INTRODUCTION TO FROBENIUS MANIFOLDS

NOTES FOR THE MRI MASTER CLASS 2009

1. FROBENIUS ALGEBRAS

Definition and examples. For the moment we fix a field k which contains $\mathbb{Q}$, such as $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$ itself. To us a k-algebra is simply a k-vector space A which comes with k-bilinear map (the product) $A \times A \to A$, $(a, b) \mapsto ab$ which is associative ($(ab)c = a(bc)$ for all $a, b, c \in A$) and a unit element $e \in A$ for that product: $e.a = e.1 = e$ for all $a \in A$. We use e to embed k in A by $\lambda \in k \mapsto \lambda e$ and this is why we often write 1 instead of e.

Definition 1.1. Let A be a k-algebra that is commutative, associative and finite dimensional as a k-vector space. A *trace map* on A is a k-linear function $I : A \to k$ with the property that the map $(a, b) \in A \times A \mapsto g(a, b) := I(ab)$ is nondegenerate as a bilinear form. In other words, the resulting map $a \mapsto I(a.-)$ is a k-linear isomorphism of A onto the space A^* of k-linear forms on A. The pair (A, I) is the called a *Frobenius k-algebra*.

Remark 1.2. This terminology can be a bit confusing: for a finite dimensional k-algebra A, one defines its trace $\mathrm{Tr} : A \to k$ as the map which assigns to $a \in A$ the trace of the operator $x \in A \mapsto ax \in A$. This is in general *not* a trace map.

Exercise 1. Prove that g then satisfies $g(ab, c) = g(a, bc)$ and that conversely, any nondegenerate k-bilinear symmetric map $g : A \times A \to k$ with that property determines a trace map on A.

Exercise 2. Let (A, I) be a Frobenius algebra. Show that the product determines the unit element: no other element than e can serve as a unit.

Examples 1.3. (a) A trace map on the field k (viewed as k-algebra) is given by a nonzero scalar λ, for it will be of the form $a \in k \mapsto \lambda a \in k$. It is *simple*, meaning that is has no nontrivial quotients.

(b) Let $A = k[t]/(t^n)$ with n a positive integer. A k-linear form $I : A \to k$ is a trace map if and only if $I(t^{n-1}) \neq 0$.

(c) Let M be a compact oriented manifold of even dimension $2d$ (for instance a compact complex manifold). Then its even degree cohomology $H^{\mathrm{even}}(M; k) = \oplus_{k=0}^{d} H^{2k}(M; k)$ is a Frobenius $\mathbb{R}$-algebra for the cup product and for integration: I is zero in degree $\neq 2d$ and sends a class in degree $2d$ to its value on the fundamental class (if $k = \mathbb{R}$, and if we use De Rham cohomology, then this is just integration of a $2d$-form); the nondegenaracy of $(a, b) \mapsto I(a, b)$ simply expresses the Poincaré duality: the fact that

$$(a, b) \in H^{2k}(M; k) \times H^{2d-2k}(M; k) \mapsto (a \cup b)[M] \in k$$

is perfect pairing. For example, for complex projective d-space, $\mathbb{P}^d$, $d \geq 1$, $H^{\text{even}}(\mathbb{P}^d; k) = H^\bullet(\mathbb{P}^d; k)$ is as an algebra generated by the hyperplane class $\eta \in H^2(\mathbb{P}^d; k)$. Clearly η^{d+1}. On the other hand, $\eta^d[\mathbb{P}^d] = 1$ (this expresses the fact that d general hyperplanes in $\mathbb{P}^d$ have exactly one point in common) and so this reproduces in fact a special case of example (b) with $n = d + 1$.

(d) (Zero-dimensional complete intersections) Here is an interesting example from commutative algebra that generalizes example (b) above. Let $f, \dots, f_m \in k[[z_1, \dots, z_m]]$ be formal power series without constant term and assume that they span an ideal in $k[[z_1, \dots, z_m]]$ of finite codimension: $A := k[[z_1, \dots, z_m]]/(f_1, \dots, f_m)$ is finite dimensional. Then one can show that the jacobian

$$\det\Big(\frac{\partial f_i}{\partial z_j}\Big)_{i,j}$$

has a nonzero image in A and generates there a one-dimensional ideal. A linear form $I : A \to k$ gives A a trace map if and only if it is nonzero on this image. (Grothendieck defined a specific trace map on A as an iterated residue; it is written

$$I(g) = \mathrm{Res}_{(f_m=0)} \cdots \mathrm{Res}_{(f_1=0)} \frac{g\,df_1 \wedge \cdots \wedge df_m}{f_1 \cdots f_m}.)$$

Exercise 3. Let $f := t^n + a_1 t^{n-1} + \cdots + a_0 \in k[t]$ be a monic polynomial and put $A = k[t]/(f)$. The images of $1, t, \dots, t^{n-1}$ form a k-basis of A. Prove that a trace map is given by the function $I : A \to k$ that takes the value 1 on the image of t^{n-1} and zero on the images of lower powers of t.

We next discuss three ways of producing new Frobenius algebras out of old: direct sums, tensor products and rescalings.

Direct sums. Let A and B be commutative k-algebras. Then the vector space $A \oplus B$ is an algebra for componentwise multiplication: $(a, b)(a', b') = (aa', bb')$ with $(1, 1)$ as its identity element. The identity element of A corresponds to $\iota = (1, 0)$, which is in $A \oplus B$ an *idempotent*: its satisfies $\iota\iota = \iota$. We can recognize that an algebra is thus obtained by its idempotents: if we are given a commutative k-algebra C with unit and a nonzero idempotent $\iota \in C$, then $C' := \iota C$ is multiplicatively closed and ι serves as a unit element for ιC. The element $1-\iota$ is also an idempotent, for $(1-\iota)^2 = 1-2\iota+\iota^2 = 1-\iota$, and so the same can be said about $C'' := (1 - \iota)C$. We have $C = C' \oplus C''$ (write $c \in C$ as $\iota c+(1-\iota)c$), not just as k-vector spaces, but even as algebras.

Lemma-definition 1.4. *Let* A *be a finite dimensional commutative* k*-algebra. Then* A *is called* semisimple *if the following equivalent conditions are fulfilled:*

(i) *A is the sum of its one-dimensional ideals,*
(ii) *A is isomorphic to k^n with componentwise multiplication.*

In that case A is in fact the direct *sum of its one-dimensional ideals.*

Proof. (ii)$\Rightarrow$(i). The one-dimensional ideals of k^n are easily seen to be the individual summands.

(i)$\Rightarrow$(ii). If I and J are distinct one-dimensional ideals of A, then we have $IJ \subset I \cap J = \{0\}$. If A is spanned by one-dimensional ideals, then we can select $I_1, \dots, I_n$ of those such that A is their direct sum as a k-vector space. Write $1 = a_1 + \cdots + a_n$ with $a_i \in I_i$. If $e_i \in I_i$ is nonzero, then $e_i = e_i.1 = \sum_j e_i a_j = e_i a_i$. It follows that $a_i \neq 0$ and so a_i generates I_i also. Upon replacing e_i by a_i, we see that a_i is an idempotent and that $(\lambda_1, \dots, \lambda_n) \mapsto \sum_i \lambda_i a_i$ is an isomorphism of algebras. □

Exercise 4. Prove that for a semisimple commutative k-algebra A, the trace $\mathrm{Tr} : A \to k$ (which assigns to a the trace of $x \in A \mapsto ax \in A$) is also a trace map. Compute it in terms of its decomposition into one-dimensional ideals.

If (A, I) and (B, J) are Frobenius k-algebras, then $(a, b) \mapsto I(a) + J(b)$ is a trace map on $A \oplus B$. It is easy to check that any trace map on the k-algebra $A \oplus B$ must be of this form. In particular, a trace map on a semisimple k-algebra of dimension n is given by n nonzero scalars. More precisely, it is given by a map from the n-element set of its one-dimensional ideals to $k^\times$.

Remark 1.5. In the situation of Lemma-definition **??**, the maximal ideals of A are the direct sums of all but one of the one-dimensional ideals. So these are also n in number and the corresponding quotient algebras are one-dimensional. If you are familiar with the language of algebraic geometry, then the above Lemma says that A is semisimple if and only if is it the coordinate ring of a reduced k-variety $\mathrm{Spec}(A)$ that has n distinct points. An idempotent of A generating one dimensional ideal is the the characteristic function of a point of $\mathrm{Spec}(A)$ (the maximal ideal defining the point is the sum of the remaining one dimensional ideals). So a trace map can be interpretated as a function $\mathrm{Spec}(A) \to k^\times$.

Tensor product. Let A and B again be commutative k-algebras. Then the vector space $A \otimes B$ is an algebra whose product is characterized by the property that $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ (beware that a general element of $A \otimes B$ is of the form $\sum_i a_i \otimes b_i$). Its unit is $1 \otimes 1$. If (A, I) and (B, J) are Frobenius algebras, then we have a trace map on $A \otimes B$ defined by $(a \otimes b) \mapsto I(a)J(b)$. This is best seen by using the associated nondegenerate symmetric bilinear forms $g(a, a') := I(aa')$ and $h(b, b') := J(bb')$. For then it reduces to seeing that $((a, b), (a', b')) \mapsto g(a, a')h(b, b')$ factors through a symmetric bilinear form $(A \otimes B) \times (A \otimes B) \to k$ that is nondegenerate whenever g and h are. This property no longer involves the algebra structure and is easily verified by choosing bases of A resp. B on which g resp. h takes a diagonal form.

For instance, if we put $A_d = k[t]/(t^{d+1})$ with $I_d : A_d \to k$ being 1 on the image of t^d and zero on the image of lower powers, then $A_{d_1} \otimes \cdots \otimes A_{d_m}$ is isomorphic to $k[t_1, \dots, t_m]/(t_1^{d_1}, \dots, t_m^{d_m})$ with trace map the function which takes the value 1 on the image of the monomial $t_1^{d_1} \cdots t_m^{d_m}$ and zero on the images of monomials of lower multidegree. Notice that this is in fact of the type mentioned in Example **??**-d. (It is also the cohomology of $\mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_m}$.)

Rescalings. Let A be a commutative finite dimensional k-algebra. For any nonzero scalar $\lambda \in k$ we can define a new commutative algebra with the same underlying vector space: replace the product by $a * b := \lambda ab$ and the unit element by $\lambda^{-1}e$. Notice that the map $\phi : x \in A \mapsto \lambda^{-1}x \in A$ is an algebra homomorphism $(A, \cdot) \to (A, *)$: $\phi(a) * \phi(b) = \lambda(\lambda^{-1}a \cdot \lambda^{-1}b) = \lambda^{-1}ab = \phi(ab)$. If we are given on A a trace map $I : A \to k$, then ϕ becomes an isomorphism of Frobenius algebras if we endow $(A, *)$ with the trace map λI.

Associativity equations. For a Frobenius algebra (A, I), we may also consider the trilinear map $T : A \times A \times A \to k$, $T(a, b, c) := I(abc)$. Notice that if we regard A just as a k-vector space and retain from the algebra structure on A only the unit element 1, then we can recover the full Frobenius algebra structure simply as follows: the product ab is the unique element of A with the property that $T(a, b, x) = T(ab, 1, x)$ for all $x \in A$. We now ask a converse question: given a k-*vector space* A, a trilinear map $T : A \times A \times A \to k$, and an element $e \in A$, what properties need we impose on order that T defines a Frobenius algebra? Clearly, T must be symmetric. We also want that the bilinear map $(a, b) \in A \times A \mapsto T(a, b, e) \in k$ is nondegenerate: this means that any linear function on A is given by $x \to T(a, x, e)$ for a unique $a \in A$. We therefore have defined a bilinear map $A \times A \to A$, $(a, b) \mapsto ab$ characterized by $T(ab, x, e) = T(a, b, x)$ for all x. Since T is symmetric, this product is commutative: $ab = ba$ for all $a, b \in A$. It is also true that e is automatically a unit: ae is characterized by $T(ae, x, e) = T(a, e, x)$ and since T is symmetric, this implies $ae = a$.

But associativity need not hold and has to be imposed: we want that $T(ab, c, x) = T(a, bc, x)$ for all $a, b, c, x \in A$, where we should remember that T also interferes in the definition of the product. So this is not so simple: if $(e = e_1, \dots, e_n)$ is a basis for A, and $T_{ijk} := T(e_i, e_j, e_k)$, then $(g_{jk} := T_{1jk})_{jk}$ is a nondegenerate matrix. If we denote by (g^{jk}) the inverse matrix, then the above recipe yields the product given by

$$e_i e_j = T_{ijk} g^{kl} e_l \quad \text{(here we use the Einstein summation convention).}$$

We want that $T(e_i e_j, e_k, e_l) = T(e_i, e_j e_k, e_l)$ and so $T(T_{ijp} g^{pq} e_q, e_k, e_l) = T(e_i, T_{jkp} g^{pq} e_q, e_l)$, or

$$\text{(Ass.)} \qquad T_{ijp} g^{pq} T_{qkl} = T_{jkp} g^{pq} T_{iql}.$$

So this system of equations must be obeyed in order that the product be associative.

Graded version. It may happen (as is the case in the case of Example **??**-c) that the algebra comes with a graded structure: $A = \oplus_{d\geq 0} A_d$ (this means that each A_d is a k-linear subspace, $A_0 = k$ and $A_k A_l \subset A_{k+l}$) and that I is nonzero in just one degree (this must then be the top degree). We then say that $(A_\bullet, I)$ is a *graded* Frobenius algebra.

