Topology Lecture Notes

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CHAPTER 1

Topological Spaces

A *metric space* is a pair (X, d) where X is a set, and d is a *metric* on X, that is a function from $X \times X$ to $\mathbb{R}$ that satisfies the following properties for all $x, y, z \in X$

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ (*symmetry*), and
3. $d(x, y) \leq d(x, z) + d(z, y)$ (*triangle inequality*).

Example 1.1. The following are all metric spaces (check this).

1. $\mathbb{R}$ with the metric $d(x, y) = |x - y|$.
2. $\mathbb{R}^d$ with the metric $d(\mathbf{x}, \mathbf{y}) = ((x_1 - y_1)^p + \cdots + (x_d - y_d)^p)^{1/p} = |\mathbf{x} - \mathbf{y}|_p$ for any $p \geq 1$.
3. $\mathbb{C}$ with the metric $d(z, w) = |x - w|$.
4. $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with the metric $d(z, w) = |\arg(z) - \arg(w)|$, where arg is chosen to lie in $[0, 2\pi)$.
5. $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with the metric $d(z, w) = |z - w|$.
6. Any set X with the metric $d(x, y) = 1$ if $x \neq y$ and 0 if $x = y$. Such a space is called a *discrete space*.
7. Let L be the set of lines through the origin in $\mathbb{R}^2$. Then each line ℓ determines a unique point ℓ^* on the $y \geq 0$ semicircle of the unit circle centered at the origin (except for the special line $y = 0$; for this line choose the point $(1, 0)$). Define a metric on L by setting $d(\ell_1, \ell_2) = |\ell_1^* - \ell_2^*|_2$.
8. Let $C[a, b]$ denote the set of all continuous functions from $[a, b]$ to $\mathbb{R}$. Define a metric on $C[a, b]$ by $d(f, g) = \sup_{t \in [a,b]} |f(t) - g(t)|$.

A function $f : X \to Y$ from the metric space (X, d_X) to the metric space (Y, d_Y) is *continuous* at the point $x_0 \in X$ if for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon.$$

The function is *continuous* if it is continuous at every point.

Definition 1.2. A set $U \subset X$ in a metric space is *open* if and only if $\forall\, x \in U\ \exists\, \epsilon_x > 0$ such that if $y \in X$ has $d(x, y) < \epsilon$ then $y \in U$. A set $C \subset X$ is *closed* if and only if its complement $C^c = X \backslash C$ is open.

A useful shorthand is the symbol for a *metric open ball*,

$$B(x; \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}.$$

As an exercise, prove the following basic result.

LEMMA 1.3. *Let X and Y be metric spaces, and $f : X \to Y$ a function. The following are equivalent:*

1. *f is continuous;*
2. *for every open set U in Y, $f^{-1}(U)$ is open in X;*
3. *for every closed set C in Y, $f^{-1}(C)$ is closed in X.*

Try to understand what this lemma is telling you about functions mapping *from* a discrete space as in Example 1.1(6) above.

Also as an exercise, prove the following.

LEMMA 1.4. *Let X be a metric space. Then*

1. *The empty set $\emptyset$ and the whole space X are open sets.*
2. *If U and V are open sets, then $U \cap V$ is an open set.*
3. *If $\{U_\alpha\}_{\alpha \in A}$ is any collection of open sets, then $\bigcup_{\alpha \in A} U_\alpha$ is an open set.*

Notice that the index set A in Lemma 1.4 does not need to be countable.

Lemma 1.4 suggests the following generalization of a metric space: think of Lemma 1.4 as defining certain properties of open sets. By Lemma 1.3 we know that the open sets tell us all about continuity of functions, so this will give us a language for talking about continuity and so on without involving metrics. This turns out to be convenient and more general – by simply dealing with open sets, we are able to define *topological spaces*, which turns out to be a strictly bigger collection of spaces than the collection of all metric spaces.

DEFINITION 1.5. If X is a set, a *topology* on X is a collection $\mathcal{T}$ of subsets of X satisfying:

1. $\emptyset, X \in \mathcal{T}$,
2. $U, V \in \mathcal{T} \implies U \cap V \in \mathcal{T}$,
3. if $U_\alpha \in \mathcal{T}$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *topological space*, and the members of $\mathcal{T}$ are called the *open sets*. If the space is also a metric space, then the open sets will be called *metric open sets* if the distinction matters. We now have a new definition of continuity – make sure you understand why this is now a *definition* and not a theorem.

DEFINITION 1.6. A function $f : X \to Y$ between topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is *continuous* if and only if $U \in \mathcal{T}_Y \implies f^{-1}(U) \in \mathcal{T}_X$.

LEMMA 1.7. *Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ be topological spaces. If functions $f : X \to Y$ and $g : Y \to Z$ are continuous, so is the composition $gf : X \to Z$.*

PROOF. If $U \in \mathcal{T}_Z$, then $g^{-1}(U) \in \mathcal{T}_Y$ since g is continuous. It follows that $f^{-1}(g^{-1}(U)) \in \mathcal{T}_X$ since f is continuous. Therefore $(gf)^{-1}(U) = f^{-1}(g^{-1}(U)) \in \mathcal{T}_X$ for all open sets U in Z. $\square$

Much of what we shall do in this course is to decide when two topological spaces are *essentially the same*.

DEFINITION 1.8. Topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are *homeomorphic* if there is a continuous bijection $f : X \to Y$ whose inverse is also continuous. The function f is called a *homeomorphism*.

EXAMPLE 1.9. (1) If (X, d) is a metric space, then by Lemma 1.4 the set of all metric open sets forms a topology on X, called the *metric topology*.
(2) If X is any set, then $\mathcal{T} = \mathbb{P}(X)$, the set of all subsets of X, forms a topology on X called the *discrete topology*. Check that this is identical to the metric topology induced by the discrete metric. Notice that any function from a discrete topological space to another topological space is automatically continuous.
(3) If X is any set then the *concrete topology* is defined to be $\mathcal{T} = \{\emptyset, X\}$. Notice that any function from a topological space to a concrete space is automatically continuous. Exercise: is the concrete topology a metric topology for some metric?
(4) If X has more than one element, $\mathcal{D}$ is the discrete topology on X, and $\mathcal{C}$ is the concrete topology on X, then $(X, \mathcal{D})$ is not homeomorphic to $(X, \mathcal{C})$.

1. The subspace topology

Given a topological space $(X, \mathcal{T}_X)$, we may induce a topology on any set $A \subset X$. Given $A \subset X$, define the *subspace topology* $\mathcal{T}_A$ on A (also called the *induced* or *relative* topology) by defining

$$U \subset A \implies U \in \mathcal{T}_A \text{ if and only if } \exists\, U' \in \mathcal{T} \text{ such that } U = U' \cap A.$$

That is, an open set in A is given by intersecting an open set in X with A. Exercise: check that this does define a topology.

LEMMA 1.10. *Let $\imath : A \to X$ be the identity inclusion map. Then, if A has the subspace topology,*

1. *$\imath$ is continuous.*
2. *If $(Y, \mathcal{T}_Y)$ is another topological space, then $f : Y \to A$ is continuous if and only if $\imath f : Y \to X$ is continuous.*
3. *If $(Y, \mathcal{T}_Y)$ is another topological space, and $g : X \to Y$ is continuous, then $g\imath : A \to Y$ is continuous.*

PROOF. (1) If $U \in \mathcal{T}_X$ then $\imath^{-1}(U) = U \cap A \in \mathcal{T}_A$, so $\imath$ is continuous.
(2) Suppose that if is continuous, and that $U \in \mathcal{T}_A$. Then there is a set $U' \in \mathcal{T}_X$ such that $U = U' \cap A = \imath^{-1}(U')$. Since if is continuous, $(if)^{-1}(U') \in \mathcal{T}_Y$, so $(if)^{-1}(U') = f^{-1}\imath^{-1}(U') = f^{-1}(U) \in \mathcal{T}_Y$ for any $U \in \mathcal{T}_A$, so f is continuous.

Conversely, if $f : Y \to A$ is continuous, then $\imath f$ is continuous since $\imath$ is.

(3) This is clear. $\square$

Exercise: do the conclusions in Lemma 1.10 *define* the subspace topology?

2. The product topology

Given topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, we want to define a natural topology on the product space $X \times Y$.

Definition 1.11. Give a set X, a *basis* is a collection $\mathcal{B}$ of subsets of X such that

1. $X = \bigcup_{B \in \mathcal{B}} B$; $\emptyset \in \mathcal{B}$.
2. $B_1, B_2 \in \mathcal{B} \implies B_1 \cap B_2 \in \mathcal{B}$.

Lemma 1.12. *Given a set X and a basis $\mathcal{B}$, let $\mathcal{T}_{\mathcal{B}}$ be the collection of subsets of X defined by*

$$U \in \mathcal{T}_{\mathcal{B}} \text{ if and only if } \exists \text{ a family of sets } \{B_\lambda\}, B_\lambda \in \mathcal{B}, \text{ with } U = \bigcup_\lambda B_\lambda.$$

Then $\mathcal{T}_{\mathcal{B}}$ is a topology on X.

Proof. (1) It is clear that $\emptyset$ and X are in $\mathcal{T}_{\mathcal{B}}$.
(2) If $U, V \in \mathcal{T}_{\mathcal{B}}$ then there are families $\{B_\lambda\}_{\lambda \in \Lambda}$ and $\{C_\mu\}_{\mu \in M}$ with

$$U = \bigcup_{\lambda \in \Lambda} B_\lambda, \quad V = \bigcup_{\mu \in M} C_\mu.$$

It follows that $U \cap V = \bigcap_{\lambda,\mu} B_\lambda \cap C_\mu \in \mathcal{T}_{\mathcal{B}}$.
(3) Closure under arbitrary unions follows similarly. $\square$

That is, there is a topology *generated* by the basis $\mathcal{B}$, and it comprises all sets obtained by taking unions of members of the basis.

Lemma 1.13. *If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, then $\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ is a basis.*

Proof. (1) $X \times Y, \emptyset \in \mathcal{B}$ clearly.
(2) Closure under finite intersections is clear: $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \times V_2)$.
(3) As an exercise, show that the basis of open rectangles is not closed under unions. (Draw a picture of $X \times Y$ and notice that the union of two open rectangles is not in general an open rectangle.) $\square$

The sets of the form $U \times V$ are called *rectangles* for obvious reasons. Exercise: show by example that the set of rectangles is not a topology.

Definition 1.14. The *product topology* on $X \times Y$ is the topology $\mathcal{T}_{\mathcal{B}}$ where $\mathcal{B}$ is the basis of rectangles.

Do not assume that W is open in the product topology if and only if it is an open rectangle.

The correct statement is: W is open in the product topology if and only if $\forall\ (x,y) \in W \subset X \times Y$ there exist sets $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ such that $(x,y) \in U \times V$ and $U \times V \subset W$.

Associated with the product space $X \times Y$ are *canonical projections* $p_1 : X \times Y \to X$, sending (x,y) to x, and $p_2 : X \times Y \to Y$, sending (x,y) to y.

LEMMA 1.15. *With the product topology:*

1. *The projections are continuous,*
2. *If $(Z, \mathcal{T}_Z)$ is another topological space, then $f : Z \to X \times Y$ is continuous if and only if $p_1 f : Z \to X$ and $p_2 f : Z \to Y$ are both continuous.*

PROOF. If $U \in \mathcal{T}_X$, then $p_1^{-1}(U) = U \times Y$ is open in $X \times Y$, so p_1 is continuous. Similarly, p_2 is continuous.
(2) If f is continuous, then $p_1 f$ and $p_2 f$ are compositions of continuous functions, hence continuous.

Conversely, suppose that $p_1 f$ and $p_2 f$ are continuous, and $U \in \mathcal{T}_X$, $V \in \mathcal{T}_Y$. Then

$$\begin{aligned} f^{-1}(U \times V) &= f^{-1}\left((U \times Y) \cap (X \times V)\right) \\ &= f^{-1}(U \times Y) \cap f^{-1}(X \times V) \\ &= f^{-1} p_1^{-1}(U) \cap f^{-1} p_2^{-1}(V) \\ &= (p_1 f)^{-1}(U) \cap (p_2 f)^{-1}(V) \in \mathcal{T}_Z, \end{aligned}$$

since $p_1 f$ and $p_2 f$ are continuous.

Now let $W = \cup U_\lambda \times V_\lambda$ be any open set in $X \times Y$. Then $f^{-1}(W) = \cup f^{-1}(U_\lambda \times V_\lambda)$ is open in Z, so f is continuous. $\square$

3. The product topology on $\mathbb{R}^n$

Recall the usual (metric) topology on $\mathbb{R}$:

$$\begin{aligned} U \in \mathcal{T}_{\mathbb{R}} &\iff \forall x \in U\ \exists\ \epsilon > 0 \text{ such that } (x-\epsilon, x+\epsilon) \subset U \\ &\iff \forall x \in U\ \exists\ a, b \in \mathbb{R} \text{ such that } x \in (a,b) \subset U. \end{aligned}$$

It follows that the product topology on $\mathbb{R}^2$, $\mathcal{T}_2$, is given by:

$$\begin{aligned} W \in \mathcal{T}_2 &\iff \forall x \in W\ \exists\ U, V \in \mathcal{T}_{\mathbb{R}} \text{ such that } x \in U \times V \subset W \\ &\iff \forall x = (x_1, x_2) \in W\ \exists\ a_1, b_1, a_2, b_2 \text{ such that} \\ &\qquad (x_1, x_2) \in (a_1, b_1) \times (a_2, b_2) \subset W. \end{aligned}$$

Similarly, the product topology on $\mathbb{R}^n$, $\mathcal{T}_n$, is given by:

$$W \in \mathcal{T}_n \iff \forall x = (x_1, \ldots, x_n) \in W \ \exists\ a_1, b_1, \ldots, a_n, b_n \text{ such that } x \in (a_1, b_1) \times \cdots \times (a_n, b_n) \subset W.$$

On the other hand, we know many metrics on $\mathbb{R}^n$, and usually use the standard Euclidean metric

$$d\left((x_1, \ldots, x_n), (y_1, \ldots, y_n)\right) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2},$$

which defines a metric topology on $\mathbb{R}^n$. Are the two topologies the same?

Lemma 1.16. *The metric topology $\mathcal{T}_d$ for the usual Euclidean metric on $\mathbb{R}^n$, and the product topology on $\mathbb{R}^n$, are identical.*

Proof. Suppose $W \in \mathcal{T}_d$, so $\forall x \in W$, $\exists\ \epsilon > 0$ such that $x \in B(x;\epsilon) \subset W$. We must find $a_1, b_1, \ldots, a_n, b_n$ such that $x \in (a_1, b_1) \times \cdots \times (a_n, b_n) \subset B(x;\epsilon)$, showing that $W \in \mathcal{T}_n$. In two dimensions, Figure 1.1 shows how to do this.

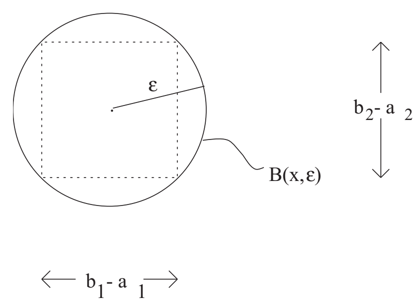


Figure 1.1. An open ball in $\mathbb{R}^2$

It follows (details are an exercise) that $\mathcal{T}_d \subset \mathcal{T}_n$.

Conversely, suppose that $W \in \mathcal{T}_n$, so that $\forall x \in W \exists a_1, b_1, \ldots, a_n, b_n$ such that $x \in (a_1, b_1) \times \cdots \times (a_n, b_n) \subset W$. We need to find ϵ positive such that $x \in B(x;\epsilon) \subset (a_1, b_1) \times \cdots \times (a_n, b_n)$. Again, Figure 1.2 in $\mathbb{R}^2$ shows how to do this.

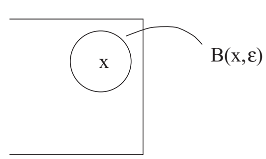


Figure 1.2. An open rectangle in $\mathbb{R}^2$

It follows that $\mathcal{T}_n = \mathcal{T}_d$. $\square$

4. The quotient topology

Given a topological space $(X, \mathcal{T}_X)$ and a surjective function $q : X \to Y$, we may define a topology on Y using the topology on X. The *quotient topology on Y induced by q* is defined to be

$$\mathcal{T}_Y = \{U \subset Y \mid q^{-1}(U) \in \mathcal{T}_X\}.$$

LEMMA 1.17. *$\mathcal{T}_Y$ is a topology on Y. The map q is continuous with respect to the quotient topology.*

As with the product topology, the quotient topology is the 'right' one in the following sense. Lemma 1.17 says that the quotient topology is not too large (does not have too many open sets); Lemma 1.18 says that the quotient topology is large enough.

LEMMA 1.18. *Let $(X, \mathcal{T}_X)$ be a topological space, with a surjection $q : X \to Y$. Let $(Z, \mathcal{T}_Z)$ be another topological space, and $f : Y \to Z$ a function. If Y is given the quotient topology, then*

1. *q is continuous;*
2. *$f : Y \to Z$ is continuous if and only if $fq : X \to Z$ is continuous.*

PROOF. (1) This is Lemma 1.17.
(2) If f is continuous, then fq is continuous since it is the composition of two continuous maps.

Assume now that fq is continuous, and that $U \in \mathcal{T}_Z$. Then

$$\begin{aligned} f^{-1}(U) \in \mathcal{T}_Y \quad & \iff q^{1}\left(f^{-1}(U)\right) \in \mathcal{T}_X \text{ (by definition)} \\ & \iff (fq)^{-1}(U) \in \mathcal{T}_X \text{ (which is true since } fq \text{ is continuous).} \end{aligned}$$

It follows that f is continuous. □

5. Three important examples of quotient topologies

EXAMPLE 1.19. [REAL PROJECTIVE SPACE] Define an equivalence relation $\sim$ on $(n+1)$ dimensional real vector space $\mathbb{R}^{n+1}$ by

$$(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1})$$

if and only if there exists $\lambda \neq 0$ such that $x_1 = \lambda y_1, \ldots, x_{n+1} = \lambda y_{n+1}$. Define n-dimensional real projective space to be the space of equivalence classes

$$\mathbb{R}P^n = \mathbb{R}^{n+1} \backslash \{0\} / \sim .$$

In $\mathbb{R}P^n$ it is convenient to use *homogeneous coordinates*, so a point is given by $[x_1, \ldots, x_{n+1}]$, where $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \backslash \{0\}$, and

$$[x_1, \ldots, x_{n+1}] = [y_1, \ldots, y_{n+1}]$$

if and only if $(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1})$.

Recall the standard notation for spheres: $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ is the n-sphere ($|\cdot|$ is the usual metric). Special cases are $S^0 =$

$\{\pm 1\}$, S^1 the circle, and S^2 the usual sphere. Make the n-sphere into a topological space by inducing the subspace topology from $\mathbb{R}^{n+1}$.

There is a natural surjection $q : S^n \to \mathbb{R}P^n$ given by $q(x_1, \ldots, x_{n+1}) = [x_1, \ldots, x_{n+1}]$. (See exercises).

Define the topology on $\mathbb{R}P^n$ to be the quotient topology defined by the function $q : S^n \to \mathbb{R}P^n$.

EXAMPLE 1.20. [THE MÖBIUS BAND] Let $X = [0,1] \times [0,1]$, the square. Define an equivalence relation $\sim$ on X by

$$(x,y) \sim (x',y') \iff \begin{array}{c} (x,y) = (x',y'), \text{ or} \\ x = 0, x' = 1, y' = 1-y, \text{ or} \\ x = 1, x' = 0, y' = 1-y. \end{array}$$

This equivalence relation is represented pictorially in Figure 1.3 – make sure you understand how this works.

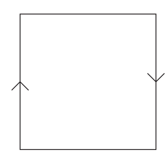


FIGURE 1.3. The relation $\sim$ on the square

The Möbius band is defined to be the quotient space $M = X/\sim$, together with the quotient topology. There is a canonical function $q : X \to M$, defined by $q(x,y) = [(x,y)]_\sim$.

Notice that $f : M \to \mathbb{R}$ is continuous if and only if $fq = g : [0,1] \times [0,1] \to \mathbb{R}$ is continuous.

A less rigorous – but more practical – construction of M is the following. Take a strip of paper, twist one half turn, then glue the ends together. Check that this gives the same topological space.

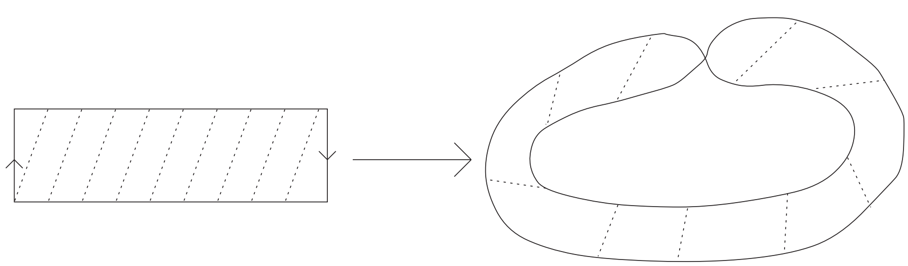


FIGURE 1.4. The Möbius band

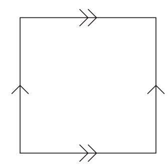


FIGURE 1.5. The torus

EXAMPLE 1.21. [THE TORUS] Let $X = [0,1] \times [0,1]$, and define an equivalence relation $\sim$ using Figure 1.5.

A convenient representation of the quotient function is $q(s,t) = (e^{2\pi is}, e^{2\pi it})$, which realizes the 2-torus as the product space $S^1 \times S^1$.

CHAPTER 2

Properties of Topological Spaces

Let $(X, \mathcal{T}_X)$ be a topological space, and $A \subset X$ a subset of X.
The *closure* of A, denoted $\bar{A}$, is the intersection of all the closed sets containing A. It follows that $\bar{A} \supset A$, $\bar{A}$ is closed, and $\bar{A}$ is the smallest set with these two properties.
The *interior* of A, denoted A°, is the union of all open sets contained in A. It follows that $A^\circ \subset A$, A° is open, and A° is the largest set with these two properties.
The *boundary* or *frontier* of A, sometimes denoted δA, is $\bar{A} \backslash A^\circ$.

Definition 2.1. Let $(X, \mathcal{T}_X)$ be a topological space, and $C \subset X$ a subset. Then C is *compact* if, given any family of open sets $\{U_\lambda\}$ which *cover* C, $C \subset \bigcup_\lambda U_\lambda$, there is a finite number $U_{\lambda_1}, \ldots, U_{\lambda_n}$ of these sets that still cover C: $C \subset U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$.

The whole space X is said to be compact if it is a compact subset of itself. Notice the terminology: an open cover of C is a collection of open sets $\{U_\lambda\}$ whose union contains C. The cover $\{V_\mu\}$ is a *subcover* of the cover $\{U_\lambda\}$ if $\{V_\mu\} \subset \{U_\lambda\}$. That is, $\forall \mu \; \exists \; \lambda$ such that $V_\mu = U_\lambda$.

If a set has a finite open cover, it does not follow that it is compact.

Recall the Heine–Borel theorem.

Theorem 2.2. *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

It follows that spheres are compact.

