# Optimality of the Heisenberg limit in quantum metrology

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dress an important subtlety. When  $\mathscr{H}$  corresponds to a proper Hamiltonian, the fact that the origin of the energy scale has no phy matting means that the actual value of  $\langle \mathscr{H} \rangle$  can be changed arbitrarily. Hence, we must fix the scale such that the ground state (which may be degenerate) has zero energy. What is the Heisenberg limit, and can it be beaten? In most cases, this is an intuitive choice. For example, most people would agree that it is natural to associate zero energy How do we define the proper resources to determine the scaling? to the vacuum state, and add the corresponding amount of energy for each added photon. Technically, this corresponds to • Given the proper resource count, the Heisenberg limit is optimal. the normal ordering of the Hamiltonian of the radiation field in order to remove the infinite vacuum energy. Slightly less • How is the Heisenberg limit related to the Uncertainty Principle? Intuitive is that the average energy of N spins in a GHZ state  $(|\uparrow\rangle^{\otimes N} + |\downarrow\rangle^{\otimes N})/\sqrt{2}$  is no longer taken to be zero, but rather N/2 times the energy splitting between  $|\uparrow\rangle$  and  $|\downarrow\rangle$ . For more general interactions  $U(\varphi)$  where we include feed-forward and arbitrary unitary gates between queries in Fig. 1, we can use an argument by Giovannetti *et al*. [10] to show that  $\langle \mathscr{H} \rangle = \langle i(\partial U(\varphi) / \partial \varphi) U^{\dagger}(\varphi) \rangle$  is unaffected by the intermediate unfitary gained wanted the seal ingit is the terms of the seal of the search of the seal of the seal of the search of the search

#### What is the Heisenberg limit?

• Mean squared error and Cramer-Rao bound:  $\delta \varphi \ge \frac{1}{\sqrt{TF(\varphi)}}$ 

• Fisher information: 
$$F(\varphi) = \int dx \frac{1}{p(x|\varphi)} \left(\frac{\partial p(x|\varphi)}{\partial \varphi}\right)^2$$

- where  $p(x|\varphi) = \text{Tr}[\hat{E}_x \rho(\varphi)]$
- Scaling of the CR bound: Standard quantum limit and Heisenberg limit:

$$\delta \varphi_{\text{SQL}} \ge \frac{1}{\sqrt{T}}$$
  $\delta \varphi_{\text{HL}} \ge \frac{1}{N}$ 

• How to compare the two (T versus F)?

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## Beating the Heisenberg limit?

- Giovannetti, Lloyd, and Maccone: Query complexity, and HL is optimal. (*Phys. Rev. Lett.* **96**, 010401, 2006).
- Boixo, Flammia, Caves, and Geremia: Nonlinear Hamiltonians can beat HL. (*Phys. Rev. Lett.* **98**, 090401, 2007).
- Roy and Braunstein: Exploit full Hilbert space to get *exponential* scaling. (*Phys. Rev. Lett.* **100**, 220401, 2008).
- Beltran and Luis: Nonlinear *classical* optics can beat the Heisenberg limit! (*Phys. Rev. A* **72**, 045801, 2005).

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## Defining the appropriate resources