Other bases. In what follows, we will often have to deal with Frobenius algebras that depend on parameters. One takes care of this algebraically by allowing k to be replaced by a $\mathbb{Q}$-algebra R. The definition of Frobenius algebra then requires minor changes: A is a now a free R-module of finite rank and $I : A \to R$ an R-linear map such that $(a, b) \in A \times A \mapsto I(ab) \in R$ is nondegenerate: the induced map $A \to \mathrm{Hom}_R(A, R)$ must be an isomorphism of R-modules. For every maximal ideal $\mathfrak{m} \subset R$, we may reduce modulo that ideal and find a Frobenius algebra in the earlier sense over the field $k_\mathfrak{m} := R/\mathfrak{m}$: $A(k_\mathfrak{m}) := A/\mathfrak{m}A$ is a $k_\mathfrak{m}$-algebra of the same dimension as the R-rank of A and I induces a trace map $I(k_\mathfrak{m}) : A(k_\mathfrak{m}) \to k_\mathfrak{m}$. For instance, if $R = \mathbb{C}[t]$, then we have for every $t_0 \in \mathbb{C}$ a Frobenius algebra $A/(t - t_0)A$.

2. Frobenius manifolds

In this section we need some basic facts from (complex) differential geometry. If M is an ordinary smooth m-manifold, then a *pseudometric* on M is a symmetric bilinear form on the tangent bundle TM of M (usually denoted g) which is *nondegenerate*. This means that g identifies the tangent bundle TM with its dual, the cotangent bundle T^*_M of M. The fundamental theorem of Riemannian geometry asserts the existence of a *Levi-Civita* connection on TM, i.e., a connection ∇ characterized by the property that if X, Y, Z are vector fields on an open $U \subset M$, then:

(i) $Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$ (flatness of the metric tensor) and

(ii) $\nabla_X Y - \nabla_Y X = [X, Y]$ (torsion freeness of the connection).

We further recall that the curvature of such a connection is given by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

It is in fact a tensor: its value at $p \in M$ only depends on the values X_p, Y_p, Z_p of the vector fields in p and thus R may be regarded as a section of $TM \otimes TM \otimes \mathrm{Hom}(TM, TM)$. Its vanishes identically if and only if there exists an atlas of charts $(U, \kappa : U \to \mathbb{R}^m)$ with the property that $g|U$ is the pull-back of a pseudometric on $\mathbb{R}^m$ with constant coefficients. We shall call such a chart *flat*, because it identifies the flat vector fields on U with the constant vector fields on $\mathbb{R}^m$.

The proof shows that this is also true in a complex-analytic setting: take for M now a complex n-manifold, for TM the holomorphic tangent bundle τ_M and for g a holomorphically varying nondegenerate symmetric bilinear form on τ_M. There is then a Levi-Civita connection which assigns to any pair of holomorphic vector fields X, Y on the same domain a holomorphic vector field $\nabla_X Y$ on that domain. Its curvature is a holomorphic tensor and that tensor vanishes identically if and only there exists an atlas of holomorphic charts $(U, \kappa : U \to \mathbb{C}^n)$ with the property that $g|U$ is the pull-back of a nondegenerate symmetric bilinear form on the tangent bundle of $\mathbb{C}^n$ with constant coefficients. Notice that in this setting g can never be a metric (for an inner product must always be complex *anti*linear in one of its arguments). On the other hand, any nondegenerate form on a complex vector space can be diagonalized with 1's on the diagonal. This means that we may prescribe the form on $\mathbb{C}^n$ as given by $\sum_\nu dz^\nu \otimes dz^\nu$: charts for which this form pulls back to the restriction of g form an atlas. Its transition maps are the restriction of an affine-linear transformations from one open subset to another with the property that its linear part respects the quadratic form $\sum_\nu (z^\nu)^2$, i.e., lies in the complex orthogonal group $SO(n, \mathbb{C})$.

We are now about to introduce the central notion of this course. Let M be a complex manifold on whose holomorphic tangent bundle is given a nondegenerate symmetric bilinear form g and a symmetric trilinear form T, both depending homolomorphically on the base point. Denote by $\cdot$ the product of this bundle characterized by $g(X \cdot Y, Z) = T(X, Y, Z)$ (we use here that g is nondegenerate). It is clear that this product is symmetric. Denote by ∇ the Levi-Civita connection on the holomorphic tangent bundle and define a one parameter family of connections ∇^λ on this bundle by

$$\nabla^\lambda_X Y := \nabla_X Y + \lambda X \cdot Y, \quad \lambda \in \mathbb{C}.$$

From the symmetry of the product we see that

$$\nabla^\lambda_X Y - \nabla^\lambda_Y X - [X, Y] = \nabla_X Y - \nabla_Y X - [X, Y] = 0$$

and so ∇^λ is torsion free. If for a local vector field X on M, μ_X denotes the multiplication operator on vector fields: $\mu_X(Y) := X \cdot Y$, then define

$$R'(X, Y) := [\nabla_X, \mu_Y] - [\nabla_Y, \mu_X] - \mu_{[X,Y]}$$

Proposition 2.1. *The operator R' defined above is a tensor; it is in fact a holomorphic 2-form that takes values in the symmetric endomorphisms of τ_M: $g(R'(X, Y)Z, W)$ is antisymmetric in (X, Y) and symmetric in (Z, W).*

Moreover, the following are equivalent:

(FM i) *∇ is flat, the product is associative and the trilinear symmetric form $T(X, Y, Z)$ is in a flat coordinate system locally like the third order derivative of a holomorphic function: M is covered by flat chart domains U on which there exists a holomorphic function $\Phi : U \to \mathbb{C}$ such that for any triple of flat vector fields X, Y, Z on U, we have $T(X, Y, Z) = \nabla_X \nabla_Y \nabla_Z \Phi$.*

(FM ii) *For every $\lambda \in \mathbb{C}$, the connection ∇^λ is flat.*

(FM iii) *∇ is flat, the product is associative and R' is identically zero.*

Proof. We compute the curvature tensor of ∇^λ. First note that

$$\nabla^\lambda_X \nabla^\lambda_Y = (\nabla_X + \lambda\mu_X)(\nabla_Y + \lambda\mu_Y) = \nabla_X \nabla_Y + \lambda(\mu_X \nabla_Y + \nabla_X \mu_Y) + \lambda^2 \mu_X \mu_Y.$$

It follows that in a straightforward manner that

$$\begin{aligned} R(\nabla^\lambda)(X,Y) = \nabla^\lambda_X\nabla^\lambda_Y - \nabla^\lambda_Y\nabla^\lambda_X - \nabla^\lambda_{[X,Y]} = \\ = R(\nabla)(X,Y) + \lambda R'(X,Y) + \lambda^2(\mu_X\mu_Y - \mu_Y\mu_X). \end{aligned}$$

Since $R(\nabla^\lambda)$ is a tensor for every λ, the coefficient of every power of λ is. In particular, R' is a tensor. It is clear from the definition that $g(R'(X,Y)Z,W)$ is antisymmetric in (X,Y). We further note that

$$\begin{aligned} g(\nabla_X(Y\cdot Z),W) - g(Y\cdot\nabla_X Z,W) = \\ = -g(Y\cdot Z,\nabla_Z W) + X(g(Y\cdot Z,W)) - g(Y\cdot\nabla_X Z,W) = \\ = -T(Y,Z,\nabla_Z W) + X(T(Y,Z,W)) - T(Y,\nabla_X Z,W) \end{aligned}$$

is symmetric in Y and Z. This remains true if we interchange Y and X. Since $g([X,Y]\cdot Z,W) = T([X,Y],Z,W)$ is also symmetric in Y and Z, it follows that $g(R'(X,Y)Z,W)$ is as well.

We now assume that ∇ is flat: $R(\nabla) = 0$. We see that ∇^λ is flat for all λ if and only if $R' = 0$ and $\mu_X\mu_Y = \mu_Y\mu_X$ for all X,Y. The last condition amounts to: $X\cdot(Y\cdot Z) = Y\cdot(X\cdot Z)$ for all X,Y,Z. But the symmetry of the product implies that the left hand side equals $X\cdot(Z\cdot Y)$ and the right hand side $(X\cdot Z)\cdot Y$. So this is just the associativity property. This proves (ii)$\Leftrightarrow$(iii).

We next prove (i)$\Leftrightarrow$(iii). Since g is flat we can use a flat chart (U,κ) such that $B = \kappa(U) \subset \mathbb{C}^n$ is an open polydisk. This allows us to pass to $B \subset \mathbb{C}^n$ and g having constant coefficients. Then ∇ is the usual derivation and the flat vector fields are the constant ones. Suppose we are given holomorphic functions $(f_{ijk} : B \to \mathbb{C})_{1\le i,j,k\le n}$. It is well-known (and not hard to prove) that these can arise as the third order partial derivatives of a holomorphic function Φ ($f_{ijk} = \partial_i\partial_j\partial_k\Phi$ for all i,j,k) if and only if $\partial_l f_{ijk}$ is symmetric in all its indices. In more intrinsic terms, if f is a trilinear form on the tangent bundle of B, then there exists a holomorphic Φ such that $f(X,Y,Z) = \nabla_X\nabla_Y\nabla_Z\Phi$ for all triples of flat vector fields X,Y,Z if and only if for any quadruple of flat vector fields (X,Y,Z,W), $X(f(Y,Z,W))$ is symmetric in its arguments. We want to apply this to $f(X,Y,Z) := g(X\cdot Y,Z)$. We already know that this expression is symmetric in its three arguments. We now compute

$$\begin{aligned} Xg(Y\cdot Z,W) &= g(\nabla_X(Y\cdot Z),W) + g(Y\cdot Z,\nabla_X W) \text{ (flatness of metric tensor)} \\ &= g(\nabla_X(Y\cdot Z),W) \text{ (since } W \text{ is flat).} \end{aligned}$$

But since X,Y,Z are flat, we have

$$R'(X,Y)Z = \nabla_X(Y\cdot Z) - \nabla_Y(X\cdot Z)$$

(the other terms vanish) and thus we see that $Xg(Y\cdot Z,W)$ is symmetric in X and Y if and only if $R' = 0$. □

Remark 2.2. A function Φ that appears in (FM-i) is called a (local) *potential function*. Since here only its third order derivatives matter, it is (in terms of flat coordinates $(z^1, \dots, z^n)$) unique up to a polynomial in of degree two. In particular, a potential function need not be defined on all of M. The associativity equations (Ass) now read as a highly nontrivial system of partial differential equations: if $(z^1, \dots, z^n)$ is a system of flat coordinates and $\partial_\nu := \frac{\partial}{\partial z^\nu}$, then we require that for all i, j, k, l,

$$\text{(WDVV)} \qquad (\partial_i \partial_j \partial_p \Phi) g^{pq} (\partial_q \partial_k \partial_l \Phi) = (\partial_j \partial_k \partial_p \Phi) g^{pq} (\partial_i \partial_q \partial_l \Phi).$$

These are known as the *Witten-Dijkgraaf-Verlinde-Verlinde* equations.

Definition 2.3. A *(complex) Frobenius manifold* is a complex manifold M whose holomorphic tangent bundle is (fiberwise) endowed with the structure of a Frobenius $\mathbb{C}$-algebra $(\cdot, I, e)$ such that (i) the equivalent conditions of Proposition **??** are fulfilled for the associated symmetric bilinear and trilinear forms g and T and (ii) the identity vector field e on M is flat for the Levi-Civita connection of g.

Dubrovin's definition of a Frobenius manifold also requires that is given an Euler field, a notion that we introduce later.

Remark 2.4. It is clear that the equivalent properties of **??** allow us to leave out or modify some of these conditions without consequence. Sometimes they are also used to introduce weaker versions of the notion of a Frobenius manifold. Manin does not assume that there exists an identity vector field. We note that if an identity vector field e exists, then it is unique: e' has also that property, then $g(X, e') = T(X, e', e) = T(X, e, e') = g(X, e)$ for all X and so $e = e'$ by the nondegeneracy of g. But then it still need not be flat. Observe that in view of the relation $I(X) = g(X, e)$, the flatness of e is equivalent to the flatness of the holomorphic differential I (relative to the connection defined by ∇ on the cotangent bundle). This means that in terms of a flat coordinate system e and I have constant coefficients. We will refer to I as the the *trace differential* I (Manin calls this the *co-unit* and denotes it ε).

Corollary 2.5. *The trace differential of a Frobenius manifold is closed.*

Proof. We just noticed that I has constant coefficients relative to a flat coordinate system. It is clear that such a differential is closed. □

So locally the trace differential can be written as the differential of a function. Such a function is called a *metric potential*.

Observe that g resp. T are given as symmetric sections of $\Omega_M \otimes \Omega_M$ resp. $\Omega_M \otimes \Omega_M \otimes \Omega_M$. So the notion of a Frobenius manifold has a contravariant character: if $h : N \to M$ is a holomorphic local isomorphism from a complex manifold N to a complex manifold M, then a Frobenius structure on M induces one on N.

Canonical coordinates for semisimple Frobenius manifolds. Let us attempt to gain some geometric understanding of a Frobenius structure on a complex manifold. If $(M; \cdot, I, e)$ a Frobenius manifold of dimension n, then the algebra structure on the holomorphic tangent space T_pM defines a subscheme in its dual T_p^*M that has $(T_pM, \cdot)$ as its coordinate ring. If this subscheme is reduced, i.e., when $(T_pM, \cdot)$ is semisimple, then this subscheme consists of an unordered basis of T_p^*M whose dual (unordered) basis is the set of idempotents of $(T_pM, \cdot)$. So the Frobenius structure on M defines a subscheme of the holomorphic cotangent bundle τ_M^*. The set of $p \in M$ for which this coordinate ring is reduced is an open subset of M (which may be empty); it is the locus where the Frobenius structure is semisimple. On that set we have locally a basis of $\dim M$ vector fields of idempotents. This does not grasp the notion in full, as we also need the trace differential as part of our data and the flatness of the metric and the identity vector field. Let us for this purpose concentrate on the case of a *semisimple Frobenius manifold*, i.e., one for which the Frobenius algebras in its tangent spaces are semisimple.