Theorem 2.3. *Let X and Y be topological spaces, and $f : X \to Y$ a continuous function. If X is compact, then $f(X) \subset Y$ is a compact subset of Y.*

1. Examples

[1] Recall that there is a continuous map $q : S^n \to \mathbb{R}P^n$. It follows that $\mathbb{R}P^n$ is compact.
[2] Similarly, the Möbius band is compact.
[3] The torus is compact.

2. Hausdorff Spaces

The next few results try to generalize the Heine–Borel theorem to topological spaces. There is one technicality, which we deal with below by considering *Hausdorff* spaces. This assumption will prevent the spaces we consider from being too pathological. In one direction there is no problem: closed subsets of compact sets are always compact.

LEMMA 2.4. *If A is a closed subset of a compact topological space, then A is compact.*

PROOF. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of A. Then $\{X\backslash A, U_\lambda\}_{\lambda\in\Lambda}$ is an open cover of all of X (since A is closed). By compactness, there is a finite subcover,

$$X \subset (X\backslash A) \cup U_{\lambda_1} \cup \cdots \cup U_{\lambda_n},$$

so $A \subset U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$ and A is therefore compact. □

DEFINITION 2.5. A topological space X is *Hausdorff* if given two points $x, y \in X$, there are open sets $U, V \subset X$ with $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Hausdorff topological spaces are in some sense not too far from being metric spaces. You may also see the Hausdorff property called T_2.

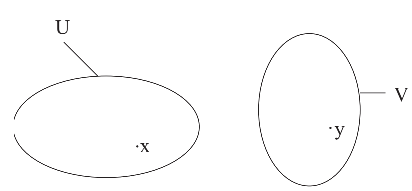


FIGURE 2.1. The Hausdorff property

EXAMPLE 2.6. (1) The metric topology on a metric space is always Hausdorff. If x and y are distinct points, then $\delta = d(x, y)$ is greater than 0. It follows that the metric open balls $B(x; \delta/3)$ and $B(y; \delta/3)$ are disjoint open sets that separate x and y.
(2) The concrete topology on any space containing at least two points is never Hausdorff (and therefore cannot be induced by any metric).
(3) Let $X = \{a, b\}$. Define a topology by $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}\}$. Then the topological space $(X, \mathcal{T})$ is not Hausdorff. Notice that the set $\{a\}$ is compact but is not closed in this topology.

THEOREM 2.7. *Suppose that X is Hausdorff, and $C \subset X$ is compact. Then C is closed.*

PROOF. It is enough to show that $X\backslash C$ is open, and this is equivalent to the following statement: for every $x \in X\backslash C$, there is an open set $W_x \ni x$ with $W_x \subset X\backslash C$.

Fix $x \in X\backslash C$, and let y be any point in C. Since $x \neq y$ and X is Hausdorff, there are open sets $U_y \ni x$ and $V_y \ni y$ with $U_y \cap V_y = \emptyset$. Now $\{V_y\}_{y \in C}$ is an open cover of C, so by compactness, there is a finite subcover

$$C \subset V_{y_1} \cup \cdots \cup V_{y_n}.$$

Let $W_x = U_{y_1} \cap \cdots \cap U_{y_n}$. Then $x \in W_x$ since $x \in U_{y_i}$ for each i. Also, W_x is open since it is a *finite* intersection of open sets. Finally, $W_x \cap C = \emptyset$ since $z \in W_x \cap C$ implies that $z \in U_{y_i}$ for $i = 1, \ldots, n$ and $z \in V_{y_k}$ for some k, so $z \in U_{y_k} \cap V_{y_k} = \emptyset$. $\square$

REMARK 2.8. There are spaces in which every compact set is closed (such spaces are usually called KC spaces) that are not Hausdorff. The simplest example of this is the *co-countable topology* $\mathcal{C}$ *on* $\mathbb{R}$. This is defined as follows: a set $A \subset \mathbb{R}$ is open (in $\mathcal{C}$) if and only if $A = \emptyset$ or $\mathbb{R}\backslash A$ is countable.

Recall that a homeomorphism is a continuous bijection whose inverse is also continuous. Also, a function is continuous if and only if the pre-image of any closed set is closed. The next result is the basic technical tool that allows us to make topological spaces by 'cutting and pasting'. From now on, we will use this result too often to mention, but try to understand when it is being used.

THEOREM 2.9. *Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous bijection. Suppose that X is compact and Y is Hausdorff. Then f is a homeomorphism.*

PROOF. Let $g = f^{-1}$: this is a well-defined map since f is a bijection. Let $A \subset X$ be closed. Since X is compact, A is compact. Also, $g^{-1}(A) = f(A)$ is the continuous image of a compact set, and is therefore a compact subset of Y. Since Y is Hausdorff, $g^{-1}(A)$ must therefore be closed – which proves that g is continuous. $\square$

3. Examples

EXAMPLE 2.10. [THE CIRCLE] We can now be a little more rigorous about the circle. As an application of Theorem 2.9, let's prove that the additive circle

$$\mathbb{T} = [0, 1]/\sim$$

where $\sim$ is defined by

$$x \sim y \iff \begin{array}{c} x = y \\ x = 0, y = 1, \text{ or} \\ x = 1, y = 0, \end{array}$$

is homeomorphic to the usual circle

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

The map $q : [0,1] \to \mathbb{T}$ defined by $q(x) = [x]$ is onto, so $\mathbb{T}$ may be given the quotient topology defined by q.

Define a map $f : [0,1] \to S^1$ by $f(t) = e^{2\pi i t}$. Then f is clearly continuous; also $f(x) = f(y)$ if and only if $x \sim y$. It follows that f defines a function $g : \mathbb{T} \to S^1$. Since $f(x) = f(y) \iff x \sim y$, g is bijective. By Lemma 1.18, the map g is continuous if and only if the composition $gq = f$ is continuous. So g is a continuous bijection from the compact space $\mathbb{T}$ (this is compact since it is the continuous image of the compact set $[0,1]$) to the Hausdorff space S^1. By Theorem 2.9, we deduce that g is a homeomorphism.

Example 2.11. [the torus] We have already sketched this construction – fill in the details as above to show that the square $X = [0,1] \times [0,1]$ with edges glued together as shown in Figure 2.2, is homeomorphic to the torus $S^1 \times S^1$.

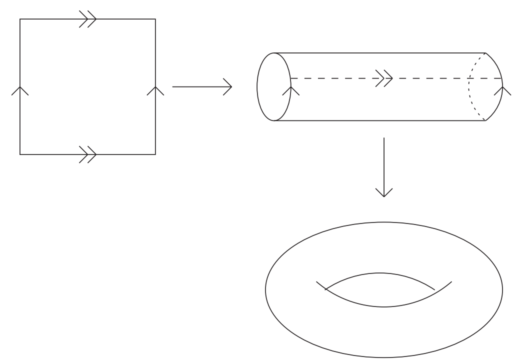


Figure 2.2. The torus obtained from the square by two glueings

Example 2.12. [the klein bottle] Introducing one half-twist in the construction of the torus gives a topological space known as the Klein bottle K, shown in Figure 2.3.

There is no subspace of $\mathbb{R}^3$ that is homeomorphic to the Klein bottle, but there is a subspace of $\mathbb{R}^4$ homeomorphic to the Klein bottle.

Example 2.13. [the projective plane again] There is one remaining way to glue the edges of a square together to make a topological space: let P be the space defined by the glueing in Figure 2.4.

Let's first show that P is homeomorphic to the Möbius band *with a disc glued onto the edge*. Look closely at the Möbius band, and

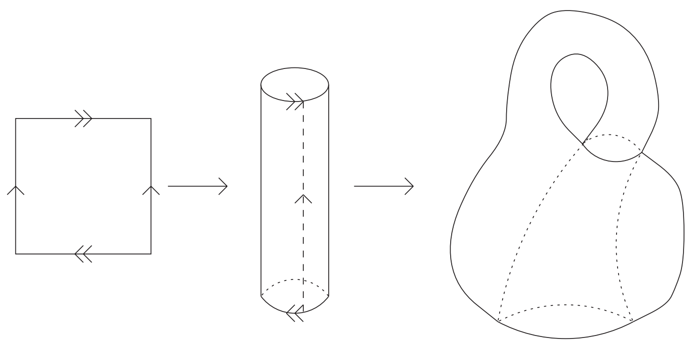


FIGURE 2.3. The Klein bottle

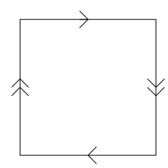


FIGURE 2.4. The surface P

notice that the *edge* is a circle. This means we can attach to it any other topological space whose edge is a circle, by simply glueing the two circles together. (A simple example of this is to glue two discs along their circular edges and obtain a sphere.)

First, cut a disc out of P and chop the resulting figure in half (Figure 2.5).

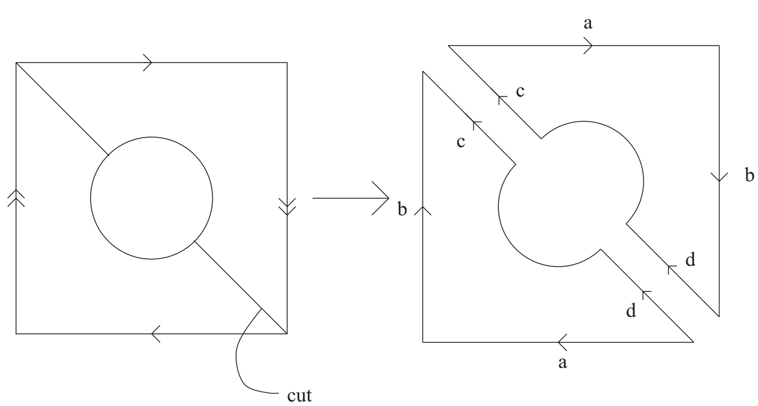


FIGURE 2.5. The surface P with a disc cut out

Notice that letters and arrows are used to keep track of how the pieces must be glued together. Now do a flip, some straightening out (all of which is simply applying certain homeomorphisms) to obtain the Möbius band. Try to convince yourself that the surgery performed in Figures 2.5, 2.6, and 2.7 may be made rigorous.

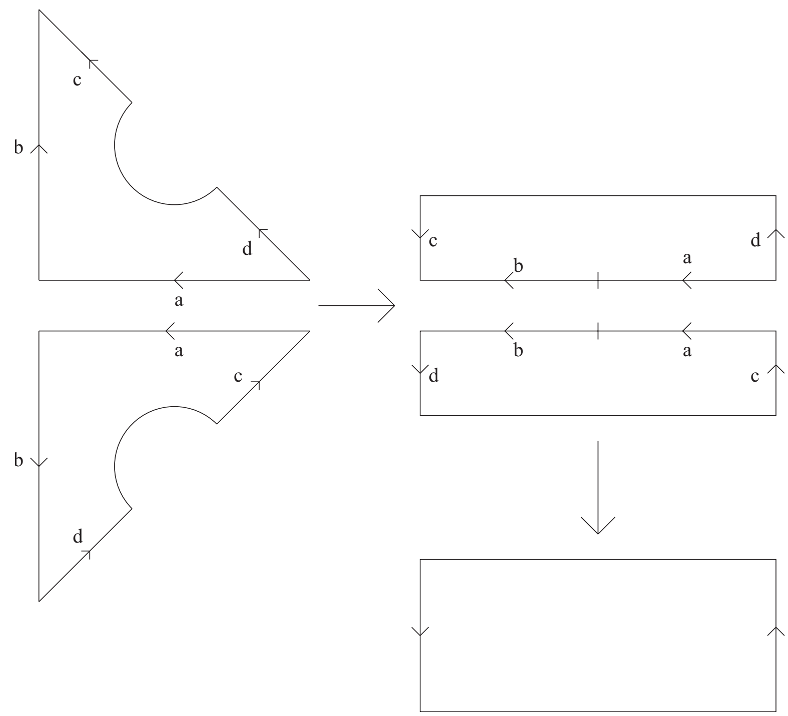


FIGURE 2.6. The Möbius band again

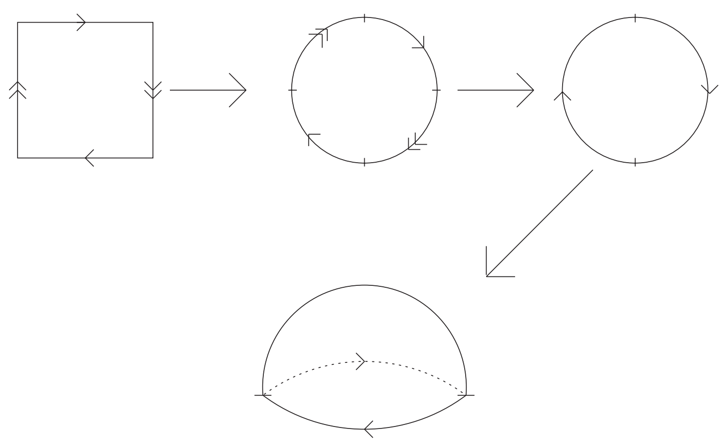


FIGURE 2.7. P is the projective plane $\mathbb{R}P^2$

Remember the map $q : S^2 \to \mathbb{R}P^2$, given by $q(x) = q(y)$ if and only if $x = \pm y$, giving a homeomorphism between $\mathbb{R}P^2$ and S^2 after x is glued to $-x$ around the equator. It follows that the surgery shown in Figure 2.7 proves that P is homeomorphic to $\mathbb{R}P^2$.

4. Connectedness

DEFINITION 2.14. A topological space X is *connected* if, given two open sets U and V with $X = U \cup V$, $U \cap V = \emptyset$, either $X = U$ or $X = V$.

LEMMA 2.15. *The following are equivalent:*

1. *X is connected.*
2. *The only subsets of X that are both open and closed are the empty set and X itself.*
3. *Every continuous function $f : X \to \{0, 1\}$ is constant.*

A subset B of a topological space X is a *connected subspace* if B is a connected space in the subspace topology.

We know from second year courses that a subset of $\mathbb{R}$ is connected if and only if it is an interval.

LEMMA 2.16. *Let X be a connected space, and $f : X \to Y$ a continuous functions.*

1. *$f(X)$ is a connected subspace of Y.*
2. *If $Y = \mathbb{R}$, then f satsfies the Intermediate Value Theorem: if $f(x) \leq f(y)$, and $c \in [f(x), f(y)]$, then there is a $z \in X$ such that $f(z) = c$.*

5. Path connectedness

DEFINITION 2.17. (1) A *path* in a topological space X is a continuous function $\gamma : [0, 1] \to X$; the *starting point* is $\gamma(0)$, the *end point* is $\gamma(1)$. The path γ *joins* the starting point to the end point.
(2) A space X is *path-connected* if for any points $x, y \in X$ there is a path joining x to y.

LEMMA 2.18. *A path-connected space is connected.*

EXAMPLE 2.19. There is a connected space that is not path-connected.

Notice that connectedness and path-connectedness are *topological properties*: if X and Y are homoemorphic spaces, then

$$X \text{ is } \left\{ \begin{matrix} \text{connected} \\ \text{path-connected} \end{matrix} \right\} \iff Y \text{ is } \left\{ \begin{matrix} \text{connected} \\ \text{path-connected} \end{matrix} \right\}$$

This gives us another genuine topological theorem: we know there are space-filling curves (continuous surjective functions from an interval to a square), but are now able to prove that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.

THEOREM 2.20. *$\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.*

PROOF. Suppose that $f : \mathbb{R} \to \mathbb{R}^2$ is a homeomorphism. Now it is clear that $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}^2\backslash\{f(0)\}$ is a homeomorphism for the induced topologies. But $\mathbb{R}^2\backslash\{f(0)\}$ is clearly connected, while $\mathbb{R}\backslash\{0\}$ is not. $\square$

CHAPTER 3

Homotopy equivalence

We have seen how to use paths in a topological space to see how well-connected a space is. The next step is to use analogues of paths in the space of maps to study the 'shape' of topological spaces.

Definition 3.1. Maps $f, g : X \to Y$ are *homotopic* if there exists a map

$$F : X \times I \to Y$$

so that $F_0 = f$, $F_1 = g$, where $F_t : \begin{array}{ccc} & \to & \\ x & \mapsto & (x,t) \end{array} X \to X \times I \xrightarrow{F} Y$.

Example 3.2. Any two maps $f, g : X \to \mathbb{R}^n$ are homotopic. To see this, notice that $\mathbb{R}^n$ is *convex*: if $\mathbf{x}$ and $\mathbf{y}$ are points in $\mathbb{R}^n$, then for any $t \in I$, the point $(1-t)\mathbf{x} + t\mathbf{y}$ is in $\mathbb{R}^n$ also. Define a homotopy F by $F(x,t) = (1-t)f(x) + tg(x)$.

Lemma 3.3. *Homotopy is an equivalence relation.*

Notice that in proving Lemma 3.3 we need the Glueing Lemma (which is on Exercise Sheet 2).

Lemma 3.4. [the glueing lemma] *Let $Z = A \cup B$, where A and B are closed subsets of Z. Suppose that $f : Z \to Y$ is any function for which $f|_A$ and $f|_B$ are maps. Then f is a map.*

Definition 3.5. Maps $f, g : X \to Y$ are *homotopic rel* A, where A is a subset of X, if $F_t(a) = F_0(a)$ for all $t \in I$ and $a \in A$. Write $f \underset{F}{\to}\sim g$ rel A for this relation.

Example 3.6. (1) Let $X = I$, and let Y be the annulus $\{x \in \mathbb{R}^2 \mid 1 \leq |x| \leq 3\}$. Let f and g be the indicated paths, both beginning at $(-2,0)$ and ending at $(2,0)$.

Then f and g are homotopic (check this). However, if $A = \{0,1\}$, then f and g are not homotopic rel A. This means that the 'hole' in the annulus can be detected by considering properties of homotopy classes of paths relative to their endpoints.

(2) This example shows that homotopy of paths relative to end points is not so good at detecting the presence of higher-dimensional 'holes'. Any two paths f and g in the 2-sphere with the same end points are homotopic rel $\{0,1\}$.

Since homotopy is an equivalence relation, we may speak of the *homotopy class* of f, denoted $[f]$. To combine homotopy classes of maps, we need to know that the obvious definition is well-defined.

LEMMA 3.7. *If there are maps* $X \xrightarrow{f} Y \xrightarrow{g} Z$*, then the rule* $[f] \circ [g] = [f \circ g]$ *gives a well-defined composition of homotopy classes.*

The point being that if $f \sim f_1$ and $g \sim g_1$, then $f \circ g \sim f_1 \circ g_1$.

DEFINITION 3.8. A map $f : X \to Y$ is a *homotopy equivalence* if there is a map $g : Y \to X$ such that $fg \sim 1_Y$ and $gf \sim 1_X$. We write $X \sim Y$, and say that X and Y have the *same homotopy type*.

As an exercise, show that this defines an equivalence relation on the set of all topological spaces, and that this equivalence is strictly weaker than that of being homeomorphic.

In order to work with Definition 3.8, we need to prove a result that allows pictorial arguments (pushing pieces of spaces around, cutting and glueing and so on) to be used.

DEFINITION 3.9. Let A be a subset of X. A map $r : X \to A$ is a *retraction* if $r|_A = 1_A$. The set A is a *strong deformation retract* of X if there is a homotopy $F_t : X \to X$ rel A such that $F_0 = 1_X$, $F_1(X) = A$ (and of course $F_t(a) = a$ for all $a \in A$ since the homotopy is rel A).

That is, X may be slid over itself into A while keeping A fixed throughout.

EXAMPLE 3.10. Let $D^2 = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$, the disc. Then $S^1 \times D^2$ is a *solid torus*, with a center circle $S^1 \times \{0\}$.

The homotopy $F_t(x, y) = (x, (1-t)y)$ shows that the solid torus can be deformed onto the center circle, so $S^1 \times \{0\} \cong S^1$ is a deformation retract of $S^1 \times D^2$.

The next lemma shows that deformation preserves the homotopy type of a space.

LEMMA 3.11. *If A is a strong deformation retract of X, then the identity map $\imath : A \hookrightarrow X$ is a homotopy equivalence.*

EXAMPLE 3.12. (1) Let $X = [0, 1]$, and $A = \{0\}$. Then $F_t(x) = (1 - t)x$ shows that A is a strong deformation retract of X.
(2) Let X be the triangle in $\mathbb{R}^2$ whose vertices are the points P, Q, R, and let A be a union of two sides.

The triangle may be written $X = \{pP + qQ + rR \mid p + q + r = 1, p, q, r \geq 0\}$. Let $w = pP + qQ + rR$ be a point in the triangle, and define $F_0(w) = w$, and $F_1(w) = w + s(P - \frac{1}{2}(Q + R))$ where s is uniquely defined by requiring that $F_1(w)$ lie on the line joining P and R or the line joining P and Q depending on which side of the line joining P and $\frac{1}{2}(Q + R)$ the point w lay.

