Complex Morse Theory

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Abstract

Let X be a projective variety and $f : X \to \mathbb{CP}^1$ nondegenerate. Then we may mimic the usual Morse theory by studying the topology of X in terms of the slices $f^{-1}(t), t \in \mathbb{CP}^1$, and in particular a generic fiber X_*. The local behavior of f is no longer described by an index, but by an action on $H_*(X_*)$ called the monodromy. The monodromy is described explicitly by the Picard-Lefschetz formula. We then restrict to the special case where f is the Lefschetz fibration associated to a Lefschetz pencil on X. The Lefschetz hyperplane theorem relates the homology of X with that of X_*. The Hard Lefschetz theorem gives a decomposition of $H_*(X)$ in terms of generic hyperplane sections; equivalently, it gives a decompositon of $H_{n-1}(X_*)$ into vanishing and invariant cycles.

1 Introduction

In real Morse theory, we study topology of a manifold X by means of a nondegenerate smooth function $f : X \to \mathbb{R}$. We find that the diffeomorphism type of the sublevel set $X^t := f^{-1}((-\infty, t])$ does not change as t varies unless t crosses at critical value; as t crosses a critical value, we find that M^t undergoes a handle attachment by a λ-cell, where λ is the Morse index of f at the corresponding critical point. The topology of M is determined completely by the local behavior of f near its critical points, and by the Morse lemma, the local behavior of f is completely determined by its Morse index.

Now consider the case when X is a smooth projective variety, and f is a holomorphic function $f : X \to \mathbb{P}^1$ (note that the obvious choice of taking $f : X \to \mathbb{C}$ would not be interesting, since for compact M the only such holomorphic functions are constant). If f has only nondegenerate critical points, then by a holomorphic version of the Morse lemma, we can choose local holomorphic coordinates near its critical points such that f is quadratic.

Since f is a complex valued (thinking of $\mathbb{P}^1$ as $\mathbb{C} \cup \{\infty\}$), we no longer have a good notion of sublevel sets, nor of its index at a critical point. In the case of a real Morse function, as we vary $t \in \mathbb{R}$ parameterizing the sublevel sets, we have no choice but to cross each critical value. In the present case, we can approach a critical value in a more interesting way. Consider and closed path γ in $\mathbb{P}^1$ which does not pass through any critical valued of f. Then we can try to understand how the topology of the fibers $f^{-1}(t)$ changes along γ. This leads to the notion of the *monodromy group*. Let $\mathbb{P}^*$ be $\mathbb{P}^1 \setminus \mathrm{Crit}(f)$. We obtain a representation of $\pi_1(\mathbb{P}^*)$ on $\mathrm{Aut}(H_*(X_*))$, where X_* is a generic fiber of f, and the monodromy group is the subgroup of $\mathrm{Aut}(H_*(X_*))$ generated by this action. The action of the monodromy group is described explicitly by the Picard-Lefschetz formula.

This leads to two types of cycles: those which are invariant under the action of the monodromy group, denoted $\mathbb{I}$, and those which vanish when pushed inside of X_*, denoted $\mathbb{V}$ for *vanishing cycles*. The Lefshetz hyperplane and Hard Lefschetz theorems describe explicitly the relationship between $\mathbb{I}$ and $\mathbb{V}$ and the topology of X and X_*.

In what follows, we will outline how to make these notions rigorous, state the theorems precisely, and give some applications. This paper will roughly follow the exposition of [9] and [6]. A treatment by Lefschetz himself can be found in [7]. Applications beyond the scope of this paper can be found in [10]. The theory we present is related to more modern developments in algebraic geometry; see for example [1].

2 Complex Morse Functions

Let X be a compact n-dimensional complex manifold and $f : X \to \mathbb{P}^1$ a holomorphic map. Suppose that f is a *nonresonant Morse function*, i.e. all of its critical points are nondegenerate and distinct critical points correspond to distinct critical values. Let $C = \{t_1, \ldots, t_r\}$ be the set of critical points of f. Since $\mathbb{P}^1 \simeq S^2$, we may decompose $\mathbb{P}^1$ as the union of two closed disks D_+ and D_- chosen so that all of the critical values lie in the interior to D_+. Let $* \in \partial D_+$ be a regular value, and set $X_+ = f^{-1}(D_+)$ and $X_* = f^{-1}(*)$. By the Ehresmann fibration theorem [3], f fibres locally trivially over $D_+ \setminus C$, so we might expect to obtain information about the topology of X_+ from that of the generic fiber X_*, together with precise information about the local behavior of f near the critical points. In fact, we have the following.