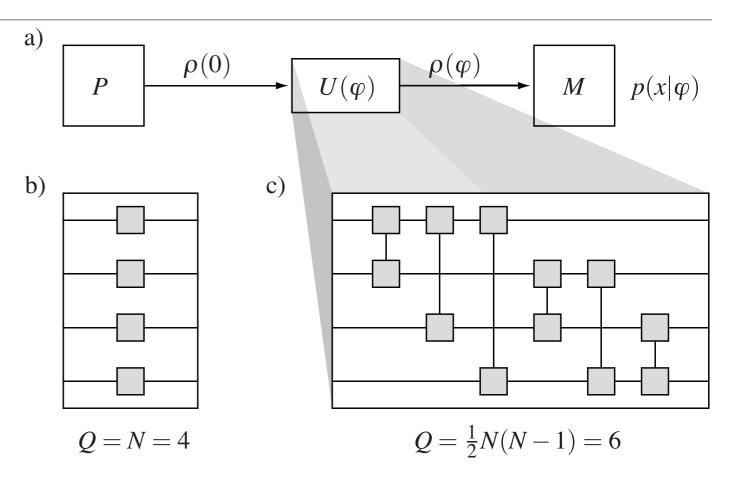
#### Resources versus query complexity

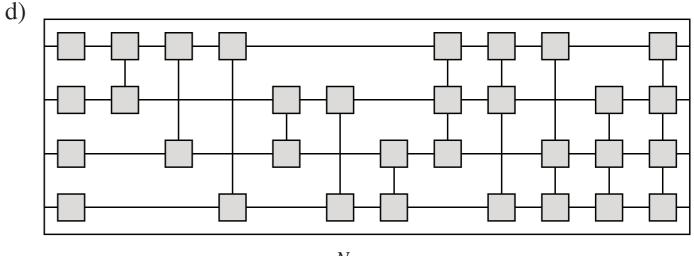
- Physicists: how does the CR bound scale with the energy resources?
- Computer scientists: how does the CR bound scale with the number of queries?
- How can we reconcile the two viewpoints?

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# Query complexity

- a) general parameter estimation setup.
- b) each grey box is a query;
- c) each vertical pair is a query;
- d) each subset is a query.
- The number of systems does not generally equal the number of queries.





 $Q = 2^N - 1 = 15$ 

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ution cappers each que ny woonsists of a foint antes action on two models. gate Example for the words, the queries metotice query the male aity gathe net e each system performs a single whete  $i \neq j$ . Since all  $H_i$  commutes this all uals the first  $f_i$  commutes the solution of the sense ator of the goint queries can be write HBFCG the number of queries From our arguments about the query of f systems and two systems can the number of queries Q is equal ne number of queries then scales tametworks, trins clear that the resource stems; d) for  $\mathcal{H}_{RB}$  all possible for  $\mathcal{H}_{RB}$  all possible for  $\mathcal{H}_{RB}$  and  $\mathcal{H}$ ral way to map this operator to a number query. The number of queries r of systems. The number of terms in  $\mathcal{H}_{BFCG}$ , and therefore the duery com-The number of terms in  $\mathcal{H}_{BFCG}$ , and therefore the duery com- $\mathcal{H}_{COM}$  con plexity with respect to the number of systems, is given by of the Hamiltonian, the fact that the origin of the re error in the parameter  $Q(N^2)$  physical meaning means that the actual v e procedure is repeated of the Roy and Braunstein is given in Figuel dust fix er in folkisation, to see that the humber of the main the corresponding ener anbergennerator blande given by 12 most, east shenry her infutive choice. when indressen for mation 1. Space whave agreet maticipation at a surce charning the sign and and Q) signs the usatestate, this defines correspo ab two symptotintally resificantly of the aretained phinen. break hically Ely. Wande Glaouvet that it and bom of the propagation of the set of the Hanil Bonian of Thursday, 19 August 2010 commencementer sold within the Heisenbergvantie infinite vacuum e

#### Expectation versus variance

• For any  $\mathscr{H}$ , the expectation value scales as:

$$\langle \mathscr{H} \rangle = \sum_{j}^{Q} \langle A_{j} \rangle \leq O(Q)$$

• This has the same asymptotic scaling behaviour as the variance:

$$(\Delta \mathscr{H})^2 = \left\langle \left( \sum_{j}^{Q} A_j \right)^2 \right\rangle - \left\langle \sum_{j}^{Q} A_j \right\rangle^2 \\ = \sum_{j}^{Q^2} \langle L_j \rangle - \sum_{j,k}^{Q} \langle A_j \rangle \langle A_k \rangle \le cQ^2$$

• The expectation value is always well-defined, but the variance is not (for example in Lorentzian or Breit-Wigner spectra).