Then the deformation F_t may be ‘filled in’ in an obvious fashion: the result is the following map.

$$F_t(w) = \begin{cases} w + 2qt(P - \frac{1}{2}(Q+R)) & \text{if } r \geq q \\ w + 2rt(P - \frac{1}{2}(Q+R)) & \text{if } r, q. \end{cases}$$

This shows that A is a deformation retract of X.
(3) Using (1) and (2) we can understand the homotopy type of simple figures:
(4) Finite connected graphs may be collapsed in a systematic way:

THEOREM 3.13. *Any finite connected graph has the homotopy type of a wedge of circles.*

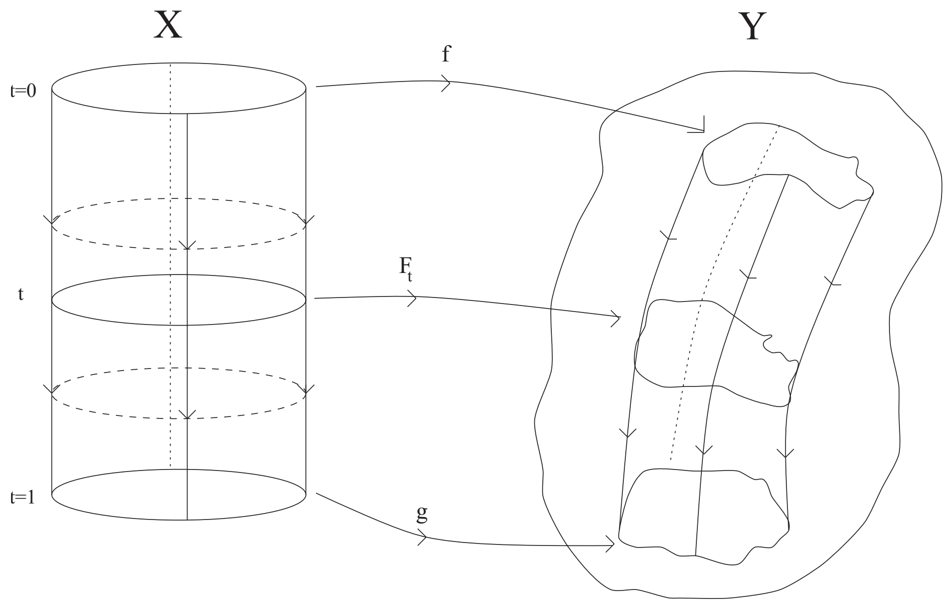


FIGURE 3.1. Maps f and g are homotopic via the homotopy F

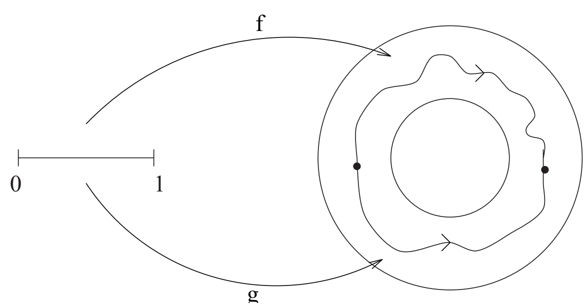


FIGURE 3.2. Homotopic paths that are not homotopic rel $\{0,1\}$

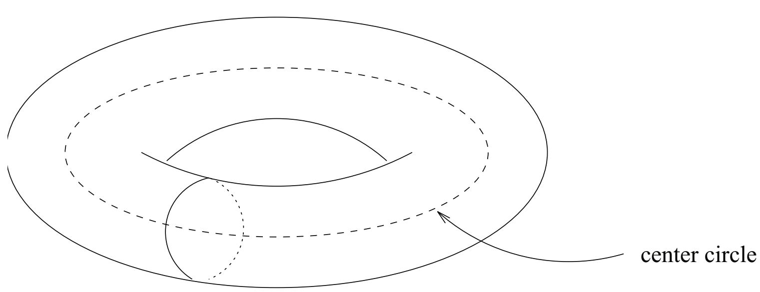


FIGURE 3.3. A deformation retract of the solid torus

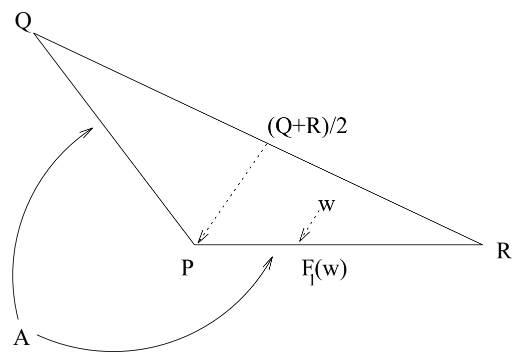


FIGURE 3.4. Deforming a triangle onto the union of two sides

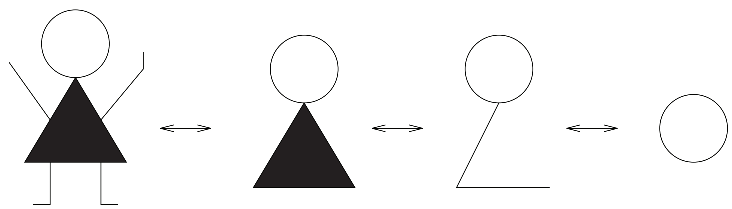


FIGURE 3.5. Homotopy equivalences

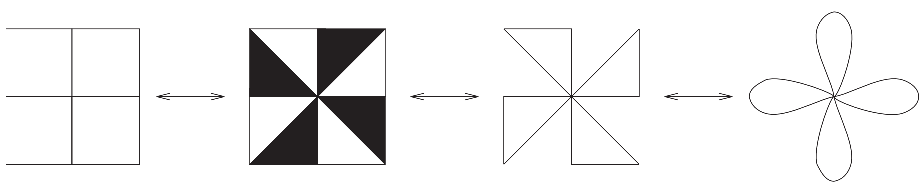


FIGURE 3.6. Homotopy type of finite connected graphs

CHAPTER 4

The Fundamental Group

In this section we define an invariant of topological spaces (that is, something preserved by homeomorphism). The invariant we describe is a certain *group*, and in principle it may be used in certain cases to show that two topological spaces are not homeomorphic. In practice we shall use it for other purposes mostly – in particular for understanding *covering spaces*, *lifting theorems*, and some interesting *fixed-point theorems*. This will all be made clear in Chapter 5.

Definition 4.1. Let (X, x_0) be a based topological space. Let $\pi_1(X, x_0)$ denote the *set* of homotopy classes of maps $\omega : I \to X$ rel $\{0,1\}$ such that $\omega(0) = \omega(1) = x_0$. That is, $\pi_1(X, x_0)$ is the set of *loops based at* x_0. Elements of $\pi_1(X, x_0)$ will be denoted

$$\langle\omega\rangle = \{\tau \mid \tau \sim \omega \text{ rel } \{0,1\}\}.$$

We now claim that there is a natural multiplication on $\pi_1(X, x_0)$ that makes it into a group.
multiplication: Define $\langle\omega\rangle\langle\sigma\rangle$ to be $\langle\omega\sigma\rangle$, where the loop $\omega\sigma$ is defined by

$$\omega\sigma(s) = \begin{cases} \omega(2s), & \text{for } 0 \le s\frac{1}{2} \\ \sigma(2s-1), & \text{for } \frac{1}{2} < s \le 1. \end{cases}$$

In order to be sure that this is well-defined, we must check two things: first that $\omega\sigma$ is a loop (the point being that we need to check it is continuous: this is an easy application of the Glueing Lemma). Secondly, we must check that the multiplication is well-defined on classes: if $\langle\omega\rangle = \langle\omega\rangle'$ and $\langle\sigma\rangle = \langle\sigma\rangle'$, then $\langle\omega\sigma\rangle = \langle\omega'\sigma'\rangle$.
associativity: Given three loops ω, σ and τ, we need to check that

$$(\omega\sigma)\tau \sim \omega(\sigma\tau) \text{ rel } \{0,1\}.$$

The motivation for the proof is given by the following diagram – make sure you understand this, as similar diagrams will be used fairly often.

If you think that there is nothing to prove here, then you should go over the definitions in this section very carefully. The following map

$$F_t(s) = \begin{cases} \omega\left(\frac{4s}{t+1}\right), & 0 \le s \le \frac{t+1}{4}, \\ \sigma\left(\frac{4s-t-1}{1}\right), & \frac{t+1}{4} \le s \le \frac{t+2}{4}, \\ \tau\left(\frac{4s-t-2}{2-t}\right), \quad \frac{t+2}{4} \le s \le 1, & \end{cases}$$

defines a homotopy rel $\{0,1\}$ between $(\omega\sigma)\tau$ and $\omega(\sigma\tau)$.

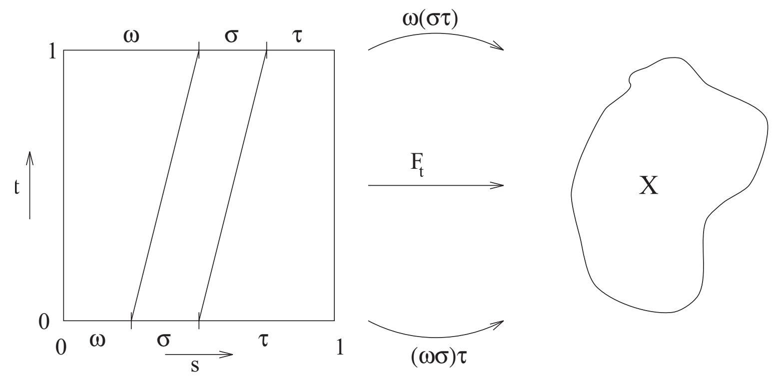


FIGURE 4.1. Associativity of loop multiplication

IDENTITY: Define the trivial loop $e(s) = x_0$ for $s \in I$. Then check that $\langle e\sigma\rangle = \langle \sigma e\rangle = \langle \sigma\rangle$ for all loops σ.
INVERSES: For any loop σ, define σ^{-1} by

$$\sigma^{-1}(s) = \sigma(1 - s).$$

Then check that $\langle \sigma^{-1}\sigma\rangle = \langle \sigma\sigma^{-1}\rangle = \langle e\rangle$.

1. Based Maps

A based map $f : (X, x_0) \to (Y, y_0)$ is a map $X \to Y$ with the property that $f(x_0) = y_0$. Given such a map, we may define a transformation

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

by setting $f_*\langle \omega\rangle = \langle f \circ \omega\rangle$.

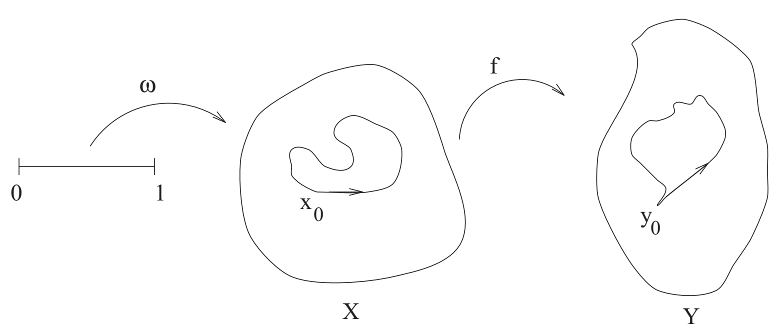


FIGURE 4.2. Image of a loop under a based map

Notice that the map f_* is well-defined by Lemma 3.7. Also, the map f_* only depends on the homotopy class of f rel $\{x_0\}$ (see exercises).

LEMMA 4.2. *The map f_* is a group homomorphism.*

We shall sometimes write $\pi_1(f)$ for f_*.

Recall (or discover) that a *functor* is a certain kind of map between *categories*. Don't worry if this does not mean anything to you: take the following discussion as an example of something we have not defined. Consider the collection $\mathfrak{T}$ of all based topological spaces together with all based maps (continuous functions between them). Let $\mathfrak{G}$ denote the collection of all groups together with all homomorphisms between them. Both $\mathfrak{T}$ and $\mathfrak{G}$ are examples of *categories*, and we may think of them as containing two kinds of things: *objects* (topological spaces X, Y and so on or groups G, H and so on) and *arrows* (continuous maps or group homomorphisms).

A *functor* from the category $\mathfrak{T}$ to the category $\mathfrak{G}$ is a mapping $F : \mathfrak{T} \to \mathfrak{G}$ with the following properties:
(1) Each topological space X is assigned to a unique group $F(X)$.
(2) Each map $f : X \to Y$ (an arrow) is assigned to a group homomorphism $F(f) : F(X) \to F(Y)$ (that is, F sends arrows to arrows).
(3) The assignment in (2) is *functorial*:
  (F1) $F(1_X) = 1_{F(X)}$,
  (F2) $F(f \circ g) = F(f) \circ F(g)$.
Property (F2) may be described as follows: commutative diagrams in $\mathfrak{T}$ are sent to commutative diagrams in $\mathfrak{G}$.

Theorem 4.3. *π_1 is a functor from $\mathfrak{T}$ to $\mathfrak{G}$.*

2. Moving the base point

So far we have been multiplying loops, with the multiplication rule being 'follow the first path then follow the second path'. It is clear that we may also multiply in this way two *paths* as long as the first one ends where the second one begins. The result will be a path from the initial point of the first path to the final point of the second path.

Let ω, σ be paths in X with the property that $\omega(1) = \sigma(0)$. Then $\omega\sigma$ is a path from $\omega(0)$ to $\sigma(1)$, and the homotopy class of $\omega\sigma$ rel $\{0, 1\}$ depends only on the homotopy class rel $\{0, 1\}$ of ω and of σ.

One may check that this multiplication of paths is associative:

$$\langle\omega\rangle(\langle\sigma\rangle\langle\tau\rangle) = (\langle\omega\rangle\langle\sigma\rangle)\langle\tau\rangle$$

whenever either side is defined.

There are also left and right identities for any path:

$$\langle e_{\sigma(0)}\rangle\langle\sigma\rangle = \langle\sigma\rangle \quad \text{left identity,}$$
$$\langle\sigma\rangle\langle e_{\sigma(1)}\rangle = \langle\sigma\rangle \quad \text{right identity.}$$

Finally (recall that a path can always be deformed back to the initial point), if we write $\sigma^{-1}(s) = \sigma(1 - s)$, then

$$\langle\sigma\rangle\langle\sigma^{-1}\rangle = \langle e_{\sigma(0)}\rangle,$$
$$\langle\sigma^{-1}\rangle\langle\sigma\rangle = \langle e_{\sigma(1)}\rangle.$$

PROPOSITION 4.4. *If α is a path in X from x_0 to x_1, then the map $\alpha_\natural : \pi_1(X, x_0) \to \pi_1(X, x_1)$ defined by $\alpha_\natural : \langle\sigma\rangle \mapsto \langle\alpha^{-1}\rangle\langle\sigma\rangle\langle\alpha\rangle$ is a group isomorphism.*

PROOF. Make sure you understand why points in the image of $\alpha_\natural$ are elements of $\pi_1(X, x_1)$. Once you understand that, it is a simple matter to see that $\alpha_\natural$ is a homomorphism, and to compute the map $(\alpha^{-1})_\natural(\alpha_\natural)$:

$$\begin{aligned}
(\alpha^{-1})_\natural(\alpha_\natural)\langle\sigma\rangle &= (\alpha^{-1})_\natural\langle\alpha^{-1}\rangle\langle\sigma\rangle\langle\alpha\rangle \\
&= \langle(\alpha^{-1})^{-1}\rangle\langle\alpha^{-1}\rangle\langle\sigma\rangle\langle\alpha\rangle\langle\alpha^{-1}\rangle \\
&= \langle\alpha\rangle\langle\alpha^{-1}\rangle\langle\sigma\rangle\langle\alpha\rangle\langle\alpha^{-1}\rangle \\
&= \langle e_{x_0}\rangle\langle\sigma\rangle\langle e_{x_1}\rangle \\
&= \langle\sigma\rangle
\end{aligned}$$

since σ is a loop based at x_0. Similarly, one checks that $\alpha_\natural(\alpha^{-1})_\natural$ is the identity on $\pi_1(X, x_1)$. □

COROLLARY 4.5. *If $f : Y \to X$ is a map with $f(y_0) = f(y_1) = x_0$, and Y is a path-connected space, then*

$$f_*(\pi_1(Y, y_0)); \qquad f_*(\pi_1(Y, y_1))$$

are conjugate subgroups of $\pi_1(X, x_0)$.

PROOF. The conjugating element is going to be the image under f of a path joining y_0 to y_1 (there must be such a path since Y is path-connected):

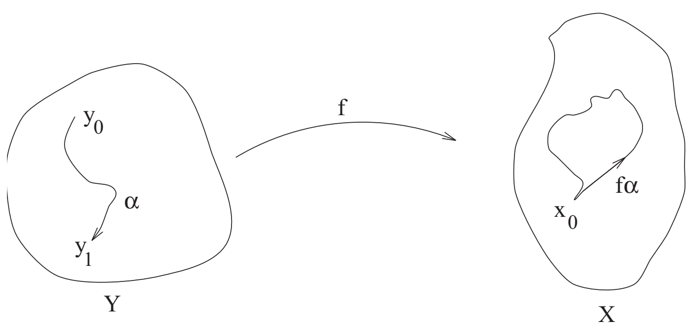


FIGURE 4.3. Image of a path under f is a loop

Let α be such a path, and then check that the following diagram commutes:

$$\begin{array}{ccc}
\pi_1(Y, y_0) & \xrightarrow{f_*} & \pi_1(X, x_0) \\
{\scriptstyle\alpha_\natural}\downarrow & & \downarrow{\scriptstyle(f\alpha)_\natural} \\
\pi_1(Y, y_1) & \xrightarrow{f_*} & \pi_1(X, x_0)
\end{array}$$

and then notice that the vertical maps are both isomorphisms. □

Finally, this gives us some indication of when two topological spaces must have the same fundamental group.

THEOREM 4.6. *If X and Y have the same homotopy type, and X is path connected, then*

$$\pi_1(X) \cong \pi_1(Y).$$

CHAPTER 5

Covering spaces

Definition 5.1. A map $p : Z \to X$ is a *covering map* if each $x \in X$ is contained in some open set $U \subset X$ such that

1. $p^{-1}(U)$ is a disjoint union of open sets in Z – the *sheets over* U,
2. each sheet is mapped homeomorphically by p to U.

We shall also say that U is *evenly covered* by $p^{-1}(U)$, and that Z is a *covering space* for X.

Lemma 5.2. *A covering map is a quotient map.*

Proof. (Included because the method and picture will be used again.) Recall that p is a quotient map if it is a map with the additional property that $p^{-1}(W)$ open implies that W is open. So let $W \subset X$ be a set with $p^{-1}(W)$ open in Z. Fix a point $x \in W$ and consider the following picture.

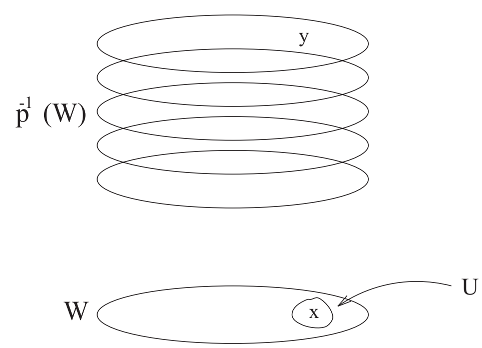


Figure 5.1. Sheets evenly covering W

Choose an open evenly covered set $U \subset X$ with $x \in U$. Let S be a sheet over U, and find $y \in S$ with $p(y) = x$. (Once you've chosen S, y is unique). Now $p^{-1}(W) \cap S$ is open, and

$$p|_S : S \to U$$

is a homeomorphism, so

$$p(S \cap p^{-1}(W)) = pS \cap W = U \cap W$$

is an open set in W containing x. $\square$

EXAMPLE 5.3. (a) Let Y be any discrete topological space, and X any topological space. Then $p : X \times Y \to X$, defined by $p(x, y) = x$ is a covering map[1]. Any open set $U \subset X$ is evenly covered by the sheets of the form $U \times \{y\}$ for $y \in Y$. Make sure you see why these are open sets in the product topology.
(b) Let $p : \mathbb{R} \to S^1$ be given by $p(t) = e^{2\pi it}$. If $\mathbb{R}$ is represented as an infinite helix, the map is vertical projection:

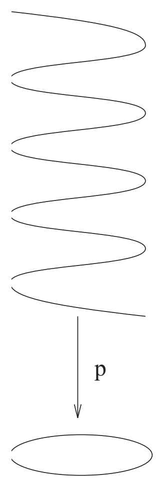


FIGURE 5.2. The reals cover the circle

For an open set $U \subset S^1$, $p^{-1}(U)$ is a disjoint union of countably many open subset of $\mathbb{R}$. An explicit construction is the following: for $x = e^{2\pi it} \in S^1$, let $U = \{y \in S^1 \mid \Re(x)\Re(y) + \Im(x)\Im(y) > 0\}$ (here $\Re$ and $\Im$ denote real and imaginary parts respectively) and $S_n = \{s \in \mathbb{R} \mid n+t-\frac{1}{4} < s < n+t+\frac{1}{4}\}$ for $n \in \mathbb{Z}$. Then U is open, S_n is open for all n, and for any n $p|_{S_n} : S_n \to U$ is a bijection. Also, $p|_{\bar{S}_n} : \bar{S}_n \to \bar{U}$ is a continuous bijection from a compact set to a Hausdorff one, so is a homeomorphism. It follows that $p|_{S_n} : S_n \to U$ is a homeomorphism.

Notice that in this case the covering space has trivial fundamental group: if this is the case, the covering is called *universal*.
(c) A triple covering. The following diagram describes a triple covering of a wedge of two circles. Study the picture carefully – the arrows ensure that pre–images of open sets are open sets.
(d) A countable universal covering of the wedge of two circles. Using the same notational conventions, the following diagram gives a cover of the wedge of two circles.

Notice that each point now has countably many pre-images. Also, the covering space Z is homotopic to a point, so this is a universal cover.

[1]Notice that the converse of this statement is not true. If $X = Y$ is a non-empty space with the concrete topology, then the projection map is a covering map.

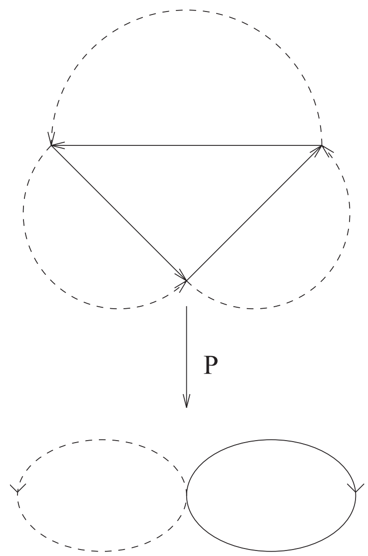


FIGURE 5.3. A triple cover of the wedge of two circles

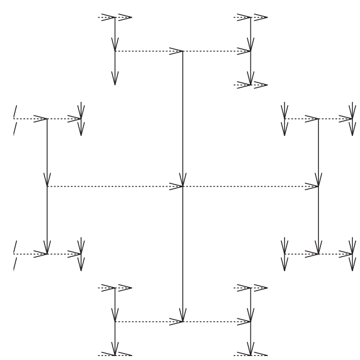


FIGURE 5.4. A countable universal covering

(e) Painting the two sides of a surface gives a double cover:

REMARK 5.4. We have not yet defined orientability, but the following seems to be the case. If X is a connected space covered by Z, and Z is *not* connected, then X is orientable. For example, painting the surface of a 2-torus gives a double cover of the Klein bottle, which suggest that the Klein bottle is not orientable.

1. Lifting maps

We next turn to the following problem: if $p : Z \to X$ is a covering map, and $f : Y \to X$ is a map, when can we expect there to be a *lift* f' of f; that is a map $f' : Y \to Z$ such that $pf' = f$.

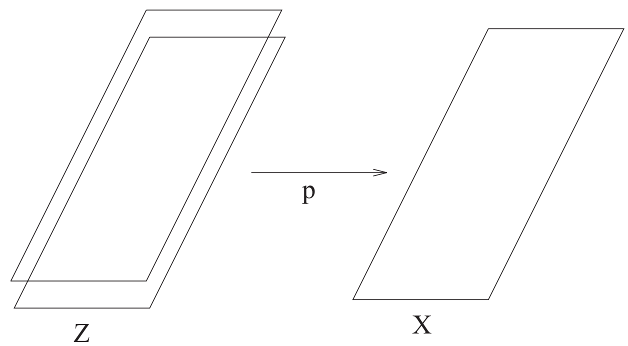


Figure 5.5. Painting sides of a surface

It will be convenient to adopt the following convention: a commutative diagram of the form

$$\begin{tikzcd} Y \arrow[r, "f'"] \arrow[d, equal] & Z \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

should be thought of as a triangle.

Theorem 5.5. *Let $p : (Z, z_0) \to (X, x_0)$ be a based covering map, and $f : Y \to X$ a map from a connected space. Then, if there is a lift f' of (a map making the diagram above commute), it is unique.*

Theorem 5.6. lifting squares *Given $F : (I \times I, (0,0)) \to (X, x_0)$, and a covering map $p : (Z, z_0) \to (X, x_0)$, there is a lift $F' : (I \times I, (0,0)) \to (Z, z_0)$.*

A similar proof gives path-lifting, where $I \times I$ is replaced by I.

