Optimality of the
Heisenberg limit in quantum metrology

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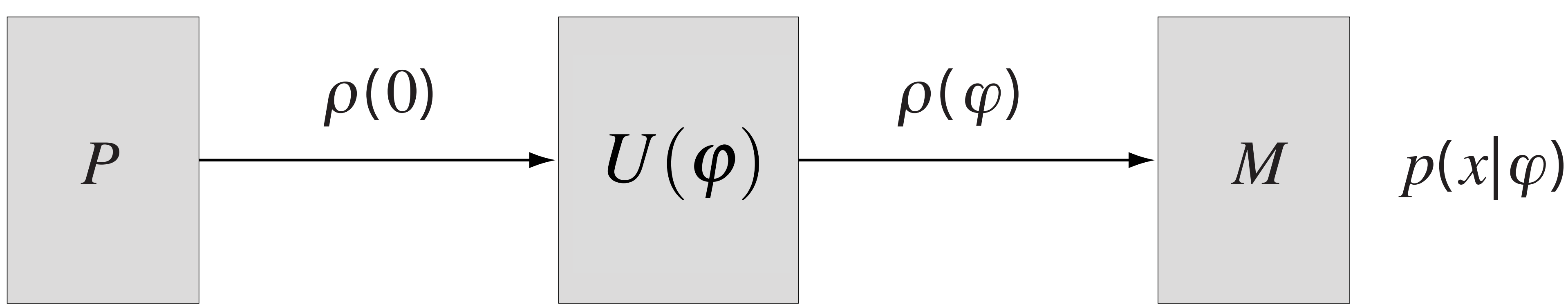
Outline

- What is the Heisenberg limit, and can it be beaten?
- How do we define the proper resources to determine the scaling?
- Given the proper resource count, the Heisenberg limit is optimal.
- How is the Heisenberg limit related to the Uncertainty Principle?



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What is the Heisenberg limit?

- Mean squared error and Cramer-Rao bound: $\delta\varphi \geq \frac{1}{\sqrt{TF(\varphi)}}$

- Fisher information: $F(\varphi) = \int dx \, \frac{1}{p(x|\varphi)} \left(\frac{\partial p(x|\varphi)}{\partial \varphi} \right)^2$

- where $p(x|\varphi) = \mathrm{Tr}[\hat{E}_x \rho(\varphi)]$

- Scaling of the CR bound: Standard quantum limit and Heisenberg limit:

$$\delta\varphi_{\mathrm{SQL}} \geq \frac{1}{\sqrt{T}} \qquad \delta\varphi_{\mathrm{HL}} \geq \frac{1}{N}$$

- How to compare the two (T versus F)?

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Beating the Heisenberg limit?

- Giovannetti, Lloyd, and Maccone: Query complexity, and HL is optimal. (*Phys. Rev. Lett.* **96**, 010401, 2006).

- Boixo, Flammia, Caves, and Geremia: Nonlinear Hamiltonians can beat HL. (*Phys. Rev. Lett.* **98**, 090401, 2007).

- Roy and Braunstein: Exploit full Hilbert space to get *exponential* scaling. (*Phys. Rev. Lett.* **100**, 220401, 2008).

- Beltran and Luis: Nonlinear *classical* optics can beat the Heisenberg limit! (*Phys. Rev. A* **72**, 045801, 2005).

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Defining the appropriate resources

Resources versus query complexity

- Physicists: how does the CR bound scale with the energy resources?
- Computer scientists: how does the CR bound scale with the number of queries?
- How can we reconcile the two viewpoints?