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to the vacuum state, and add the corresponding amount of energy for each added photon. Technically, this corresponds to the normal ordering of the Hamiltonian of the radiation field in order to remove the infinite vacuum energy. Slightly less intuitive is that the average energy of N spins in a GHZ state  $(|\uparrow\rangle^{\otimes M} = (|\downarrow\rangle^{\otimes M})/\sqrt{2}$  is no longer taken to be zero, but rather is described by N/2 times the energy splitting between  $|\uparrow\rangle$  and  $|\downarrow\rangle$ . For more general interactions  $U_{f}(\varphi)$  where we include feedforward and arbitrary unitary gates between queries in Fig. 1, we can use an argument by Giovannetti *et al*. [10] to show that  $\langle \mathscr{H} \rangle = \langle i(\partial U(\varphi) / \partial \varphi) U^{\dagger}(\varphi) \rangle$  is unaffected by the intermedi-with  $U(\varphi) = \exp(-i\varphi \mathscr{H})$ , the proper resource count is given by ate unitary gates, and the scaling is therefore still determined by Q. This is always well-defined, even when the query complexity itself may not Finally, one may argue that the resource count should be defined in terms of the variance or semi-norm of  $\mathcal{H}$ . Indeed,

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## Optimality of the Heisenberg limit

#### Fisher information and statistical distance

• We can define a distance between two probability distributions:

$$ds^2 = \int dx \, \frac{1}{p(x)} [dp(x)]^2$$

• This is directly related to the Fisher information:

$$\left(\frac{ds}{d\varphi}\right)^2 = \int dx \frac{1}{p(x|\varphi)} \left(\frac{\partial p(x|\varphi)}{\partial \varphi}\right)^2 = F(\varphi)$$

• In Hilbert space, the natural statistical distance is the Wootters distance:

$$s(\boldsymbol{\psi}, \boldsymbol{\phi}) = \arccos(|\langle \boldsymbol{\psi} | \boldsymbol{\phi} \rangle|)$$

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#### Bound on the derivative of s

- Include all relevant systems in parameter estimation procedure: pure states.
- The evolution can be written as

$$|\psi(\varphi)\rangle = \exp(-i\varphi\mathscr{H})|\psi(0)\rangle$$

and the statistical distance is evaluated as

$$s(\varphi) = \arccos |\langle \psi(0) | \psi(\varphi) \rangle|$$

• It can then be shown that (Jones and Kok, arXiv:1003.4870)

$$\frac{ds}{d\varphi} \leq \langle \mathscr{H} \rangle$$

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#### Bound on the derivative of s ( $\varphi \rightarrow \theta$ and $\mathscr{H} \rightarrow K$ )

• To prove this last bound, we attempt to take the derivative directly.

$$\frac{ds}{d\theta} = \frac{d}{d\theta} \arccos\left(|\langle \psi_0 | \psi_\theta \rangle|\right) = -\frac{1}{\sqrt{1 - |\langle \psi_0 | \psi_\theta \rangle|^2}} \frac{d}{d\theta} |\langle \psi_0 | \psi_\theta \rangle|$$

• This means 
$$\frac{ds}{d\theta} \leq -\frac{d}{d\theta} |\langle \psi_0 | \psi_\theta \rangle|$$
, which gives  $-\frac{d}{d\theta} |\langle \psi_0 | \psi_\theta \rangle| \leq \left| \frac{d}{d\theta} \langle \psi_0 | \psi_\theta \rangle \right|$ 

• The final step is then

$$\frac{ds}{d\theta} \le \left|\frac{d}{d\theta} \langle \psi_0 | \psi_\theta \rangle\right| \le \frac{\left|\langle \psi_0 | K | \psi_\theta \rangle\right|}{\hbar} \le \frac{\left|\langle \psi_0 | K | \psi_0 \rangle\right|}{\hbar} \equiv \frac{\left|\langle K \rangle\right|}{\hbar}$$

• which proves the inequality (see Jones & Kok, PRA 82, 022107, 2010).

