\tilde{x}^{ts}$ maximizes $u^t(x^{ts})$ subject to $x^{ts} \geq 0$ and

$$\sum_{j=1}^{m+1} \tilde{p}_j(x_j^{ts} - a_j^{ts}) = 0.$$

It should be noted that the NE approaches the CE "from below," that is, through outcomes that are not in general Pareto optimal. This contrasts with the convergence of cooperative solutions like the core and the value, which are by definition Pareto optimal all the way (see, e.g., Debreu and Scarf 1963; Shapley and Shubik 1969*b*; Debreu 1975; Owen 1975; and Shapley 1975).

The type of convergence revealed in Theorem 2 depends crucially on having interior solutions, with no trader up against his spending limit. All must have enough cash (or credit, in an extended model), where the meaning of "enough" depends on the payment commodity's marginal utility relative to that of the other goods. Of course there can never be "enough" fiat money in this sense (in a one-period model), for the NE will always have everyone spending the limit and wishing he could spend more. If the CE is considered socially desirable (as one road to the Pareto optimum, if for no other reason), then a society whose trading system resembles our replicated model should make sure that the means of payment is something that is both widely available and generally desirable, either as a consumption item or as a means of payment in future periods.

VI. Variants and Extension

It is not difficult to modify the basic game so that the goods do not necessarily all pass through the market before consumption. A number of different considerations and problems arise, however, depending on just how this is done. We shall review some of them briefly here; for further details see Shapley (1976).

Suppose, first, that a trader can hold back any part of his initial endowment. A strategy for i is now a *pair* of m-dimensional vectors (b^i, q^i), with the b_j^i representing cash bids, as before, and the q_j^i representing quantities of goods sent to the respective trading posts. Thus the amounts $(a_j^i - q_j^i)$ are held back. Of course we require that

$$0 \leq q_j^i \leq a_j^i, \qquad j = 1, \ldots, m.$$

The prices are given by

$$p_j = \bar{b}_j / \bar{q}_j \qquad (\text{if } \bar{q}_j > 0),$$

and we add the convention that if $\bar{q}_j = 0$ any cash sent to the jth trading post is lost. Sales and purchases are calculated as before, but we must remember to add back the amounts $(a_j^i - q_j^i)$ before entering the utility function.

The nomogram in the Edgeworth box works essentially as before, except that the domain of strategies S^i is now the full rectangle $M^iRG^iO^i$ instead of the line segment M^iO^i (see fig. 2). It is easy to see that all outcomes in the northwest and southeast quadrants can now be reached without the aid of credit. But when there are more than two traders, we find the feasible set still sharply restricted by the requirement of trading at fixed prices, as discussed previously in Sections IIC and IVD.

Despite the naturalness of the "hold-back" option, it has one major disadvantage. It turns out that the enlarged strategy spaces cause the conditions that define the noncooperative equilibrium to be *under*determined, making the set of solutions usually infinite.[30] This circumstance leads us to search for ways to sharpen either the solution concept or the rules of the game, without entirely giving up the hold-back feature. The simplest expedient is to prohibit a trader from simultaneously buying *and* selling at the same trading post, by imposing the condition

$$b_j^i q_j^i = 0, \qquad j = 1, \ldots, m.$$

The strategy domains at each trading post are now L-shaped, with a corner at the initial point R. Despite this corner, a trader in an active trading post should not experience any discontinuity in changing roles from buyer to seller, as his budget set is still the familiar, smooth hyperbola through R, as long as there are other traders on both sides of the market. The two-person case is rather trivial, however, there being no meaningful competition to motivate price formation.

This modification does restore some semblance of uniqueness to the NE, but it has its own drawback: the game is *not additive* over players. A coalition has options not available to an individual. Thus, if a trader could split himself into two legal persons, he could buy and sell simultaneously at the same trading post (e.g., in an attempt to stabilize the price). This suggests that the restriction $b_j^i q_j^i = 0$ might be unrealistic in some applications.

Any version of the rules that permits traders to stay out of a market has another peculiarity, namely, the possibility of trading posts that are

[30] This multiplicity is not merely a matter of simultaneous buying and selling by the same trader that could simply be canceled out. Indeed, if at equilibrium Trader i is sending both goods and cash to the same trading post and if the price there is p_j, then he might consider decreasing both q_j^i and b_j^i, in the ratio of 1 to p_j. This would not change his final outcome (assuming that he does not spend the extra cash elsewhere) or the price p_j, but it would change the marginal cost of good j to the other traders and so destroy the equilibrium.

completely inactive. Such a situation is very stable, as no trader would want to enter a market where $\bar{b}_j = \bar{q}_j = 0$. He would either lose his goods or money or, at best, have them returned to him to no advantage. Thus the "null" strategy for all traders is always an equilibrium point, in both of the models above. Moreover, we can arbitrarily declare any subset of trading posts to be inactive and solve the remaining subeconomy —any NE of that subeconomy will be also a NE of the economy as a whole.