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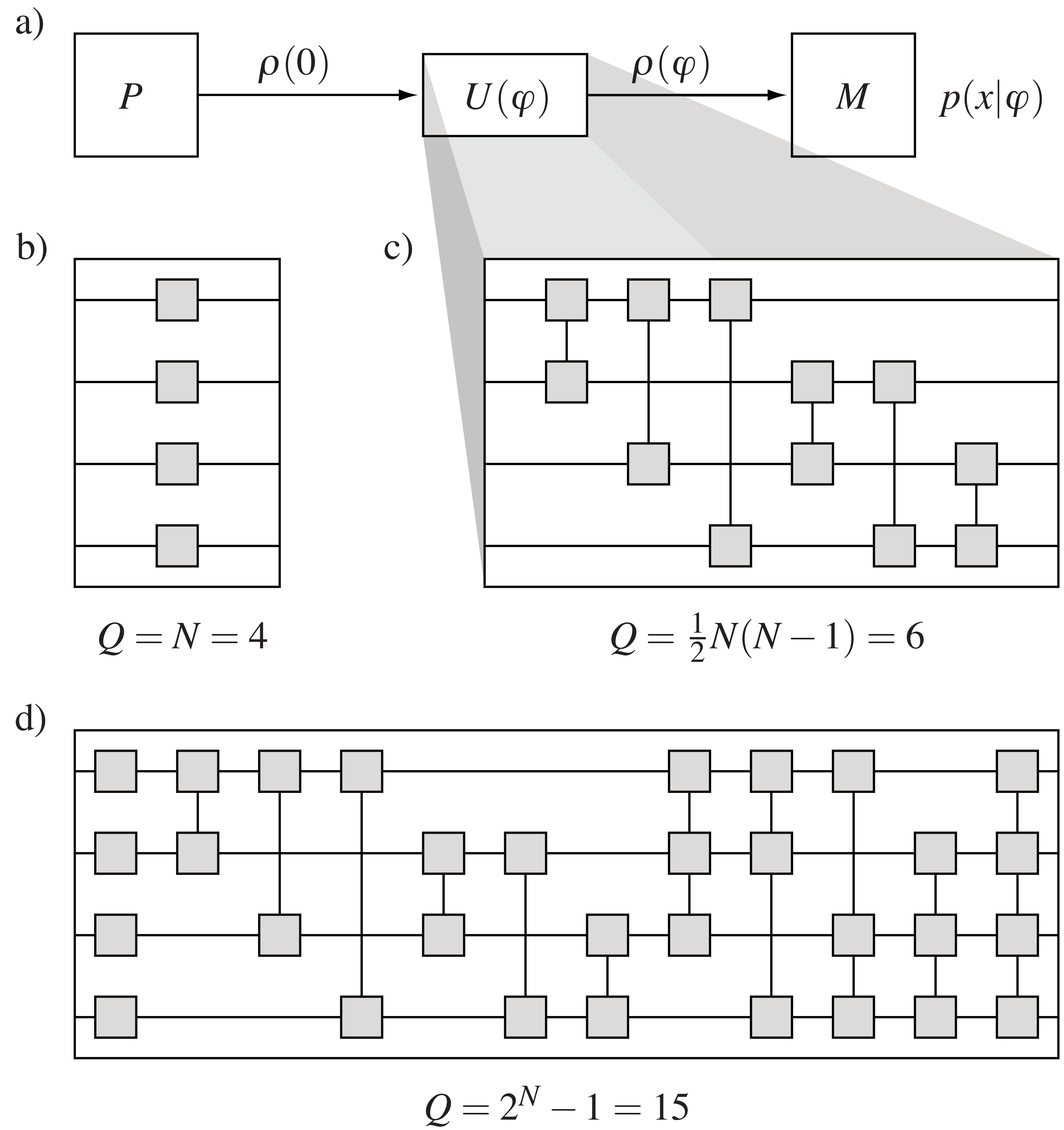
Query complexity

- a) general parameter estimation setup.
- b) each grey box is a query;
- c) each vertical pair is a query;
- d) each subset is a query.
- The number of systems does not generally equal the number of queries.



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Queries and resources

- The number of queries Q is equal to the number of terms in the Hamiltonian:

$$\mathscr{H}_{BFCG} = \sum_{i>j} H_i \otimes H_j$$

- The proper resource count that compares to Q is expectation value: $\langle \mathscr{H} \rangle$.