Lemma 2.1. *The homology $H_q(X_+, X_*)$ vanishes for $q \neq n$. For $q = n$, $H_n(X_+, X_*)$ is free of rank r, with one generator C_j for each critical point.*

Sketch of proof. Let x_i be the critial points and t_i be the critical values. Let $D_j \subset D_+$ be disks of radius ρ centered at the critical values with ρ small enough so that they are all disjoint. Let l_i be a path in D_+ connecting $*$ to ∂D_j. Let l be the union of the paths l_i and k be the union of l and $\cup D_J$ (see figure 1). Set $L = f^{-1}(l)$ and $K = f^{-1}(k)$. Using the fact that k is a strong deformation retract of D_+ and that $*$ is a strong deformation retract of l, together with the Ehresmann fibration theorem and the homotopy lifting property of fibrations, it is not hard to show that X_* is a deformation retract of L and that K is a deformation retract of X_+. Then we have

$$H_q(X_+, X_*) = H_q(X_+, L) = H_q(K, L).$$

Now let $X_{D_j} = f^{-1}(D_j)$ and $X_j = f^{-1}(t_j + \rho)$. Let $U = f^{-1}(l - \cup\partial D_j)$, i.e. U is the open subset of L which does not lie above any of the boundaries ∂D_j. Then we have

$$\begin{aligned} K - U &= \bigsqcup X_{D_j}, \\ L - U &= \bigsqcup X_j. \end{aligned}$$

Thus by excision we have

$$H_q(K, L) = H_q(K - U, L - U) = \bigoplus_{j=1}^{r} H_q(X_{D_j}, X_j),$$

and hence $H_q(X_+, X_*) = \oplus_j H_q(X_{D_j}, X_j)$. Now let B_j be a small closed ball of radius R in X_+ centered at x_j. Set $E_j = X_{D_j} \cap B_j$, $F_j = X_j \cap B_j$. Once again using the Ehresmann fibration theorem, one may deduce that F_j is a deformation retract of $F_j \cup (X_{D_j} \setminus \mathrm{int}(B_j))$. Using this fact, and excising $X_{D_j} \setminus \mathrm{int}(B_j)$, gives

$$H_q(X_{D_j}, X_j) = H_q(X_{D_j}, F_j \cup (X_{D_j} \setminus \mathrm{int}(B_j))) = H_q(E_j, F_j).$$

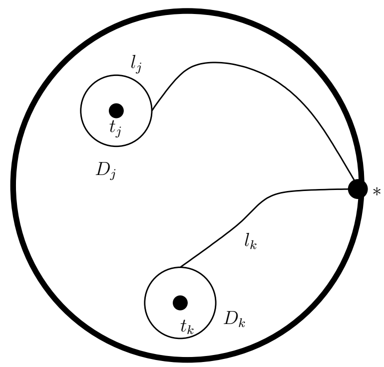

Figure 1: *The sets l and k and the paths generating $\pi_1(D \setminus \{t_1, \ldots, t_r\})$.*

Thus the homology of the pair (X_+, X_*) is completely localized near the critical points of f. Since f is a nondegenerate, we may choose local (holomorphic) Morse coordinates near B_j so that

$$f(x) = t_j + \sum_i x_i^2.$$

Using these local coordinates (and dropping the index j), we have

$$\begin{aligned} E &= \{x \in \mathbb{C}^n : \sum_i |x_i|^2 \leq R^2, |\sum_i x_i^2| \leq \rho\}, \\ F &= \{x \in E : \sum_i z_i^2 = \rho\}. \end{aligned}$$

From this description, it is clear that E is contractible, and hence the connecting homomorphism $H_q(E,F) \to H_{q-1}(F)$ is an isomorphism when $q \neq 0$. Furthermore, $H_0(E,F) = 0$. It is relatively straightforward to check that F is diffeomorphic to the disk bundle of the tangent bundle TS^{n-1}. Hence for $q \neq 0$,

$$H_q(E,F) = H_{q-1}(F) = H_{q-1}(TS^{n-1}) = H_{q-1}(S^{n-1}) = \begin{cases} 0, & q \neq n, \\ \mathbb{Z}, & q = n. \end{cases}$$

□

It will be useful to have a geometric description of the generators of $H_n(X_+, X_*)$. For each critical point, we consider the local model (E,F). As noted in the proof of the lemma, F is diffeomorphic to the disk bundle of the tangent bundle $TS^{n-1}_{\sqrt{\rho}}$, where $S^{n-1}_{\sqrt{\rho}}$ is the sphere of radius $\sqrt{\rho}$. For each (sufficiently small) $\rho > 0$, $H_{n-1}(F_\rho)$ is generated by the sphere S_ρ (once we choose an orientation on it). However, as we approach a critical point, $\rho \to 0$ and F_ρ degenerates into the cone

$$\sum x_i^2 = 0.$$

For this reason, we call S_ρ a *vanishing sphere*. Its class in $H_{n-1}(F_\rho)$, denoted Δ, is called a *vanishing cycle*. As $\rho \to 0$, the family of vanishing cycles Δ trace out an n-cycle Z. More precisely, since $\partial : H_n(E,F) \to H_{n-1}(F)$ is an isomorphism, there exists a unique relative cycle $Z \in H_n(E,F)$ such that $\partial Z = \Delta$. Such a relative cycle is called the *Lefschetz thimble* associated with the vanishing cycle Δ (see figures 2 and 3).