We should not completely rule out such inactive or partially inactive solutions as unrealistic on their face. It can plausibly be argued that the real world is full of "latent" markets awaiting discovery which can only become active by an act of faith on the part of the first entrants. But we are interested in active NE as well, and the possibility of inactive ones makes the main existence theorem, comparable to Theorem 1 above, considerably more delicate both to state and to prove. Sometimes markets are "legitimately" inactive, as when the traders start with a Pareto-optimal distribution. The resolution of the problem seems to lie in demanding that inactive trading posts have "virtual prices" at which all traders find it in their best interest neither to buy nor to sell (Shapley 1976).

A. Multiperiod Extensions

Many applications of our general approach can benefit from—and some will require—the adoption of a multiperiod framework. Specifically, we may mention the study of (*a*) credit and bankruptcy, (*b*) nonsymmetric information conditions, (*c*) uncertainty and insurance, (*d*) cyclical variation of endowments and the money market, (*e*) the derived utility of fiat money, (*f*) interest rates and inflation, and (*g*) the role of capital goods and ownership shares. Shubik and others have devised exploratory game models for most of these situations and have worked out a number of tutorial examples (see Shubik 1972, 1973, 1977; Shubik and Whitt 1973; and Dubey and Shubik 1976). There remains, however, much more that can and should be done.

There remain also some serious conceptual problems with the general mathematical theory for multistage models of this type. One kind of difficulty is already apparent if we try merely to extend our basic prototype, without added features, to two or more periods. We should first recall that a vital ingredient in the proof of Theorem 1 (the existence theorem) is the *concavity* of the payoff functions $\Pi^i(b^1, \ldots, b^n)$, with respect to their respective b^i. The natural way to attack the two-period case is to solve the second stage parametrically, as a function of the cash distribution at the end of the first stage. But then the traders' total payoffs will include a term that depends on this cash distribution, and there seems no way to assure that this term will be concave.

VII. Conclusion

In this paper, in an admittedly simplified and abstract setting, we have explored some elementary implications of an explicit market mechanism for the formation of price. We believe that this species of game-theoretic model, by being well defined independently of equilibrium conditions or behavioral assumptions (though capable of accommodating such conditions and assumptions) and by reflecting the decentralized decision making attainable through the use of a tangible money, is far more flexible as an investigatory and explanatory tool than the usual Walrasian model, with its ill-defined causal linkage between individual actions and the action of the system as a whole.

References

Arrow, K. J., and Debreu, G. "Existence of an Equilibrium for a Competitive Economy." *Econometrica* 22 (July 1954): 265–90.

Arrow, K. J., and Hahn, F. H. *General Competitive Analysis.* San Francisco: Holden Day, 1971.

Aumann, R. J., and Peleg, B. "Von Neumann–Morgenstern Solutions to Cooperative Games without Side Payments." *Bull. American Math. Soc.* 66 (May 1960): 173–79.

Aumann, R. J., and Shapley, L. S. *Values of Non-atomic Games.* Princeton, N.J.: Princeton Univ. Press, 1974.

Clower, R. "A Reconsideration of the Microfoundations of Monetary Theory." *Western Econ. J.* 5 (March 1967): 1–9.

Cournot, A. A. "Recherches sur les principes mathématiques de la théorie des richesses." Paris: Hachette, 1838. Translated by N. T. Bacon. New York: Macmillan, 1897.

Debreu, G. "A Social Equilibrium Existence Theorem." *Proc. Nat. Acad. Sci.* 38 (October 1952): 886–93.

———. "Economies with a Finite Set of Equilibria." *Econometrica* 38 (May 1970): 387–92.

———. "The Rate of Convergence of an Economy." *J. Math. Econ.* 2 (March 1975): 1–7.

Debreu, G., and Scarf, H. E. "A Limit Theorem on the Core of an Economy." *Internat. Econ. Rev.* 4 (September 1963): 253–46.

Dubey, P., and Shubik, M. "Trade and Prices in a Closed Economy with Exogenous Uncertainty, Different Levels of Information, Money and Special Futures Markets." Cowles Foundation Discussion Paper no. 410, Yale Univ., January 1976.

Edgeworth, F. Y. *Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences.* London: Kegan Paul, 1881.

Hildenbrand, W. *Core and Equilibria of a Large Economy.* Princeton, N.J.: Princeton Univ. Press, 1974.