- We must choose $E_0 = 0$

- Alternatively, we can evaluate $\langle \mathscr{H} - h_{\min} I \rangle$, where $h_{\min}$ is the smallest eigenvalue of $\mathscr{H}$.

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Expectation versus variance

- For any $\mathscr{H}$, the expectation value scales as:

$$\langle \mathscr{H} \rangle = \textstyle\sum_j^Q \langle A_j \rangle \leq O(Q)$$

- This has the same asymptotic scaling behaviour as the variance:

$$\begin{aligned}(\Delta\mathscr{H})^2 &= \left\langle \left(\sum_j^Q A_j \right)^2 \right\rangle - \left\langle \sum_j^Q A_j \right\rangle^2 \\ &= \sum_j^{Q^2} \langle L_j \rangle - \sum_{j,k}^{Q} \langle A_j \rangle \langle A_k \rangle \leq cQ^2\end{aligned}$$

- The expectation value is always well-defined, but the variance is not (for example in Lorentzian or Breit-Wigner spectra).

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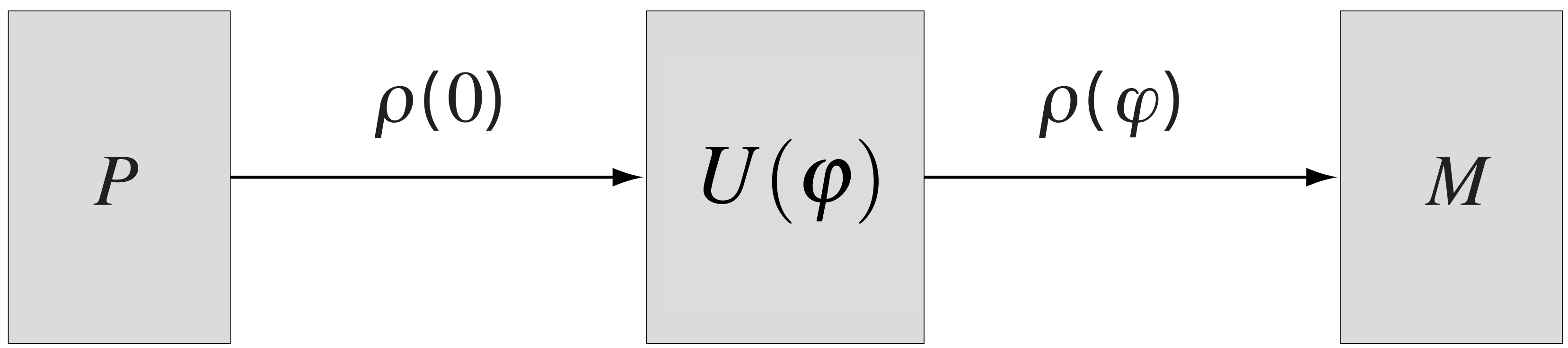
The resource count in parameter estimation

- In conclusion, when the parameter estimation procedure is described by



with $U(\varphi) = \exp(-i\varphi\mathscr{H})$, the proper resource count is given by $\langle \mathscr{H} \rangle$.

This is always well-defined, even when the query complexity itself may not be.

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Optimality of the Heisenberg limit

Fisher information and statistical distance

- We can define a distance between two probability distributions:

$$ds^2 = \int dx \, \frac{1}{p(x)} [dp(x)]^2$$

- This is directly related to the Fisher information:

$$\left(\frac{ds}{d\varphi}\right)^2 = \int dx \, \frac{1}{p(x|\varphi)} \left(\frac{\partial p(x|\varphi)}{\partial \varphi}\right)^2 = F(\varphi)$$

- In Hilbert space, the natural statistical distance is the Wootters distance:

$$s(\psi, \phi) = \arccos(|\langle \psi | \phi \rangle|)$$

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Bound on the derivative of *s*