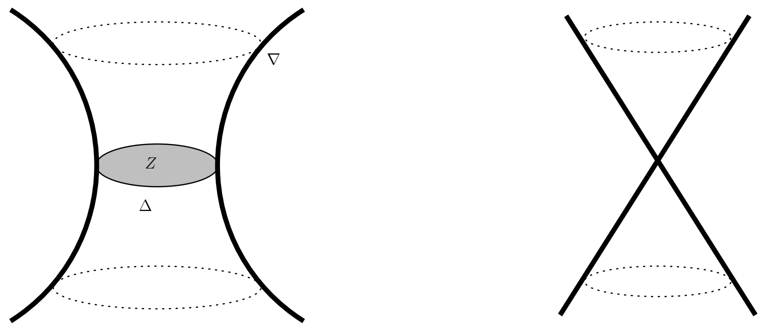

Figure 2: *The thimble Z bounding the vanishing sphere Δ. As the radius goes to 0, the disk bundle collapses to a cone and Δ vanishes. The disk ∇ is a generator of the infinite cyclic group $H_{n-1}(F, \partial F)$.*

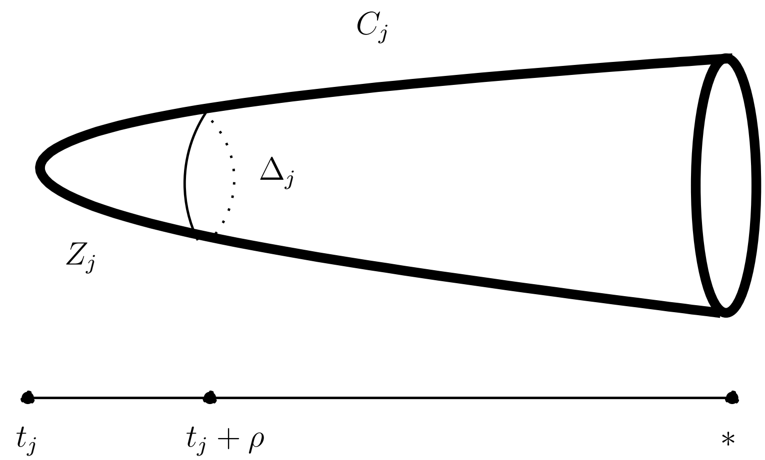

Figure 3: *The geometric representation of the thimble Z_j in $H_n(X_+, X_*)$.*

Let Δ_j be the vanishing cycle corresponding to the critical point x_j. Now consider the path l_j connecting $x_j + \rho$ to the basepoint $*$ of D_+. Thinking of X_+ as a fiber bundle over D_+^*, we may restrict it to the path l_j. Since l_j is contractible, the restriction $X_+|_{l_j}$ is a trivial fiber bundle, and hence there is a homeomorphism $\phi : F_* \times l_j \to X_+|_{l_j}$. Using this homeomorphism, we see that the image of Δ_j in $H_n(X_+, X_*)$ may be represented geometrically by the thimble $C_j = \Delta_j \cup \phi(S^{n-1} \times l_j)$ (see figure 3).

We will denote by $\mathbb{V}(X_*)$ the submodule of $H_{n-1}(X_*)$ generated by the vanishing cycles. $\mathbb{V}(X_*)$ is called the module of *vanishing cycles*. Since each vanishing cycle is the boundary of a thimble C_j, we have