Corollary 5.7. *Let $C \subset I \times I$ be connected, let $I : C \hookrightarrow I \times I$ be the inclusion map, and let $f : (C, (0,0)) \to (Z, z_0)$ and $F : (I \times I, (0,0)) \to (X, x_0)$ be maps with $pf = Fi$, where p is the based covering. Then there exists a unique $F' : (I \times I, (0,0)) \to (Z, z_0)$ such that $f = F'i$ and $pF' = F$. That is, there exists a unique diagonal map F' making the following commutative square into two commutative triangles.*

$$\begin{tikzcd} (C, (0,0)) \arrow[r, "f"] \arrow[d, "i"'] & (Z, z_0) \arrow[d, "p"] \\ (I \times I, (0,0)) \arrow[r, "F"] & (X, x_0) \end{tikzcd}$$

These lifting results provide the tools we need to understand homotopy classes of loops in topological spaces.

COROLLARY 5.8. *If $p : (Z, z_0) \to (X, x_0)$ is a based covering, then $p_* : \pi_1(Z, z_0) \to \pi_1(X, x_0)$ is a monomorphism.*

2. The action on the fibre

The lifting results above now give very easily one of the most important observations in the course: there is a natural action of $\pi_1(X)$ on the *fibres* of a covering map.

DEFINITION 5.9. A *right action* of a group G on a set S is a function $S \times G \to S$, written $(s, g) \mapsto sg$, with the properties
(A1) $s1 = s$ for all $s \in S$;
(A2) $(sg_1)g_2 = s(g_1 g_2)$.

If $p : (Z, z_0) \to (X, x_0)$ is a based covering, then the set $F = p^{-1}(x_0)$ is called the *fibre* of p. Define an action

$$F \times \pi_1(X, x_0) \to F$$

of $\pi_1(X, x_0)$ on F by setting $(z, \langle\sigma\rangle) \mapsto z\langle\sigma\rangle = \sigma'(1)$ where $\sigma' : (I, 0) \to (Z, z)$ is the lift of the path σ (so in particular, $p\sigma' = \sigma$). Now $pz = x_0$ since z is in the fibre of p.

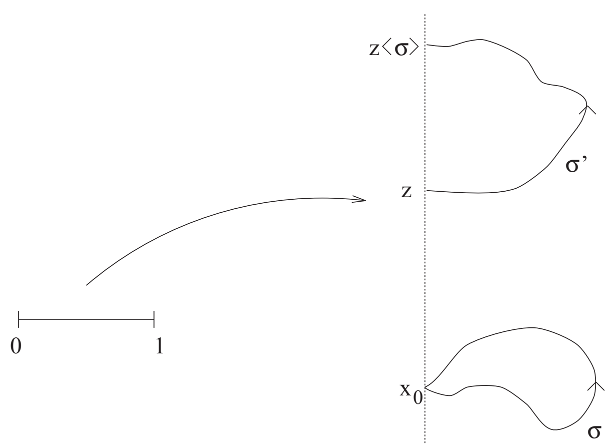


FIGURE 5.6. Action on the fibre

We must check that the action is well-defined, and that it satisfies (A1) and (A2).

THEOREM 5.10. *Let $p : (Z, z_0) \to (X, x_0)$ be a based covering, with Z path connected and $F = p^{-1}(x_0)$. Then the action of $\pi_1(X, x_0)$ on F induces a bijection*

$$\delta : p_*\pi_1(Z, z_0)\backslash\pi_1(X, x_0) \longrightarrow F$$

where $p_*\pi_1(Z, z_0)\backslash\pi_1(X, x_0)$ is the set of cosets of the form $p_*\pi_1(Z, z_0)\langle\sigma\rangle$, and the map is given by

$$\delta\left((p_*\pi_1(Z, z_0))\langle\sigma\rangle\right) = z_0\langle\sigma\rangle.$$

That is, $\delta[\langle\sigma\rangle] = \sigma'(1)$*, where* $p\sigma' = \sigma$ *and* $\sigma'(0) = z_0$*.*

Proof. Before going through the proof in your lecture notes, understand what needs to be checked.
(1) δ is well-defined: if $[\langle\sigma\rangle] = [\langle\tau\rangle]$ then $z_0\langle\sigma\rangle = z_0\langle\tau\rangle$.
(2) δ is onto: for any $z \in F$, there is a path σ' joining z_0 to z whose image under p is a loop based at x_0 (this is obvious).
(3) δ is injective: if $\delta[\langle\sigma\rangle] = \delta[\langle\tau\rangle]$ then $\langle\sigma\rangle = \langle\gamma\rangle\langle\tau\rangle$ for some $\langle\gamma\rangle \in p_*\pi_1(Z, z_0)$. $\square$

Corollary 5.11. *If* Z *is path connected and a universal cover (i.e.* $\pi_1(Z, z_0) = 0$*), then* δ *defines a bijection between* $\pi_1(X, x_0)$ *and* F*.*

The bijection δ now allows us to define the *degree* of a loop in the circle. Let $p : (\mathbb{R}, 0) \to (S^1, 1)$ be the covering map $t \mapsto e^{2\pi it}$. Then $\mathbb{R}$ is path-connected and a universal cover, the fibre is $p^{-1}(1) = \mathbb{Z}$. So the above result gives a bijection $\delta = \deg : \pi_1(S^1, 1) \to \mathbb{Z}$. This function is the 'degree' function, and it measure how often the path winds around the circle. Notice that at this point we do not know that δ is a group homomorphism.

Example 5.12. The map $\deg : \pi_1(S^1, 1) \to \mathbb{Z}$ is a group isomorphism.

All that remains to be checked is that $(\sigma\tau)'(1) = \sigma'(1) + \tau'(1)$.

Other fundamental groups can now be computed.

Example 5.13. Let $fr\{a, b\}$ be the free group on generators a and b, and let S^1_a, S^2_b be two circles. Write $S^1_a \vee S^1_b$ for the wedge of the two circles joined at x_0. Then there is an isomorphism from $fr\{a, b\}$ to $\pi_1(S^1_a \vee S^1_b, x_0)$.

Example 5.14. Recall that there is a double cover $p : S^n \to \mathbb{R}P^n$, and (for $n \geq 2$), this cover is universal. The open hemispheres $U_- = \{y \in S^n \mid x \cdot y < 0\}$ and $U_+ = \{y \in S^n \mid x \cdot y > 0\}$ are even sheets over the open neighbourhood of $p(x)$ given by $U = pU_- = pU_+$. The bijection given by the action on the fibre gives

$$\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}.$$

This is our first example of *torsion* in a fundamental group, and it turns out to be very important. An easy application of Example 5.14 is the *Borsuk–Ulam Theorem*.

Theorem 5.15. *There is no map* $f : S^2 \to S^1$ *with the property that* $f(-x) = -f(x)$ *for all* $x \in S^2$*.*

Proof. Suppose there is such a map. Then f induces a well-defined map $f' : \mathbb{R}P^2 \to \mathbb{R}P^1$, giving a commutative diagram

$$\begin{tikzcd} S^2 \arrow[r, "f"] \arrow[d, "p_2"'] & S^1 \arrow[d, "p_1"] \\ \mathbb{R}P^2 \arrow[r, "f'"] & \mathbb{R}P^1 \end{tikzcd}$$

where $f'(\{x, -x\}) = \{f(x), f(-x)\} = \{f(x), -f(x)\}$. Since the verticals are quotient maps, f' is clearly continuous. Now $\mathbb{R}P^1 = S^1$ (via the homeomorphism $\{z, z^{-1}\} \mapsto z^2 \in \mathbb{C}$), so f'_* is a homomorphism

$$f'_* : \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z} \longrightarrow \pi_1(\mathbb{R}P^1) \cong \mathbb{Z}.$$

It follows that $f'_* = 0$.

On the other hand, let $z_0 \in S^2$, and choose a path σ' in S^2 from z_0 to $-z_0$. Then $p_2\sigma' = \sigma$ is a *loop* in $\mathbb{R}P^2$ based at $x_0 = \{z_0, -z_0\}$.

We have the following diagram:

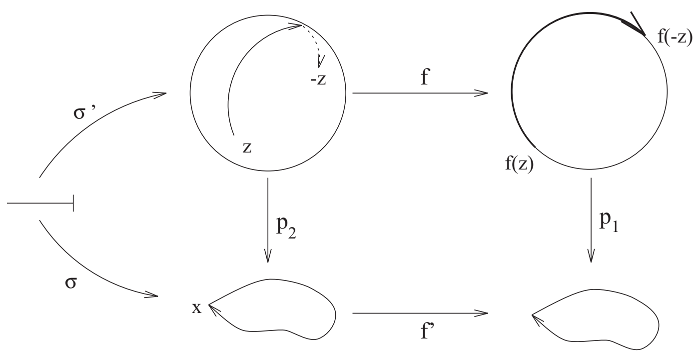


Figure 5.7. Borsuk–Ulam Theorem

Now $\langle\sigma\rangle$ is not trivial because $z_0\langle\sigma\rangle = \sigma'(1) = -z_0 \neq z_0$. Also, $f'_*\langle\sigma\rangle = \langle f'\sigma\rangle$ is not trivial since

$$\begin{aligned} f(z_0)\langle f'(\sigma)\rangle &= (f\sigma')(1) \\ &= f(-z_0) \\ &= -f(z_0) \neq f(z_0). \end{aligned}$$

It follows that $f'_*\langle\sigma\rangle$ is not trivial, contradicting the fact that f'_* is trivial. □

Corollary 5.16. *If $f : S^2 \to \mathbb{R}^2$ has the property that $-f(x) = f(-x)$, then there exists an x such that $f(x) = 0$.*

Corollary 5.17. *If $f : S^2 \to \mathbb{R}^2$ then there exists x such that $f(-x) = f(x)$.*

COROLLARY 5.18. *If the Earth's surface is represented by* S^2*, and*

$$f(x) = (\textit{temp. at } x, \textit{ humidity at } x),$$

then at any moment there are an antipodal pair of points with the same temperature and humidity.

COROLLARY 5.19. HAM AND CHEESE SANDWICH THEOREM (STONE–TUKEY) *Let* A, B, C *be open bounded sets in* $\mathbb{R}^3$*. Then there exists a plane* $P \subset \mathbb{R}^3$ *dividing each of them exactly in half.*

CHAPTER 6

Classification of surfaces

In this section we aim to classify all the topological spaces that have the property that each point has a neighbourhood homeomorphic to an open disc in $\mathbb{R}^2$. Of course $\mathbb{R}^2$ is such a space, but we shall restrict attention to *compact* spaces.

DEFINITION 6.1. A (compact) *surface* is a (compact) Hausdorff topological space X with the property that every point $x \in X$ has an open neighbourhood $U \ni x$ such that U is homeomorphic to an open disc in $\mathbb{R}^2$.

EXAMPLE 6.2. (1) The 2-sphere $S^2 \subset \mathbb{R}^3$ is a compact surface.
(2) The torus $T^2 \subset \mathbb{R}^3$ is a surface; we know that T^2 may be described as an identification space of a square – with an obvious notational device, the identification may be described by the symbol $aba^{-1}b^{-1}$.

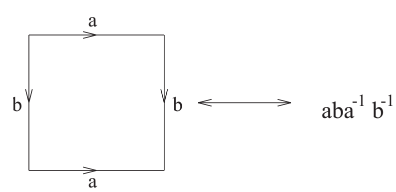


FIGURE 6.1. The torus as an identification space

(3) Now we have the symbol notation above, we can ask questions like the following: what surface is represented by the symbol $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}$? The following cut-and-paste argument shows that this is the surface of a two-holed torus.
(4) What surface is represented by the symbol $abca^{-1}cb^{-1}$?

We first approach the problem from the surface end – starting with a surface, is it possible to give a representation of the surface as an identification space of a polygon, and thereby as a symbol?

DEFINITION 6.3. A *triangulation* of a compact surface S is a finite family of closed subsets $\{T_1, \dots, T_n\}$ that cover S, and homeomorphisms $\phi_i : T_i' \to T_i$, where each T_i' is a triangle in $\mathbb{R}^2$, such that the sets T_i satisfy the following intersection condition:
(IC) Any sets T_i, T_j, $i \neq j$ are either disjoint, or they have a vertex in common, or they have an entire edge in common.

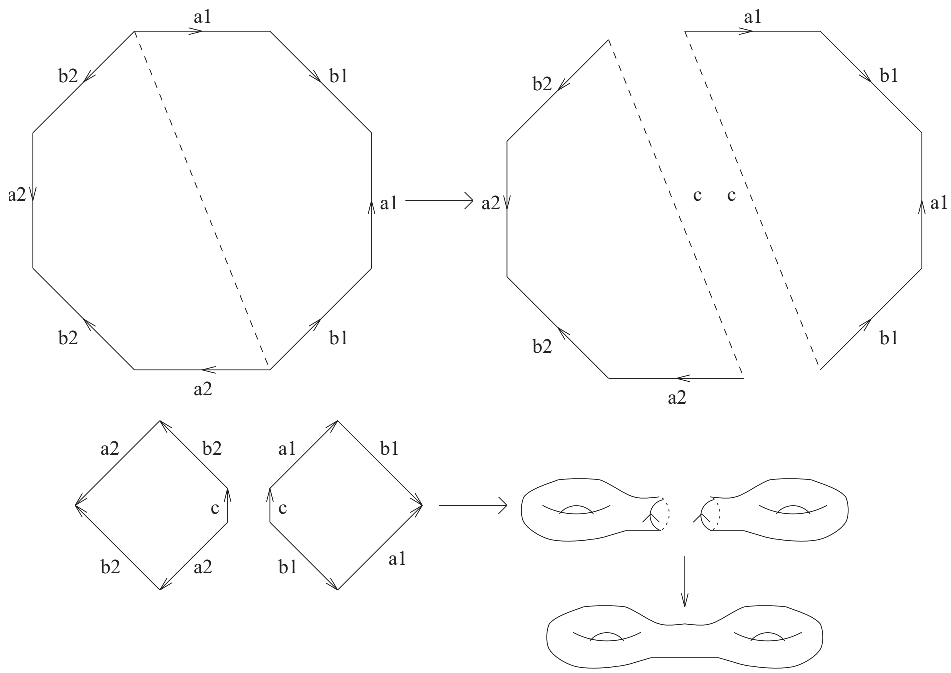


Figure 6.2. The torus with two holes

A result due to Rado in 1925 says that any compact surface has a triangulation. Moreover, in the triangulation each edge is an edge of exactly two triangles, and for each vertex v the triangles with v as a vertex may be arranged in cyclic order $T_0, T_1, \ldots, T_n$ such that T_i and T_{i+1} have an edge in common.

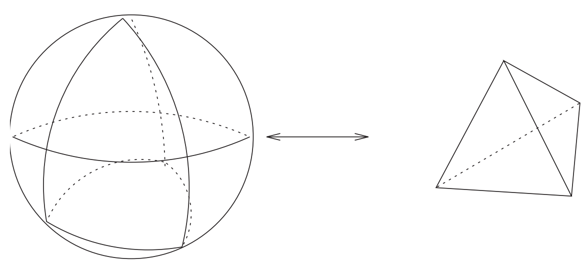


Figure 6.3. A triangulation of the 2-sphere

The triangulation of a surface gives a purely combinatorial redefinition of a surface. Let M be a collection of triangles with (IC). Call M *connected* if there is a path along the edges from any vertex to any other vertex. The edges opposite the vertex v in the triangles of M having v as a vertex form a graph called the *link* of v.

Definition 6.4. A (combinatorial) *closed compact surface* is a collection M of triangles such that

1. M has (IC),
2. M is connected, and
3. for every vertex v of a triangle in M, the link of v is a simple closed polygon.

For example, (1) below is a surface while (2) is not.

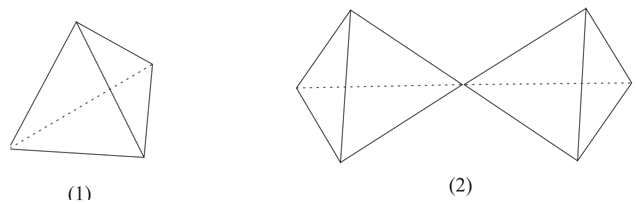


FIGURE 6.4. (1) is a surface while (2) is not

(3) The sphere can be triangulated, and then described as an identification space:

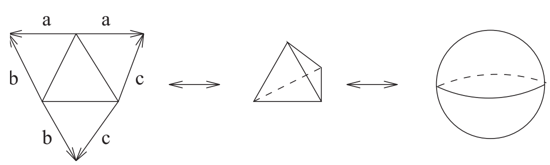


FIGURE 6.5. The combinatorial sphere

(4) Another example is a triangulation of the 2-torus: the link of the vertex v is drawn in bold.

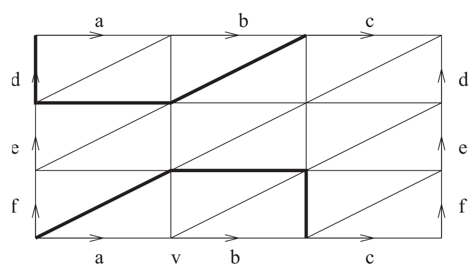


FIGURE 6.6. The torus as a combinatorial surface

(5) A triangulation of the Klein bottle.
(6) A triangulation of Projective space.
(7) If t^1 and t^2 are two triangles of a closed surface M, then it is possible to construct a sequence of triangles connecting t^1 and t^2, with consecutive triangles having an edge in common. (This is proved via the condition on links in the triangulation).
(8) The following diagram is not a closed surface: the link of the vertex v is not a simple closed polygon.

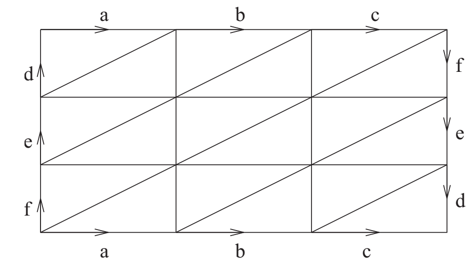


FIGURE 6.7. The Klein bottle

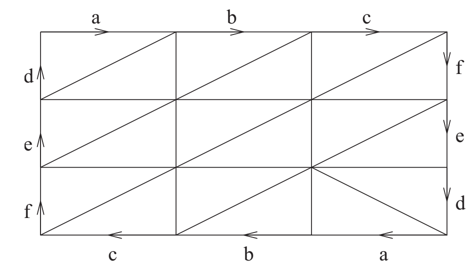


FIGURE 6.8. A triangulation of projective space

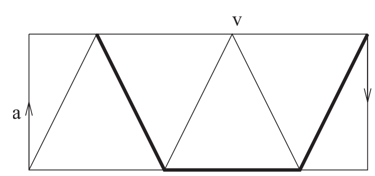


FIGURE 6.9. The Möbius band is not a closed surface

1. Orientation

In a combinatorial surface, introduce an orientation: a clockwise or anti-clockwise arrow in each triangle. The orientation is *coherent* across an edge joining two triangles if the orientations are the same:

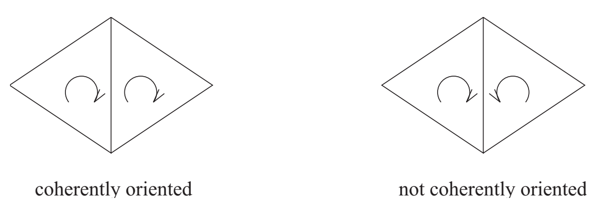


FIGURE 6.10. Coherent and incoherent orientations

DEFINITION 6.5. The surface M is *orientable* if the triangles can be given an orientation so that all neighbouring triangles are coherently oriented.

Example 6.6. (1) The torus is orientable.

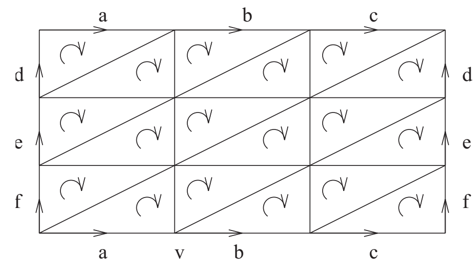


Figure 6.11. The torus is orientable

(2) The Klein bottle is non-orientable. The shaded region is a Möbius band – and the space left after removing this Möbius band is again a Möbius band. So K is 'twice' as non-orientable as the projective plane: in example (3) below we shall see that the projective plane also contains a Möbius band, but removing it leaves an orientable surface.

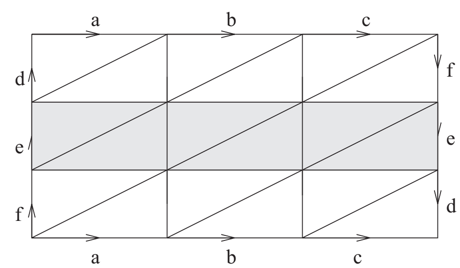


Figure 6.12. The Klein bottle is non-orientable

(3) The projective plane is non-orientable.

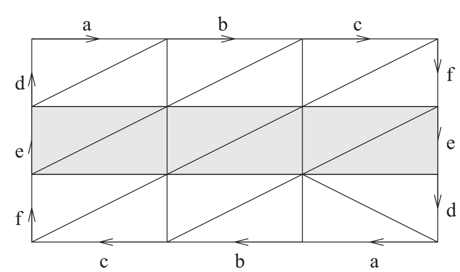


Figure 6.13. The projective plane is non-orientable

The shaded Möbius band, when removed, leaves something that certainly has no more Möbius bands in it.

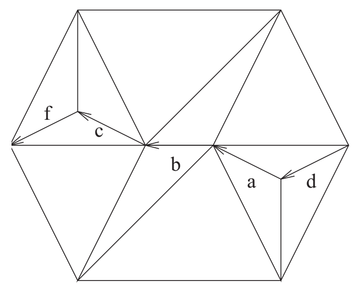


FIGURE 6.14. The projective plane contains only one Möbius band

2. Polygonal representation

Let M be a closed surface in the sense of Definition 6.4. Orient the edges $e^1, \dots, e^m$ of M arbitrarily, and then label the triangles in M as $t^1, \dots, t^n$. Notice that $3n = 2m$, and so n is even. The information contained in the surface is now n triangles, and for each triangle the three labelled oriented edges which belong to it.

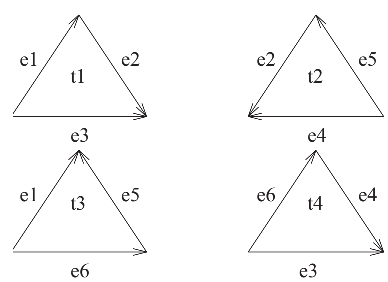


FIGURE 6.15. The triangles in the tetrahedron

It will be useful to have a more convenient representation, and this is done by partially assembling the surface while remaining in the plane.

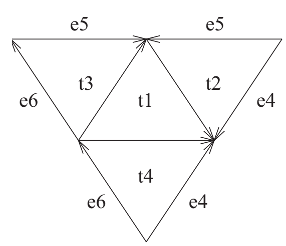


FIGURE 6.16. A model of the tetrahedron

The same procedure may be followed for any surface M. The triangles are assembled one at a time. At each stage, glue an edge of an unused triangle to an edge of a used triangle. The boundary at each stage of the resulting figure is a simple closed polygon. In order to keep the figure planar, triangles may need to be shrunk and squeezed a bit. Also, we may assume that the resulting polygon is convex.

However, we need to be sure that until all the triangles are used, there is an unused triangle with an edge in common with the boundary polygon of the used triangles.