- Include all relevant systems in parameter estimation procedure: pure states.
- The evolution can be written as

$$|\psi(\varphi)\rangle = \exp(-i\varphi\mathscr{H})|\psi(0)\rangle$$

- and the statistical distance is evaluated as

$$s(\varphi) = \arccos|\langle\psi(0)|\psi(\varphi)\rangle|$$

- It can then be shown that (Jones and Kok, arXiv:1003.4870)

$$\frac{ds}{d\varphi} \leq \langle\mathscr{H}\rangle$$

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Bound on the derivative of s ($\varphi \to \theta$ and $\mathscr{H} \to K$)

- To prove this last bound, we attempt to take the derivative directly.

$$\frac{ds}{d\theta} = \frac{d}{d\theta}\arccos\left(|\langle\psi_0|\psi_\theta\rangle|\right) = -\frac{1}{\sqrt{1-|\langle\psi_0|\psi_\theta\rangle|^2}}\frac{d}{d\theta}|\langle\psi_0|\psi_\theta\rangle|$$

- This means $\frac{ds}{d\theta} \leq -\frac{d}{d\theta}|\langle\psi_0|\psi_\theta\rangle|$, which gives $-\frac{d}{d\theta}|\langle\psi_0|\psi_\theta\rangle| \leq \left|\frac{d}{d\theta}\langle\psi_0|\psi_\theta\rangle\right|$

- The final step is then

$$\frac{ds}{d\theta} \leq \left|\frac{d}{d\theta}\langle\psi_0|\psi_\theta\rangle\right| \leq \frac{|\langle\psi_0|K|\psi_\theta\rangle|}{\hbar} \leq \frac{|\langle\psi_0|K|\psi_0\rangle|}{\hbar} \equiv \frac{|\langle K\rangle|}{\hbar}$$

- which proves the inequality (see Jones & Kok, PRA **82**, 022107, 2010).

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The Heisenberg limit is optimal

- We can use this bound in the Fisher information:

$$(\delta\varphi)^2 \geq \frac{1}{T}\left(\frac{ds}{d\varphi}\right)^{-2} = \frac{1}{T\langle\mathscr{H}\rangle^2}$$

- This is the Heisenberg limit (for $T = 1$):

$$\delta\varphi \geq \frac{1}{\langle\mathscr{H}\rangle}$$

- This reconciles the number of queries with the physical resource count, and allows us to compare T and F.

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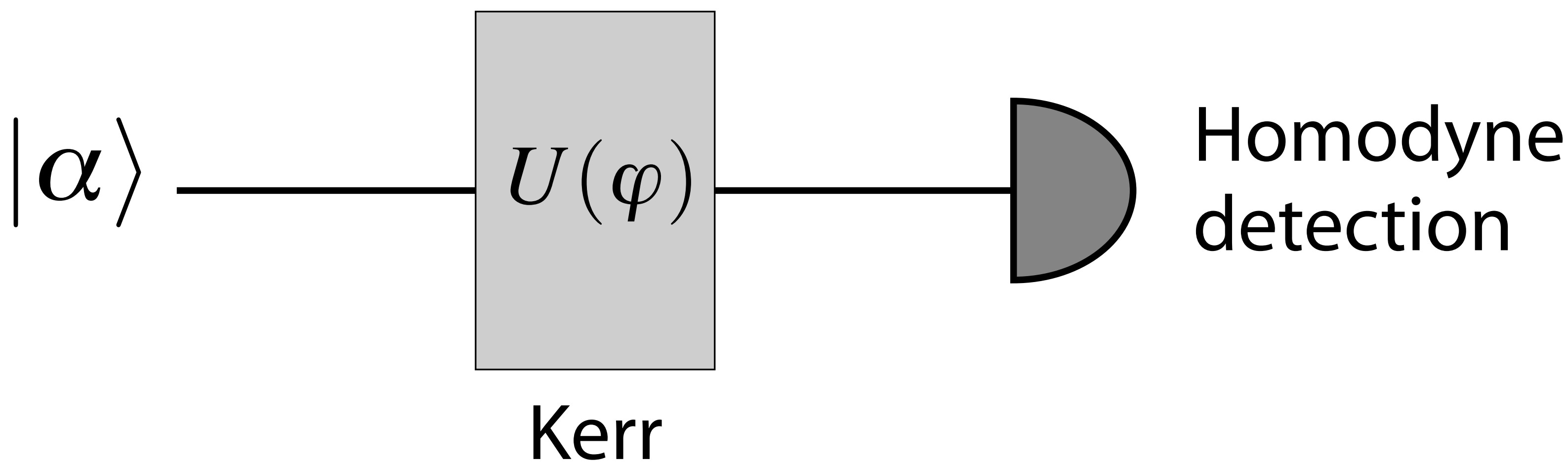
An example