$$\mathbb{V}(X_*) = \text{image}\,(\partial : H_n(X_+, X_*) \to H_{n-1}(X_*))$$

3 Monodromy and the Picard-Lefschetz Theory

We will begin by introducing the general notions of *geometric* and *algebraic monodromy*. Suppose we have a map $f : A \to B$, $B^* \subset B$, such that f fibers $E = f^{-1}(B^*)$ locally trivially over B^*. Let $\gamma : [0,1] \to B^*$ be a path from a to b. Since $[0,1]$ is contractible, the pullback bundle $\gamma^* E$ is trivial, i.e. there exists a map $\Gamma : F_a \times [0,1] \to E \hookrightarrow A$ which lifts γ. In detail, $f \circ \Gamma = \gamma$, $\Gamma(x,0) = x$ $\forall x \in F_a$, and $x \mapsto \Gamma(x,t)$ is a homeomorphism $F_a \cong F_{\gamma(t)}$ for all t. Then for $L \subset A$ such that $F_a \cup F_b \subset L$, Γ is a map between pairs $\Gamma : F_a \times (I, \partial I) \to (A, L)$, where $I = [0,1]$. Furthermore, since $\Gamma_1 := \Gamma(-,1)$ gives a homeomorphism $F_a \cong F_b$, it induces an isomorphism $(\Gamma_1)_* : H_*(F_a) \to H_*(F_b)$. This map depends only on the homotopy class of γ (relative to ∂I), so we denote it by $\gamma_* = (\Gamma_1)_*$. If γ is a closed path, then Γ is called a *geometric monodromy* and γ_* is called the *algebraic monodromy*. The main goal of this section is to describe γ_* when f is a complex Morse function.

We must introduce the notion of *extension along* γ. Let ι be the canonical generator of $H_1(I, \partial I)$. Then the *extension along* γ is

$$\begin{array}{rcl} \tau_\gamma : H_q(F_a) & \to & H_{q+1}(A, L) \\ x & \mapsto & \Gamma_*(x \times \iota). \end{array}$$

Now suppose $x \in H_q(F_a)$. Then we have

$$\begin{array}{rcl} \partial \tau_\gamma(x) & = & \partial \Gamma_*(x \times \iota) \\ & = & \Gamma_*(\partial(x \times \iota)) \\ & = & \Gamma_*((-1)^q (x \times \{1\} - x \times \{0\}) \\ & = & (-1)^q \left((\Gamma_1)_* x - (\Gamma_0)^* x\right) \\ & = & (-1)^q \left(\gamma_* x - x\right). \end{array}$$

That is,

$$\gamma_*(x) = x + (-1)^q \partial \tau_\gamma(x).$$

Thus we will understand γ_* completely if we understand τ_γ.

Now restrict to the case where f is a complex Morse function. As before, f gives a fibration $f : X^* \to \mathbb{P}^*$ where $\mathbb{P}^* = \mathbb{P}^1 \setminus \{t_1, \ldots, t_r\}$ and $X^* = f^{-1}(\mathbb{P}^*)$. The fundamental group $\pi_1(\mathbb{P}^*)$ is generated by the paths γ_j, where γ_j is a path starting at the basepoint $*$, traveling along a path l_j to a neighborhood of t_j, circling t_j once counterclockwise, and then returning to $*$ back along l_j. The extension τ_γ satisfies a certain naturality condition (i.e. it preserves certain commutative diagrams) and a composition law, which allow us to reduce global computations to local ones (full details can be found in §5 of [6]). Thus we need only compute the monodromy in the local model (E, F).

However, in the local model we have an explicit description of everything. F is the disk bundle of the tangent bundle TS^{n-1}, hence $H_{n-1}(F) = \mathbb{Z}$, generated by the vanishing cycle $\Delta = [S^{n-1}]$. The connecting homomorphism $\partial : H_n(E, F) \to H_{n-1}(F)$ is an isomorphism, hence $H_n(E, F) = \mathbb{Z}$, generated by the thimble Z satisfying $\partial Z = \Delta$. The group $H_{n-1}(F, \partial F)$ is infinite cyclic, generated by any fiber of the disk bundle F. Denote by ∇ any such disk (see figures 2 and 4). Poincaré-Lefschetz duality gives an isomorphism $PD : H^{n-1}(F) \to H_{n-1}(F, \partial F)$, so we may define an intersection pairing $H_{n-1}(F, \partial F) \times H_{n-1}(F) \to \mathbb{Z}$ by

$$\langle x, y \rangle := \langle PD^{-1}x, y \rangle,$$

where the bracket on the right hand side is the usual Kronecker pairing. Here we use the orientation on F induced by its complex structure. With this orientation, we have

$$\langle \nabla, \Delta \rangle = (-1)^{n(n-1)/2}.$$

Now f is a fibration $E^* \to D^*$, and $\pi_1(D^*)$ is generated by the path $\gamma(t) = \exp(2\pi i t)$. Thus it suffices to compute $(\tau_\gamma)_*(\nabla)$. This may be done explicitly in the local model (see §6 of [6] or §4.4 of [9]). The result is

$$\tau_\gamma(\nabla) = -Z$$

Combining this with the previous computation relating $\gamma_*(x)$ and $\tau_\gamma(x)$, we have

$$\begin{aligned} \gamma_*(\nabla) - \nabla &= (-1)^{n-1}\partial\tau_\gamma(\nabla) \\ &= -(-1)^{n-1}\partial Z \\ &= (-1)^n\Delta. \end{aligned}$$

For any $c \in H_{n-1}(F, \partial F)$ we have $c = (-1)^{n(n-1)/2}\langle c, \Delta\rangle\nabla$. Thus $\gamma_*(c) - c = (-1)^{n+n(n-1)/2}\langle c, \Delta\rangle[\Delta] = (-1)^{n(n+1)/2}\langle c, \Delta\rangle[\Delta]$. Thus we have the following theorem.

