C2 Fundamental Theory of Dynamical Systems

Handout 5

The Jordan Normal Form

Recall that by linearizing around hyperbolic fixed points and periodic orbits of nonlinear systems, one can obtain a locally accurate picture of the dynamics. Hence, it is important to understand the different kinds of evolution that can occur in a linear system. The JNF *allows a complete representation of the* eigenstructure *of linear systems, with an associated picture of linear dynamics. The* JNF *is often called the* canonical *form of a matrix, because it is essentially the "simplest possible" representation after a change of variables. In particular, powers of a matrix and exponentials are easy to compute in canonical form. This handout summarizes the Jordan Normal Form (*JNF*) of a real matrix, and some of its implications for the dynamics of linear systems. In particular, it shows how to compute the* JNF *of a given matrix:*

Structure of real $n \times n$ matrices

Recall that an *eigenvalue* λ and its corresponding *eigenvector* $v \neq 0$ are defined by

$$Av = \lambda v$$

From this definition, we see that A has an eigenvalue λ if and only if the matrix $(A - \lambda I)$ is non-invertible. Hence $\chi(\lambda) = \det(A - \lambda I) = 0$. Clearly χ is a degree n polynomial in λ, and hence has n real or complex roots, some possibly repeated. Since χ has real coefficients, complex roots come in conjugate pairs, that is if $\chi(\lambda) = 0$, then $\chi(\bar{\lambda}) = \overline{\chi(\lambda)} = \bar{0} = 0$. Note that if v is an eigenvector, then so is cv for any scalar $c \neq 0$. If λ is an eigenvalue of multiplicity r (*ie* $(x-\lambda)^r$ divides $\chi(x)$), then there are at most r linearly independent eigenvectors corresponding to λ (recall that r vectors $v_1, \ldots, v_r$ are linearly independent if $c_1v_1 + \ldots + c_rv_r = 0$ implies $c_1 = \ldots = c_r = 0$).

The n eigenvalues of A may therefore arise in the following combinations:

i) A real eigenvalue λ of multiplicity r with r independent eigenvectors.

ii) A real eigenvalue λ of multiplicity r with less than r independent eigenvectors.

iii) A complex eigenvalue λ of multiplicity r with r independent complex eigenvectors.

iv) A complex eigenvalue λ of multiplicity r with fewer than r independent eigenvectors.

Jordan Normal Form of A

The *JNF* of A is block diagonal, with a block corresponding to each distinct eigenvalue (or complex conjugate pair). Each block in turn is made up of sub-blocks, each corresponding to an *independent* eigenvector. Because of the block structure, we can treat each of the above cases separately, and will in fact just consider the following three scenarios: a) A has n independent eigenvectors (which may correspond to distinct or repeated eigenvalues), b) A has λ as an eigenvalue of multiplicity n and only one independent eigenvector and c) $n = 2$ and A has a complex conjugate pair of eigenvalues. From these we can build up all possible combinations of i)-iii) above; we do not treat iv) explicitly since this is just a combination of cases ii) and iii).

DIAGONAL BLOCKS

Suppose that A has n independent eigenvectors. Then it is *diagonalizable*, that is there exists an invertible P such that $P^{-1}AP = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = B$, where $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. The matrix P is the matrix whose columns are the n independent eigenvectors $v_1, v_2, \ldots, v_n$ of A. We denote this as $P = [v_1| v_2| \ldots| v_n]$. Note that the fact that $v_1, v_2, \ldots, v_n$ are independent implies that P is independent (excercise).

To see that P diagonalizes A, observe that

$$\begin{aligned} AP &= A[v_1| v_2| \ldots| v_n] \\ &= [Av_1| Av_2| \ldots| Av_n] \\ &= [\lambda_1 v_1| \lambda_2 v_2| \ldots| \lambda_n v_n] \\ &= [v_1| v_2| \ldots| v_n]\ \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \\ &= PB \end{aligned}$$

NON-TRIVIAL BLOCKS

Suppose now that A has a single eigenvalue λ of multiplicity n, so that $\chi(x) = (x-\lambda)^n$, but only one independent eigenvector v_1. We define n-1 *generalized eigenvectors* obtained by solving

$$\begin{aligned} (A - \lambda I)v_2 &= v_1 \\ &\vdots \\ (A - \lambda I)v_i &= v_{i-1} \\ &\vdots \\ (A - \lambda I)v_n &= v_{n-1} \end{aligned}$$

with $v_i \neq 0$ for $i = 2, \ldots, n$. The term generalized eigenvector is motivated by the fact that $(A - \lambda I)^k v_k = 0$ for a generalized eigenvector, whereas $(A - \lambda I)v = 0$ for an ordinary eigenvector. Note that $v_1, v_2, \ldots, v_n$ are linearly independent (exercise, use the fact that $(A - \lambda I)^k v_k = 0$) and hence $P = [v_1| v_2| \ldots| v_n]$ is invertible. Using the same argument as above (see Q4, Ex. Sheet 4), it is easy to see that $AP = PB$, where B is the $n \times n$ Jordan block