Can nonlinear optics beat the Heisenberg limit?

- A very interesting proposal by Beltran and Luis suggests that the Heisenberg limit can be beaten when we used nonlinear optics:



- A straightforward calculation then shows that

$$\delta\varphi \simeq \frac{1}{4\langle\hat{n}\rangle^{3/2}} = \frac{1}{4|\alpha|^3}$$

which beats the $1/\langle\hat{n}\rangle$ limit.

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Can nonlinear optics beat the Heisenberg limit?

- We can resolve this paradox by noting that $\langle \mathscr{H} \rangle = \langle \hat{n}^2 \rangle$. The Heisenberg limit is not given by $1/\langle \hat{n} \rangle$, but rather by $1/\langle \hat{n}^2 \rangle$.

- Consequently, the Heisenberg limit is not broken, and it is not even attained!

- Consider $|\psi(\varphi)\rangle = \exp(-i\varphi\hat{n}^2)|\psi(0)\rangle = \frac{|0\rangle + e^{-i\varphi N^2}|N\rangle}{\sqrt{2}}$

- Measuring the operator $X = |0\rangle\langle N| + |N\rangle\langle 0|$ then gives the limit

$$\delta\varphi = \frac{\Delta X}{|d\langle X\rangle/d\varphi|} = \frac{1}{N^2}$$

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Relation to the Uncertainty Principle

Heisenberg limit and the uncertainty principle

- Holland and Burnett introduced the term *Heisenberg limit*, and referred to the uncertainty principle in the book by Heitler.

- However, we argued that the Heisenberg limit is given in terms of the expectation value, and not the variance.

- Formally integrate the bound on the derivative of the statistical distance:

$$\int_0^{\varphi} d\varphi' \geq \frac{1}{\langle \mathscr{H} \rangle} \int_0^{\pi/2} ds \quad \Rightarrow \quad \varphi \geq \frac{\pi}{2} \frac{1}{\langle \mathscr{H} \rangle}$$

- This is the Margolus-Levitin bound!
  (see Jones & Kok, *Physical Review A* **82**, 022107, 2010).

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Heisenberg limit and the uncertainty principle

- We can also bound the Fisher information by the variance:

$$F(\varphi) \leq 4(\Delta \mathscr{H})^2$$

- This leads to the famous Mandelstam-Tamm bound:

$$\delta\varphi \geq \frac{1}{2\Delta \mathscr{H}}$$

- This is a form of Heisenberg's Uncertainty Principle.

- So the Heisenberg limit is really an example of the Margolus-Levitin bound, rather than Heisenberg's uncertainty principle.

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Conclusions

- We reconciled the physical resources in a parameter estimation procedure with the query complexity of the corresponding quantum network.
- Using this definition of the resources, we proved that the Heisenberg limit is optimal.
- Quantum mechanical procedures beat nonlinear optical procedures.
- The Heisenberg limit is not a form of the Uncertainty Principle, but rather a manifestation of the Margolus-Levitin bound.

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