Theorem 3.1. The Picard-Lefschetz formula. *For $q \neq n-1$, $\pi_1(\mathbb{P}^*, *)$ acts trivially on $H_q(X_*)$. For $q = n-1$, the generator γ_j acts by*

$$(\gamma_j)_*(c) - c = (-1)^{n(n+1)/2}\langle c, \Delta_j\rangle\Delta_j.$$

Let $\mathfrak{G}$ be the subgroup of $\mathrm{Aut}(H_{n-1}(X_*, \mathbb{Z}))$ generated by the monodromies. $\mathfrak{G}$ is called the *monodromy group*. We may consider the submodule $\mathbb{I}(X_*) \subset H_{n-1}(X_*, \mathbb{Z})$ which is invariant under the action of $\mathfrak{G}$. $\mathbb{I}(X_*)$ is called the *module of invariant cycles*. From the Picard-Lefschetz formula, it is clear that $\mathbb{I}(X_*)$ is the orthogonal complement to $\mathbb{V}(X_*)$ with respect to the intersection pairing. That this gives a direct sum decomposition of $H_{n-1}(X_*)$ will be the main result of §4.

4 Pencils and the Lefschetz Theorems

So far we have studied the local behavior of a general nonresonant holomorphic Morse function $f : X \to \mathbb{P}^1$. Such a function gives a family of slices $(X_t)_{t\in\mathbb{P}^1}$. Using the results of sections §2 and §3, we would like to obtain information about the global topology of X in terms of the topology of the slices X_t. Before we do this, we must first restrict our attention to the special case where $f : X \to \mathbb{P}^1$ is the *Lefschetz fibration* associated to a *Lefschetz pencil*, to be defined below.

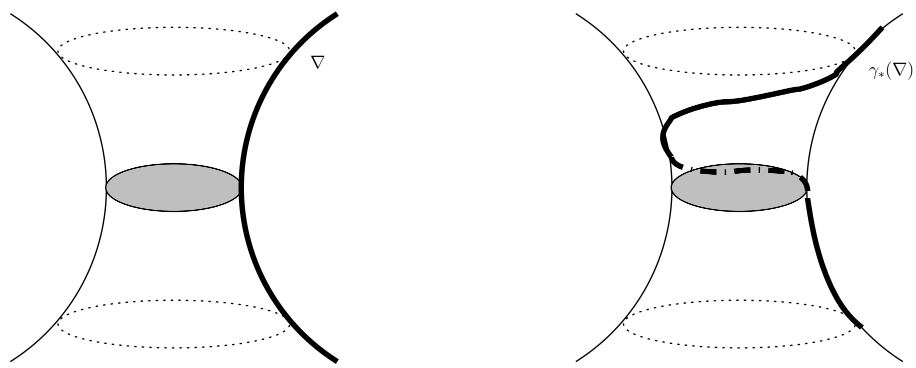

Figure 4: *Monodromy around a critical value.*

Let $P(d,N)$ be the space of all degree d polynomials in $\mathbb{C}[x_0,\ldots,x_N]$ and let $\mathbb{P}(d,N)$ be its projectivization. Let U be any subvariety of $\mathbb{P}(d,N)$. Then each $t \in U$ defines a degree d hypersurface $H_t \subset \mathbb{P}^N$, and this defines a family of slices $X_t := X \cap H_t$ on X. The family $(X_t)_{t\in U}$ is called an *(ample) linear system.* When $\dim U = 1$, it is called a *pencil.* The *base locus* of the linear system is the intersection

$$B = \bigcap_{t\in U} X_t,$$

and its elements are called *base points.* For any $x \in X \setminus B$, we obtain a hyperplane $H_x \subset U$ defined by

$$H_x := \{f \in U : \ f(x) = 0\},$$

i.e. the set of all functions in U which vanish at x. H_x is just a hyperplane in U, and thus can be identified as a point in the dual projective space $\hat{U}$, and in this way we obtain a map