There certainly exists an unused triangle with a *vertex* v in common with some used triangle, since the construction ensures that all vertices are on the boundary polygon at each stage.

Now each link of v is a simple closed polygon, so there is an unused triangle with v and another vertex (and hence an edge) in common with a used triangle. (The triangle you end up adding is not automatically the one you started with having a vertex in common.)

(Notice how this argument would break down for example (2) after Definition 6.4).

The resulting figure is a *polygonal representation of* M. It has n triangles, and the boundary has $n+2$ edges in equally labelled pairs. The identifications given by the edge labellings form a *symbol*: read around the figure, starting anywhere, and use $^{-1}$ to denote reverse directions. For the tetrahedron in Figure 6.16, the symbol is

$$e_5^{-1}e_4e_4^{-1}e_6^{-1}e_6e_5.$$

Notice that the triangles in a polygonal representation can always be oriented coherently, but this may not give a coherent orinetation to M: if the symbol contains $\dots a \dots a^{-1} \dots$ then the orientation will be coherent across the edge a; if the symbol contains $\dots b \dots b \dots$ then the orientation cannot be carried across the edge b coherently.

LEMMA 6.7. *The surface M is orientable if and only if for every letter in a symbol for M, the inverse also occurs.*

3. Transformation to standard form

In this section, we show how any symbol may be re-written in a standard way. This list of standard symbols, together with the Euler characteristic, gives the classification of surfaces.

We shall use 'word' to mean a string of letters like *abc*. Words will sometimes be denoted by capital letters X, Y and so on.

Along the way, we may need to increase the number of triangles: a *barycentric subdivision* of a triangulation replaces every triangle in it by six triangles:

RULE [1] If a word appears in the symbol consistently (for example, as $abc \dots abc$ or as $abc \dots c^{-1}b^{-1}a^{-1}$), then replace it with a letter. For example,

$$\dots abc \dots c^{-1}b^{-1}a^{-1} \dots \longrightarrow \dots x \dots x^{-1} \dots$$

RULE [2] If a symbol has at least 4 letters, then aa^{-1} can be cancelled.

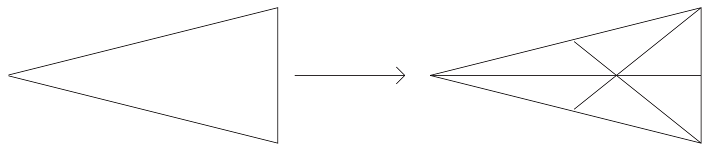


FIGURE 6.17. Barycentric subdivision

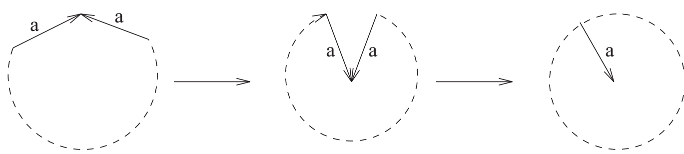


FIGURE 6.18. Cancelling aa^{-1}

RULE [3] If a pair appears in the form $\ldots a \ldots a \ldots$, it may be replaced by $\ldots bb \ldots$ as follows.

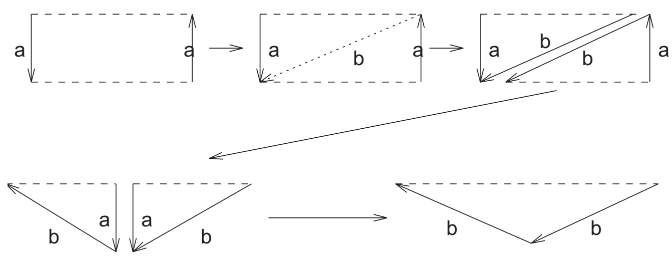


FIGURE 6.19. Pairs of the kind $\ldots a \ldots a \ldots$

Notice that part of the remainder of the symbol will have been reversed. Thus, the rule is more accurately represented as $XaYaZ \longrightarrow XbbY^{-1}Z$.

RULE [4] If a pair appears in the form $\ldots b \ldots b^{-1} \ldots$, and without an interlocking pair of the same kind (that is, without a pair $a \ldots a^{-1}$ appearing in the order $b \ldots a \ldots b^{-1} \ldots a^{-1}$), then it may be replaced by $\ldots b \ldots b \ldots$. By applying previous rules, we may assume that the pair appears as $\ldots bXddb^{-1} \ldots$ where $X = d_1d_1 \ldots d_nd_n$.

The diagram above shows that $\ldots bXddb^{-1} \cdots \longrightarrow \ldots bXe^{-1}be^{-1} \ldots$, and then by [3], $\ldots bXe^{-1}be^{-1} \cdots \longrightarrow \ldots bbeX^{-1}e^{-1} \ldots$. That is, we have reduced the original pair separated by

$$d_1d_1 \ldots d_nd_ndd$$

to a similar pair separated by

$$d_n^{-1}d_n^{-1} \ldots d_1^{-1}d_1^{-1}.$$

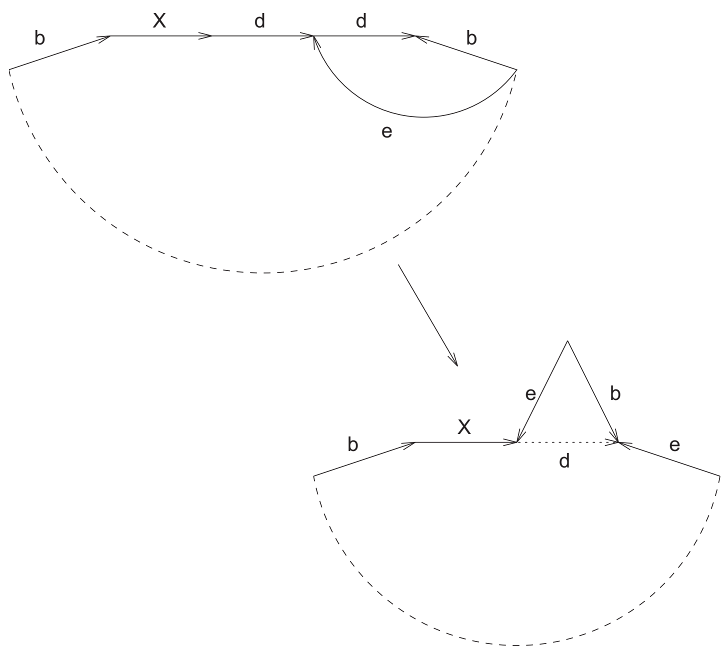


FIGURE 6.20. Pairs of the kind $\dots b \dots b^{-1} \dots$

After n such steps, the symbol has the form $\dots bddb^{-1} \dots \longrightarrow \dots be^{-1}be^{-1} \dots$ (by the same cut and paste with $X = \emptyset$), and then $\dots be^{-1}be^{-1} \dots \longrightarrow \dots ff \dots$ by [1], with $f = be^{-1}$.

Notice that the existence of such a pair $\dots b \dots b^{-1} \dots$ means that the vertices at the start and end of the edge b^{-1} are not identified, and our reduction has produced a situation in which the vertices are identified.

RULE [5] Assume we have carried out [3] and [4] enough to produce a symbol comprising pairs of the form aa and interlocking pairs of the form

$$\dots b \dots c \dots b^{-1} \dots c^{-1} \dots .$$

I claim first that the pairs aa can all be grouped at the start of the symbol. To see this, notice that the following diagram shows that $aaXbbY \longrightarrow aadX^{-1}dY$, which by [3] gives $aaddXY$.

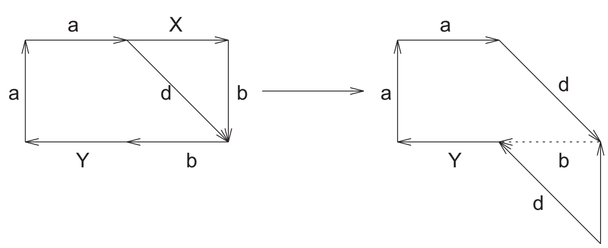


FIGURE 6.21. Assembling crosscaps

RULE [6] After applying the above rules, the symbol has the form

$$a_1a_1 \ldots a_pa_pX$$

and with all vertices identified. The word X (if it is not empty) must consist of interlocking pairs. Choose the closest interlocking pairs $\ldots a \ldots b \ldots a^{-1} \ldots b^{-1} \ldots$, and replace them with $\ldots cdc^{-1}d^{-1} \ldots$ by the following argument.

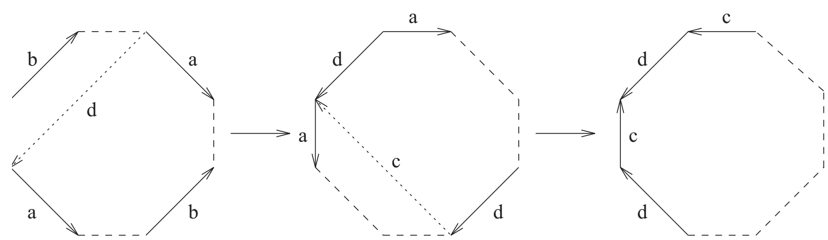


FIGURE 6.22. Pairs of the kind $\ldots a \ldots a \ldots$

Notice that no new pairs of the form $a \ldots a$ are produced, because no section of the figure is turned over before glueing.

By applying the above rules, we may reduce any symbol to one of the form

$$a_1a_1 \ldots a_pa_pc_1d_1c_1^{-1}d_1^{-1} \ldots c_qd_qc_q^{-1}d_q^{-1},$$

in which all vertices are identified. The words of the form $cdc^{-1}d^{-1}$ are called *handles* (we'll see why shortly), and the pairs of the form aa are called *crosscaps*.

RULE [7] If there is at least one pair aa, then every handle $cdc^{-1}d^{-1}$ may be turned into two crosscaps.

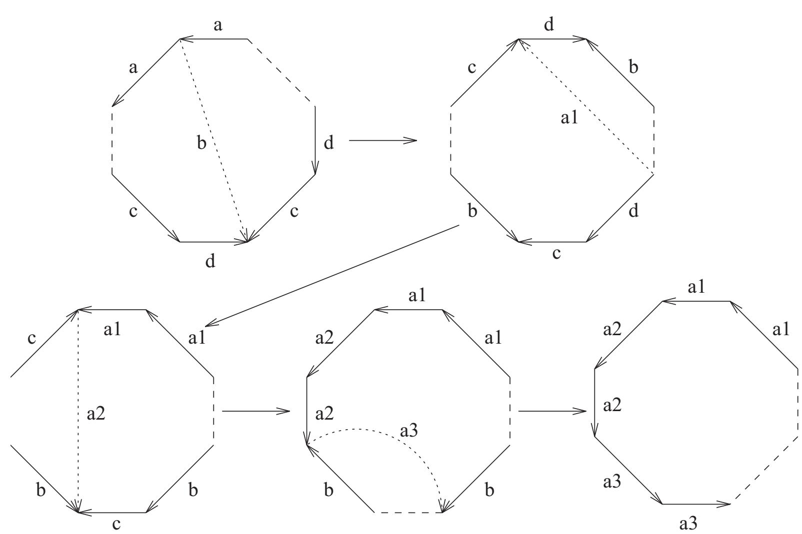


FIGURE 6.23. Turning a handle into two crosscaps

The above discussion has proved the following theorem.

THEOREM 6.8. *Every closed surface may be represented by one of the following:*
(1) aa^{-1} *(the sphere; orientable), (2)* $a_1a_1 \dots a_ka_k$ *(the sphere with* k *cross-caps; non-orientable), or (3)* $c_1d_1c_1^{-1}d_1^{-1} \dots c_qd_qc_q^{-1}d_q^{-1}$ *(the sphere with* q *handles; orientable).*

The number k or q is called the *genus* of the surface. Any surface with standard form aa^{-1} is a *triangulation* of the sphere (and so on for the other types).

4. Juxtaposition of symbols

Let M and N be closed surfaces with symbols X and Y. What surface has the symbol XY? This is not uniquely determined, but we shall see that the standard form of XY is uniquely determined by X and Y.

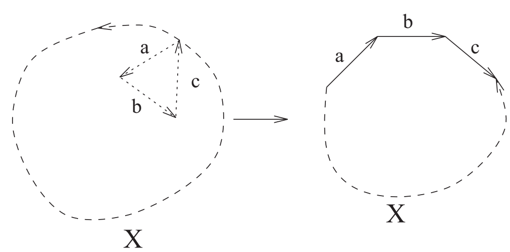


FIGURE 6.24. Opening the triangle

First choose a triangle in M meeting the boundary polygon of the plane representation with symbol X, and open out along the edge of the triangle as shown in Figure 6.24 (perform a barycentric subdivision first if need be). Do the same thing in Y,

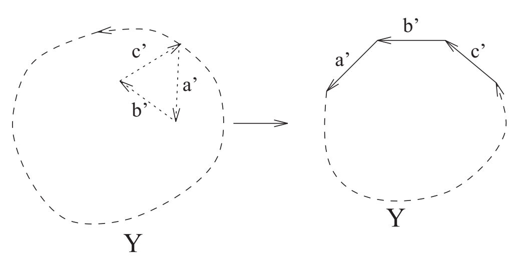


FIGURE 6.25. Opening the triangle in Y

and then glue to obtain a polygon with symbol XY.
The result is a closed surface, orientable if and only if both M and N are.

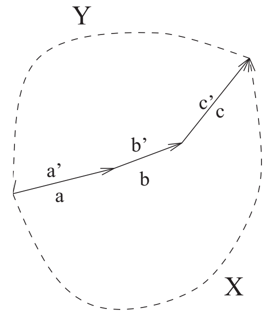


FIGURE 6.26. The 'sum' of X and Y

LEMMA 6.9. *By Theorem 6.8, any two connected sums of M and N have the same standard from, so we may define $M\sharp N$ to be the connected sum of M and N, unique up to homeomorphism.*

Notice that some of the 'up to homeomorphisms' may be non-trivial. For example, the following diagram gives two possible connected sums of tori.

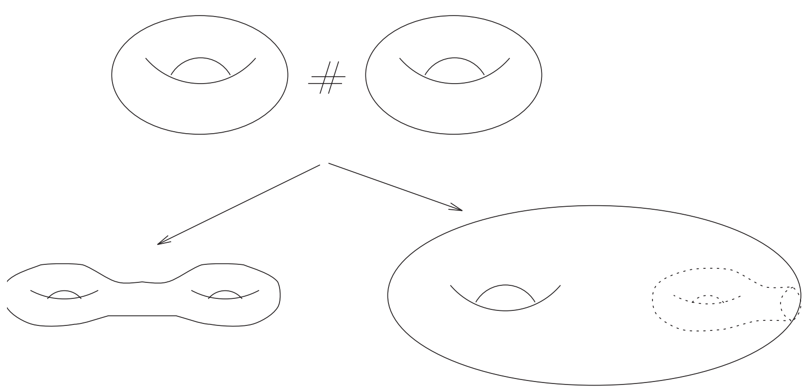


FIGURE 6.27. The connected sum of two tori

For multiple connected sums, write $kN = N\sharp N\sharp\ldots\sharp N$. Also, write S for the sphere, P for the projective plane and T for the torus. This notation allows Theorem 6.8 to be restated as: any closed surface is homeomorphic to one of S, kP, or kT.

EXAMPLE 6.10. (1) If M is any closed surface, then $M\sharp S \cong S\sharp M \cong M$.
(2) $P\sharp 2P \cong P\sharp T$.
(3) $2P$ can be replaced by T if there is at least one P, so Theorem 6.8 may be rewritten as follows: every closed surface is homeomorphic to one of S, hT, $P\sharp\left(\frac{k-1}{2}\right)T$ (k odd) or $K\sharp\left(\frac{k-2}{2}\right)T$ ($k \geq 2$ even), where $K = 2P$ is a Klein bottle.

5. Euler characteristic

Let K be a set of triangles with the intersection condition (IC). Let $\alpha_0(K)$ denote the number of vertices in K, $\alpha_1(K)$ the number of edges, and $\alpha_2(K)$ the number of triangles.

DEFINITION 6.11. The number

$$\chi(K) = \alpha_(K) - \alpha_1(K) + \alpha_2(K)$$

is the *Euler characteristic* of K.

We wish to show that the Euler characteristic is preserved under the reduction rules and under barycentric subdivision.

LEMMA 6.12. *The Euler characteristic is preserved by the symbol reductions and by barycentric subdivision.*

PROOF. The first assertion has a long proof – simply check all the cases. To start you off, notice that in rule [2] (cancellation of $\dots aa^{-1}\dots$ we lose 1 edge and 1 vertex, preserving χ. The other rules are similar.

Under barycentric subdivision, let K' be the combinatorial surface obtained from K. Then $\alpha_0(K') = \alpha_0(K) + \alpha_1(K) + \alpha_2(K)$ (one new vertex for each edge and each triangle in K), $\alpha_1(K') = 2\alpha_1(K) + 6\alpha_2(K)$ (six new edges in each triangle of k and each edge of k split into two), and $\alpha_2(K') = 6\alpha_2(K)$. It follows that

$$\begin{aligned}\chi(K') &= \alpha_0(K') - \alpha_1(K') + \alpha_2(K') \\ &= \alpha_0(K) + \alpha_1(K) + \alpha_2(K) - 2\alpha_1(K) - 6\alpha_2(K) + 6\alpha_2(K) \\ &= \alpha_0(K) - \alpha_1(K) + \alpha_2(K) = \chi(K).\end{aligned}$$

□

(Notice we need barycentric subdivision on things like the sumbol aa^{-1} to really produce a combinatorial surface).

THEOREM 6.13. *Let M be a closed surface, with a symbol containing n letters (n is even), and represented by a plane polygon bounded by a simple closed polygon with $n + r$ sides (notice that a letter may stand for several edges). Let M have m distinct vertices appearing at the start or end of the letters. Then*

$$\chi(M) = m - \tfrac{1}{2}n + 1.$$

This is proved by a simple counting argument. It may be used to show that the Euler characteristic is a topological invariant, which shows in turn that the list of surfaces in Theorem 6.8 are all distinct.

PROOF. (of Theorem 6.13) Let D denote the polygonal region representing M but without any edges identified. By a simple induction, we have that

$$\chi(D) = 1.$$

On the other hand, each side of M that does not use a new letter appears twice and generates one new vertex, so

$$\alpha_0(M) - \alpha_0(D) = m + \tfrac{1}{2}r - (n+r) = m - n - \tfrac{1}{2}r.$$

The edges of M are glued in pairs, so that

$$\alpha_1(M) - \alpha_1(D) = \tfrac{1}{2}(-n-r).$$

Finally there are equal numbers of triangles, so

$$\alpha_2(M) = \alpha_2(D).$$

Adding up we get

$$\chi(M) - \chi(D) = m - \tfrac{1}{2}n,$$

so $\chi(M) = m - \frac{1}{2}n + 1$. □

To deduce that any combinatorial surface representing a given surface has the same Euler characteristic a further argument is needed (see below).

Corollary 6.14. *(1)* $\chi(S) = 2$*;*
(2) $\chi($*sphere with* k *cross-caps*$) = 2 - k$*;*
(3) $\chi($*sphere with* h *handles*$) = 2 - 2h$*.*

So the standard form of any closed surface M is unique, and it is determined by $\chi(M)$ and the orientability of M.

Example 6.15. (1) $abcbca = 2P$.
(2) $abca^{-1}cb^{-1} = 3P$.
(3) $abcdefe^{-1}db^{-1}afc = 6P$.
(4) $ae^{-1}a^{-1}bdb^{-1}ced^{-1}c^{-1} = 2T$.

6. Invariance of the characteristic

It remains to prove that different triangulations of the same surface cannot give rise to different Euler characteristics.

Solution 1: Consider subdivisions of the surface into polygons (not just triangles). Notice that the characteristic is unchanged by (a) subdividing edges, or if only two edges meet at a vertex, by removing that vertex. (b) subdividing an n-gon by connecting two of the vertices by a new edge, or amalgamtaing two regions into one by removing an edge). (c) introducing a new edged and vertex running into a region from a vertex on the edge, or removing such an edge.

Now IF two triangulations of the same surface have the property that any edge from the first triangulation meets any edge from the second in only finitely many points, then it is not hard to see that the above moves preserve the characteristic and allow us to move from the first triangulation to the second.

The problem is that there may be edges intersecting each other infinitely often... This can be avoided by moving one of the edges slightly without altering the combinatorics. However that proof is long - it does not need new ideas but it is not easy.

Solution 2: If we develop a little more theory about how to compute fundamental groups (mainly the Seifert-van Kampen theorem) then you can compute the fundamental group of each standard model of a surface. Some of these groups are large and complicated, but their abelianizations can be computed and are all different.

Solution 3: With the machinery we develop later, it is possible to compute the first homology groups of each surface, and these are all different. The problem with this is that we only know that the homology group depends only on the surface because of an unproven theorem!

CHAPTER 7

Simplicial complexes and Homology groups

For now, we shall deal mainly with dimensions $0, 1, 2$ and 3. An implicit exercise throughout this section is to generalize all the definitions and proofs to higher dimensions.

DEFINITION 7.1. An *oriented 0-simplex* is a point P.
An *oriented 1-simplex* is a directed line segment P_1P_2.
An *oriented 2-simplex* is a triangle $P_1P_2P_3$ with a prescribed order.

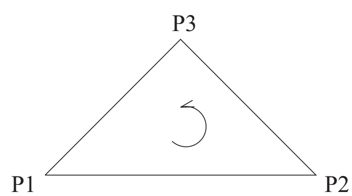


FIGURE 7.1. An oriented 2-simplex

Notice that the simplex of opposite orientation is (defined to be) the negative of the simplex:

1. $P_1P_2 = -P_2P_1 \neq P_2P_1$,
2. $P_1P_2P_3 = P_2P_3P_1 = P_3P_1P_2 = -P_1P_3P_2 = -P_3P_2P_1 = -P_2P_1P_3$.

An *oriented 3-simplex* is a tetrahedron $P_1P_2P_3P_4$ with a prescribed orientation.

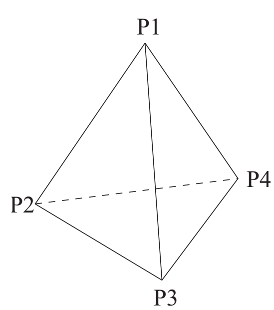


FIGURE 7.2. An oriented 3-simplex

Notice that for 2-simplexes,

$$P_iP_jP_k = \operatorname{sign}\begin{bmatrix} 1 & 2 & 3 \\ i & j & k \end{bmatrix} P_1P_2P_3,$$

and this extends to higher dimensions,

$$P_iP_jP_kP_\ell = \operatorname{sign}\begin{bmatrix}1 & 2 & 3 & 4\\ i & j & k & \ell\end{bmatrix} P_1P_2P_3P_4.$$

DEFINITION 7.2. The *boundary* of an oriented simplex is defined as follows.
The boundary of a 0-simplex P is 0:

$$\partial_0(P) = 0.$$

The boundary of a 1-simplex P_1P_2 is

$$\partial_1(P_1P_2) = P_2 - P_1,$$

(the formal difference of the end point and the starting point of the simplex).
The boundary of a 2-simplex $P_1P_2P_3$ is

$$\partial_2(P_1P_2P_3) = P_2P_3 - P_1P_3 + P_1P_2.$$

The boundary of a 3-simplex $P_1P_2P_3P_4$ is

$$\partial_3(P_1P_2P_3P_4) = P_2P_3P_4 - P_1P_3P_4 + P_1P_2P_4 - P_1P_2P_3.$$

Each summand of the boundary is called a *face* of the simplex. Thus, (for the 3-simplex in Figure 7.2), $P_2P_3P_4$, $P_1P_4P_3 = -P_1P_3P_4$ are faces, while $P_1P_3P_4$ is not.