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#### The Heisenberg limit is optimal

• We can use this bound in the Fisher information:

$$(\delta \varphi)^2 \ge \frac{1}{T} \left( \frac{ds}{d\varphi} \right)^{-2} = \frac{1}{T \langle \mathscr{H} \rangle^2}$$

• This is the Heisenberg limit (for T = 1):

$$\delta \varphi \geq rac{1}{\langle \mathscr{H} 
angle}$$

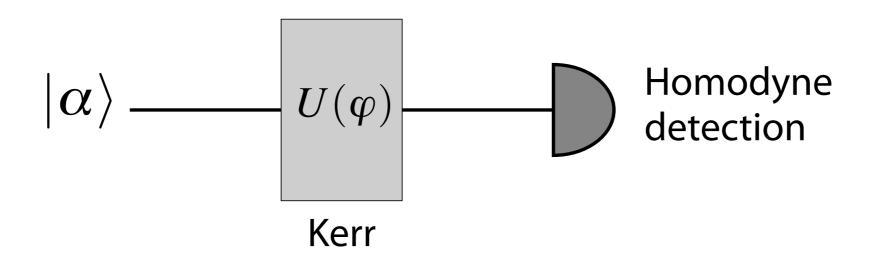
• This reconciles the number of queries with the physical resource count, and allows us to compare *T* and *F*.

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## An example

## Can nonlinear optics beat the Heisenberg limit?

• A very interesting proposal by Beltran and Luis suggests that the Heisenberg limit can be beaten when we used nonlinear optics:



• A straightforward calculation then shows that

$$\delta \varphi \simeq rac{1}{4\langle \hat{n} 
angle^{3/2}} = rac{1}{4|lpha|^3}$$

which beats the  $1/\langle \hat{n} \rangle$  limit.

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#### Can nonlinear optics beat the Heisenberg limit?

- We can resolve this paradox by noting that  $\langle \mathscr{H} \rangle = \langle \hat{n}^2 \rangle$ . The Heisenberg limit is not given by  $1/\langle \hat{n} \rangle$ , but rather by  $1/\langle \hat{n}^2 \rangle$ .
- Consequently, the Heisenberg limit is not broken, and it is not even attained!

• Consider 
$$|\psi(\phi)\rangle = \exp(-i\phi\hat{n}^2)|\psi(0)\rangle = \frac{|0\rangle + e^{-i\phi N^2}|N\rangle}{\sqrt{2}}$$

• Measuring the operator  $X=|0
angle\langle N|+|N
angle\langle 0|$  then gives the limit

$$\delta \varphi = \frac{\Delta X}{|d\langle X\rangle/d\varphi|} = \frac{1}{N^2}$$

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## Relation to the Uncertainty Principle

## Heisenberg limit and the uncertainty principle

- Holland and Burnett introduced the term *Heisenberg limit*, and referred to the uncertainty principle in the book by Heitler.
- However, we argued that the Heisenberg limit is given in terms of the expectation value, and not the variance.
- Formally integrate the bound on the derivative of the statistical distance:

$$\int_0^{\varphi} d\varphi' \ge \frac{1}{\langle \mathscr{H} \rangle} \int_0^{\pi/2} ds \quad \Rightarrow \quad \varphi \ge \frac{\pi}{2} \frac{1}{\langle \mathscr{H} \rangle}$$

• This is the Margolus-Levitin bound! (see Jones & Kok, *Physical Review A* 82, 022107, 2010).

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## Heisenberg limit and the uncertainty principle

• We can also bound the Fisher information by the variance:

$$F(\boldsymbol{\varphi}) \leq 4(\Delta \mathscr{H})^2$$

• This leads to the famous Mandelstam-Tamm bound:

$$\delta \varphi \geq \frac{1}{2\Delta \mathscr{H}}$$

- This is a form of Heisenberg's Uncertainty Principle.
- So the Heisenberg limit is really an example of the Margolus-Levitin bound, rather than Heisenberg's uncertainty principle.

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## Conclusions

- We reconciled the physical resources in a parameter estimation procedure with the query complexity of the corresponding quantum network.
- Using this definition of the resources, we proved that the Heisenberg limit is optimal.
- Quantum mechanical procedures beat nonlinear optical procedures.
- The Heisenberg limit is not a form of the Uncertainty Principle, but rather a manifestation of the Margolus-Levitin bound.

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