Let M be a semisimple Frobenius manifold of dimension n. So every $p \in M$ has a neighborhood U on which we have a basis of idempotent vector fields $e_1, \dots, e_n$. Notice that the symmetric bilinear form g takes on this basis the diagonal form: $g(e_i, e_j)$ is zero unless $i = j$ in which case we get $I(e_i)$.

Proposition 2.6. *The idempotent vector fields commute:* $[e_i, e_j] = 0$ *for all* i, j.

Proof. Bearing in mind that $e_k \cdot e_l = \delta_{kl} e_k$ and that $(e_k)_k$ is orthogonal for g (so that $g(e_k, e_l) = \delta_{kl} g(e_k, e_k)$), we compute

$$\begin{aligned}
&g(R'(e_i, e_j)e_k, e_l) = \\
&= g\big(\nabla_{e_i}(e_j \cdot e_k) - e_j \cdot \nabla_{e_i} e_k - \nabla_{e_j}(e_i \cdot e_k) + e_i \cdot \nabla_{e_j} e_k - [e_i, e_j] \cdot e_k, e_l\big) = \\
&\quad = g\big(\delta_{jk} \nabla_{e_i} e_k - \delta_{jl} \nabla_{e_i} e_k - \delta_{ik} \nabla_{e_j} e_k + \delta_{il} \nabla_{e_j}(e_k) - [e_i, e_j] \cdot e_k, e_l\big) = \\
&\quad\quad = (\delta_{jk} - \delta_{jl}) g(\nabla_{e_i} e_k, e_l) - (\delta_{ik} - \delta_{il}) g(\nabla_{e_j} e_k, e_l) - \delta_{kl} g([e_i, e_j], e_k).
\end{aligned}$$

If we take $k = l$, then we see that the last expression becomes $-g([e_i, e_j], e_k)$ and so the vanishing of R' implies that $[e_i, e_j] = 0$. (So we only needed the computation for $k = l$, but the case $k \neq l$ is useful for later reference.) $\square$

Now recall that a basis of n commuting vector field on a manifold is (at least locally) a set of coordinate vector fields. So near p we have a set of coordinates $(U; z^1, \dots, z^n)$ such that $e_i = \partial_\nu$ is idempotent. We call this a *canonical coordinate system* at p. (NB: this is in general *not* a flat coordinate system.) We can also assume that I is exact on U: $I = d\eta$ for some holomorphic function $\eta : U \to \mathbb{C}$. Then g takes on U the form $\sum_\nu \partial_\nu \eta . dz^\nu \otimes dz^\nu$.

Exercise 5. It is clear that M has an atlas of canonical charts. What form do the transition functions of this atlas take?

A local description. We now aim for a local characterization of a semisimple Frobenius structure in terms of canonical coordinates. To this end, we start out with a holomorphic function η on an open $U \subset \mathbb{C}^n$ with $d\eta$ nowhere zero and ask: what conditions should η satisfy in order that the coordinate vector fields $\partial_1, \dots, \partial_n$ are the idempotent vector fields of a semisimple Frobenius structure that has η as a metric potential? This means that for the associated nondegenerate symmetric bilinear form on the tangent bundle of U is (what is called in the Riemannian setting) of Darboux-Egoroff type: $g := \sum_\nu \partial_\nu \eta . dz^\nu \otimes dz^\nu$ and that we want that (i) the Levi-Civita connection of g be flat, (ii) the tensor R' be identically zero and (iii) the identity field $\sum_\nu \partial_\nu$ to be flat. It so happens that (ii) is automatically satisfied. We abbreviate $\partial_i\eta$ by η_i, $\partial_i\partial_j\eta$ by η_{ij} etc..

Lemma 2.7. *For any* g *as above, the tensor* R' *is identically zero.*

Before we begin the proof, we compute the Christoffel symbols of the Levi-Civita connection ∇. The *second Christoffel identity* says that these are given by:

$$g(\nabla_{\partial_i}\partial_j, \partial_k) = \tfrac{1}{2}\big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\big).$$

In this case, $g_{ij} = \delta_{ij}\eta_i$ and so we find

$$g(\nabla_{\partial_i}\partial_j, \partial_k) = \tfrac{1}{2}\big((\delta_{jk} + \delta_{ik})\eta_{ij} - \delta_{ij}\eta_{ik}\big),$$

in other words,

$$g(\nabla_{\partial_i}\partial_j, \partial_k) = \begin{cases} -\frac{1}{2}\eta_{ik} & \text{if } i = j \neq k, \\ \frac{1}{2}\eta_{ij} & \text{if } k \in \{i, j\}, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 6. Prove that our candidate identity vector field $\sum_j \partial_j$ is flat precisely if $\sum_j \eta_{ij} = 0$ for all i.

Proof of Lemma **??**. We prove that $g(R'(\partial_i, \partial_j)\partial_k, \partial_l) = 0$ for all i, j, k, l. The formula in **??** shows that $g(R'(\partial_j, \partial_j)\partial_k, \partial_k) = 0$. So we may assume that $i \neq j$ and $k \neq l$. The same formula in **??** also shows that we need to check that then

$$(\delta_{jk} - \delta_{jl})g(\nabla_{e_i} e_k, e_l) = (\delta_{ik} - \delta_{il})g(\nabla_{e_j} e_k, e_l).$$

This follows by substituting here the formula we found for the Christoffel symbols. $\square$

The flatness of the Levi-Civita connection is expressed by the *Darboux-Egoroff equations*. We will not derive these but just mention them (the derivation is straightforward). They are best expressed in terms of the symmetric matrix

$$\gamma_{ij} := \frac{1}{2}\frac{\eta_{ij}}{\sqrt{\eta_i\eta_j}},$$

where we suppose being given on the coordinate chart a square root of each η_i. (Notice that $(\sqrt{\eta_i}\partial_i)_i$ is then an orthonormal basis (frame) of the tangent bundle of U.) One computes that ∇ is flat precisely when

$$\partial_k\gamma_{ij} = \gamma_{ik}\gamma_{kj} \text{ for all } i, j, k \text{ and } \sum_k \partial_k\gamma_{ij} = 0.$$

Example 2.8. The trivial example is $M = \mathbb{C}^n$ (with coordinates $z^1, \dots, z^n$), $g = \sum_\nu (dz^\nu)^2$ and product $\partial_\nu \cdot \partial_\nu = \partial_\nu$. A potential function is $\Phi(z) = \sum_\nu \frac{1}{6}(z^\nu)^3$ and the pencil of connections is given by $\nabla^\lambda_{\partial_\nu}\partial_\mu = \lambda\delta_{\mu,\nu}\partial_\nu$.

Exercise 7 (One-dimensional case). Let M be a connected one-dimensional complex manifold. Prove that a structure of a Frobenius algebra on the tangent bundle on M amounts to giving on M a holomorphic vector field e and a holomorphic differential form I such that $I(e)$ is never zero. Prove that we have the structure of a Frobenius manifold if and only if $I(e)$ is constant and that the manifold is locally like the trivial Frobenius manifold. Which differential equation characterizes a potential function?

Example 2.9 (Two-dimensional case)**.** In this case we need not be concerned with associativity, because a commutative product on a vector space A of dimensional two with nonzero unit e is automatically associative: if $a \in A$ is such that $\{1, a\}$ is a basis, then μ_a and $\mu_1 = 1$ commute with each other (and obviously with themselves) and we have seen in the proof of Proposition **??** that this implies associativity.

[We use the occasion to prove that A is then isomorphic to the semisimple $\mathbb{C} \oplus \mathbb{C}$ or to the nonsemisimple $\mathbb{C}[y]/(y^2)$: since a^2 must be a linear combination of 1 and a: $a^2 = \lambda_1 x + \lambda_2$. Then $f \in \mathbb{C}[x] \mapsto f(a)$ defines an algebra isomorphism $\mathbb{C}[x]/(x^2 - \lambda_1 x - \lambda_2) \cong A$. Let x_1, x_2 be the roots of $x^2 - \lambda_1 x - \lambda_2$. In case they are distinct, the evaluation map $f \in \mathbb{C}[x] \mapsto (f(x_1), f(x_2))$ descends to an algebra isomorphism $A \cong \mathbb{C} \oplus \mathbb{C}$ and if $x_1 = x_2$, the algebra homomorphism

$$f \in \mathbb{C}[x] \mapsto f(x_1 + y) \equiv f(x_1) + yf'(x_1) \in \mathbb{C}[y]/(y^2)$$

identifies A with $\mathbb{C}[y]/(y^2)$.]

We continue to investigate the situation locally. Let e be the unit vector field and I the trace differential. Since e is flat, $g(e, e) = I(e \cdot e)$ is constant, say equal to $c \in \mathbb{C}$. We first do the case $c = 0$. Then we can find flat coordinates (z, w) such that $e = \partial_z$ and $g = dz \otimes dw + dw \otimes dz$. These coordinates are unique up to constants. The full structure will be given by a holomorphic potential function $\Phi(z, w)$ given up to quadratic terms. Since we have $g(\partial_z \cdot \partial_z, \partial_z) = g(\partial_z, \partial_z) = 0$ and $g(\partial_z \cdot \partial_z, \partial_w) = g(\partial_z, \partial_w) = 1$, it follows that $\Phi_{zzz} = 0$ and $\Phi_{zzw} = 1$. But since $\partial_z \cdot \partial_w = \partial_w$, we must also have $\Phi_{zwz} = 1$, $\Phi_{zww} = 0$. It follows that up to quadratic terms, $\Phi(z, w)$ equals $\frac{1}{2}z^2 w + f(w)$, where f is holomorphic. Conversely, with these choices of e and g, any Φ of this form defines a Frobenius manifold.

If $c \neq 0$, then we can find flat coordinates (z, w) such that $e = \partial_z$ and $g = c dz \otimes dz + c dw \otimes dw$. This coordinate system is unique up to constants and a sign for w. If $\Phi(z, w)$ is a potential function, then we want that $\Phi_{zzz} = c$, $\Phi_{zzw} = 0$, $\Phi_{zwz} = 0$, $\Phi_{zww} = c$ and so up to quadratic terms, $\Phi(z, w)$ equals $\frac{1}{6}cz^3 + \frac{1}{2}czw^2 + f(w)$, where f is holomorphic. Here too we have converse: with these choices of e and g, any Φ of this form defines a Frobenius manifold.

The most important class of examples (that in fact motivated the definition in the first place) is furnished by quantum cohomology. We prefer to discuss this notion later, but as it seems a good idea to encounter at least one interesting nontrivial example early in the game, we give the following one, which originates in work of Kyoji Saito (and which was put in this setting by Dubrovin, who also conceived the very notion we are discussing now). This example comes here a bit out of the blue, but we shall later provide context.

Example 2.10. We take for M the space of polynomials in $\mathbb{C}[z]$ of the form

$$p(z) := z^{n+1} + a_2 z^{n-1} + a_3 z^{n-2} + \cdots + a_n z + a_{n+1}.$$

So these are the monic polynomials of degree $n+1$ with the property that the sum of their roots equals zero. We can also picture M as an orbit space: if H denotes the hyperplane of $\mathbb{C}^{n+1}$ defined by $z_1 + \cdots + z_{n+1} = 0$, then H is clearly invariant under the symmetric group $\mathcal{S}_{n+1}$ acting by permuting coordinates. Then if we assign to $(z_1, \cdots, z_{n+1}) \in H$ the polynomial $p(z) = (z+z_1)\cdots(z+z_{n+1})$, we notice that p lies in M; in fact a_i is the ith symmetric function of $(z_1, \cdots, z_{n+1})$. Conversely, any $p \in M$ is factored in this way, with the factoring being unique up to order and so we may identify M with the $\mathcal{S}_{n+1}$-orbit space of H.

We regard $a_2, \ldots, a_{n+1}$ as a set of coordinates for M. Notice that M is in fact an affine space over the vector space V of polynomials of degree $\leq n-1$ (the difference of any two elements of M lies in V and any vector in V thus occurs). In particular, the holomorphic tangent space of M at p may be identified with V (with basis $\frac{\partial}{\partial a_2}, \ldots, \frac{\partial}{\partial a_{n+1}}$). For $p \in M$ we consider the algebra

$$A_p := \mathbb{C}[z]/(p') = \mathbb{C}[z]/((n+1)z^n + (n-1)a_2 z^{n-2} + \cdots + a_n).$$

The roots of p' are the critical points of p. So if they are all distinct, then we may think of A_p as the coordinate ring of this set and A_p is semisimple (an idempotent is the characteristic function of a critical point of p). Otherwise A_p has nilpotent elements. A basis of A_p consists of the images $[z^k] \in A_p$ of z^k for $k = 0, 1, \ldots, n-1$. Since these monomials also form a basis of V, the natural map $V \to A_p$ is an isomorphism of n-dimensional vector spaces.