DEFINITION 7.3. A *simplicial complex* is a subset of $\mathbb{R}^n$ comprising a union of simplexes such that
(1) each point of the set belongs to at least one and only finitely many simplexes;
(2) two different simplexes in the complex either are disjoint or one is (up to orientation) a face of the other, or a face of a face of the other (and so on), or the set of points in common is (up to orientation) a face (or a face of a face or...) of each simplex.

1. Chains, cycles and boundaries

The surface of the oriented tetrahedron is an example of a simplicial complex, built out of four 0-simplexes, six 1-simplexes, and four 2-simplexes.

For a simplicial complex K, let $C_n(K)$ be the free abelian group generated by the oriented n-simplexes in K. The elements of $C_n(K)$ are the *n-chains of K*.

For example, if K is the complex in Figure 7.3, then (write m_i for general elements of $\mathbb{Z}$):

$$C_2(K) = \{m_1P_2P_3P_4 + m_2P_1P_3P_4 + m_3P_1P_2P_4 + m_4P_1P_2P_3\},$$

$$C_1(K) = \{m_1P_1P_2+m_2P_1P_3+m_3P_1P_4+m_4P_2P_3+m_5P_2P_4+m_6P_3P_4\},$$

and

$$C_0(K) = \{m_1P_1 + m_2P_2 + m_3P_3 + m_4P_4\}.$$

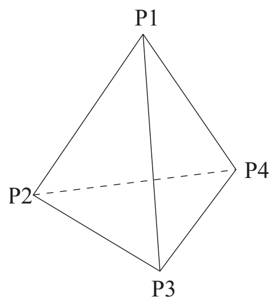


FIGURE 7.3. The tetrahedron is a simplicial complex

Now if σ is an n-simplex in K, then $\partial_n(\sigma)$ is an element of $C_{n-1}(K)$. If we extend the definition to make $C_{-1}(K) = 0$, then ∂_n extends to the *boundary homomorphism*

$$\partial_n : C_n(K) \longrightarrow C_{n-1}(K).$$

EXAMPLE 7.4. A sample calculation of the boundary of a 1-chain:

$$\begin{aligned}\partial_1(3P_1P_2 - 4P_1P_3 + 5P_2P_4) &= 3\partial_1(P_1P_2) - 4\partial_1(P_1P_3) + 5\partial_1(P_2P_4)\\ &= 3(P_2 - P_1) - 4(P_3 - P_1) + 5(P_4 - P_2)\\ &= P_1 - 2P_2 - 4P_3 + 5P_4.\end{aligned}$$

DEFINITION 7.5. The group of *n-cycles* in K, is defined to be $Z_n(K) = \ker(\partial_n)$.

EXAMPLE 7.6. The 1-chain $z = P_1P_2 + P_2P_3 + P_3P_1$ is a 1-cycle since $\partial_1(z) = 0$, so $z \in Z_1(K)$.

DEFINITION 7.7. The group of $(n-1)$-*boundaries* in K is defined by $B_{n-1}(K) = \text{image}(\partial_n)$.

EXAMPLE 7.8. We know that $w = P_1P_2 + 2P_2P_3 + P_3P_1$ is a 1-chain, so $\partial_1(w) = P_3 - P_2$ is a 0-boundary.

EXAMPLE 7.9. Compute directly the groups $Z_n(K)$ and $B_n(K)$ where K is the simplicial complex shown in Figure 7.3.
Since the highest-dimensional simplex in K is 2-dimensional, $C_3(K) = 0$, so $B_2(K) = \partial_3(C_3(K)) = 0$.
Also, $C_{-1}(K) = 0$, so $Z_0(K) = C_0(K)$. Therefore $Z_0(K)$ is free abelian on the generators P_1, P_2, P_3, P_4.
Now the image of a group under a homomorphism is generated by the images of the generators. Since $C_1(K)$ is generated by

$$P_1P_2, P_1P_3, P_1P_4, P_2P_3, P_2P_4, P_3P_4,$$

$B_0(K)$ must be generated by

$$P_2 - P_1, P_3 - P_1, P_4 - P_1, P_3 - P_2, P_4 - P_2, P_4 - P_3.$$

Of course there is no reason to expect these to *freely* generate $B_0(K)$, and it is clear that they do not. We claim that $B_0(K)$ is free abelian on the generators $P_2 - P_1, P_3 - P_1, P_4 - P_1$. (This is easy: these are independent over $\mathbb{Z}$, and the other generators are in the group generated by these three).

Now let's try to find $Z_1(K)$. Any element of $C_1(K)$ is of the form $c = \sum m_{ij} P_i P_j$. Now $\partial_1(c) = 0$ if and only if each vertex at the beginning of r edges (counted with multiplicity) is also at the end of r edges (again counted with multiplicity). It follows that

$$\begin{aligned} z_1 &= P_2P_3 + P_3P_4 + P_4P_2, \\ z_2 &= P_1P_4 + P_4P_3 + P_3P_1, \\ z_3 &= P_1P_2 + P_2P_4 + P_4P_1, \\ z_4 &= P_1P_3 + P_3P_2 + P_2P_1 \end{aligned}$$

are all 1-cycles (notice that these are the boundaries of the individual 2-simplexes). We claim that these four cycles generate $Z_1(K)$: to prove this we must do some work as Z_1 is defined not as the image under a homomorphism but as the *kernel* of a homomorphism. So, choose an arbitrary element $z \in Z_1(K)$. Consider the vertex P_1, and let the coefficient of P_1P_j in $Z_(K)$ be m_j. Then

$$z' = z + m_2 z_4 - m_4 z_2$$

is a cycle that does not contain edges P_1P_2 or P_1P_4. It follows that the only edge having P_1 as a vertex in the cycle z' is P_1P_3, so this must appear with coefficient 0 (otherwise there would be nothing to cancel the P_1 appearing in the boundary). So z' is a cycle consisting only of the edges of the 2-simplex $P_2P_3P_4$. Since in a 1-cycle each of the vertices P_2, P_3 and P_4 must appear the same number of times as a beginning and as an end of an edge,

$$z' = z + m_2 z_4 - m_4 z_2 = r z_1$$

for some r. So $Z_1(K)$ is generated by z_1, z_2, z_4. Since these are boundaries of the 2-simplex,

$$Z_1(K) = B_1(K).$$

Finally, let us compute $Z_2(K)$. The chain group $C_2(K)$ is generated by the simplexes $P_2P_3P_4, P_3P_1P_4, P_1P_2P_4, P_2P_1P_3$. If $P_2P_3P_4$ has coefficient r_1, and $P_3P_1P_4$ has coefficient r_2 in a 2-cycle, then the common edge P_3P_4 has coefficient $r_1 - r_2$, so $r_1 = r_2$. Similarly, in a cycle each of the 2-simplexes appear with the same coefficient, so $Z_2(K)$ is generated by the single element

$$P_2P_3P_4 + P_3P_1P_4 + P_1P_2P_4 + P_2P_1P_3.$$

It follows that $Z_2(K)$ is infinite cyclic.

Notice that in the above example we find that B_n is a subgroup of Z_n. This is true in general, because of a simple but deep equation dealt with in the next section.

2. The equation $\partial^2 = 0$

THEOREM 7.10. *Let K be a simplicial complex. Then the homomorphism*

$$\partial_{n-1}\partial_n : C_n(K) \to C_{n-2}(K)$$

is trivial. That is, '$\partial^2 = 0$'.

PROOF. First show that it is enough to check this on simplexes, since these generate C_n. Then simply calculate: for example, in dimension 2 we have

$$\begin{aligned}\partial_1\left(\partial_2(P_1P_2P_3)\right) &= \partial_1\left(P_2P_3 - P_1P_3 + P_1P_2\right)\\ &= (P_3 - P_2) - (P_3 - P_1) + (P_2 - P_1)\\ &= 0.\end{aligned}$$

□

COROLLARY 7.11. $B_n(K) = \partial_{n+1}(C_{n+1}(K))$ *is a subgroup of* $Z_n(K) = \ker(\partial_n)$.

DEFINITION 7.12. The nth *homology group* of the simplicial complex K is defined to be the quotient group

$$H_n(K) = \frac{Z_n(K)}{B_n(K)}.$$

Look back at Example 7.9: we may now write down the homology groups of the tetrahedron:

$C_3(K) = 0$, so $Z_3(K) = B_3(K) = 0$ and $H_3(K) = 0$.

$Z_2(K) \cong \mathbb{Z}$, $B_2(K) = 0$ so $H_2(K) \cong \mathbb{Z}$.

$Z_1(K) = B_1(K)$, so $H_1(K) = 0$.

$Z_0(K)$ is free abelian on P_1, P_2, P_3, P_4, while $B_0(K)$ is generated by $P_2 - P_1, P_3 - P_1, P_4 - P_1, P_3 - P_2, P_4 - P_2, P_4 - P_3$ We now claim that each coset of $B_0(K)$ in $Z_0(K)$ contains exactly one term of the form rP_1. To prove this, let $z \in Z_0(K)$ be of the form

$$z = s_2P_2 + s_3P_3 + s_4P_4 + *P_1,$$

so that

$$z - [s_2(P_2 - P_1) + s_3(P_3 - P_1) + s_4(P_4 - P_1)] = rP_1,$$

showing that $z \in rP_1 + B_0(K)$. (At this point we have proved that $H_0(K)$ is cyclic: the next thing to find out is whether or not it has torsion). If $r'P_1 \in rP_1 + B_0(K)$, then $(r' - r)P_1 \in B_0(K)$, so $r' = r$. It follows that $H_0(K) \cong \mathbb{Z}$.

CHAPTER 8

More homology calculations

In this section we go through a few more homology calculations, and find spaces with torsion in their homology groups. We also interpret what the homology groups are saying about the 'shape' of the simplicial complex.

First, we state the basic result that shows how homology groups are related to the topology of the simplicial complex.

THEOREM 8.1. *If X and Y are homeomorphic topological spaces with the property that they are homeomorphic to some simplicial complex, then they may be triangulated to form simplicial complexes K and L (homeomorphic to X and Y respectively, and $H_*(K) \cong H_*(L)$.*

Notice the notation H_* is shorthand for H_n for all n. In light of Theorem 8.1, we may now talk about the homology groups of a topological space.

EXAMPLE 8.2. If S is the 2-sphere, then $H_0(S) \cong \mathbb{Z}$, $H_1(S) = 0$, $H_2(S) \cong \mathbb{Z}$, and $H_{n\geq 3}(S) = 0$.

Recall that $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ is the n-sphere, and $B^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ is the n-ball.

Also notice that for a simplicial complex, connected implies path-connected. We shall therefore use the term 'connected' to mean both these properties.

THEOREM 8.3. *Let K be a simplicial complex. Then $H_0(K) \cong \mathbb{Z}^m$, where m is the number of connected components of K.*

PROOF. $C_0(K)$ is free abelian on the finite number of vertices P_i in K, and $B_0(K)$ is generated by the edges $P_{i_2} - P_{i_1}$, where $P_{i_1}P_{i_2}$ is an edge in K. Fix a vertex P_{i_1}. Then any P_{i_r} in the same connected component of K as P_{i_1} can be joined to P_{i_r} by a path of edges

$$P_{i_1}P_{i_2}, P_{i_2}P_{i_3}, \ldots, P_{i_{r-1}}P_{i_r},$$

so

$$P_{i_r} = P_{i_1} + (P_{i_2} - P_{i_1}) + \cdots + (P_{i_r} - P_{i_{r-1}})$$

which shows that $P_{i_r} \in P_{i_1} + B_0(K)$.

On the other hand, if P_{i_s} is *not* in the same component as P_{i_1}, then by reversing the above argument we have that $P_{i_s} \notin P_{i_1} + B_0(K)$. To see this, assume that $P_{i_s} \in P_{i_1} + B_0(K)$. Then $P_{i_s} - P_{i_1}$ is an integer combination of boundaries of 1-simplexes in K. In this expression, find if possible a boundary $(P_{i_2} - P_{i_1})$ with P_{i_2} distinct from P_{i_s} and P_{i_1}. Subtract $(P_{i_2} - P_{i_1})$ from both sides to see that $P_{i_s} - P_{i_2}$ is an integer combination of boundaries of 1-simplexes in K. Continue for finitely many steps: the process stops only when the next vertex found is P_{i_s}, at which point the list of boundaries obtained gives a path from P_{i_1} to P_{i_s}. □

THEOREM 8.4. *If K is a collapsible simplicial complex, then $H_n(K) = 0$ for all $n \geq 1$, and $H_0(K) \cong \mathbb{Z}$.*

PROOF. 'Collapsible' will be defined in the lectures, and is illustrated by the allowed collapsings below (these are not collapsible, but the moves indicated are collapses):

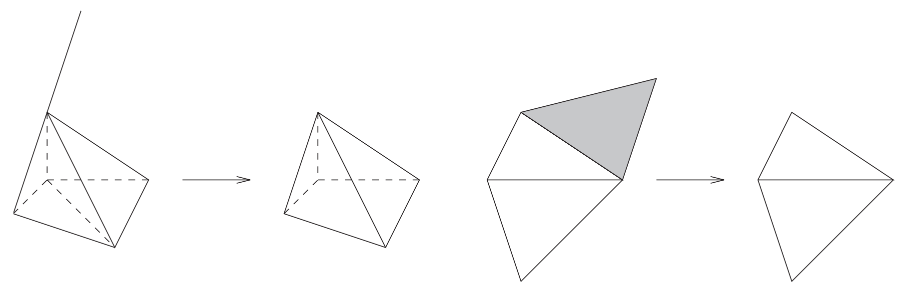


FIGURE 8.1. Collapsing a simplicial complex

'Collapsible' then means that the simplicial complex may be collapsed to a point. The proof is completed by showing that collapsing does not change the homology groups. □

COROLLARY 8.5. *Any triangulation of S^n is not collapsible, but any triangulation of B^n is collapsible.*

1. Geometrical interpretation of homology

We have seen already that $H_0(K)$ is given by $\mathbb{Z}^m$, where m is the number of connected components in K.

The 1-cycles are generated by closed curves along edges of K.

The 2-cycles are generated by closed 2-dimensional surfaces in K (and so on...).

Now $H_1(K) = \frac{Z_1(K)}{B_1(K)}$ amounts roughly to counting closed curves that appear in K which are not there simply because they are boundaries of 2-dimensional pieces.

$H_2(K)$ counts the closed 2-dimensional surfaces in the space which are not boundaries of 3-simplexes, and so on.

Thus, the homology groups are in some sense counting the 'holes' in higher dimensions. We saw already that the fundamental group

π_1 detects the presence of 2-dimensional holes well, but fails to detect 3-dimensional holes (like the inside of a 2-sphere). However, the homology group of the right dimension will detect the hole.

For example, $H_1(S^2) = 0$ since any closed surface on S^2 bounds a 2-dimensional piece of the sphere. On the other hand, $H_2(S^2) \neq 0$ since the 2-dimensional surface S^2 is not the boundary of anything in the sphere.

EXAMPLE 8.6. We expect $H_1(S^1)$ to be isomorphic to $\mathbb{Z}$ by the above argument. Let's prove this.

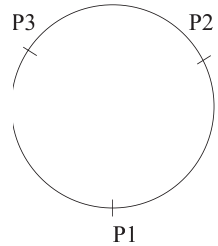


FIGURE 8.2. A triangulation of the circle

Let K be the indicated triangulation of the circle. Then $C_1(K)$ is generated by P_1P_2, P_2P_3, and P_3P_1.

If a 1-chain is a cycle, then it must contain P_1P_2 and P_2P_3 the same number of times (that is, with the same coefficient), since the boundary cannot contain a non-zero multiple of P_2. A similar argument works for any pair of edges, so $Z_1(K)$ is generated by the cycle

$$P_1P_2 + P_2P_3 + P_3P_1.$$

$B_1(K) = \partial_2(C_2(K)) = 0$ (there are no 2-simplexes in K), so $H_1(K) \cong \mathbb{Z}$.

In fact a higher-dimensional argument shows that $H_n(S^n) \cong H_0(S^n) \cong \mathbb{Z}$ for all n, while $H_{j \neq n,0}(S^n) = 0$.

DEFINITION 8.7. The elements of $H_n(K)$ (i.e. the cosets of $B_n(K)$ in $Z_n(K)$) are called *homology classes*. Cycles in the same homology class are called *homologous*.

EXAMPLE 8.8. The annulus (or the cylinder). Triangulate the annulus as shown in Figure 8.3.

$H_0(K) \cong \mathbb{Z}$ since K is connected.

Let $z = rP_1P_2 + \dots$ be a 1-cycle. Then $z - r\partial_2(P_1P_2Q_1)$ is a cycle homologous to z and without P_1P_2. Continue in the same way to find a cycle homologous to z containing no edge on the inner circle. Now subtract multiples of $\partial_2(Q_iP_iQ_j)$ to get a cycle z' homologous to z with no terms Q_iP_i either. Now if the edge Q_5P_1 appears in z' then P_1 would

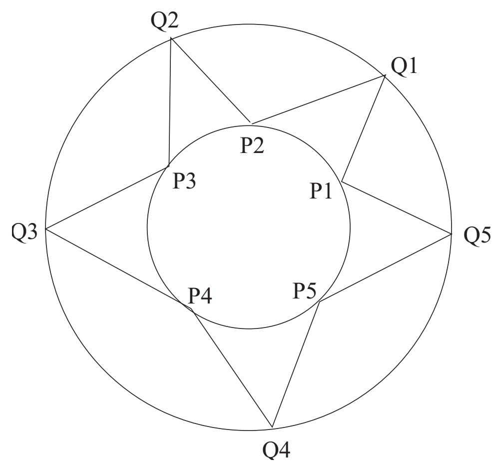


FIGURE 8.3. A triangulation of the annulus

appear in $\partial_1(z')$ which is impossible (and similarly for the other edges going from the outer to the inner circle). So z is homologous to a cycle made up of edges along the outer circle only. By a familiar argument, it follows that z is homologous to a cycle of the form

$$n(Q_1Q_2 + Q_2Q_3 + Q_3Q_4 + Q_4Q_5 + Q_5Q_1).$$

It follows that $H_1(K) \cong \mathbb{Z}$.

$H_2(K) = 0$ since $Z_2(K) = 0$ (any 2-simplex has in its boundary an edge on the inner or the outer circle, which appears in no other 2-simplex).

EXAMPLE 8.9. The torus. Triangulate the torus as indicated.

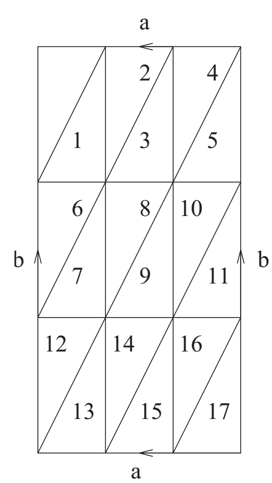


FIGURE 8.4. A triangulation of the torus

Since K is connected, $H_0(K) \cong \mathbb{Z}$.

Let z be a 1-cycle. Change by a multiple of the boundary of $\triangle 1$ to get a homologous cycle not containing the side $\diagup$ of $\triangle 1$. Repeat with boundary of $\triangle 2$ to eliminate the side $|$ of $\triangle 2$. Continue, eliminating $\diagup$ of $\triangle 3$, $|$ of $\triangle 4$, $\diagup$ of $\triangle 5$, $-$ of $\triangle 6$, $\diagup$ of $\triangle 7$, $|$ of $\triangle 8$, $\diagup$ of $\triangle 9$, $|$ of $\triangle 10$, $\diagup$ of $\triangle 11$, $-$ of $\triangle 12$, $\diagup$ of $\triangle 13$, $|$ of $\triangle 14$, $\diagup$ of $\triangle 15$, $|$ of $\triangle 16$, and $\diagup$ of $\triangle 17$.

So z is homologous to a cycle z' containing only edges of the circles a and b and the edges 1,2,3,4 below.

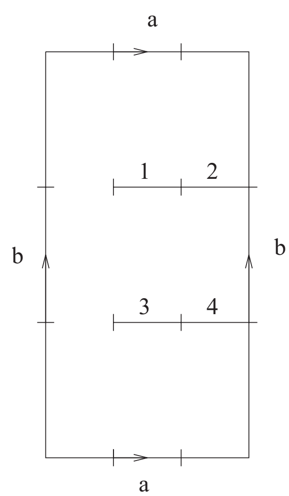


FIGURE 8.5. Any 1-cycle is homologous to a cycle here

Since z' is a cycle, it cannot containg any of the edges 1,2,3 or 4. So z is homologous to a cycle having edges only on the circle a or the circle b. The usual argument implies that each edge on each of the two circles must appear the same number of times.

At this point we know that $H_1(K)$ is a *quotient* of $\mathbb{Z} \oplus \mathbb{Z}$: the next little argument finds out which quotient.

If a 2-chain is to have a boundary containing just a's and b's, then all the triangles (give them all a clockwise orientation) must appear with the same coefficient so that the inner edges will cancel out. The boundary of such a triangle is therefore 0. It follows that every homology class contains exactly one element of the form $ra + sb$, and that

$$H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z},$$

with generators a and b.

For $H_2(K)$, a 2-cycle must contain $\triangle 2$ with clockwise orientation the same number of times it contains $\triangle 3$ (also with clockwise orientation), to make the $\diagup$ edge cancel, and similarly for any adjacent triangles. So every triangle appears with the same coefficient in a 2-cycle, and $Z_2(K) \cong \mathbb{Z}$. Since there are no 3-simplexes, $B_2(K) = 0$, so

$$H_2(K) \cong \mathbb{Z}.$$

Finally, it is clear that $H_{n\geq 3}(K) = 0$.

EXAMPLE 8.10. The Klein bottle. Let K be the triangulation of the Klein bottle indicated below.

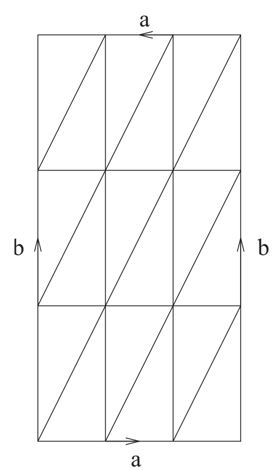


FIGURE 8.6. A triangulation of the Klein bottle

As usual, $H_0(K) \cong \mathbb{Z}$.

By the argument used in Example 8.8, any 1-cycle is homologous to a cycle of the form $ra + sb$. Again, this shows that $H_1(K)$ is a *quotient* of $\mathbb{Z} \oplus \mathbb{Z}$.

If a 2-chain is to have a boundary containing just a and b, then again all triangles oriented clockwise must appear with the same coefficient (so the common edges inside will cancel out). Now the boundary of such a 2-chain is $k(2a)$, where k is the number of times each triangle appears. It follows that $H_1(K)$ is the abelian group generated by a and b with the relation $2a = 0$, so

$$H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Since there are no 2-cycles, $H_2(K) = 0$.