$$f : X \setminus B \to \hat{U}$$

by $x \mapsto H_x$. For convenience, let $X^* = X \setminus B$. Next, define

$$\hat{X} = \{(x,H) \in X \times \hat{U} : \ f(x) = 0 \ \forall p \in H\},$$

i.e. $\hat{X}$ is the closure in $X \times \hat{U}$ of the graph of f. $\hat{X}$ is called the *modification of* X, and is just a blow-up of X along the base locus B. The projection onto the second factor gives a holomorphic map

$$\hat{f} : \hat{X} \to \hat{U}.$$

To see the connection with complex Morse functions, we now retrict to the case where U is one-dimensional, so that $U \cong \hat{U} \cong \mathbb{P}^1$. The modification gives us a holomorphic map $\hat{f} : \hat{X} \to U \cong \mathbb{P}^1$. In this case, the family $(X_t)_{t\in U}$ is called a *Lefschetz pencil* if the following conditions are satisfied:

1. The base locus B is either empty or smooth of complex codimension 1 in X.
2. The modification $\hat{X}$ is a smooth manifold.
3. The map $\hat{f} : \hat{X} \to U$ is a nonresonant Morse function.

The map $\hat{f} : \hat{X} \to U$ is called the *Lefschetz fibration* associated to the Lefschetz pencil on X. We have the following genericity result (§1 and §2 of [6], §4.1 of [9]).

Theorem 4.1. *For an generic projective line $U \subset \mathbb{P}(d, N)$, the pencil $(X_t)_{t \in U}$ is Lefschetz.*

We now give an important construction of Lefschetz pencils. Fix a complex codimension 2 subspace $A \subset \mathbb{P}^N$ called the *axis*. Let U be the set of hyperplanes in $\mathbb{P}^N$ containing A. U can be identified with any line in $\mathbb{P}^N$ not containing A. Let S be any such line. S is called the *screen*. Any point $s \in S$ determines a unique hyperplane $H(s)$ passing through s and containing A. The base locus is

$$B = \bigcap_{s \in S} X_s = \bigcap_{s \in S} X \cap H(s) = X \cap A.$$

For a generic A, this determines a Lefschetz pencil $(X_s)_{s \in S}$ on X.

Now suppose $X \subset \mathbb{P}^N$ and H is a hyperplane in $\mathbb{P}^N$ which intesects X transversally (i.e. H is generic). We will now use the results of §2 and §3 to obtain information about the relationship between the topology of X and the topology of $X \cap H$. Fix any generic codimension two subspace $A \subset \mathbb{P}^N$ so that A defines a Lefschetz pencil on X as above. Then $X \cap H$ is diffeomorphic to a generic fiber of the associated Lefschetz fibration $\hat{f} : \hat{X} \to S$. As in §2, write $\mathbb{P}^1 = D_+ \cup D_-$ such that the interior of D_+ contains all of the critical values of $\hat{f}$. Let $\hat{X}_\pm = \hat{f}^{-1}(D_\pm)$, and let B be the base locus of the pencil on X. Let $p : \hat{X} \to X$ be the projection map from the modification of X, and let $\hat{B} = f^{-1}(B)$. Then $\hat{B}$ is diffeomorphic to $B \times \mathbb{P}^1$, and since $\mathbb{P}^1 \cong S^2$ the Künneth theorem gives us

$$H_q(\hat{B}) \cong H_q(B) \oplus H_{q-2}(B).$$

We have a natural injection $H_{q-2}(B) \to H_q(\hat{B})$ given by $c \mapsto c \times [S^2]$, and composing this with the inclusion $\hat{B} \to \hat{X}$ gives a map

$$\kappa : H_{q-2}(B) \to H_q(\hat{X}).$$

Combining this with the projection map $p_* : H_q(\hat{X}) \to H_q(X)$, one may show the following

Lemma 4.2. *The sequence*

$$0 \to H_{q-2}(B) \xrightarrow{\kappa} H_q(\hat{X}) \xrightarrow{p_*} H_q(X) \to 0.$$

is exact and splits for every q.

We now state the first theorem relating X and $X \cap H$.

Theorem 4.3. Lefschetz Hyperplane Theorem. *If H is a hyperplane intersecting X transversely, then*

$$H_q(X, X \cap H) = 0, \ q \leq n - 1.$$

Sketch of proof. Consider the following isomorphisms

$$\begin{aligned} H_q(\hat{X}, \hat{X}_+ \cap \hat{B}) &\cong H_{q-2}(X_*, B), \\ H_q(\hat{X}, \hat{X}_* \cup \hat{B}) &\cong H_q(X, X_*), \\ H_q(\hat{X}_+ \cup \hat{B}, \hat{X}_* \cup \hat{B}) &\cong H_q(\hat{X}_+, \hat{X}_*). \end{aligned}$$