A trace map on A_p will defined by means of a residue. Let us compute for any $f \in \mathbb{C}[z]$ the residue of the differential $\frac{f dz}{p'}$ at ∞. If f is divisible by p': $f = f_1 p'$, with $f_1 \in \mathbb{C}[z]$, then clearly the differential (which equals $f_1 dz$) has no finite poles. In view of the residue theorem (which asserts that the sum of the residues of a rational function in a single variable is always zero) the residue at infinity will be zero. This shows that the map

$$f \in \mathbb{C}[z] \mapsto \operatorname{Res}_{z=\infty} \frac{f\,dz}{p'}$$

factors through a linear function $I_p : A_p \to \mathbb{C}$. Now let us compute its value on the monomials z^k, $k = 0, 1, \dots, n-1$ that map to a basis of A_p. Upon making the substitution $\zeta = z^{-1}$, we get

$$\operatorname{Res}_{z=\infty} \frac{z^k dz}{p'} = \operatorname{Res}_{\zeta=0} \frac{\zeta^{-k}. - \zeta^{-2} d\zeta}{(n+1)\zeta^{-n} + (n-1)a_2\zeta^{-(n-2)} + \cdots + a_n} =$$
$$= \operatorname{Res}_{\zeta=0} \frac{-\zeta^{n-k-2} d\zeta}{(n+1) + (n-1)a_2\zeta^2 + \cdot + a_n\zeta^n}.$$

We see that we get zero unless $k = n-1$, in which case the residue is $\frac{-1}{n+1}$. So the trace differential is given by $\frac{-1}{n+1} da_2$. In particular, $\eta := \frac{-1}{n+1} a_2$ may serve as a metric potential. We verify that the associated bilinear form on A_p is nondegenerate. The matrix of g_p on the basis $([z^k])_{k=0}^{n-1}$ of $V \cong T_pM$ has as its (i,j)-entry $\frac{-1}{n+1}$ times the coefficient of z^{i+j} on $[z^{n-1}]$ in A_p. So these entries are on the antidiagonal $(i+j = n-1)$ all equal to $\frac{-1}{n+1}$ and zero above the antidiagonal. Such a matrix is clearly nonsingular.

We now have constructed on the tangent bundle of M the structure of a Frobenius algebra by identifying T_pM via V with A_p.

Next we produce flat coordinates on a dense open subset. Since z has a pole of order one at ∞, we shall (at some point) identify the tangent space $T_pM \cong V$ with the space of polar parts at ∞ (including constants) given up to order $n-1$. In what follows we use that identification. The trick is now based on the fact that for $|z|$ large, we can extract from $p(z)$ an $(n+1)$th root: $p(z) = w^{n+1}$ with w holomorphic and w/z close to 1 for $|z|$ large:

$$(\dagger) \qquad z(w,b)^{n+1} + a_2 z(w,b)^{n-1} + \cdots + a_{n+1} \equiv w^{n+1}.$$

Since z has a pole of order one at ∞ and w^{-1} serves there as a local coordinate, it is clear that $z(w,b)$ will have at ∞ a Laurent series expansion of the form

$$z(w,b) = w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \frac{b_3}{w^3} + \cdots,$$

This expresses the coefficients b_i in terms of the coefficients $(a_2, \dots, a_{i+1})$. For instance, we see easily that $b_0 = 0$ and $b_1 = -\frac{1}{n+1} a_2$ so that db_1 is the trace differential. Conversely, in order that $(\dagger)$ holds, a_i is recursively expressed in terms of $b_1, \dots, b_{i-1}$ so that $b_1, \dots, b_n$ may serve as a new coordinate system for M. We show that this coordinate system is flat. For this we differentiate both sides of $(\dagger)$ with respect to b_i. We find

$$p'(z(w,b))w^{-i} + \frac{\partial a_2}{\partial b_i} z(w,b)^{n-1} + \cdots + \frac{\partial a_{n+1}}{\partial b_i} \equiv 0.$$

This amounts to saying that $\frac{\partial}{\partial b_i}\big|_p = -w^{-i}p'$, $i = 1, \dots, n$, where the right hand side is understood as a polar part at ∞ including the constant term (to be precise, in terms of the z-coordinate we get a priori a series $\sum_{i\le n-1} c_i z^i$ which converges for $|z|$ large and we then retain $\sum_{i=0}^{n-1} c_i z^i$), which is subsequenty identified with $V \cong T_pM$. But then

$$g(\frac{\partial}{\partial b_i}\Big|_p, \frac{\partial}{\partial b_j}\Big|_p) = \mathrm{Res}_{z=\infty} \frac{w^{-i}p'.w^{-j}p'}{p'}dz = \mathrm{Res}_{z=\infty} \frac{dp}{w^{i+j}} =$$
$$= \mathrm{Res}_{w=\infty} \frac{d(w^{n+1})}{w^{i+j}} = (n+1)\,\mathrm{Res}_{w=\infty} \frac{w^n dw}{w^{i+j}}$$

and the last residue is zero unless $i+j = n+1$, in which case we get $n+1$. So this is a flat coordinate system indeed. The trace differential db_1 is clearly flat as well.

It remains to verify that the tensor R' vanishes identically. According to Lemma **??** this is the case wherever M admits a system of canonical coordinates. We only need to verify this last property on an open-dense subset. Let $U \subset M$ be an open subset where p' has n distinct roots $u_1, \dots, u_n$ that depend holomorphically on $(a_2, \dots, a_n)$ so that

$$p'(z) = (n+1)(z-u_1)\cdots(z-u_n)$$

on U. Now consider the functions $w_i := p(u_i)$, $i = 1, \dots, n$, on U. We have

$$dw_i = p'(u_i)du_i + \sum_{k=0}^{n-1} u_i^k da_{n+1-k} = \sum_{k=0}^{n-1} u_i^k da_{n+1-k},$$

and so $dw_1 \wedge \cdots \wedge dw_n = \Delta da_{n+1} \wedge \cdots \wedge da_2$, where Δ is the determinant of the matrix $(u_i^j)_{i=1,\dots,n;j=0,\dots,n-1}$. This matrix is singular if two of the $u_1, \dots, u_n$ coincide and hence Δ is divisible by $\prod_{i<j}(u_i - u_j)$. But both Δ and $\prod_{i<j}(u_i - u_j)$ are homogeneous of the same degree $\frac{1}{2}n(n-1)$ and so must be proportional to one another (in fact, they are equal up to sign). Since Δ is nonzero on U, it follows that at every point of U, $(w_1, \dots, w_n)$ may serve as a local coordinate system. In particular, we have defined coordinate vector fields $\frac{\partial}{\partial w_i}$ on U. Notice that $\frac{\partial}{\partial w_j}\big|_p \in T_pM$, when viewed as an element of A_p, is just the image of $\sum_{k=0}^{n-1} \frac{\partial a_{n+1-k}}{\partial w_j} z^k$ in A_p. This is in fact the j-th idempotent $e_j(p)$, i.e., the characteristic function of the jth root u_j among all roots of p', for the identity above shows that its value in u_i equals

$$\sum_{k=0}^{n-1} \frac{\partial a_{n+1-k}}{\partial w_j} u_i^k = dw_i(\frac{\partial}{\partial w_j}) = \delta_{ij}.$$

We have thus proved that the coordinate vector fields $\frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_n}$ are in fact the idempotent vector fields. So $(w_1, \dots, w_n)$ is (locally) a canonical coordinate system.

This completes the proof that the thus endowed M is a Frobenius manifold.

3. EULER FIELDS

Observe that Example **??** is polynomial (rather than complex analytic) in nature. More specifically, there is a $\mathbb{C}^\times$-symmetry present which gives rise to 'weighted homogeneous structure'. Let us begin with observing (or recalling) that in algebraic geometry a $\mathbb{C}^\times$-action on a geometric object has as its algebraic counterpart a grading of the corresponding algebraic object. For instance, if V is a complex vector space, then its coordinate ring $\mathbb{C}[V]$ is the $\mathbb{C}$-algebra of polynomial functions on V. If $h_\lambda : V \to V$ denotes multiplication by the scalar $\lambda \in \mathbb{C}^\times$, then the induced action on $\mathbb{C}[V]$ (given by $f \mapsto fh_\lambda$) yields a grading $\mathbb{C}[V] = \oplus_{d=0}^\infty \mathbb{C}[V]_d$ into eigenspaces: $\mathbb{C}[V]_d$ is the vector space of homogeneous polynomials of degree d and h_λ acts on this space as multiplication by λ^d. The infinitesimal generator of this $\mathbb{C}^\times$-action is the vector field E on V which in $p \in V$ is the vector p itself, but now regarded as an element of the tangent space of V at p (if we have chosen coordinates $(t^1, \dots, t^m)$, then $E = \sum_\nu t^\nu \frac{\partial}{\partial t^\nu}$; this is classically called the Euler field on $\mathbb{C}^m$). The grading is also the eigenspace decomposition of this vector field: $E(f) = d.f$ if $f \in \mathbb{C}[V]_d$.

In short, a degree can (and often should) be understood as an eigenvalue relative to an infinitesimal $\mathbb{C}^\times$-action.

Let us return to Example **??**. The multiplicative group $\mathbb{C}^\times$ acts on $\mathbb{C}^{n+1}$ by scalar multiplication. It leaves H invariant and commutes with the action of $\mathcal{S}_{n+1}$. It therefore descends to an action on M: if $\lambda \in \mathbb{C}^\times \mapsto h_\lambda$ is the action, then $h_\lambda(a_i) = \lambda^i a_i$ (remember that a_i is in fact an ith symmetric function, and so it is not surprising that it has degree i). This $\mathbb{C}^\times$-action propagates throughout: we obtain an algebra isomorphism $A_{h_\lambda(p)} \to A_p$ if we substitute λz for z, as follows from the identity

$$\begin{aligned}(n+1)(\lambda z)^n + (n-1)\lambda^2 a_2(\lambda z)^{n-2} + \cdots + \lambda^n a_n) &= \\ &= \lambda^n\big((n+1)z^n + (n-1)a_2 z^{n-2} + \cdots + a_n\big).\end{aligned}$$

In other words, the isomorphism of vector spaces $D_p h_\lambda : T_p M \to T_{h_\lambda(p)}M$ pulls back the multiplication tensor on $T_{h_\lambda(p)}M$ to λ times the one on T_pM. The trace differential is $I = \frac{-1}{n+1} da_2$ and we notice that $D_p h_\lambda^* I_{h_\lambda(p)} = \lambda^2 I_p$.

Since this kind of structure is often present (or assumed) on a Frobenius manifold, it deserves a definition. In order to help to understand it, we first introduce an auxiliary notion. Let $(M; \cdot, I)$ and $(N; \cdot, J)$ be Frobenius manifolds and let $h : M \to N$ be a holomorphic local isomorphism. We say that h is *conformal* if there exist constants c, c' such that for every $p \in M$, the map $D_p h : T_pM \to T_{h(p)}N$ is such that $D_p h(v) \cdot D_p h(v') = c D_p h(v \cdot v')$ and $J_{h(p)} D_p h = c' I_p$ (briefly: h multiplies the product by c and the trace map by c'). Notice that then D_ph sends the identity of T_pM to c^{-1} times the identity of $T_{h(p)}N$.

Definition 3.1. A vector field E on M is called an *Euler field* if its local flows are conformal. Precisely, E is an *Euler field* of bidegree (d_o, d) (where d_o and d are in principle complex numbers) if for any local flow h_λ generated by E (recall that the domain of h_λ is usually an open subset of M), h_λ multiplies the product by $e^{\lambda d_o}$ and the trace differential by $e^{\lambda(d_o+d)}$.

(We use Hitchin's convention; for Dubrovin the bidegree is here $(d_o, d_o + d)$.) So in case $d = 0$, then both product and trace differential are rescaled by the same factor $e^{\lambda d_o}$. In view of our earlier discussion, this means that the Frobenius algebra structure on $T_{h_\lambda(p)}M$ pulled back to T_pM via D_ph_λ is obtained from the given Frobenius algebra structure on T_pM after a rescaling by $e^{\lambda d_o}$.

Since $g(X,Y) = I(X \cdot Y)$ and $T(X,Y,Z) = I(X \cdot Y \cdot Z)$, it follows that $h_\lambda^* g = e^{\lambda(2d_o+d)}g$ and $h_\lambda^* T = e^{\lambda(3d_o+d)}T$ and from $X = e \cdot X$, we see that $h_\lambda^* X = h^*(e \cdot X) = e^{\lambda d_o} h_\lambda^* e \cdot h_\lambda^* X$ and so $h_\lambda^* e = e^{-\lambda d_o} e$.

The last property of Definition **??** is equivalent to its infinitesimal version (which is in fact the customary formulation):

$$[E, X \cdot Y] - [E,X] \cdot Y - X \cdot [E,Y] = d_o X,$$
$$[E, I(X)] - I([E,X]) = (d_o + d)(X).$$

Notice that the first identity implies $[E, e] = -d_o e$. If we regard $\cdot$ and I as sections of $\tau_M^* \otimes \tau_M^* \otimes \tau_M$ and τ_M^*, then this is best stated in terms of Lie derivatives:

$$\mathcal{L}_E(\cdot) = d_o(\cdot), \quad \mathcal{L}_E(I) = (d_o + d)I,$$

where we recall that $\mathcal{L}_X$ is on vector fields given by the Lie bracket $\mathrm{ad}_X = [X, \ ]$ and on differentials by $\mathcal{L}_X = \iota_X d + \iota_X d$ (here d is of course exterior derivation and not the degree defined above). Since I is closed, it follows that $d\iota_E(I) = (d_o + d)I$. We also obeserve that

$$\mathcal{L}_E(g) = (2d_o + d)g, \quad \mathcal{L}_E(T) = (3d_o + d)T.$$

It is clear that the Euler vector fields (of unspecified bidegree) form a vector space (closed under the bracket, and hence in fact a Lie algebra–it is indeed the Lie algebra of a group of local flows). We shall always normalize a nonzero Euler vector field E such that $d_o = 1$. In particular, $[E, e] = -e$.