Notice that the Klein bottle is non-orientable, and has torsion in its homology. These two properties go together for closed surfaces, but do not for surfaces with boundary, as the next example shows.

EXAMPLE 8.11. Let K be a triangulation of the Möbius band. Then $H_0(K) \cong \mathbb{Z}$, $H_1(K) \cong \mathbb{Z}$, and $H_{n\geq 2}(K) = 0$. The reason is that the Möbius band retracts onto the circle (or, K collapses to a triangulation of the circle).

2. Euler characteristic

Recall the structure theorem for finitely generated abelian groups: any such group G is given uniquely by

$$G \cong \mathbb{Z}^m \oplus \text{torsion}.$$

The number m is called the *Betti number* of G. By the structure theorem, the Betti number is well-defined (isomorphic groups have the same Betti numbers).
Let β_j be the Betti number of $H_j(K)$ for a given simplicial complex K. Define the *Euler characteristic* of K by

$$\chi(K) = \sum_{i \geq 0} (-1)^i (\# \text{ of } n\text{-simplexes in K}).$$

THEOREM 8.12. $\chi(K) = \sum_{j \geq 0} (-1)^j \beta_j$.

This is a remarkable fact: of course there is no reason to have β_j equal to the number of j-simplexes: in fact one may change the triangulation (by barycentric subdivision for instance), which changes all the summands in the Euler characteristic but by Theorem 8.12 must preserve the Euler characteristic.

CHAPTER 9

Simplicial approximation and an application

So far we have seen how to construct abelian groups $H_n(K)$ (for $n \geq 0$) for a simplicial complex K. We also triangulated topological spaces, to allow $H_n(X)$ to be defined for a topological space X.

Remark 9.1. Problem Can $H_n(\cdot)$ be extended to a functor from the category of topological spaces and continuous maps to the category of abelian groups and homomorphisms?

To understand what is involved here, consider the following diagram.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle triangulate}\downarrow & & \downarrow{\scriptstyle triangulate} \\ K & \xrightarrow{\bar{f}} & L \\ {\scriptstyle H_n}\downarrow & & \downarrow{\scriptstyle H_n} \\ H_n(K) & \xrightarrow{H_n(\bar{f})} & H_n(L) \end{array}$$

Thus we need to make sense of *simplicial maps* between simplicial complexes. This is fairly involved and we shall initially do this without proofs.

Definition 9.2. If K is a simplicial complex of dimension n (a combinatorial-geometric object) denote by $|K|$ (called the *polyhedron of* K) the set of points in $\mathbb{R}^n$ that lie in at least one of the simplexes of K, topologized as a subset of $\mathbb{R}^n$.

Lemma 9.3. *(a) $|K|$ is a closed compact subset of $\mathbb{R}^n$.*
(b) Every point of $|K|$ is in the interior of exactly one simplex of K.
(c) If L is a subcomplex of K, then $|L|$ is a closed subset of $|K|$.

Definition 9.4. If K and L are simplicial complexes, a *simplicial map* $f : |K| \to |L|$ is a function from $|K|$ to $|L|$ with the following properties.

1. If a is a vertex of a simplex of K, then $f(a)$ is a vertex of a simplex of L.
2. If $(a^0, a^1, \ldots, a^n)$ is a simplex of K, then $f(a^0), \ldots, f(a^n)$ span a simplex of L (possibly with some repeats).

3. If $x = \sum \lambda_i a^i$ is in a simplex $(a^0, a^1, \ldots, a^n)$ of K, then $f(x) = \sum \lambda_i f(a^i)$ (i.e. f is linear on each simplex).

Of course a *simplicial map of simplicial pairs* $f : (|K|, |L|) \to (|M|, |N|)$ is just a simplicial map $f : |K| \to |M|$ such that $f(|L|) \subset |N|$.

REMARK 9.5. (1) Simplicial maps are the natural structure-preserving maps between simplicial complexes, allowing us to define the category of simplicial complexes and simplicial maps.
(2) If f is a simplicial map, it is automatically continuous.
(3) Definition 9.1 allows us to be more precise about triangulations: a triangulation of a topological space X is a simplicial complex K and a homeomorphism $h : |K| \to X$. We shall usually ignore h and treat $|K|$ itself as the triangulation of X.

THEOREM 9.6. *If $f : X \to Y$ is a map, then there exist triangulations K and L of X and Y respectively, such that the induced map $|K| \to |L|$ is homotopic to a simplicial map $\bar{f} : |K| \to |L|$.*

The proof of this result will be found in the references.

DEFINITION 9.7. If $f : X \to Y$ is a map, then define the induced map in homology, $H_n(f) : H_n(X) \to H_n(Y)$, as follows. If z is an n-cycle in (a suitable triangulation K of) X, then

$$H_n(f)(z + B_n(K)) = \bar{f}(z) + B_n(L),$$

where L is a (suitable) triangulation of Y.

THEOREM 9.8. *$H_n(\cdot)$ is a functor from the category of topological spaces and maps to the category of abelian groups and homomorphisms.*

Notice that a proof of this result involves constructing the map $\bar{f}$ and showing that homotopic maps induce the same maps in homology. As an application of these ideas, we now prove the Brouwer Fixed-Point theorem. In comparing this result to the Borsuk–Ulam theorem, notice how homology allows us to work easily in higher dimensions.

THEOREM 9.9. *Any map $f : B^n \to B^n$ has a fixed point for $n \geq 1$.*

PROOF. First see that the case $n = 1$ is trivial (draw a picture).

Recall that $B_n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ is the n-ball. Assume that $f : B^n \to B^n$ is a map with no fixed point, so $f(x) \neq x$ for all $x \in B^n$. For each $x \in B^n$, there is then a uniquely defined line segment from $f(x)$ to x, which may be extended *in the direction $f(x)$ to x* until it meets the boundary S^{n-1} of B^n at some point y. This assignment defines a map $g : B^n \to S^{n-1}$, by $g(x) = y$. By Theorem 9.8, there is an induced homomorphism of abelian groups

$$H_{n-1}(g) : H_{n-1}(B^n) \to H_{n-1}(S^{n-1}).$$

We know that $H_{n-1}(B^n) = 0$, while $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$.

On the other hand, if K and L are the triangulations of B^n and S^{n-1} used to define $H_{n-1}(g)$, then the $(n-1)$-cycle given by the subcomplex of K that is a triangulation of S^{n-1} (with the proper orientation) represents the homology class of 0 in $H_{n-1}(B^n)$ but is sent under g to a generator of $H_{n-1}(S^{n-1})$ (since g fixes the boundary of B^n), which is a contradiction. $\square$

CHAPTER 10

Homological algebra and the exact sequence of a pair

In this section we develop more sophisticated machinery to compute homology groups, and to construct the homology groups of a pair of spaces. These turn out to be stronger invariants than the homology groups alone. The first step is to develop a better understanding of the algebra that arises from chain groups and boundary homomorphisms. If K is a simplicial complex, then it has naturally associated to it a sequence of abelian groups and homomorphisms

$$\cdots \longrightarrow C_n(K) \stackrel{\partial_n}{\to} \longrightarrow C_{n-1}(K) \stackrel{\partial_{n-1}}{\to} \longrightarrow \dots \stackrel{\partial_2}{\to} \longrightarrow C_1(K) \stackrel{\partial_1}{\to} \longrightarrow C_0(K) \stackrel{\partial_0}{\to} \longrightarrow 0,$$

with the additional critical property that $\partial_{k-1}\partial_k = 0$.

It is convenient to consider the purely algebraic portion of this situation – and to extend the chain to the right (negative values of k).

1. Chain complexes and mappings

DEFINITION 10.1. A *chain complex* $\langle A, \partial \rangle$ is a doubly infinite sequence

$$A = \{\dots, A_2, A_1, A_0, A_{-1}, A_{-2}, \dots\}$$

of abelian groups A_k, together with a collection

$$\partial = \{\partial_k \mid k \in \mathbb{Z}\}$$

of homomorphisms such that $\partial_k : A_k \to A_{k-1}$ and $\partial_k \partial_{k-1} = 0$.

For brevity, we shall sometimes denote the chain complex $\langle A, \partial \rangle$ by A. In a chain complex, it is clear that the image of ∂_k is a subgroup of the kernel of ∂_{k-1}. By analogy with the topological situation, we make the following definitions.

DEFINITION 10.2. If A is a chain complex, then the kernel $Z_k(A)$ of ∂_k is the *group of k-cycles*, and the image $B_k(A) = \partial_{k+1}(A_{k+1})$ is the *group of k-boundaries*. The factor group $H_k(A) = Z_k(A)/B_k(A)$ is the *kth homology group of A*.

In the topological setting, a mapping f from X to Y gives, for suitable triangulations of X and Y, a homomorphism f_k from $C_k(X)$ to $C_k(Y)$ which commutes with ∂_k. This gives an anologous notion of maps for chain complexes.

THEOREM 10.3. *Let $\langle A, \partial\rangle$ and $\langle A', \partial'\rangle$ be chain complexes, and suppose there is a collection of homomorphisms $f_k : A_k \to A'_k$ giving the following commutative diagram*

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{\partial_{k+2}} & A_{k+1} & \xrightarrow{\partial_{k+1}} & A_k & \xrightarrow{\partial_k} & A_{k-1} & \xrightarrow{\partial_{k-1}} & \dots \\
 & & \Big\downarrow{\scriptstyle f_{k+1}} & & \Big\downarrow{\scriptstyle f_k} & & \Big\downarrow{\scriptstyle f_{k-1}} & & \\
\dots & \xrightarrow{\partial'_{k+2}} & A'_{k+1} & \xrightarrow{\partial'_{k+1}} & A'_k & \xrightarrow{\partial'_k} & A'_{k-1} & \xrightarrow{\partial'_{k-1}} & \dots
\end{array}$$

Then f_k induces a natural homomorphism $f_{*k} : H_k(A) \to H_k(A')$.

In the above situation, we say that $f = \{f_k \mid k \in \mathbb{Z}\}$ *commutes* with ∂.

DEFINITION 10.4. A chain complex $\langle A', \partial'\rangle$ is a *subcomplex* of a chain complex $\langle A, \partial\rangle$ if, for all k, A'_k is a subgroup of A_k, and $\partial'_k(c) = \partial_k(c)$ for all $c \in A'_k$.

EXAMPLE 10.5. Let A be a chain complex, and let A' be a subcomplex of A. Let i be the collection of injection mappings $i_k : A'_k \to A_k$ given by $i_k(c) = c$. It is clear that i commutes with ∂, so there are induced homomorphisms $i_{*k} : H_k(A') \to H_k(A)$. Despite the fact that i is simply the identity inclusion, the induced map in homology may be non-trivial. For instance, we may view (a triangulation of) the 2-sphere S^2 as a subcomplex of (a triangulation) of the 3-ball B^3: the induced map

$$i_{*2} : H_2(S^2) \to H_2(B^3)$$

is not an isomorphism.

2. Relative homology

Let A' be a subcomplex of A (for instance, arising from a simplicial subcomplex Y of a simplicial complex X). We can regard $C_k(Y)$ as a subgroup of $C_k(X)$, and $\partial_k(C_k(Y)) \le C_{k-1}(Y)$. We may therefore form the collection A/A' of factor groups A_k/A'_k, and we claim that A/A' gives rise to a chain complex in a natural way. To do this, a collection $\bar{\partial}$ of homomorphisms

$$\bar{\partial}_k : (A_k/A'_k) \to \left(A_{k-1}/A'_{k-1}\right)$$

such that $\bar{\partial}_{k-1}\bar{\partial}_k = 0$ must be constructed. Define $\bar{\partial}_k$ by setting

$$\bar{\partial}_k(c + A'_k) = \partial_k(c) + A'_{k-1}$$

for $c \in A_k$.

$\bar{\partial}_k$ IS WELL-DEFINED. If $c_1 \in c + A'_k$, then $c_1 - c \in A'_k$, so $\partial_k(c_1 - c) \in A'_{k-1}$. Thus

$$\partial_k(c_1) \in \partial_k(c) + A'_{k-1}$$

also, so $\bar{\partial}_k$ is well-defined.

It is clear that $\bar{\partial}_k$ is a homomorphism and $\bar{\partial}_{k-1}\bar{\partial}_k = 0$.

THEOREM 10.6. *If A' is a subcomplex of the chain complex A, then the collection A/A', together with the collection $\bar{\partial}$ of homomorphisms defined by*

$$\bar{\partial}_k(c + A'_k) = \partial_k(c) + A'_{k-1}$$

for $c \in A_k$, is a chain complex.

Since A/A' is a chain complex, it has associated to it homology groups $H_k(A/A')$.

DEFINITION 10.7. The homology group $H_k(A/A')$ is the *kth relative homology group of A modulo A'*.

In the topological setting, where Y is a subcomplex of a simplicial complex X, the usual notation for the kth relative homology group arising from the subcomplex $C(Y)$ of the chain complex $C(X)$ is $H_k(X, Y)$. That is, all the chains of Y are 'set equal to zero'. This corresponds to the geometric process of shrinking the subcomplex Y to a point.

EXAMPLE 10.8. Let X be the one-dimensional simplicial complex shown in Figure 10.1,

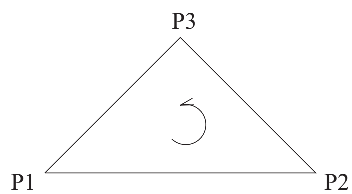


FIGURE 10.1. The simplicial complex X

and let Y be the subcomplex consisting of the edge P_2P_3.

We know that $H_1(X) \cong \mathbb{Z}$. Geometrically, shrinking P_2P_3 to a point collapses the rim of the triangle, as shown in Figure 10.2.

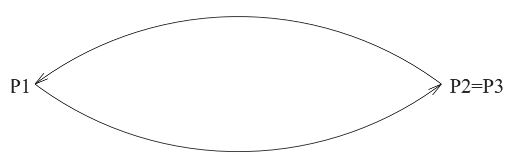


FIGURE 10.2. The simplicial complex X with edge P_2P_3 shrunk to a point

The result is still topologically a circle, so we expect that $H_1(X, Y) \cong \mathbb{Z}$.

Generators for $C_1(X)$ are P_1P_2, P_2P_3, and P_3P_1. Since $P_2P_3 \in C_1(Y)$, generators of $C_1(X)/C_1(Y)$ are

$$P_1P_2 + C_1(Y), \text{ and } P_3P_1 + C_1(Y).$$

To find $Z_1(X,Y)$, compute

$$\begin{aligned}\bar{\partial}_1(nP_1P_2 + mP_3P_1 + C_1(Y)) &= \partial_1(nP_1P_2) + \partial_1(mP_3P_1) + C_0(Y)\\ &= n(P_2 - P_1) + m(P_1 - P_3) + C_0(Y)\\ &= (m-n)P_1 + C_0(Y),\end{aligned}$$

since P_2 and P_3 are in $C_0(Y)$. Thus, for a cycle, we must have $m = n$, which shows that a generator for $Z_1(X,Y)$ is $(P_1P_2 + P_3P_1) + C_1(Y)$. Since $B_1(X,Y) = 0$, this shows that

$$H_1(X,Y) \cong \mathbb{Z}.$$

Since $P_1 + C_0(Y)$ generates $Z_0(X,Y)$ and

$$\bar{\partial}_1(P_2P_1 + C_1(Y)) = (P_1 - P_2) + C_0(Y) = P_1 + C_0(Y),$$

so $H_0(X,Y) = 0$.

Example 10.9. Consider the circle S^1 as a subcomplex of the ball B^2. Geometrically, shrinking the edge of the ball B^2 to a point results in a topological 2-sphere (make sure you understand why it does not result in a point). So, we expect that despite the fact that $H_2(B^2) = 0$, the relative homology group $H_2(B^2, S^1)$ should be $\mathbb{Z}$.

To compute H_2, regard B^2 as the triangular region inside the triangle of Figure 63, and S^1 as the rim of the triangle. Then $C_2(B^2, S^1)$ is generated by $P_1P_2P_3 + C_2(S^1)$, and

$$\begin{aligned}\bar{\partial}_2(P_1P_2P_3 + C_2(S^1)) &= \partial_2(P_1P_2P_3) + C_1(S^1)\\ &= (P_2P_3 - P_1P_3 + P_1P_2) + C_1(S^1).\end{aligned}$$

Now $(P_2P_3 - P_1P_3 + P_1P_2) \in C_1(S^1)$, so

$$\bar{\partial}_2(P_1P_2P_3 + C_2(S^1)) = 0.$$

Therefore $P_1P_2P_3 + C_2(S^1) \in Z_2(B^2, S^1)$. Since $B_2(B^2, S^1) = 0$, we have as expected that

$$H_2(B^2, S^1) \cong \mathbb{Z}.$$

3. The exact homology sequence of a pair

In this section there are many routine calculations using quotient groups which are left as exercises.

Lemma 10.10. *Let A' be a subcomplex of a chain complex A, and let j be the collection of canonical homomorphisms $j_k : A_k \to A_k/A'_k$. Then, for every k,*

$$j_{k-1}\partial_k = \bar{\partial}_k j_k,$$

so j commutes with ∂.

An immediate consequence is the following important result.

Theorem 10.11. *The map j_k induces a natural homomorphism*

$$j_{*k} : H_k(A) \to H_k(A/A').$$

Let now A' be a subcomplex of a chain complex A. Let $h \in H_k(A/A')$, so that $h = z + B_k(A/A')$ for some $z \in Z_k(A/A')$ and $z = c + A'_k$ for some $c \in A_k$. Notice that passing from h to c involves choosing a representative of a representative. Now $\bar{\partial}_k(z) = 0$, which implies that $\partial_k(c) \in A'_{k-1}$ and so $\partial_k(c) \in Z_{k-1}(A')$ (since $\partial_{k-1}\partial_k = 0$). Define a map

$$\partial_{*k} : H_k(A/A') \to H_{k-1}(A')$$

by setting $\partial_{*k}(h) = \partial_k(c) + B_{k-1}(A')$.

That is, start with an element of $H_k(A/A')$. Such an element is represented by a relative k-cycle modulo A'. That means its boundary is in A'_{k-1}. Since its boundary is in A'_{k-1} and is also the boundary of something in A_k, this boundary must be a $(k-1)$-cycle in A'_{k-1}. So, from $h \in H_k(A/A')$ we have produced a $(k-1)$-cycle representing a homology class in $H_{k-1}(A')$.

Lemma 10.12. *The map $\partial_{*k} : H_k(A/A') \to H_{k-1}(A')$ is well-defined, and is a homomorphism.*

Recall from Example 10.5 the maps i_{*k}, and from Lemma 10.10 the maps j_{*k}: these give a chain of maps and homomorphisms,

$$\dots \qquad \xrightarrow{\partial_{*k+1}} H_k(A') \xrightarrow{i_{*k}} H_k(A) \xrightarrow{j_{*}k} H_k(A/A')$$
$$\xrightarrow{\partial_{*k}} H_{k-1}(A') \xrightarrow{i_{*k-1}} H_{k-1}(A) \xrightarrow{j_{*k-1}} H_{k-1}(A/A') \xrightarrow{\partial_{*k-1}} \dots$$

Lemma 10.13. *The sequence of groups and homomorphisms above form a chain complex.*

Now we have a chain complex, we could of course find the homology groups... Fortunately there are no further complications to be unearthed in that direction: *all the homology groups of the sequence are trivial*.

Recall that a chain complex $A = \langle A, \partial \rangle$ is called a (long) *exact sequence* if all the homology groups are 0, i.e. for every k the image of ∂_k is exactly equal to the kernel of ∂_{k-1}.

Theorem 10.14. *The chain complex (10.1) is exact.*

The exact sequence is called the *exact homology sequence of the pair* (A, A'). If the chain complexes arise from a pair of topological spaces, we obtain the exact homology sequence of a pair of spaces. In this case, we obtain for a subcomplex L of the simplicial complex K,

$$\dots \qquad \xrightarrow{\partial_{*k+1}} H_k(L) \xrightarrow{i_{*k}} H_k(K) \xrightarrow{j_{*}k} H_k(K, L)$$
$$\xrightarrow{\partial_{*k}} H_{k-1}(L) \xrightarrow{i_{*k-1}} H_{k-1}(K) \xrightarrow{j_{*k-1}} H_{k-1}(K, L) \xrightarrow{\partial_{*k-1}} \dots$$
$$\xrightarrow{j_{*}1} H_1(K, L) \xrightarrow{\partial_{*1}} H_0(L) \xrightarrow{i_{*0}} H_0(K) \xrightarrow{j_{*0}} H_0(K, L) \to 0$$

EXAMPLE 10.15. Let's assume the result that $H_k(B^n) = 0$ for $k > 0$, and use Theorem 10.14 to deduce the homology of spheres. Choose and fix a triangulation K of B^{n+1}, with the subcomplex L corresponding to the boundary S^n. Form the exact homology sequence of the pair (K, L):

$$\underbrace{H_{n+1}(L)}_{=0} \overset{i_{*n+1}}{\longrightarrow} \underbrace{H_{n+1}(K)}_{=0} \overset{j_{*n+1}}{\longrightarrow} \underbrace{H_{n+1}(K,L)}_{\cong\mathbb{Z}} \overset{\partial_{*n+1}}{\longrightarrow}$$

$$\underbrace{H_n(L)}_{=?} \overset{i_{*n}}{\longrightarrow} \underbrace{H_n(K)}_{=0} \overset{j_{*n}}{\longrightarrow} \underbrace{H_n(K,L)}_{=0} \overset{\partial_{*n}}{\longrightarrow} \dots \overset{j_{*k+1}}{\longrightarrow}$$

$$\underbrace{H_{k+1}(K,L)}_{=0} \overset{\partial_{*k+1}}{\longrightarrow} \underbrace{H_k(L)}_{=?} \overset{i_{*k}}{\longrightarrow} \underbrace{H_k(K)}_{=0} \overset{j_{*k}}{\longrightarrow} \dots$$

for any k, $1 \le k < n$. Our assumption is that $H_k(K) = 0$ for $k \ge 1$, as indicated. Since L is subcomplex of K, we have that $C_k(K) \le C_k(L)$ (for $k \le n$, and using the standard $(n+1)$-simplex as K; since all the k-chains must live on the faces of K). It follows that $H_k(K, L) = 0$ for $k \le n$, which is again indicated on the exact sequence. As in Example 10.9, we see that $H_{n+1}(K, L) \cong \mathbb{Z}$, with a generating homology class containing the representative

$$P_1P_2 \dots P_{n+2} + C_{n+1}(L).$$

For $1 \le k < n$, the exact sequence in the last row of the diagram above tells us that $H_k(L) = 0$: from $H_k(K) = 0$, we see that

$$\ker i_{*k} = H_k(L).$$

On the other hand, from $H_{k+1}(K, L) = 0$, we see that $\text{image}\partial_{*k+1} = 0$. From exactness, $\ker i_{*k} = \text{image}\partial_{*k+1}$, so $J_k(L) = 0$ for $1 \le k < n$.