Jones, R. A. "The Origin and Development of Media of Exchange." *J.P.E.* 84, no. 4, pt. 1 (August 1976): 757–75.

Levitan, R. E., and Shubik, M. "Noncooperative Equilibria and Strategy Spaces in an Oligopolistic Market." In *Differential Games and Related Topics*, edited by H. Kuhn and G. Szegö. Amsterdam: North-Holland, 1971.

———. "Price Duopoly and Capacity Constraints." *Internat. Econ. Rev.* 13 (February 1972): 111–22.

Maňas, M. "A Linear Oligopoly Game." *Econometrica* 40 (September 1972): 917–22.

Nash, J. F., Jr. "Noncooperative Games." *Ann. Math.* 54 (September 1951): 286–95.

Ostroy, J. M., and Starr, R. M. "Money and the Decentralization of Exchange." *Econometrica* 42 (November 1974): 1093–1113.

Owen, G. "On the Core of Linear Production Games." *Math. Programming* 9 (December 1975): 358–70.

Patinkin, D. *Money, Interest, and Prices: An Integration of Monetary and Value Theory.* Evanston, Ill.: Row, Peterson, 1956.

Postlewaite, A., and Schmeidler, D. "Approximate Efficiency of Non-Walrasian Nash Equilibria." Discussion Paper, Univ. Illinois at Urbana-Champaign, October 1975.

Samuelson, P. A. *Foundations of Economic Analysis.* Cambridge, Mass.: Harvard Univ. Press, 1947.

Scarf, H. E. "The Core of an *n* Person Game." *Econometrica* 35 (January 1967): 50–69.

———. *The Computation of Economic Equilibria.* New Haven, Conn.: Yale Univ. Press, 1973.

Shapley, L. S. "An Example of a Slow Converging Core." *Internat. Econ. Rev.* 16 (June 1975): 345–51.

———. "Noncooperative General Exchange." In *Theory and Measurement of Economic Externalities*, edited by S. A. Y. Lin. New York: Academic Press, 1976.

Shapley, L. S., and Shubik, M. "Quasi-Cores in a Monetary Economy with Nonconvex Preferences." *Econometrica* 34 (October 1966): 805–27.

———. "On Market Games." *J. Econ. Theory* 1 (June 1969): 9–25. (*a*)

———. "Pure Competition, Coalitional Power, and Fair Division." *Internat. Econ. Rev.* 10 (October 1969): 337–62. (*b*)

———. "The Assignment Game. I. The Core." *Internat. J. Game Theory* 1, no. 2 (1972): 111–30.

———. "Characteristic Function, Core, and Stable Sets." Rand Corporation Report R-904/6, Santa Monica, Calif., July 1973.

———. "Competitive Outcomes in the Cores of Market Games." *Internat. J. Game Theory* 4, no. 4 (1976): 229–37.

Shubik, M. "Edgeworth Market Games." In *Contributions to the Theory of Games*, edited by R. D. Luce and A. W. Tucker. Vol. 4. Annals of Mathematics Study no. 40. Princeton, N.J.: Princeton Univ. Press, 1959.

———. "Extended Edgeworth Bargaining Games and Competitive Equilibrium." *Metroeconomica* 20 (September–December 1968): 299–312.

———. "Pecuniary Externalities: A Game Theoretic Analysis." *A.E.R.* 61 (September 1971): 713–18.

———. "A Theory of Money and Financial Institutions: Fiat Money and Noncooperative Equilibrium in a Closed Economy." *Internat. J. Game Theory* 1, no. 4 (1972): 243–68.

———. "Commodity Money, Oligopoly, Credit and Bankruptcy in a General Equilibrium Model." *Western Econ. J.* 11 (March 1973): 24–38.

———. "A Dynamic Economy with Fiat Money, without Banking, but with Ownership Claims to Production Goods." In *Mélanges F. Perroux*, edited by H. Bartoli. Paris, 1977, in press.

Shubik, M., and Whitt, W. "Fiat Money in an Economy with One Nondurable Good and No Credit." In *Topics in Differential Games*, edited by A. Blaquière. Amsterdam: North-Holland, 1973.

Telser, L. G. *Competition, Collusion, and Game Theory.* Chicago: Aldine-Atherton, 1972.

Viner, J. "Cost Curves and Supply Curves." *Zeitschrift Nationalökonomie* 3 (1931): 23–46.

von Neumann, J. "Zur Theorie der Gesellschaftsspiele." *Math. Ann.* 100 (1928): 295–320. (Or see the English translation by Sonya Bargmann in *Contributions to the Theory of Games*, edited by R. D. Luce and A. W. Tucker. Vol. 4. Annals of Mathematics Study no. 40. Princeton, N.J.: Princeton Univ. Press, 1959.)