Notice that in the Euler field in Example **??** is normalized (we have $d_o = 1$) and $d = 1$.

Exercise 8. Consider the potential functions that appear in Example **??**. Investigate for which analytic functions $f(w)$ in that Example there exists an nonzero Euler field.

Lemma 3.2. *Suppose M is a semisimple Frobenius manifold endowed with a (normalized) Euler field E. Then we can normalize the canonical coordinates*

$(z^1, \dots, z^n)$ *(by adding a constant) so that they become homogeneous of degree one relative to* E*:* $E(z^\nu) = z^\nu$*, or equivalently,* E *takes the form* $\sum_\nu z^\nu \partial_\nu$ *and the idempotent vector fields* ∂_ν *have degree* -1*. This coordinate system is unique up to a permutation of coordinates. Moreover, we can take a metric potential* η *to be of degree* $d+1$ *(so* $I = \sum_\nu \eta_\nu dz^\nu$ *and* $g = \sum_\nu \eta_\nu dz^\nu \otimes dz^\nu$ *with* η_ν *of degree* d*) and we can take our potential function* Φ *to be of degree* $d+3$*.*

Proof. In the same way we argued that $[E, e] = -e$, it follows that for an idempotent vector field e_ν, $[E, e_\nu] = -e_\nu$. So if $(z^1, \dots, z^n)$ is a system of canonical coordinates, then $[E, \partial_\nu] = -\partial_\nu$. If we write $E = \sum_\mu E^\mu \partial_\mu$, then we see that this means that $dE^\nu = dz^\nu$. So $E^\nu - z^\nu$ is constant. By adding this constant to z^ν, we end up with a canonical coordinate system as desired. The other assertions are easy or straightforward. □

Remark 3.3. The origin of this coordinate system need not be in its domain. This makes it possible for d not to be an integer. However, the cases for which η is a homogeneous polynomial are of special interest.

Let us now investigate how a normalized Euler field E is expressed in terms of a flat coordinate system $(z^1, \dots, z^n)$. We assume here that $g = \sum_\nu dz^\nu \otimes dz^\nu$. Any transformation h which respects g is affine-linear: it is of the form $z \mapsto \sigma(z) + z_o$, with $z_o \in \mathbb{C}^n$ and $\sigma \in SO(n, \mathbb{C})$. If h is only conformal, then we must allow σ to be a nonzero scalar times an element of $SO(n, \mathbb{C})$: $\sigma \in \mathbb{C}^\times . SO(n, \mathbb{C})$. Hence E, being the infinitesimal form of such a transformation will be a vector field that has the shape

$$E = \sum_\nu \Big(\sum_\mu a^\nu_\mu z^\mu + b^\nu\Big)\partial_\nu,$$

where $a = (a^\nu_\mu) \in \mathrm{Lie}(\mathbb{C}^\times . SO(n, \mathbb{C})) = \mathbb{C} + \mathfrak{so}(n, \mathbb{C})$. In other words, $(a^\nu_\mu + a^\mu_\nu)$ is multiple of the identity matrix. Notice that the linear part $\sum_{\nu,\mu} a^\nu_\mu z^\mu \partial_\nu$ of E is intrinsically defined as a covariant derivative: we have $\nabla_{\partial_\mu} E = \sum_\mu a^\nu_\mu \partial_\nu$ and so the endomorphism Q_p of T_pM which assigns to a tangent vector $v \in T_pM$ the vector $\nabla_v E \in T_pM$ yields an endomorphism Q of the tangent bundle that is covariantly constant. Dubrovin calls this the *grading operator*. If M is connected, then its conjugacy class (and in particular, its chacteristic polynomial) is an invariant of the Frobenius manifold.

We compute m in the identity $(a^\nu_\mu + a^\mu_\nu) = m\delta_{\nu,\mu}$. Since $\mathcal{L}_E g = (2+d)g$, we have for any pair X, Y of *flat* vector fields,

$$(2+d)g(X,Y) = E(g(X,Y)) - g([E,X],Y) - g(Y,[E,X])) =$$
$$= 0 + g(\nabla_X E, Y) + g(X, \nabla_Y E) = g(Q(X),Y) + g(X, Q(Y)) = g((Q+Q^*)X, Y),$$

where Q^* denotes the g-transpose. It follows that $Q + Q^* = (2+d)\mathrm{Id}$, or equivalently, that $Q' := Q - (1 + \frac{1}{2}d)\mathrm{Id}$ is skew symmetric in the sense that $(Q')^* = -Q'$.

Exercise 9. Prove that $Q(e) = e$. Conclude that $g(e,e) = 0$ unless $d = 0$.

Let M be a Frobenius manifold with normalized Euler field E. Consider the product $M \times \mathbb{C}^\times$ and denote projection on the first factor by π_M and the second coordinate by λ. It is clear that $\pi_M^*\tau_M$ is a direct summand of $\tau_{M\times\mathbb{C}^\times}$ and that a supplement is generated by the vector field $-\lambda\frac{\partial}{\partial\lambda}$ (notice that $-\lambda\frac{\partial}{\partial\lambda}$ is the infinitesimal generator of the $\mathbb{C}^\times$-action in $\mathbb{C}^\times$ by inverse multiplication: $t \in \mathbb{C}^\times$ sends λ to $t^{-1}\lambda$) and hence also by $E - \lambda\frac{\partial}{\partial\lambda}$.

Proposition 3.4. *The connection $\hat\nabla$ on $\pi_M^*\tau_M$ characterized by the fact that for* flat *vector fields X, Y on M,*

$$\hat\nabla_X(Y) = \lambda X \cdot Y, \quad \hat\nabla_{E-\lambda\frac{\partial}{\partial\lambda}}(Y) = [E, Y]$$

is a flat connection. (Here the vector field X, defined on the open $U \subset M$, say, is regarded as a section of $\pi_M^*\tau_M$ over $U \times \mathbb{C}^\times$; likewise for Y and E.)

Proof. We must show that the curvature form $R(\hat\nabla)$ is identically zero. Let X and Y be flat vector fields on an open $U \subset M$. Then the value of $R(\hat\nabla)(X, Y)$ in (p, λ) is $R(\nabla^\lambda)(X_p, Y_p)$ and hence zero. It remains to show that $R(\hat\nabla)(X, E - \lambda\frac{\partial}{\partial\lambda}) = 0$. For this, we notice $\hat\nabla_{E-\lambda\frac{\partial}{\partial\lambda}} Y = [E, Y]$ is also flat and that $[E - \lambda\frac{\partial}{\partial\lambda}, Y] = [E, Y]$. So

$$\begin{aligned} R(\hat\nabla)(X, E - \lambda\frac{\partial}{\partial\lambda})(Y) &= \hat\nabla_X\hat\nabla_{E-\lambda\frac{\partial}{\partial\lambda}}(Y) - \hat\nabla_{E-\lambda\frac{\partial}{\partial\lambda}}\hat\nabla_X(Y) - \hat\nabla_{[X,E-\lambda\frac{\partial}{\partial\lambda}]}(Y) = \\ &= \lambda X \cdot [E, Y] - \big([E, \lambda X \cdot Y] - \lambda X \cdot Y\big) - \lambda[X, E] \cdot Y = \\ &= \lambda\Big(X \cdot Y - [E, X \cdot Y] + [E, X] \cdot Y - X \cdot [E, Y]\Big) = 0. \quad \square \end{aligned}$$

There is a partial converse, which we present in the form of an exercise.

Exercise 10. Let M be a complex manifold with on its tangent bundle the structure of a holomorphically varying structure of a Frobenius manifold $(\cdot, I)$ and a holomorphic vectorfield E such that

(i) $(\cdot, I)$ is of bidegree $(1, 1 + d)$: $\mathcal{L}_E(\cdot) = (\cdot)$ and $\mathcal{L}_E(I) = (1 + d)I$,
(ii) the connection $\hat\nabla$ on $\pi_M^*\tau_M$ as defined above is flat.

Prove that we then have the structure of a Frobenius manifold with E as Euler field, except that the identity need not be flat relative to the Levi-Civita connection.

4. Other examples

The Dunkl connection. We describe a remarkable class of examples of that fail to be Frobenius manifolds in only aspect: their identity vector field is not flat. We shall refer to such structures as *almost-Frobenius manifolds*.

Let be given a complex vector space V of finite dimension n endowed with a nondegenerate complex bilinear form $g : V \times V \to \mathbb{C}$. We assume also given a finite collection $\mathcal{H}$ of linear hyperplanes in V with the property that for every $H \in \mathcal{H}$ the restriction of g to $H \times H$ is nondegenerate (so that V is the direct sum of H and its g-orthogonal complement $H^\perp$, a complex line in V) and for every $H \in \mathcal{H}$ a nonzero self-adjoint linear map $\rho_H : V \to V$ with kernel H. So ρ_H has the form $\rho_H(v) = \alpha_H(v)\check{\alpha}_H$, where $\alpha_H \in V^*$ has zero set H and $\check{\alpha}_H \in H^\perp$. In particular, the trace of ρ_H is equal to $\alpha_H(\check{\alpha}_H)$ and is hence nonzero. We take the occasion to note that the meromorphic differential ω_H on V defined by

$$\omega_H = \frac{d\alpha_H}{\alpha_H} \qquad (\text{so } \omega_H(X_p) = \frac{\alpha_H(X_p)}{\alpha_H(p)})$$

only depends on H and not on the choice of α_H.

Let $V^\circ := V - \cup_{H\in\mathcal{H}} H$. Every tangent space T_pV° can be identified with V and via this identification we regard g as a nondegenerate symmetric bilinear form g on the holomorphic tangent bundle of V°. The Levi-Civita connection of g is the standard one on a vector space (its flat vector fields are the constant vector fields). We define a product on the tangent bundle of V° by

$$X \cdot Y := \sum_H \omega_H(X)\rho_H(Y).$$

Since $\omega_H(X_p)\rho_H(Y_p) = \alpha_H(p)^{-1}\alpha_H(X_p)\alpha_H(Y_p)\check{\alpha}_H$ we see that this product is commutative. For $\lambda \in \mathbb{C}$ we define a connection ∇^λ on the holomorphic tangent bundle of V° as usual:

$$\nabla^\lambda_X(Y) := \nabla_X(Y) + \lambda X \cdot Y = \nabla_X(Y) + \sum_H \lambda\omega_H(X)\rho_H(Y).$$

Proposition 4.1. *If $\sum_H \rho_H$ is the identity transformation, then the Euler vector field E on V, characterized by $E_p = (p,p) \in \{p\} \times V = T_pV$, serves as identity element for the above product (but is not flat for g); the product has E-degree 1 and g has E-degree 2.*

If the system $(V, g, \{\rho_H\}_H)$ has the Dunkl property*, meaning that for every linear subspace $L \subset V$ of codimension 2 obtained as an intersection of members of $\mathcal{H}$ the sum $\sum_{H\in\mathcal{H}, H\supset L} \rho_H$ commutes with each of its terms, then the product is associative, the connection ∇^λ is flat for every $\lambda \in \mathbb{C}$, and a potential function is given by*

$$\Phi := \sum_{H\in\mathcal{H}} \frac{g(\check{\alpha}_H, \check{\alpha}_H)}{2\,\mathrm{Tr}(\rho_H)} \alpha_H^2 \log \alpha_H.$$

(So if both conditions hold, then V° is an almost-Frobenius manifold and E is a normalized Euler field of bidegree $(1,1)$.)

Central in the proof is the notion of a logarithmic form and the residue of such a form and so we discuss this first. A meromorphic k-form ω on a complex manifold V° is said to be *logarithmic*, if locally it can be written as a sum of meromorphic forms of the following simple type: $f_0\frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_k}{f_k}$ with $f_0, \dots, f_k$ holomorphic. Notice that the exterior derivative of a logarithmic form or the wedge product of two such is again a logarithmic form. For example, the form ω_H defined above is a logarithmic form on the projective completion of V and so $\Omega\wedge\Omega$ is a logarithmic 2-form with values in $\mathrm{End}(V)$. If $H \subset V^\circ$ is complex submanifold of codimension one, then ω determines on H a meromorphic $(k-1)$-form, called its *residue* $\mathrm{Res}_H\,\omega$ along H: this residue is zero if ω does not have a pole along H, but if in $f_0\frac{df_1}{f_1}\wedge\cdots\wedge\frac{df_k}{f_k}$, f_k vanishes to order one on H and the other f_i have a meromorphic restriction to H, then $\mathrm{Res}_H(f_0\frac{df_1}{f_1}\wedge\cdots\wedge\frac{df_k}{f_k})$ is the restriction of $f_0\frac{df_1}{f_1}\wedge\cdots\wedge\frac{df_{k-1}}{f_{k-1}}$ to H as a meromorphic form. For example, the form ω_H defined above has residue constant 1 along H and residue -1 along the hyperplane at infinity.