A similar argument gives $H_n(L) \cong \mathbb{Z}$:

(1) Since $H_{n+1}(K) = 0$, we have $\text{image} j_{*n+1} = 0$.
(2) By (1) and exactness, $\ker \partial_{*n+1} = \text{image} j_{*n+1} = 0$, so ∂_{*n+1} is an isomorphism.
(3) By (2), $\text{image}\partial_{*n+1} \cong \mathbb{Z}$.
(4) Since $H_n(K) = 0$, $\ker i_{*n} = H_n(L)$.
(5) By exactness again, $\text{image}\partial_{*n+1} = \ker i_{*n}$, so $H_n(L) \cong \mathbb{Z}$.

Thus, $H_n(L) = H_n(S^n) \cong \mathbb{Z}$ and $H_k(L) = H_k(S^n) = 0$ for $1 \le k < n$.

Since S^n is connected, $H_0(S^n) \cong \mathbb{Z}$. This fact can also be read off from the last three terms of the exact sequence,

$$\underbrace{H_1(K,L)}_{=0} \overset{\partial_{*1}}{\longrightarrow} \underbrace{H_0(L)}_{=?} \overset{i_{*0}}{\longrightarrow} \underbrace{H_0(K)}_{\cong\mathbb{Z}} \overset{j_{*0}}{\longrightarrow} \underbrace{H_0(K,L)}_{=0}.$$

EXAMPLE 10.16. Consider the following diagram representing a triangulation K of the Möbius band, and a subcomplex L that triangulates the boundary circle of the Möbius band.

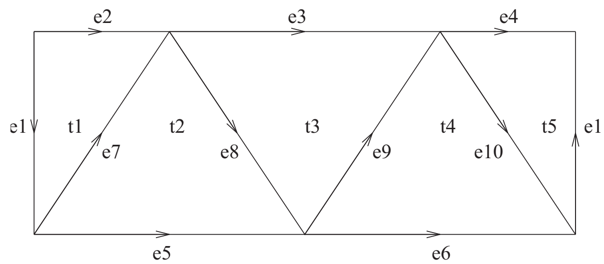


FIGURE 10.3. The Möbius band and edge

If z is any 1-cycle in K, then it may be written in the form

$$z = \lambda_1 z^1 + \cdots + \lambda_6 z^6$$

where z^i is the 1-cycle associated with the unique loop on the interior edges of K together with the vertex e^i which contains e^i with coefficient $+1$. Thus, $z^1 = e^1 + e^7 + e^8 + e^9 + e^{10}$, $z^2 = e^2 + e^8 + e^9 + e^10$ and so on.

Now it is easy to check that $H_1(L) \cong \mathbb{Z}$, with generator $e^2 + e^3 + e^4 + e^5 + e^6$, and $H_1(K) \cong \mathbb{Z}$, with generator $z^1 = e^1 + e^7 + e^8 + e^9 + e^{10}$. Let's compute the homomorphism i_{*1}. First, more details on $H_1(K)$. $B_1(K)$ is generated by the following elements of $Z_1(K)$:

$\partial_2 t^1 = e^2 - e^7 - e^1 = z^2 - z^1$,
$\partial_2 t^2 = e^7 + e^8 - e^5 = -z^5$,
$\partial_2 t^3 = e^3 - e^9 - e^8 = z^3$,
$\partial_2 t^4 = e^9 + e^{10} - e^6 = -z^6$,
$\partial_2 t^5 = e^4 - e^1 - e^{10} = z^4 - z^1$,

so that $H_1(K)$ is the abelian group generated by $z^1, z^2, z^3, z^4, z^5, z^6$ with the relations $z^2 - z^1 = -z^5 = z^3 = -z^6 = z^4 - z^1 = 0$.

Working in $H_1(K)$ (so equals means homologous), we have that

$$\begin{aligned} e^2 + e^3 + e^4 + e^5 + e^6 &= z^2 + z^3 + z^4 + z^5 + z^6 \\ &= z^1 + 0 + z^1 + 0 + 0 \\ &= 2z^1. \end{aligned}$$

Now the homomorphism i_{*1} is determined by its effect on the generator of $H_1(L)$, and we have seen that i_{*1} applied to $e^2 + e^3 + e^4 + e^5 + e^6$ in L gives $e^2 + e^3 + e^4 + e^5 + e^6$ in K, but in $H_1(K)$ we know that

$$e^2 + e^3 + e^4 + e^5 + e^6 = 2z^1$$

in $H_1(K)$. Thus i_{*1} takes a generator to twice a generator: it is multiplication by two from $\mathbb{Z}$ to $\mathbb{Z}$.

We are now in a position to write down the exact sequence for the pair (K, L):

$$\underbrace{H_2(L)}_{=0} \to \underbrace{H_2(K)}_{=0} \to \underbrace{H_2(K,L)}_{=?} \to \underbrace{H_1(L)}_{\cong\mathbb{Z}} \xrightarrow{i_{*1}=\times 2} \underbrace{H_1(K)}_{\cong\mathbb{Z}} \xrightarrow{j_{*1}} \underbrace{H_1(K,L)}_{=?}$$

$$\to \underbrace{H_0(L)}_{\cong\mathbb{Z}} \overset{i_{*0}}{\to}\to \underbrace{H_0(K)}_{\cong\mathbb{Z}} \to \underbrace{H_0(K,L)}_{=?} \to 0.$$

Now apply exactness to deduce the following.
$H_2(K,L) \cong \ker i_{*1} = 0$.
$H_1(K,L) \cong H_1(K)/\mathrm{image} i_{*1} \cong \mathbb{Z}/2\mathbb{Z}$, with generator the image under j_{*1} of a generator of $H_1(K)$.
Since both K and L are connected and non-empty, the map i_{*0} is an isomorphism (check this!). It follows that $H_0(K,L) = 0$.

Example 10.17. In a similar way we can compute the relative homology group of the pair (cylinder, both ends). Let K be the triangulation of the cylinder indicated below, and let L be the subcomplex compirising the two ends (i.e. e^7, e^8, e^9 and e^{10}, e^{11}, e^{12}).

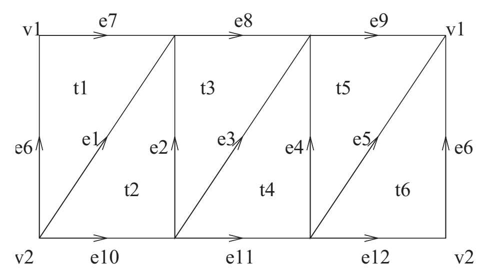


Figure 10.4. The cylinder and its two ends

In this case $i_{*0} : H_0(L) \to H_0(K)$ has infinite cyclic kernel, generated by the element $v^1 - v^0$ in $H_0(L)$. (Notice that L is not connected but K is, so i_{*0} cannot be an isomorphism in contrast to the situation of Example 10.16). Also, i_{*1} takes each of the generators $e^7 + e^8 + e^9$ and $e^{10} + e^{11} + e^{12}$ of $H_1(L)$ to a generator of $H_1(K)$: it follows in particular that i_{*1} is surjective.

As an exercise, apply the method of Example 10.16 to compute the groups $H_k(K,L)$ for $k = 0, 1, 2$.

Notice that these examples show that using homology of pairs produces strictly more information: for example, the Möbius band and the circle have the same homology groups. However, the Möbius band has a triangulation K containing a subcomplex L with the property that $H_1(K,L) \cong \mathbb{Z}/2\mathbb{Z}$, which shows that it is not homeomorphic to the circle.

APPENDIX A

Finitely generated abelian groups

Proofs for the results on this sheet may be found in any group theory book. We shall deal with abelian (commutative) groups, and therefore use additive notation (so the binary operation of the group is $+$).

Definition A.1. Let G be an abelian group. A set of elements $\{g_i\}_{i\in I}$ in G is a *generating set* for G if every $g \in G$ can be written in the form

$$g = \sum_{i\in I} n_i g_i$$

($n_i \in \mathbb{Z}$) in which sum all but finitely many n_i are zero. If G has a generating set with finitely many elements, then G is said to be *finitely generated*.

Examples of finitely generated groups are $\mathbb{Z}$, $C_4 \times C_4$ (C_n is the cyclic group of order n.) An example of an abelian group that is *not finitely generated* is $\mathbb{Q}$.

A group that is generated by a single element is called a *cyclic group*.

A group G is *torsion-free* if for each $g \in G\backslash\{0\}$, $ng = 0 \implies n = 0$. A group G is *torsion* if every $g \in G$ has an $n \in \mathbb{N}$ for which $ng = 0$.

Example A.2. The set of torsion elements in an abelian group forms a subgroup, called the *torsion subgroup*.

Definition A.3. Let G be a finitely-generated torsion-free group. Then G has a *basis*: a generating set $\{g_1, \dots, g_m\}$ that is also *independent*:

$$n_1 g_1 + \cdots + n_m g_m = 0 \implies n_1 = \cdots = n_m = 0.$$

Abelian groups that are torsion-free are called *free abelian groups*. A basis for a finitely-generated torsion-free abelian group is not unique (for example, $\{(1,0),(0,1)\}$ and $\{(3,4),(4,5)\}$ are both bases for $\mathbb{Z}\times\mathbb{Z}$), but the cardinality of a basis is determined by the group.

Definition A.4. If G is a finitely-generated torsion-free group, then the number of elements in a basis for G is the *rank* of G.

It makes sense to talk about an independent generating set for a finitely generated abelian group that is not free: a set $\{b_1, \dots, b_n\}$ is a basis in this sense if $\{b_1, \dots, b_n\}$ generates G and $\sum n_i b_i = 0$ if and only if $n_i b_i = 0$ for each i. However, the number of elements in such a set is NOT well-defined. As an exercise, show that $\{1\}$ and $\{2, 3\}$ are both independent generating sets for C_6. Because of this, we shall only use words like basis for free abelian groups.

A general finitely-generated abelian group with a generating set of m elements is a quotient of a free abelian group of rank m. Equivalently, it is given by *generators and relations*,

$$G = \langle g_1, \dots, g_m \mid R \rangle,$$

where R is a set of relations of the form $n_1 g_1 + \dots + n_m g_m = 0$. Notice that we are always assuming that the generators commute, so this is not quite the same notation as used in presentations of groups.

Example A.5. Work through the following:

$$\mathbb{Z} \cong \frac{\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}}{\langle(3,6,-9),(2,4,-1),(1,2,11)\rangle}$$
$$\cong \langle a, b, c \mid 3a + 6b - 9c = 0, 2a + 4b - c = 0, a + 2b + 11c = 0\rangle.$$

To see this, work with the generator-relation expression. By adding integer multiples of the relation $a + 2b + 11c = 0$ to the other two relations, we arrive at

$$a + 2b + 11c = 0, -23c = 0, 24c = 0.$$

It follows that

$$\langle a, b, c \mid 3a + 6b - 9c = 0, 2a + 4b - c = 0, a + 2b + 11c = 0\rangle$$
$$\cong \langle a, b, c \mid a + 2b + 11c = 0, -23c = 0, 24c = 0\rangle.$$

It follows that in the group $c = 24c - 23c = 0$, and so $a = -2b$:

$$\langle a, b, c \mid 3a+6b-9c = 0, 2a+4b-c = 0, a+2b+11c = 0\rangle \cong \langle a, b \mid a+2b = 0\rangle.$$

Finally, this shows that the group is generated by the single element b, and that b is torsion-free:

$$\langle a, b, c \mid 3a + 6b - 9c = 0, 2a + 4b - c = 0, a + 2b + 11c = 0\rangle \cong \langle b\rangle.$$

1. The Fundamental Theorem

Lemma A.6. *If G is a finitely-generated abelian group with torsion subgroup T, then G is an internal direct product $T \times F$ where $F \cong \mathbb{Z}^m$ is a free subgroup of G.*

Notice that if G itself is torsion then F is trivial. The number m is called the *Betti number of G*, and it is uniquely determined by G.

Theorem A.7. *Every finitely-generated abelian group G is isomorphic to a direct product of cyclic groups in the form*

$$C_{p_1^{r_1}} \times \dots \times C_{p_n^{r_n}} \times \mathbb{Z}^m,$$

where the p_i are primes, not necessarily distinct. Equivalently, it may be written in the form

$$C_{m_1} \times \dots C_{m_r} \times \mathbb{Z}^m,$$

in which m_i divides m_{i+1}.

The number m is the Betti number of G. The numbers m_i are the *torsion coefficients* of G; they are uniquely determined by G. The prime powers $p_i^{r_i}$ are also uniquely determined by G.

Example A.8. (1) $C_2 \times C_2 \times C_2 \times C_3 \times C_3 \times C_5$ is isomorphic to $C_2 \times C_6 \times C_{30}$.
(2) The abelian groups of order $360 = 2^3 3^2 5$ may be determined using Theorem A.7. They are (writing them in terms of torsion coefficients): $C_2 \times C_6 \times C_{30}$, $C_6 \times C_{60}$, $C_2 \times C_2 \times C_{90}$, $C_2 \times C_{180}$, $C_3 \times C_{120}$, and C_{360}.

Finally, an example taken from topology: the following example is the kind of calculation that you need to be able to do.

Example A.9. Let Z be the free abelian group generated by $\{P_1, P_2, P_3, P_4\}$, and let B be the subgroup of G generated by

$$\{P_2 - P_1, P_3 - P_1, P_4 - P_1, P_3 - P_2, P_4 - P_2, P_4 - P_3\}.$$

Determine the structure of Z/B.

Consider a general element $z + B$ of the quotient group Z/B. Then $z = s_1P_1 + s_2P_2 + s_3P_3 + s_4P_4$ say, so

$$z - [s_2(P_2 - P_1) + s_3(P_3 - P_1) + s_4(P_4 - P_1)] = rP_1$$

for some $r \in Z$, so

$$z + B = rP_1 + B.$$

That is, every coset of B (i.e. every element of Z/B) may be written in the form $rP_1 + B$. *This means that Z/B is cyclic – all that remains is to decide if it has torsion.*

To decide if Z/B is torsion, assume that $sP_1 + B = rP_1 + B$. This implies that $(s - r)P_1 \in B$, and it is clear from the generators of B that this requires $r = s$. It follows that Z/B is torsion-free and cyclic, so $Z/B \cong \mathbb{Z}$.

(We have also found a generator for Z/B, namely the coset $P_1 + B$.)

2. Exact sequences

A sequence of groups and homomorphisms

$$F \xrightarrow{\alpha} G \xrightarrow{\beta} H$$

is *exact at G* if $\mathrm{image}(\alpha) = \ker(\beta)$, where $\mathrm{image}(\alpha) = \alpha(F)$ and $\ker(\beta) = \{g \in G \mid \beta(g) = 0\}$.

EXAMPLE A.10. Prove the following.
(1) If both α and β are trivial (zero homomorphisms) then $G = 0$.
(2) If $\alpha = 0$ then β is injective.
(3) If $\beta = 0$ then α is surjective.
(4) If $F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 0$ is exact at G and at H then

$$G/\alpha(F) \cong H.$$

The result (4) is sometimes written $G/F \cong H$ by a slight abuse of notation: F is isomorphic to $\alpha(F)$.

A sequence of groups and homomorphisms is called *exact* if it is exact at every group.

The study of exact sequences is a part of *homological algebra*, about which we shall say more later in the course.

APPENDIX B

Review problems

These problems are set at the approximate level of understanding required to do well in the examination. The real exam questions will of course be more carefully written. No attempt has been made to make these questions similar in length to exam questions.

[1] Define the homomorphism f_* from $\pi_1(X; x_0)$ to $\pi_1(Y; y_0)$ associated with a based map $f : (X; x_0) \to (Y; y_0)$. Prove that your definition gives a well–defined function, and that it is a homomorphism.

Give examples to show that:

(i) f may be injective with f_* not injective;

(ii) f may be surjective with f_* not surjective.

[2] Let X and Y be topological spaces.

(a) Define the concept of a *continuous function* from X to Y.

(b) Define the term *compact* for a topological space.

(c) Suppose $f : X \to Y$ is continuous and surjective. Show that if X is compact then Y must also be compact.

(d) Using (b) (or another method) show that the circle S^1 is compact. (You may assume that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded).

(e) Is it true that the pre–image of a compact set under a continuous map is compact?

[3](a) Define what it means for two continuous maps to be *homotopic*, and what it means for two spaces to be (i) *homeomorphic* and (ii) *homotopy equivalent*. Give an example (with brief proofs) of two spaces that are homotopy equivalent without being homeomorphic. Can the reverse happen?

(b) Regard S^1 as the set $\{z \in \mathbb{C} \mid |z| = 1\}$, and let $p : I \to S^1$ be a loop with $p(0) = p(1) = 1$. Define the *degree* of p, written $\deg(p)$. State carefully but do not prove any preliminary results you need. Calculate the degree of the map $q_n(t) = e^{2\pi i n t}$.

(c) Define $f : S^1 \to S^1$ by $f(z) = z^n$. Assuming that deg defines an isomorphism from $\pi_1(S^1; 1) \to \mathbb{Z}$, calculate the homomorphism f_*. Prove that if $n \neq 0$ there can be no map $g : D^2 \to S^1$ such that g restricted to S^1 is equal to f.

[4](a) Explain briefly (without giving proofs) how the *fundamental group* $\pi_1(X, x_0)$ of a space X at a point x_0 is defined.

(b) State carefully a theorem relating the size of the fibres of a based covering of a path–connected space to the number of cosets of a subgroup of the fundamental group of the space.
(c) A certain path–connected topological space L has the sphere S^n as its universal cover, and the covering map $P : S^n \to L$ has 5 points in each fibre. Find if possible the structure of the group $\pi_1(L, \ell_0)$ for any $\ell_0 \in L$. (Indicate clearly any assumptions you make about n).
[5] Let X be a topological space, and suppose that U and V are two open, path–connected subsets of X such that $X = U \cup V$, and $U \cap V$ is non–empty and path–connected. Label the various inclusions as in the following diagram:

$$\begin{array}{ccc} U \cap V & \xrightarrow{k} & U \\ \ell\downarrow & & \downarrow i \\ V & \xrightarrow{j} & X \end{array}$$

Pick a base point $x_0 \in U \cap V$. Show that if the homomorphisms $k_* : \pi_1(U \cap V; x_0) \to \pi_1(U; x_0)$ and $\ell_* : \pi_1(U \cap V; x_0) \to \pi_1(V; x_0)$ are both onto, then the homomorphisms $i_* : \pi_1(U; x_0) \to \pi_1(X; x_0)$ and $j_* : \pi_1(V; x_0) \to \pi_1(X; x_0)$ are also onto.
[6](a) State how the projective plane $\mathbb{R}P^n$ may be defined as a quotient space of the sphere S^n. Compute the fundamental group of the projective plane $\mathbb{R}P^n$ (you may assume that $\pi_1(S^n)$ is known). Indicate carefully how the value of n affects your argument. If X is a space with $\pi_1(X) = \mathbb{Z}/3\mathbb{Z}$, explain why there can be no covering map $p : \mathbb{R}P^2 \to X$.
[7] State carefully and prove the *Borsuk–Ulam Theorem* concerning maps $f : S^2 \to S^1$ with the property that $f(-x) = -f(x)$ for all $x \in S^2$. (state but do not prove any standard results you need on fundamental groups).
[8] State clearly the classification of closed compact surfaces. Explain the importance of orientability and the Euler characteristic in the classification.

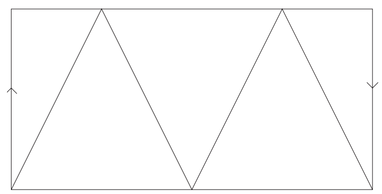
Find the standard form of the closed surfaces represented by the following symbols:

(a) $abca^{-1}cb^{-1}$

(b) $abcdefe^{-1}db^{-1}afc$

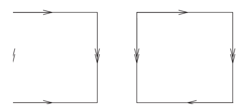
[9] Let K be a simplicial complex of dimension n and let L be a subcomplex of K. Define the *homology sequence* of the pair (K, L). State (but do not prove) the main theorem about this sequence.

Let K be the Möbius band triangulation shown above, and let L be the edge. Show that $H_1(L) \cong \mathbb{Z}$, and given that $H_2(K) = 0$, $H_1(K) \cong \mathbb{Z}$, show that if $\imath : L \hookrightarrow K$ is the inclusion map, then the induced map in homology $\imath_* : H_1(L) \to H_1(K)$ is multiplication by 2.



Now write down the homology sequence of the pair (K, L) from $H_2(K, L) = 0$ to $\tilde{H}_0(L) = 0$ and deduce the relative homology group $H_1(K, L)$.

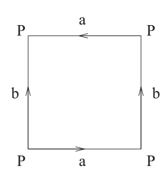
[10] Write down the fundamental (or homotopy) groups of the spaces obtained by identifying the edges of a square as in the following diagrams:



You need not prove your assertion. Prove that the two spaces are not homeomorphic.

[11] Find the Euler characteristic of a 2–sphere with n handles attached to it.

[12] Compute directly the homology groups of the space X obtained by identifying the edges in the following diagram.



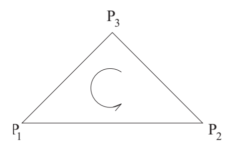
(a) Define a map f from X onto the topological circle give by the b edges by sending a point Q in the square to a point on either b edge horizontally. Compute the induced maps $f_* : H_n(X) \to H_n(b)$ for $n = 0, 1, 2$, by describing the images of generators.

(b) Do the same with the map g from X defined by sending a point Q to a point on the a edge directly above it.

[13] Describe without proofs how the *exact homology sequence of a pair* is defined.

[14] Let X be the simplicial complex consisting of the edges of the triangle below,

and let Y be the subcomplex consisting of the edge P_2P_3. Compute from first principles the relative homology groups $H_0(X, Y)$ and $H_1(X, Y)$.



[15] Consider the edge Y (a circle) of the disk X as a subcomplex. Compute $H_2(X, Y)$ directly.
[16] Let K be a simplicial complex with the following property: there is a vertex v_0 of K such that if σ is a q–simplex of K and v_0 is not a vertex of σ, then there is a (unique) $(q+1)$–simplex of K with σ as one of its q–dimensional faces and v_0 as one of its vertices. Prove that $H_q(K) = 0$ if $q > 0$. (Hint: how do you prove that $H_q(\Delta^n) = 0$ for $q > 0$ where Δ^n is the simplicial complex consisting of all the faces of an n–simplex?)
[17] Let K be a simplicial complex. Define the Euler characteristic of K and prove that if K and L are two simplicial complexes with $|K|$ homeomorphic to $|L|$ then the Euler characteristics of K and of L are equal. (You may state without proving any properties of the rank of an abelian group that you need).
[18](a) Let X be the space consisting of two circles with a single common point. Briefly describe the steps in the calculation of $\pi_1(X)$.
(b) Indicate with a sketch the covering space of X corresponding to the subgroup generated by a loop which runs three times round one circle and once round the other circle.
[19](a) What is meant by saying "spaces Y and Z have the same homotopy type"?
(b) Prove that the projective plane with one point removed does not have the same homotopy type as a torus with one point removed.
[20] State and prove the Brouwer Fixed Point Theorem concerning maps of closed balls.
[21] Let K be a simplicial complex. Define the terms *p–chain*, *p–boundary*, *p–cycle* and the *p^{th} homology group $H_p(K)$*. Prove that $H_0(K) \cong \mathbb{Z}^n$ where n is the number of components of K.
[22] And of course make sure you can do all the exercises...

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