§3: ALTERNATIVE CHARACTERIZATIONS OF TOPOLOGICAL SPACES

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0.1. **Closed sets.** In a topological space (X, τ), define a **closed** subset to be a subset whose complement is open. Evidently specifying the open subsets is equivalent to specfying the closed subsets.

The closed subsets of a topological space satsify the following properties:

(CTS1) $\emptyset$, X are closed.
(CTS2) Finite unions of closed sets are closed.
(CTS3) Arbitrary intersections of clsoed sets are closed.

Conversely, given such a family of subsets of X, then taking the open sets as the complements of each element in this family, we get a topology.

0.2. **Closure.** If S is a subset of a topological space, we define its **closure** $\overline{S}$ to be the intersection of all closed subsets containing S. Since X itself is closed containing S, this intersection is nonempty, and a moment's thought reveals it to be the minimal closed subset containing S.

The following is a simple but ubiquitously useful characterization of closure in terms of open sets.

Proposition 1. *Let X be a topological space, $Y \subset X$ and $x \in X$. TFAE:*
(i) $x \in \overline{Y}$.
(ii) For any open set U containing x, $U \cap Y \neq \emptyset$.

Proof: If U is an open set containing x and disjoint from Y, then $X \setminus U$ is closed, contains Y and does not contain x, so $x \in \overline{Y}$. The converse is quite similar, and we leave it to the reader.

A subset Y of a topological space X is **dense** if $\overline{Y} = X$. For example, both the rational numbers and the irrational numbers are dense in $\mathbb{R}$; in a discrete space no proper subset is dense; in an indiscrete space any nonempty subset is dense.

The **density** $d(X)$ of a topological space is defined to be the minimal cardinality of a dense subset.[1]

[1] This is an example of a **cardinal invariant** of a topological space, i.e., a mapping which assigns to each topological space a cardinal number such that homeomorphic spaces get assigned the same cardinal number. Here we are making use of the fact that any set of cardinal numbers is well-ordered, which is known to be equivalent to the Axiom of Choice (AC). As in many areas of modern mathematics, one does not get very far in general topology without assuming AC, and it will certainly be a default assumption for us.

Some simple obervations on density:

a) $d(X) \leq |X$.
b) $d(X) = 0$ iff $X = \emptyset$.
c) For every discrete space we have $d(X) = |X|$.
d) For every indiscrete space we have $d(X) = 1$.
e) If X is infinite and all singleton sets are closed, then $d(X) \geq \aleph_0$.

Notice that the converses of c) and d) do not hold. For instance the set $\mathbb{Q}$ of rational numbers with the Euclidean topology is not discrete and has density and cardinality both equal to $\aleph_0$. Evidently a space X has density 1 iff there exists $x \in X$ such that $\overline{\{x\}} = X$. Such a point is said to be a **generic point** of X. For instance the point $\circ$ in the DVR space $\{\circ, \bullet\}$ is generic.

A topological space X with $d(X) \leq \aleph_0$ is called **separable**. Obviously any countable spcae is separable, as are many uncountable spaces, e.g. $\mathbb{R}^N$ for any $N \in \mathbb{N}$.

Exercise X.X*: Let X be a Hausdorff topological space.
a) Show that $|X| \leq 2^{2^{|X|}}$.
b) For each infinite cardinal κ, exhibit a Hausdorff space X with density κ and cardinality $2^{2^{\kappa}}$.

Viewing closure as a mapping c from 2^X to itself, it satisfies the following properties, the **Kuratowski closure axioms**:

(KC1) $c(\emptyset) = \emptyset$.
(KC2) For $A \in 2^X$, $A \subset c(A)$.
(KC3) For $A \in 2^X$, $c(c(A)) = c(A)$.
(KC4) For $A, B \in 2^X$, $c(A \cup B) = c(A) \cup c(B)$.

Note that (KC4) implies the following axiom:

(KC5) If $B \subset A$, $c(B) \subset c(A)$.

Indeed, $c(A) = c((A \setminus B) \cup B) = c(A \setminus B) \cup c(B)$.

A function $c : 2^X \to 2^X$ satisfying (KC1)-(KC4) is called an "abstract closure operator." Kuratowski noted that any such operator is indeed the closure operator for a topology on X:

Theorem 2. *(Kuratowski) Let X be a set, and let $c : 2^X \to 2^X$ be an operator satisfying axioms (KC1), (KC2) and (KC4).*
a) The subsets $A \in 2^X$ satisfying $A = c(A)$ obey they axioms (CTS1)-(CTS3) and hence are the closed subsets for a unique topology τ_c on X.
b) If c also satisfies (KC3), then closure in τ_c corresponds to closure with respect to c: for all $A \subset X$ we have $\overline{A} = c(A)$.