Proof of Proposition **??**. We first observe that $\omega_H(E) = 1$ for every H and so

$$E\cdot X = \sum_H \omega_H(E)\rho_H(X) = \sum_H \rho_H(X) = X.$$

We already noticed that E is a normalized Euler vector field for the product. That I has degree $(1,1)$ follows from Problem **??**. Hence, it remains to show that ∇^λ is flat for all λ, for according to Proposition **??**, this implies the associativity of the product. Upon replacing ρ_H by $\lambda\rho_H$, we see that it is even enough to prove the flatness of ∇^1. Now if we identify the holomorphic tangent bundle of V° with the trivial bundle with fiber V, then we see that the connection form of ∇^1 is simply $\Omega := \sum_H \omega_H\otimes\rho_H$. Since this is a closed 1-form on V° with values in $\mathrm{End}(V)$, the curvature form is given by

$$R(\nabla^1) = \Omega\wedge\Omega = \tfrac{1}{2}\sum_{H,H'\in\mathcal{H}} \omega_{H'}\wedge\omega_H\otimes[\rho_H,\rho_{H'}].$$

(Notice that the term associated to (H,H') is the same as the one associated to (H',H) and is zero if $H=H'$. So the factor $\frac{1}{2}$ can be avoided if we put a total order on $\mathcal{H}$ and only sum over the pairs $(H,H')\in\mathcal{H}^2$ with $H<H'$.) In order to prove that $\Omega\wedge\Omega = 0$, we first focus on its behavior along any codimension two intersection L of members of $\mathcal{H}$. We shall use the simple fact that the complex projective space $\mathbb{P}^n$ has no nonzero holomorphic k-forms for $k>0$. (Proof: a holomorphic k-form on $\mathbb{C}^n$ is of the form $\sum_I f_I dz^I$, with f_I holomorpic and I running over the k-element subsets of $\{1,\dots,n\}$. In order that it has no pole at infinity, we must still have a holomorphic k-form after making the substitution $(w^1,\dots,w^n) = (\frac{1}{z^1},\frac{z^2}{z^1},\dots,\frac{z^n}{z^1})$. If you work this out, you find that $f_{(n+1-k,\dots,n)} = 0$, and similarly that the other f_I are zero.)

Let $H\in\mathcal{H}$ contain L. We first take the residue of $\Omega\wedge\Omega$ on H (that will be a differential on H with poles of order one along any intersection $H'\cap H$, $H'\neq H$) and subsequently take the residue on L (a priori a 0-form on an open-dense subset of L, but we will see it is actually constant). We get

$$\operatorname{Res}_L \operatorname{Res}_H \big(\Omega \wedge \Omega\big) = \operatorname{Res}_L \Big(\sum_{H' \neq H} \omega_{H'}\big|_H \otimes [\rho_H, \rho_{H'}]\Big) = \\ = \sum_{H' \neq H, H' \supset L} [\rho_H, \rho_{H'}] = [\rho_H, \sum_{H' \supset L} \rho_{H'}]$$

and the last expression is zero by assumption. If we now fix $H \in \mathcal{H}$ and let L run over all the codimension two intersections in H, then we see that this implies that the $\operatorname{End}(V)$-valued differential $\operatorname{Res}_H \big(\Omega \wedge \Omega\big)$ on H has no poles. The latter has at most a first order pole along the hyperplane at infinity of L, and the residue there is (by the residue theorem) the sum of the residues along all other hyperplanes in L and hence zero. So $\operatorname{Res}_H \big(\Omega \wedge \Omega\big)$ is a regular $\operatorname{End}(V)$-valued differential on the projective completion of H. Hence it is zero. But since this true for every $H \in \mathcal{H}$, it follows that follows that $\Omega \wedge \Omega$ is a holomorphic $\operatorname{End}(V)$-valued 2-form. It has a pole of order one at most along the hyperplane at infinity and its residue there is a holomorphic differential, hence zero. So $\Omega \wedge \Omega$ is holomorphic on the projective completion of V and hence zero.

Finally, a straightforward computation shows that if X, Y, Z are flat (= constant) vector fields on V°, then for Φ as in the statement we have

$$\nabla_X \nabla_Y \nabla_Z \Phi = \sum_{H \in \mathcal{H}} \frac{g(\check{\alpha}_H, \check{\alpha}_H)}{\operatorname{Tr}(\rho_H)\alpha_H} \alpha_H(X)\alpha_H(Y)\alpha_H(Z).$$

Now is $g(\check{\alpha}_H, \ )$ a linear form whose zero set is H and which is therefore proportional to α_H. By evaluating both on $\check{\alpha}_H$ we see that

$$g(\check{\alpha}_H, \ ) = \frac{g(\check{\alpha}_H, \check{\alpha}_H)}{\operatorname{Tr}(\rho_H)} \alpha_H$$

and so

$$\nabla_X \nabla_Y \nabla_Z \Phi = \sum_H \alpha_H(X)\alpha_H(Y) \frac{g(\check{\alpha}_H, Z)}{\alpha_H} = g(X \cdot Y, Z).$$

This proves that Φ is a potential function for V°. □

Exercise 11. Show that the trace differential I on V° is given by $I(X_p) = g(p, X_p)$ and prove that $\frac{1}{2} g(z, z)$ is a metric potential.

Lauricella manifolds. We yet have to find explicit examples satisfying the conditions of Proposition **??**. Here is one, to which we shall refer as the class of *Lauricella manifolds*, because of its relation to the Lauricella hypergeometric functions.

Fix nonzero complex numbers $\mu_0, \dots, \mu_n$ such that $\sum_i \mu_i = 1$ and $\sum_{i \in I} \mu_i \neq 0$ for any nonempty subset $I \subset \{0, \dots, n\}$ with $|I| \leq 3$. We define a symmetric nondegenerate bilinear form on $\mathbb{C}^{n+1}$ (with basis vectors $\varepsilon_0, \dots, \varepsilon_n$) by $g(z, w) = \sum_i \mu_i z^i w^i$. The vector $v := \sum_i \varepsilon_i$ has self-product $g(v, v) = \sum_i \mu_i \neq 0$ and so its g-orthogonal complement is nondegenerate. This latter orthogonal complement will be our V. We take as our collection of hyperplanes the $H_{ij} \cap V$, where H_{ij} is defined by $z^i = z^j$, $i < j$. Notice that $v \in H$. The g-orthogonal complement of H_{ij} in $\mathbb{C}^{n+1}$ is spanned by the vector $\varepsilon_{ij} = \mu_j \varepsilon_i - \mu_i \varepsilon_j$, which indeed lies in V. Furthermore, $g(\varepsilon_{ij}, \varepsilon_{ij}) = 2\mu_i\mu_j(\mu_i + \mu_j) \neq 0$ and so g is nondegenerate on H_{ij}. Let $\tilde{\rho}_{ij} \in \mathrm{End}(\mathbb{C}^{n+1})$ be the self-adjoint linear map with kernel H_{ij} and trace $\mu_i + \mu_j$: $\tilde{\rho}_{ij}(z) = (z^i - z^j)\varepsilon_{ij})$. Since $\tilde{\rho}_{ij}$ fixes v, it preserves V; the induced map in V will be our ρ_{ij}.

We verify that the triple $(V, g, \{\rho_{ij}\})$ satisfies the Dunkl property. We first check that $\sum_{i<j} \rho_{ij}$ is the identity. If $z = (z^0, \dots, z^n) \in V$, then

$$\sum_{i<j} \rho_{ij}(z) = \sum_{i<j} (z^i - z^j)(\mu_j \varepsilon_i - \mu_i \varepsilon_j) =$$

$$= \sum_{i \neq j} z^i \mu_j \varepsilon_i - \sum_{i \neq j} z^i \mu_i \varepsilon_j = \sum_i \big(\sum_{j \neq i} \mu_j) z^i \varepsilon_i - \sum_j \big(\sum_{i \neq j} z^i \mu_i) \varepsilon_j =$$

$$= \sum_i \big(1 - \mu_i) z^i \varepsilon_i - \sum_j -z^j \mu_j . \varepsilon_j = \sum_i \varepsilon_i z^i = z,$$

where we used that $\sum_i \mu_i = 1$ and $\sum_i z^i \mu_i = 0$.

The Dunkl property is verified in the same way. Any codimension two intersection of this hyperplane arrangement is the intersection with V of H_{ij} and H_{kl} with i, j, k, l distinct or is of the form H_{ijk}: $z_i = z_j = z_k$ with i, j, k distinct. In the first case ρ_{ij} and ρ_{kl} commute and so their sum commutes with each term. In the second case the hyperplanes involved are H_{ij}, H_{jk} and H_{ki} and the above derivation (applied to to the case $n = 2$) shows that $\rho_{ij} + \rho_{jk} + \rho_{ki}$ is the selfadjoint map whose kernel is $H_{ijk} \cap V$ and which acts as multiplication by $\mu_i + \mu_j + \mu_k$ in the g-orthogonal complement of $H_{ijk} \cap V$ in V. This clearly commutes with any selfadjoint map which preserves H_{ijk}, in particular with ρ_{ij}, ρ_{jk} and ρ_{ki}.

Remark 4.2. The connection with the theory of Lauricella functions is as follows. We no longer assume that $\sum_i \mu_i = 1$, only that the real part of each μ_i lies in the interval $(0, 1)$. An element $(z^0, \dots, z^n) \in V^\circ$ is best thought of as a numbered $(n + 1)$-element subset of $\mathbb{C}$ of which each element z^i is given a weight μ_i; the fact that $(z^0, \dots, z^n) \in V$ means that the weighted barycenter $\sum_i \mu_i z^i$ is the origin. For such a $(z^0, \dots, z^n) \in V^\circ$, we choose in $\mathbb{C}$ an piecewise linear arc γ from z^i to z^j whose relative interior avoids $(z^0, \dots, z^n)$ (but we allow $i = j$). Then the integral

$$F_\gamma(z^0, \dots, z^n) := \int_\gamma \prod_{i=0}^{n} (z^i - \zeta)^{-\mu_i} \, d\zeta$$

makes sense and converges, once we have chosen a determination of each factor $(z^i - \zeta)^{-\mu_i}$ (it is enough to give a determination of $\log(z^i - \zeta)$, for

$(z^i - \zeta)^{-\mu_i} = \exp(-\mu_i \log(z^i - \zeta)))$. These determinations (and hence F_γ) subsist as holomorphic functions when we let $(z^0, \dots, z^n)$ vary over an open disk in V°. It can be shown that dF_γ is flat for the connection described above, and that for certain choices of arcs $\gamma_1, \dots, \gamma_n$, the corresponding $(F_{\gamma_1}, \dots, F_{\gamma_n})$ make up a flat coordinate system for that connection. Functions F_γ of this type are called *Lauricella functions* and were first considered (at least when the weights μ_i are all real) by Lauricella and Appell in the last decade of the 19th century. Some of these are hypergeometric functions in several variables. Deligne-Mostow determined some 90 years later the rational values of the weights for which they have a geometric interpretation as *automorphic forms* on a complex ball (in this case $\sum_i \mu_i > 1$).

Reflection groups. The first class of examples found to satisfy the Dunkl conditions are furnished by the real finite reflection groups (and were discovered by Dunkl).

If $V_\mathbb{R}$ is a real finite dimensional vector space (as a rule, we omit the subscript $_\mathbb{R}$, when we complexify), then a linear transformation of $V_\mathbb{R}$ is called *reflection* if is of order two and fixes a hyperplane in $V_\mathbb{R}$ pointwise. So there exist $\alpha \in V_\mathbb{R}^*$ and $\check{\alpha} \in V_\mathbb{R}$ with $\alpha(\check{\alpha}) = 2$ such that it has the form $x \mapsto x - \alpha(x)\check{\alpha}$. A finite subgroup $GL(V_\mathbb{R})$ said to be a *reflection group* if it is generated by reflections. (Any Weyl group is a reflection group, but there are more; the group of automorphisms of an icosahedron is an example.) We say that W is *irreducible* if it leaves no proper subspace invariant.

Examples 4.3. (A_n) The symmetric group $\mathcal{S}_{n+1}$ acting on $\mathbb{R}^{n+1}$ as permutation group of the basis elements is a reflection group: as is well-known, $\mathcal{S}_{n+1}$ is generated by its transpositions (ij), $i \le i < j \le n+1$, and such a transposition acts in $\mathbb{R}^{n+1}$ as the reflection perpendicular to $\varepsilon_i + \varepsilon_j$ (having the hyperplane $x^i = x^j$ as its fixed point set). Notice that $\mathcal{S}_{n+1}$ leaves invariant the hyperplane in $\mathbb{R}^{n+1}$ defined by $x^1 + \cdots + x^{n+1} = 0$, and still acts on this hyperplane as a reflection group $\mathcal{S}_{n+1}$. That reflection group is usually denoted $W(A_n)$.

(B_n) The group $W(B_n)$ of transformations of $\mathbb{R}^n$ that send each basis vector to a basis vector up to sign: This is also a reflection group with reflection vectors the basis vectors ε_i and the $\varepsilon_i \pm \varepsilon_j$, $1 \le i < j \le n$. The subgroup generated by the reflections in the basis vectors is normal in $W(B_n)$ (it accounts for the changes in sign of basis vectors) and has the symmetric group $\mathcal{S}_n$ as quotient group: $W(B_n)$ is thus identified with a semidirect product of $\mathcal{S}_n \ltimes (\mathbb{Z}/2)^n$.

(D_n) Here $n \ge 2$. The group $W(D_n)$ is the subgroup of $W(B_n)$ of index two generated by the reflections in the vectors $\varepsilon_i \pm \varepsilon_j$ only. The composition of the (commuting) reflections in $\varepsilon_1 + \varepsilon_2$ and $\varepsilon_1 - \varepsilon_2$ is minus the identity on the plane spanned by ε_1 and ε_2 (and the identity on the space spanned by all the other basis vectors), but it will not contain the reflection in a single basis vector. We have $W(B_n) \cong \mathcal{S}_n \ltimes K_n$, with $K_n = (\ker(\mathbb{Z}/2)^n \xrightarrow{\text{sum}} \mathbb{Z}/2)$.

(I_n) Here $n \geq 3$. These consist of all the reflection groups in dimension two: I_n is the dihedral group of automorphisms of a regular n-gon.