$$B = \begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \qquad (1)$$

COMPLEX CONJUGATE EIGENVALUES

Suppose that A is a 2×2 matrix with a complex conjugate pair of eigenvalues $\lambda = \alpha + i\beta, \bar{\lambda} = \alpha - i\beta$, with $\beta \neq 0$. Let $u + iv$ be the complex eigenvector corresponding to λ, so that

$$A(u+iv) = (\alpha+i\beta)(u+iv)$$

Equating real and imaginary parts, we get

$$Au = \alpha u - \beta v \qquad (2a)$$

$$Av = \alpha v + \beta u \qquad (2b)$$

We first show that this means that u and v are linearly independent as real valued vectors. Suppose not, so that there exists constants c and d such that

$$cu + dv = 0 \tag{3}$$

with either $c \neq 0$ or $d \neq 0$. Applying A to (3) we obtain

$$(c\alpha + d\beta)u + (d\alpha - c\beta)v = 0 \tag{4}$$

From (3) we have $dv = -cu$. Substituting this into (4) gives $\alpha cu + \beta du - \alpha cu - \beta cv = 0$, so that $\beta du = \beta cv$. Since $\beta \neq 0$, this implies that $du = cv$, and hence $d^2u = cdv$. However, by (3) we have $cdv = -c^2u$ and hence $d^2u = -c^2u$. Since c and d are real and at least one is assumed to be non-zero, we cannot have $d^2 = -c^2$, and hence $u = 0$. From (2b), we obtain $Av = \alpha v$ and since α is not an eigenvalue of A, we have $v = 0$. But this is a contradiction since this means that $u+iv = 0$, but $u+iv$ is meant to be an eigenvector of A. Hence u and v are linearly independent.

Thus if we let $P = [u|v]$, then P is an invertible matrix. Proceeding as above, we have

$$\begin{aligned} AP &= A[u|v] \\ &= [Au|Av] \\ &= [\alpha u - \beta v|\alpha v + \beta u] \\ &= [u|v]B \end{aligned}$$

where

$$B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \tag{5}$$

IMPLICATIONS FOR LINEAR DYNAMICAL SYSTEMS

The long-term evolution of a linear system is best analysed in the canonical form provided by the *JNF* decomposition. As above, let the matrix P be such that if B is the *JNF* of A then

$$AP = PB$$

As we have seen in lectures the study of the system

$$x_{m+1} = Ax_m$$

is reduced to that of

$$y_{m+1} = By_m$$

by changing variables from x to $y = P^{-1}x$. Similarly, in the case of differential equations

$$\dot{x} = Ax$$

is transformed into

$$\dot{y} = By$$

POWERS OF *B*

i) If $B = \text{diag}(\lambda_1, \ldots, \lambda_n)$ then $B^m = \text{diag}(\lambda_1^m, \ldots, \lambda_n^m)$.

ii) If B is a non-trivial block given by (1), then by induction it is straightforward to show that

$$B^m = \begin{pmatrix} \lambda^m & m\lambda^{m-1} & \binom{m}{2}\lambda^{m-2} & \cdots & \binom{m}{n-1}\lambda^{m-(n-1)} \\ 0 & \lambda^m & m\lambda^{m-1} & \cdots & \binom{m}{n-2}\lambda^{m-(n-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda^m & m\lambda^{m-1} \\ 0 & 0 & \cdots & 0 & \lambda^m \end{pmatrix}$$

where

$$\binom{m}{j} = \frac{m!}{j!(m-j)!}$$

with the convention that

$$\binom{m}{j} = 0$$

for $j \geq m$.

iii) If B corresponds to a complex conjugate pair of eigenvalues, and so is given by (5), let r and θ be given by $\alpha = r\cos\theta$, $\beta = r\sin\theta$. Then by induction, it can be shown

$$B^m = r^m \begin{pmatrix} \cos m\theta & \sin m\theta \\ -\sin m\theta & \cos m\theta \end{pmatrix}$$

EXPONENTIALS OF *B*

Recall that the solution of $\dot{y} = By$ is given by $y(t) = \exp Bt.y(0)$. As shown in lectures, the matrix $\exp Bt$ can be easily computed when B is in *JNF*.

i) If $B = \text{diag}(\lambda_1, \ldots, \lambda_n)$ then $\exp Bt = \text{diag}(\exp\lambda_1 t, \ldots, \exp\lambda_n t)$.

ii) If B is a non-trivial block given by (1), then

$$\exp Bt = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{1}{2!}t^2 & \cdots & \frac{1}{(n-1)!}t^{n-1} \\ 0 & 1 & t & \cdots & \frac{1}{(n-2)!}t^{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & t \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

This formula is most easily verified using the decomposition given in lectures.

iii) If B corresponds to a complex conjugate pair of eigenvalues, and so is given by (5), then, as shown in lectures

$$\exp Bt = e^{\alpha t} \begin{pmatrix} \cos t\beta & \sin t\beta \\ -\sin t\beta & \cos t\beta \end{pmatrix}$$