The first may be deduced by excision together with the Ehresmann fibration theorem. The second is induced by the projection map $\hat{X} \to X$. The third is by excision. Using these isomorphisms, the long exact sequence for the triple $(\hat{X}, \hat{X}_+ \cup \hat{B}, \hat{X}_* \cup \hat{B})$ becomes

$$\cdots \to H_{q-1}(X_*, B) \to H_q(\hat{X}_+, \hat{X}_*) \to H_q(X, X_*) \to H_{q-2}(X_*, B) \to \cdots.$$

By the lemma 2.1, $H_q(\hat{X}_+, \hat{X}_*) = 0$ for $q \leq n-1$ and we immediately obtain isomorphisms

$$H_q(X, X_*) \cong H_{q-2}(X_*, B), q \neq n, n+1,$$

and the exact sequence

$$\begin{aligned} 0 \to H_{n+1}(X, X_*) &\to H_{n-1}(X_*, B) \to H_n(\hat{X}, \hat{X}_*) \to \\ &\to H_n(X, X_*) \to H_{n-2}(X_*, B) \to 0. \end{aligned}$$

The proof may now be completed by induction on n. For $n = 1$, the result is obviously true. Now suppose that it is true for $n-1$. Then $\dim X_* = n-1$ and the base locus B intersects it transversely, so by the induction hypothesis we have $H_q(X_*, B) = 0$ $\forall q \leq n-2$. Using the isomorphism $H_q(X, X_*) \cong H_{q-2}(X_*, B)$ for $q \neq n, n+1$, we obtain the desired result. □

It is actually possible to prove the stronger statement that the pair $(X, X \cap H)$ is $(n-1)$-connected (see for example [2] and [8]). This may be proved using ordinary Morse-Bott theory, and has the advantage that it may be exteneded to the equivariant case. However, for our purposes the above proof is sufficient, and makes explicit the role of vanishing cycles.

So far we have not made use of the monodromy of the Lefschetz fibration. Recall the module of module of *vanishing cycles*:

$$\mathbb{V}(X_*) = \ker\left(H_{n-1}(\hat{X}_*), H_{n-1}(\hat{X})\right),$$

which is spanned by the vanishing cycles Δ_j. Also recall the module of *invariant cycles*, defined as the submodule of $H_{n-1}(X_*)$ which is invariant under the action of the monodromy group. By the Picard-Lefschetz formula, a cycle $c \in H_{n-1}(X_*)$ is invariant if and only if $\langle c, \Delta_j \rangle = 0$ $\forall j$. Thus $\mathbb{I}(X_*)$ is the orthogonal complement to $\mathbb{V}(X_*)$ with respect to the intersection pairing. We have the following deep theorem.

Theorem 4.4. Hard Lefschetz Theorem. *For field coefficients, we have*

$$H_{n-1}(X_*) = \mathbb{V}(X_*) \oplus \mathbb{I}(X_*).$$

Before indicating how this theorem is proved, we must give an alternative but equivalent statement. The Lefschetz hyperplane theorem implies that the map $i_* : H_{n-1}(X_*) \to H_{n-1}(X)$ is an epimorphism sending $\mathbb{V}(X_*)$ to 0. Hence the Hard Lefschetz theorem is equivalent to the statement that the restriction $i_* : \mathbb{I}(X_*) \to H_{n-1}(X)$ is an isomorphism. Let $i^!$ be the Gysin morphism $H_{n+1}(X) \to H_{n-1}(X \cap H)$, i.e. $i^! = PD_{X_*} \circ i^* \circ PD_X^{-1}$, where PD is the Poincaré dual map. The Lefschetz hyperplane theorem implies that i^* is injective, and hence $i^!$ is injective. Furthermore, its image is $\mathbb{I}(X_*)$. Then it follows that the Hard Lefshetz theorem is equivalent to the statement that the composition $i_* \circ i^! : H_{q+1}(X) \to H_{q-1}(X)$ is an isomorphism. Let $\omega \in H^2(X)$ be the Poincaré dual of the hyperplane section $X_* = X \cap H$, i.e. $[X_*] = \omega \cap [X]$. Then we have $i_* \circ i^! = \omega \cap$.

Thus the Hard Lefschetz theorem is equivalent to the statement that $\omega\cap$ is an isomorphism. Now suppose $x \in H_{n+q}(X)$, $0 \leq q \leq n$. Then x is called *primitive* if $\omega^{q+1} \cap x = 0$. Then repeated application of the isomorphism $\omega\cap$, i.e. considering a sequence of hyperplane sections $X \supset X' \supset X'' \supset \ldots \supset \emptyset$, yields the following, equivalent to the Hard Lefschetz theorem.