Exercise 12. Prove the following isomorphisms of reflection groups $A_2 \cong I_3$, $B_2 \cong I_4$, $D_2 \cong A_1 \times A_1$, $D_3 \cong A_3$ (you first need to decide what 'isomorphism' means here—there is an obvious definition).

Exercise 13. Prove that $W(A_n)$ is the automorphism group of the n-simplex spanned by the vectors $(n+1)\varepsilon_i - (\varepsilon_1 + \cdots + \varepsilon_{n+1})$ (these are the orthogonal projections in the hyperplane $x^1 + \cdots + x^{n+1} = 0$ of the vectors $(n+1)\varepsilon_i$).

Exercise 14. Prove that $W(B_n)$ is the automorphism group of the generalization of the octahedron to dimension n: the convex hull in $\mathbb{R}^n$ of the basis vectors and their opposites.

Exercise 15. Prove that the reflection groups $(A_n)_{n\geq 1}$, $(B_n)_{n\geq 2}$, $(D_n)_{n\geq 3}$, $(I_n)_{n\geq 3}$ are irreducible.

Remark 4.4. These examples yields all but finitely many isomorphism classes of irreducible finite reflection groups. The remaining ones, denoted E_6, E_7, E_8, F_4, H_3, H_4 and are harder to describe, except H_3, which is the automorphism group of the icosahedron. The other cases admit a similar characterization as the automorphism group of higher dimensional polyhedron. A very central result in Lie theory says that the classification of the finite dimensional complex simple Lie algebras is the same as that of the Weyl groups. These Weyl groups are Coxeter groups a bit of extra structure and their isomorphism groups are denoted $(A_n)_{n\geq 1}$, $(B_n)_{n\geq 2}$, $(C_n)_{n\geq 3}$, $(D_n)_{n\geq 4}$, E_6, E_7, E_8, F_4, where C_n has the same underlying reflection group as B_n.

Let $W \subset \mathrm{GL}(V_\mathbb{R})$ be a finite reflection group. If $g_\mathbb{R} : V_\mathbb{R} \times V_\mathbb{R} \to \mathbb{R}$ is an inner product, then the the sum of the W-translates of $g_\mathbb{R}$ is still an inner product that has the virtue of being W-invariant as well. Thus, W becomes a group of $g_\mathbb{R}$-orthogonal transformation of $V_\mathbb{R}$. We assume without much loss of generality that W is irreducible in the sense that it leaves no proper subspace of $V_\mathbb{R}$ invariant. This is equivalent to: any W-invariant symmetric bilinear form is a multiple of $g_\mathbb{R}$ (why?).

A reflection s in W defines a hyperplane $H_\mathbb{R} \subset V_\mathbb{R}$ as its fixed point set and that hyperplane determines s (for it is the $g_\mathbb{R}$-orthogonal reflection in that hyperplane). We denote that (complexified) reflection $s_H \in \mathrm{GL}(V)$. Then $1 - s_H$ is a g-selfadjoint map whose kernel is H and whose trace is 2.

Denote by $\mathcal{H}$ the collection of reflection hyperplanes in V. Notice that W leaves $\mathcal{H}$ invariant: for $H \in \mathcal{H}$ and $w \in W$, ws_Hw^{-1} is the orthogonal reflection in the hyperplane $w(H)$ (and hence equal to $s_{w(H)}$). A very basic property of a finite reflection group is that the stabilizer of any point is also a reflection group. This implies that W acts freely on $V^\circ := V - \cup_{H\in\mathcal{H}} H$.

Proposition 4.5. *Let $H \in \mathcal{H} \mapsto \kappa_H \in \mathbb{C}^\times$ be a W-invariant function with the property that $\sum_H \kappa_H = \frac{1}{2}\dim V$. If we put $\rho_H := \kappa_H(1 - s_H)$, then $\sum_H \rho_H$ is the identity and $(V, g, \{\rho_H\})$ satisfies the Dunkl condition. Thus V° acquires the* *structure of an almost-Frobenius manifold invariant under W (which therefore descends to such a structure on $W\backslash V^\circ$).*

Proof. Consider the form $h(v,v') := g(\sum_H \rho_H(v), v')$. Since each ρ_H is self-adjoint, h is symmetric. It is also easy to see h is W-invariant and so h must be a multiple of g: there exists a $\lambda \in \mathbb{C}$ such that for all $v, v' \in V$, $g(\sum_H \rho_H(v), v') = \lambda g(v,v')$. This implies that $\sum_H \rho_H$ is scalar multiplication by λ. Comparing traces, we see that $\sum_H 2\kappa_H = \lambda \dim V$ and so $\lambda = 1$. This proves that $\sum_H \rho_H$ is the identity of V.

Now let $L \subset V$ be a codimension two intersection of members of $\mathcal{H}$. Then the reflections s_H, $H \supset L$, generate a reflection subgroup W_L of W which leaves L pointwise fixed. It acts effectively in the two-dimensional g-orthogonal complement $L^\perp$.

If W_L acts irreducibly in $L^\perp$, then the argument that showed that $\sum_H \rho_H$ is a scalar gives here that $\sum_{H\in\mathcal{H}, H\supset L} \rho_H$ acts in $L^\perp$ as a scalar. Since each term is zero on L and preserves $L^\perp$, it follows that $\sum_{H\in\mathcal{H}, H\supset L} \rho_H$ commutes with each of its terms.

If the action of W_L in $L^\perp$ is not irreducible, then W_L leaves invariant a line in $L^\perp_{\mathbb{R}}$ (and hence also the line orthogonal to this line in $L^\perp_{\mathbb{R}}$). This can only happen when $\mathcal{H}$ has only has two members, H_1, H_2, say, which contain L which intersect $L^\perp$ in the two lines. These will be perpencular to each other and so ρ_{H_1} and ρ_{H_2} commute with each other and hence also with their sum. □

Saito's Coxeter examples. There is another way to attach a (this time, genuine) Frobenius manifold to a real finite reflection group. For the reflection group of type A_n this reproduces the polynomial example **??**.

Let $W \subset GL(V)$ be the complexification of a real finite reflection group and put $n := \dim V$. We need a few facts about these groups, for the proof of which we refer to the literature. The first remarkable fact is that the orbit space $W\backslash V$ is an affine space. We present it here as a *Fact*, meaning that we will not give a proof (see for instance N. Bourbaki: Groupes et Algèbres de Lie, Ch. V).

Fact 4.6 (Chevalley). *The W-invariant polynomial functions on V make up a polynomial algebra: $\mathbb{C}[V]^W$ is as a subalgebra of $\mathbb{C}[V]^W$ has $n = \dim V$ homogeneous generators $f^1, \dots, f^n$ that are algebraically independent. Moreover, the degrees of the generators are symmetrically distributed with respect to their average: if $d_k := \deg(f^k)$ is such that $d_1 \le d_2 \le \cdots \le d_n$, then $d_k + d_{n+1-k}$ is independent of k.*

The first part of this algebraic result has a surprising geometric content. Let us first observe that any two vectors $v, v' \in V$ that lie not in the same W-orbit can be separated by a W-invariant polynomial: choose $f \in \mathbb{C}[V]$ which vanishes in v', but does not vanish in any point of the W-orbit of v. Then the product of all of its W-transforms: $\prod_{w\in W} w^* f$ is as desired. Since such a W-invariant polynomial is an expression in the $f^1, \dots, f^n$, it follows that the map $F = (f^1, \dots, f^n) : V \to \mathbb{C}^n$ factors through an *injective* map $(u^1, \dots, u^n) : W\backslash V \to \mathbb{C}^n$. A finiteness argument from commutative algebra implies that this map is also surjective. In other words, we may identify the W-orbit space $W\backslash V$ of V (which we shall denote by V_W) with $\mathbb{C}^n$ and the choice of the generating set $(f^1, \dots, f^n)$ amounts to a choice of a system of coordinates $(u^1, \dots, u^n)$ on V_W. Beware however that *a priori* V_W does *not* have the structure of a vector space: it is just an algebraic variety and is (only) as such isomorphic to $\mathbb{C}^n$. (Yet, we will find that in the present case—indeed, *a posteriori*—V_W does have a vector space structure.) Not however that this variety comes with a $\mathbb{C}^\times$-action, inherited from scalar multiplication in V: if d_i is the degree of f^i, then $F(\lambda v) = (\lambda^{d_1} f^1(v), \dots, \lambda^{d_n} f^n(v))$.

The $\mathbb{C}^\times$-action V_W inherits from V is via the isomorphism $V_W \to \mathbb{C}^n$ the diagonal action with weights $d_1, \dots, d_n$.

From on we assume that W is irreducible. We shall always suppose that the degrees of the generators f^i are ordered by size: $d_1 \le d_2 \le \cdots \le d_n$.

Example 4.7. In the case A_n, we may take $f^k = \sigma_{1+k}$, $k = 1, \dots, n$.

In the case B_n, we may take $f^k = \sigma_{1+k}((z^1)^2, \dots, (z^n)^1)$, $k = 1, \dots, n$.

In the case D_n, we may take

Lemma 4.8. *We have* $d_1 = 2$ *and* $d_2 > 2$.

Proof. We cannot have $d_1 = 1$ (otherwise W would leave invariant a hyperplane, which would contradict our assumptions that it be irreducible). On the other hand, we already observed that W leaves invariant a complexified inner product and so $d_1 = 2$. If we would have $d_2 = 2$, then f^1 and f^2 are linearly independent symmetric W-invariant bilinear forms. Some nontrivial linear combination $\lambda_1 f^1 + \lambda_2 f^2$ will be degenerate. Its nilspace is W-invariant and since W is irreducible this is not possible. So $d_2 > 2$. □

Corollary 4.9. *We have* $d_n > d_{n-1}$ *and so the subalgebra of* $\mathbb{C}[V]^W$ *generated by the invariant polynomials of degree* $< d_n$ *is also a polynomial algebra, namely the one generated by* $(f^1, \dots, f^{n-1})$.

Proof. If we combine Fact **??** with the previous lemma, then we see that $d_n > d_{n-1}$. The second statement is clear. □

This highest degree d_n is also called the *Coxeter number* of W and is often denoted h. So we can now say that $d_k + d_{n+1-k} = h + 2$ for all k. Part of the geometric content of the above corollary can be stated as follows: we have naturally defined the affine space $B := \operatorname{Spec} \mathbb{C}[u^1, \dots, u^{n-1}]$ so that the inclusion of $\mathbb{C}[u^1, \dots u^{n-1}] \subset \mathbb{C}[u^1, \dots, u^n]$ defines a natural projection $\pi : V_W \to B$. We keep on referring to the image of $0 \in V$ in V_W or B by the origin (and denote it accordingly by 0).

An invariant polynomial of particular interest is the discriminant, which is defined as follows. For every reflection hyperplane $H_\mathbb{R}$, there are just two normal vectors $\pm\check{\alpha}_H$ perpendicular to H (relative to a W-invariant inner product). So $\Delta(z) := \prod_H g(\check{\alpha}_H, z)$ is a homogenenous polynomial that is not W-invariant, but its square Δ^2 is. So we have a unique polynomial $D(u^1,\dots,u^n)$ such that $\Delta^2 = D(f^1,\dots,f^n)$. Its zero set is called the *discriminant*, a name that we also use for D itself.

Lemma 4.10. *Up to a constant multiple, Δ equals the jacobian determinant of F: $df^1\wedge df^2\wedge\cdots\wedge df^n$ equals $F^*\Delta$ times a constant n-form on V.*

Proof. The map $F : V \to V_W \cong \mathbb{C}^n$ is a local isomorphism when restricted to V°. So $df^1\wedge\cdots\wedge df^n$ is nonzero on V°. On the other hand, if p is a general point of a hyperplane H, then there is a s_H-invariant neighborhood U of p, such that its s_H-orbit space is mapped isomorphically onto an open subset of $V \cong \mathbb{C}^n$. In local coordinates, the map is like

$$(\zeta^1,\zeta^2,\dots,\zeta^n)\mapsto((\zeta^1)^2,\zeta^2,\dots,\zeta^n),$$

(with the reflection being given by $(\zeta^1,\zeta^2,\dots,\zeta^n)\mapsto(-\zeta^1,\zeta^2,\dots,\zeta^n)$) and so $df^1\wedge\cdots\wedge df^n$ is a unit times $2\zeta^1 d\zeta^1\wedge\cdots\wedge d\zeta^n$. This proves that $df^1\wedge\cdots\wedge df^n$ vanishes with order one on the union of hyperplanes. We conclude that $df^1\wedge\cdots\wedge df^n$ is divisible by Δ with quotient a rational n-form without poles and zeroes. This is then necessarily a constant n-form on V. □

Corollary 4.11. *The number of reflection hyperplanes in W equals $\frac{1}{2}nh$. Moreover, D is weighted homogeneous of degree nh.*

Proof. The degree of $df^1\wedge df^2\wedge\cdots\wedge df^n$ is $\sum_i d_i$ and since we have $d_k + d_{n+1-k} = h+2$ for all k, it follows that this sum is equal to $\frac{1}{2}n(h+2) = \frac{1}{2}nh + n$. So the degree of Δ is $\frac{1}{2}nh$. □

Fact 4.12. *The restriction of D to $\pi^{-1}(0)$ is not identically zero.*

(The standard proof (cf. *op. cit.*) uses a socalled Coxeter transformation. This is an element $c\in W$ leaving invariant a line L not contained in any reflection hyperplane on which c has an eigen value of the form $e^{2\pi(\sqrt{-1}d_k/h}$. Then for $k+1,\dots,n$, we have $f_k|L = f_k\circ c|L = e^{2\pi\sqrt{-1}d_k/h}f_k|L$ and hence $f_k|L = 0$ unless d_k divides h, or equivalently, $k\neq n$. So L maps to $\pi^{-1}(0)$ and is non contained in D.) Since D is weighted homogeneous of degree hn and u^n has weight $d_n = h$, it follows that $D(u^n,0,0,\dots,0)$ is a constant times $(u^n)^n$.