Proof: a) Call a set c-closed if $A = c(A)$. By (KC1) the empty set is c-closed; by (KC2) X is c-closed. By (KC2) finite unions of c-closed sets are closed. Now let

$\{A_\alpha\}_{\alpha \in I}$ be a family of $c-$closed sets, and put $A = \cap A_\alpha$. Then for all α, $A \subset A_\alpha$, so by (KC5), $c(A) \subset c(A_\alpha)$ for all α, so

$$c(A) \subset \cap\, c(A_\alpha) = \cap A_\alpha = A.$$

Thus the c-closed sets satisfy (CTS1)-(CTS3), so that the family τ_c of complements of c-closed sets form a topology on X.

Now assume (KC3); we wish to show that for all $A \subset X$, $c(A) = \overline{A}$. We have $\overline{A} = \cap_{C=c(C)\supset A} C$, the intersection extending over all closed subsets containing A. By (KC3), $c(A) = c(c(A))$ is a closed subset containing A we have $\overline{A} \subset c(A)$. Conversely, since $A \subset \cap_C C$, $c(A) \subset \cap_C c(C) = \cap_C C = \overline{A}$. So $c(A) = \overline{A}$.

Remark: Later we will see an interesting example of an operator which satisfies (KC1), (KC2), (KC4) but not necessarily (KC3): the **sequential closure**.

The following result characterizes continuous functions in terms of closure.

Theorem 3. *(Hausdorff) Let $f : X \to Y$ be a map of topological spaces. TFAE:*
(a) f is continuous.
(b) For every subset S of X, $f(\overline{S}) \subset \overline{f(S)}$.

Proof: Suppose f is continuous, S is a subset of X and $\overline{A} = A \supset f(S)$. If $x \in X$ is such that $f(x) \in Y \setminus A$, then, since f is continuous and $Y \setminus A$ is open in Y, $f^{-1}(Y \setminus A)$ is an open subset of X containing x and disjoint from S. Therefore x is not in the closure of S.

Conversely, if f is not continuous, then there exists some open $V \subset Y$ such that $U := f^{-1}(V)$ is not open in X. Thus, there exists a point $x \in U$ such that every open set containing x meets $S := X \setminus U$. Thus $x \in \overline{S}$ but $f(x)$ is in V hence not in $Y \setminus V$, which is a closed set containing $f(\overline{S})$.

If $x \in \overline{Y}$ and U is an open set containing x, then $A := X \setminus U$

0.3. **Interior operator.** The dual notion to closure is the **interior** of a subset A in a topological space: A° is equal to the union of all open subsets of A. In particular a subset is open iff it is equal to its interior. We have

$$A^\circ = X \setminus \overline{X \setminus A},$$

and applying this formula we can mimic the discussion of the previous subsection in terms of axioms for an "abstract interior operator" $A \mapsto i(A)$, which one could take to be the basic notion for a topological space. But this is so similar to the characterization using the closure operator as to be essentially redundant.

0.4. **Boundary operator.** For a subset A of a topological space, one defines the **boundary**[2]

$$\partial A = \overline{A} \setminus A^\circ = \overline{A} \cap \overline{X \setminus A}.$$

Evidently ∂A is a closed subset of A, and, since $\overline{A} = A \cup \partial A$, A is closed iff $A \supset \partial A$. A set has empty boundary iff it is both open and closed, a notion which is important in connectedness and in dimension theory.

[2]Alternate terminology: **frontier**.

Example: Let X be the real line, $A = (-\infty, 0)$ and $B = [0, \infty)$. Then $\partial A = \partial B = \{0\}$, and

$$\partial(A \cup B) = \partial \mathbb{R} = \emptyset \neq \{0\} = (\partial A) \cup (\partial B);$$
$$\partial(A \cap B) = \partial \emptyset = \emptyset \neq \{0\} = (\partial A) \cap (\partial B).$$

Thus the boundary operator is not as well-behaved as either the closure or interior operators. According to Willard, p. 28: "It is possible, but unrewarding, to characterize a topology completely by its frontier [boundary] operation."

0.5. **Neighborhoods.** Let x be a point of a topological space, and let N be a subset of X. We say that N is a **neighborhood** of x if $x \in N^{\circ}$. Open sets are characterized as being neighborhoods of each point they contain.

Let $\mathcal{N}_x$ be the family of all neighborhoods of x. It satisfies the following nice properties:

(NS1) $N \in \mathcal{N}_x \implies x \in N$.
(NS2) $N,\ N' \in \mathcal{N}_x \implies N \cap N' \in \mathcal{N}_x$.
(NS3) $N \in \mathcal{N}_x$, $N' \supset N \implies N' \in \mathcal{N}_x$.
(NS4) For $N \in \mathcal{N}_x$, there exists $U \in \mathcal{N}_x$, $U \subset N$, such that $y \in V \implies V \in \mathcal{N}_y$.

Suppose we are given a set X and a function which assigns to each $x \in X$ a family $\mathcal{N}(x)$ of subsets of X satisfying (NS1)-(NS3). Then the collection of subsets U such that $x \in U \implies U \in \mathcal{N}(x)$ form a topology on X. If we moreover impose (NS4), then $\mathcal{N}(x) = \mathcal{N}_x$ for all x.