Theorem 4.5. Lefschetz Decomposition Theorem. *Every $x \in H_{n+q}(X)$ can be written uniquely as*

$$x = x_0 + \omega \cap x_1 + \omega^2 \cap x_2 + \ldots,$$

and each $x \in H_{n-q}(X)$ can be written uniquely as

$$x = \omega^q \cap x_0 + \omega^{q+1} \cap x_1 + \ldots,$$

where the $x_i \in H_{n+q+2i}(X)$ are primitive, and $q \geq 0$.

The Hard Lefschetz Theorem and Lefschetz Decomposition theorems may be proved using Hodge theory. Using Poincaré duality, the Hard Lefschetz theorem is equivalent to the statement that $\cup\omega : H^{n-1}(X) \to H^{n+1}(X)$ is an isomorphism. If we work over field coefficients, then the singular cohomology is isomorphic to the de Rham cohomology, so we may represent ω by a 2-form. Since $H^2_{\mathrm{dR}}(\mathbb{P}^N)$ is one-dimensional and spanned by $[\omega_{FS}]$, where ω_{FS} is the standard (Fubini-Study) Kähler form on $\mathbb{P}^N$, $[\omega_{FS}]$ is the Poincaré dual to the hyperplane $H \subset \mathbb{P}^N$. Thus $[\omega_{FS}|_X]$ represents $\omega \in H^2_{\mathrm{dR}}(X)$, i.e. ω can be represented by the Kähler form on X.

Thus for complex coefficients, the Hard Lefschetz theorem is equivalent to the statement that $L : H^{n-1}_{\mathrm{dR}}(X) \to H^{n+1}_{\mathrm{dR}}(X)$, given by $L(\eta) = \eta \wedge \omega_{FS}$, is an isomorphism. Let $L^\dagger$ be the adjoint of L and $h = [L, L^\dagger]$. Using standard results of Hodge theory (see for example [4]), one may show that these maps define a representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ on $H^*_{\mathrm{dR}}(X)$. The Lefschetz Decomposition theorem is then a consequence of standard results in the representation theory of $\mathfrak{sl}(2,\mathbb{C})$.

5 Applications

The study of complex Morse functions has already led us to a rather deep theorem concerning the topology of algebraic varieties. We now give some immediate corollaries to the results of the previous sections.

To begin with, we saw that for a complex Morse function $f : X \to \mathbb{P}^1$, the generic fibers are all diffeomorphic (this is a consequence of the Ehresmann fibration theorem). Now consider this result applied to $\mathbb{P}^N$ itself. We have the following.

Proposition 5.1. *All smooth degree d hypersurfaces of $\mathbb{P}^N$ are diffeomorphic.*

In the proof of the Lefschetz hyperplane theorem, we found that $H_{q+1}(X, X_*) = H_q(X_*)$ for $q \neq n, n-1$. Then if $n \geq 2$, we have that $H_0(X, X_*) = 0$ and $H_1(X, X_*) = 0$. Then considering the long exact sequence for the pair (X, X_*), we see $H_0(X, X_*) = H_0(X)$. Thus we have the following version of Bertini's theorem:

Proposition 5.2. *For $n \geq 2$, a generic hyperplane section of X is smooth and connected.*

Next we deduce an interesting consequence of the Hard Lefschetz theorem. Consider the restriction of the intersection form on $H_{n-q}(X)$ to $\mathbb{I}_q(X)$. Here, we have a sequence of hyperplane sections $X^{(i+1)} \subset X^{(i)}$ and $\mathbb{I}_q(X) := \mathbb{I}(X^{(q)})$. We have already seen that $\mathbb{I}_q(X)$ is the orthogonal complement of $\mathbb{V}_q(X)$ with respect to the intersection form. Thus it is easy to see that the Hard Lefschetz theorem is equivalent to the statement that the restriction of the intersection form to $\mathbb{I}_q(X)$ is nondegenerate. When $n - q$ is odd, the intersection form is skew. But a skew form on a vector space can be nondegenerate only if the space is even dimensional. Thus we obtain the following result.

Proposition 5.3. *If X is a smooth projective variety, then the odd-dimensional Betti numbers of X are even.*

Now consider the manifold $X = S^3 \times S^1$. X can be identified with a quotient of $\mathbb{C}^2 \setminus \{0\}$ by a holomorphic group action (see [5]), and thus inherits a complex structure, so that X is a compact complex manifold. This manifold is called the *Hopf surface*. On the other hand, by the Künneth theorem we have $b_1(X) = 1$. Since $b_1(X)$ is odd, we see by the previouis proposition that X is cannot be biholomorphically equivalent to a projective variety, and thus by Chow's theorem [4] is not algebraic. Thus we have the following highly nontrivial result:

Proposition 5.4. *There exist compact complex manifolds which are not algebraic.*

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