Exercise 16. Prove that D has the form $c_0(u^n)^n + c_1.(u^n)^{n-1} + c_2.(u^n)^{n-1} + \cdots + c_n$ with c_0 a nonzero constant and c_k weighted homogeneous degree of degree kh in $u^1,\dots u^{n-1}$. Conclude that for $k>0$, c_k has order $>k$ at the origin (so that the multiplicity of D at the origin is exactly n).

We denote by g the (constant) nondegenerate symmetric bilinear form on the tangent bundle of V defined by f^2. We choose a cobasis $(z^1,\dots,z^n)$ on V for which $f^2=\sum_k (z^k)^2$ so that $g=\sum_k dz^k\otimes dz^k$.

This form is W-invariant and hence descends to one on the complement of the discriminant V_W°. We shall refer to that form as a *Chevalley form* and denote it g^C. For the purposes of computation is much more convenient to pass to the form of the cotangent bundle, $\check{g} = \sum_\nu \frac{\partial}{\partial z^k} \otimes \frac{\partial}{\partial z^k}$. Then $\check{g}_C^{ij} = \check{g}_C(du^i, du^j)$ and so

$$F^*\check{g}_C^{ij} = \check{g}(df^i, df^j) = \sum_k \frac{\partial f^i}{\partial z^k}\frac{\partial f^j}{\partial z^k}.$$

Notice that the right hand side is homogeneous of degree d_i+d_j-2. The left hand side, being obviously W-invariant, must be a polynomial in $u^1, \dots, u^n$. Since $d_i + d_j - 2 \le 2d_n - 2$, it follows that the degree of $\check{g}_C^{ij}$ in u^n has degree 1 at most. So we can write

$$\check{g}_C^{ij} = \check{g}_o^{ij} + u^n \check{g}_S^{ij},$$

with $\check{g}_o^{ij}$ and $\check{g}_S^{ij}$ independent of u^n (the subscript S refers to Kyoji Saito, who was the first to consider this form).

Lemma 4.13. *The determinant* $\det(\check{g}_S^{ij})$ *is a nonzero constant and hence* $\check{g}_S^{ij}$ *defines nondegenerate form* $g^S = g^S_{ij} du^i \otimes du^j$ *(the* Saito metric*) on the tangent bundle of* V_W*. The Saito metric is weighted homogeneous of degree* $h+2$*.*

Proof. We observe that $\check{g}_S^{ij}$ has degree $d_i + d_j - 2 - h$, whenever it is nonzero. Since a zero polynomial has any degree, it follows that $\check{g}_S^{ij}$ is constant whenever $d_i + d_j \le h + 2$. So if for any positive integer k that appears as some d_i, $I(k)$ denotes the set of indices i for which $d_i = k$, then the submatrix $(\check{g}_S^{ij})_{i\in I(k), j\in I(h+2-k)}$ is constant. An elementary linear algebra exercise, then shows that by modifying the $(f^i)_{i\in I(k)}$ and the $(f^i)_{i\in I(h+1-k)}$ by linear combinations thereof we can arrange that this submatrix is such that $\check{g}_S^{ij} = 0$ $i+j \le n$ and $\check{g}_S^{ij}$ is constant for $i+j = n+1$ (some care is needed only when $k = h+1-k$). If we do this for all k, then $(\check{g}_S^{ij}(0))$ acquires the diagonal form. It follows that

$$\det(\check{g}_C^{ij}) = \det(\check{g}_o^{ij} + u^n \check{g}_S^{ij}) \equiv (u^n)^n \det(\check{g}_S^{ij}(0)) \pmod{(u^1, \dots, u^{n-1})}.$$

The left hand side is a nonzero scalar times $(u^n)^n$, and so $\det(\check{g}_S^{ij}(0)) \neq 0$. Since $\det(\check{g}_S^{ij})$ is weighted homogeneous, it follows that $\det(\check{g}_S^{ij})$ is constant nonzero everywhere. This prove that g^S is defined. It is a priori clear that $\check{g}_S(0)$ has the form $\sum_{i=1}^n c_i du^i \otimes du^{n+1-i}$ with $c_i = c_{n+1-i} \neq 0$. It follows that g^S is weighted homogeneous of degree $h+2$ as asserted. □

Corollary 4.14 (Saito's flat coordinates)**.** *The Saito metric is flat. In fact, we can choose our homogeneous basic invariants* $f^1, \dots, f^n$ *in such a manner that* g^S *has* constant *coefficients:* $g^S = \sum_k du^k \otimes du^{n+1-k}$.

Proof. Since $\check{g}_C$ is flat[1], so is its pull-back under the translation $(u^1, \dots, u^n) \mapsto (u^1 + \lambda, \dots, u^n)$. In other words the form $(\check{g}_o^{ij} + (u^n + \mu)\check{g}_S^{ij})$ is flat for all

[1]But beware that $\check{g}_C$ is nondegenerate on V_W° only.

$\mu \in \mathbb{C}$. If we take $\mu \neq 0$, divide by μ and let $\mu \to \infty$, we get $(\check{g}_S^{ij})$. This procedure preserves flatness and hence the Saito form is flat. Now observe that the flatness of a metric on a manifold M implies that a given basis at a given cotangent space T_p^*M extends uniquely to flat coordinate system at p whose differentials at p give that basis. If we apply this to $M = V_W$ endowed with the Saito metric, $p = 0$ and the $du^i(0)$, then we find a flat local coordinate system $('u^1, \dots, 'u^n)$ for V_W at 0 with $d'u^k(0) = du^k(0)$, $k = 1, \dots, n$. But this is unique and so it must be invariant under all the symmetries present here, in particular under the $\mathbb{C}^\times$-action. This means that $'u^k$ transforms under the $\mathbb{C}^\times$-action as $du^k(0)$: the pull-back of $'u^k$ under $\mu \in \mathbb{C}^\times$ is $\mu^{d_k}\, 'u^k$. In other words, $'u^k$ is defined on all of V_W and is weighted homogeneous of degree d_k. So $'u^k$ is a polynomial and $'u^k - u^k$ is a weighted homogeneous polynomial of degree d_k of order ≥ 2 (i.e., without linear or constant part). With induction on the degree, we then find that u^k is a polynomial in the $('u^1, \dots, 'u^n)$. So if we replace $f^k = F^*u^k$ by $F^*('u^k)$, then we get an alternate set of generators of $\mathbb{C}[V]^W$ which now in addition produces a flat coordinate system in V_W. □

Remark 4.15. The essential part of the above argument amounts to saying that we have a natural identification of V_W with its tangent space at the origin (actually given by an 'exponential map'). As a consequence (and perhaps to our surprise), V_W has the natural structure of a vector space.

From now on we assume that $(f^1, \dots, f^n)$ *has been chosen in accordance with the previous proposition, so that* $g^S = \sum_k du^k \otimes du^{n+1-k}$. We write ∂_k for the coordinate vector field $\frac{\partial}{\partial u^k}$.

Denote by $(A_j^{C,i})$ the connection form of the Levi-Civita connection of g^C on the tangent bundle of V_W° relative to the coordinate vector fields $(\partial_1, \dots, \partial_n)$. Then $(\check{A}^j_{C,i} = -A_j^{C,i})$ is the connection form of g^C on the cotangent bundle of V_W° relative to the basis $(du^1, \dots, du^n)$. The latter pulls back under F to the connection form of $\sum_k dz^k \otimes dz^k$ on the cotangent bundle of V° relative to the basis $(df^1, \dots, df^n)$ and hence is characterized by

$$F^*\check{A}^j_{C,i} = -\sum_k d\Big(\frac{\partial f^i}{\partial z^k}\Big)\frac{\partial f^j}{\partial z^k} = -\sum_{k,l} \frac{\partial^2 f^i}{\partial z^l \partial z^k}\frac{\partial f^j}{\partial z^k} dz^l.$$

This form has degree $d_i + d_j - 2$ as well, and so a similar argument as for g^C shows that we can write

$$F^*\check{A}^j_{C,i} = F^*(\check{A}^j_{o,i} + u^n F^*\check{A}^j_{S,i})$$

with $\check{A}^j_{o,i}$ and $\check{A}^j_{S,i}$ independent of u^n and with du^n not occurring in $\check{A}^j_{S,i}$. The form $(\check{A}^j_{o,i} + (\mu + u^n F^*\check{A}^j_{S,i})$ is the connection matrix of the pull-back of g^C over translation over μ in the u^n-direction. If we divide by μ and let $\mu \to \infty$ we get $(\check{A}^j_{S,i})$, the connection matrix of the Saito metric. But the latter is zero by our judicious choice of the generators. So $A^j_{C,i} = A^j_{o,i}$ is independent of u^n.

The symmetric bilinear form $\check{g}^S + \lambda\check{g}^C$ on the tangent bundle of V_W is for $\lambda \neq 0$ a multiple of $\check{g}^C + \lambda^{-1}\check{g}^C$. The latter is in fact the form $\check{g}^C$ takes if we displace it over λ^{-1} in the u^n-direction. So the Levi-Civita connection of $\check{g}^S + \lambda\check{g}^C$ is flat. This connection is given by $\lambda A^{C,i}_j$. In view of Proposition **??** this suggests that (V_W, g^S) admits the structure of a Frobenius manifold with product

$$\partial_i \cdot \partial_j = \textbf{(hier een constante?)} \sum_k A^k_{C,i}(\partial_j)\partial_k.$$

This product is commutative (for $\check{g}^S + \lambda\check{g}^C$ is torsion free) and Proposition **??** implies that it is associative as well. On V°_W we may define this product more invariantly as follows; if X and Y are vector fields on an open simply connected $U \subset V^\circ_W$ *that are flat for the Saito metric* and $\tilde{X}$ and $\tilde{Y}$ are lifts to the preimage of U in V°, then the lift of $X \cdot Y$ is the usual covariant derivative on V, $\nabla_{\tilde{X}}\tilde{Y}$.

Theorem 4.16. *The triple* $(V_W, \cdot, g^S)$ *is a Frobenius manifold with unit vector field* $e := \partial_n$*, trace differential* $I := du^1$ *and* $E := h^{-1}\sum_k d_k u^k \partial_k$ *as normalized Euler field.*

Notice that this Frobenius manifold has polynomial structure constants. A theorem of Hertling asserts that this essentially exhausts all the Frobenius manifolds that are generically semisimple and admit a normailzed Euler field.

Proof of Theorem **??**. **(nog te voltooien)** Let $X = \sum_k X^k \frac{\partial}{\partial z^k}$ be a local vector field on V. The covariant derivative of $F^*du^1 = \sum_k 2z^k dz^k$ with respect to X is $2\sum_k X^k dz^k$. $\square$

5. Moduli spaces of curves

A *smooth rational curve* is an algebraic curve (as is always tacitly assumed here, defined over the complex field) isomorphic to the Riemann sphere $\mathbb{P}^1$. This may serve as a definition (it is more usual characterize such a curve as a smooth connected projective curve having no nonzero regular (=holomorphic) differentials). The automorphism group of $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$ is $\mathrm{PGL}(2,\mathbb{C})$, which acts by fractional linear transformations: $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ sends $z \in \mathbb{C} \subset \mathbb{P}^1$ to $(az+b)(cz+d)^{-1}$. This group acts simply transitively on the set of ordered distinct triples in $\mathbb{P}^1$: if $z_1, z_2, z_3 \in \mathbb{P}^1$ are distinct, then there is unique $\sigma \in \mathrm{PGL}(2,\mathbb{C})$ which sends (z_1, z_2, z_3) to $(0, 1, \infty)$. Equivalently, if C is a smooth rational curve and $p_1, p_2, p_3 \in C$ are distinct, then there is a unique isomorphism $\phi : C \to \mathbb{P}^1$ which maps (p_1, p_2, p_3) onto $(0, 1, \infty)$. So if we are given a fourth point $p_4 \in C - \{p_1, p_2, p_3\}$, then $\phi(p_4) \in \mathbb{P}^1 - \{0, 1, \infty\} = \mathbb{C} - \{0, 1\}$ is a complete (and classical) invariant of the system $(C; p_1, p_2, p_3, p_4)$, called its *cross ratio*. The question arises what it means for this invariant to be zero. At first sight, this seems to mean that p_1 and p_4 coalesce. But note that if $t = \phi(p_4)$, then multiplication by t^{-1} sends $(0, \infty, 1, t)$ to $(0, \infty, t^{-1}, 1)$ and so letting $t \to 0$ can also be understood as the coalescing of the complementary pair p_2 and p_3. We give an interpretation which reconciles these two: consider the product $\mathbb{P}^1 \times \mathbb{P}^1$ with the porjection $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ on the second factor. It has the following four sections: the constant sections $\sigma_1, \sigma_2, \sigma_3$ which assign to $t \in \mathbb{P}^1$ the pairs $(0, t)$, (∞, t), $(1, t)$ respectively and the section $\sigma_4 : t \mapsto (t, t)$. The latter meets the first three sections in the points $(0, 0)$, (∞, ∞) and $(1, 1)$ respectively. Let $\mathcal{C} \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blowup of these three points and denote by the $\tilde{\pi} : \mathcal{C} \to \mathbb{P}^1$ be the composite of the map $\mathcal{C} \to \mathbb{P}^1 \times \mathbb{P}^1$ with π.