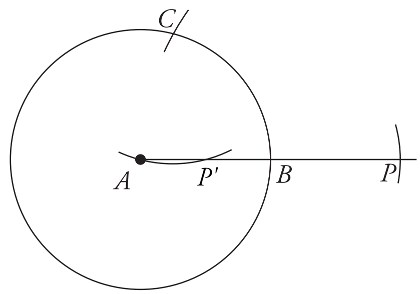
1 Inversion

Finding the midpoint of a segment using only a compass

Problem 1.1. *Given a line through A and B, find the midpoint of the segment AB using only a compass.*



With centre A, and radius $r = AB$, draw $\mathcal{C}(A, r)$. Cut off the point P on AB so that B is the midpoint of AP. With centre P, draw $\mathcal{C}(P, AP)$ cutting the first circle at C. Draw $\mathcal{C}(C, r)$ cutting the line AB at P'. Then P' is the midpoint of AB.

Proof. The triangles $AP'C$ and ACP are similar isosceles triangles, so

$$\begin{aligned} & \frac{AP'}{AC} = \frac{AC}{AP}, \\ \implies \quad & \frac{AP'}{r} = \frac{r}{2r}, \\ \implies \quad & AP' = \frac{r}{2} = \frac{AB}{2}. \end{aligned}$$

□

Note that with $AP = 2r$ and $AP' = \frac{1}{2}r$, we have $AP \cdot AP' = r^2$. This relationship between P and P' is called an *inversion*. More generally, we have the following definition:

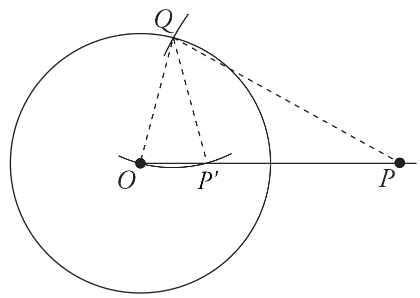
Given a circle $\mathcal{C}(O, r)$, and a point P other than O, the point P' on the ray $\overrightarrow{OP}$ is the ***inverse*** of P if $OP \cdot OP' = r^2$. The circle $\mathcal{C}(O, r)$ is called the ***circle of inversion***, the point O is called the ***centre of the inversion***, r is called the ***radius of inversion***, and r^2 is called the ***power of the inversion***.

Remarks: Suppose P is a point other than the centre of inversion. If P is outside the circle of inversion, then its inverse, P', is interior to the circle of inversion. If P is on the circle, then it is its own inverse. If P is inside the circle, then its inverse is exterior to the circle of inversion.

Compass method of finding the inverse (Ogilvy p. 29)

Note that given the ray OP, and given the circle $\mathcal{C}(O, r)$, the compass-only construction as given above works to find the inverse of P when P outside the circle of inversion:

With centre P draw an arc cutting the $\mathcal{C}(O, r)$ at Q. With centre Q and radius OQ, draw an arc cutting OP at P'. Then P' is the inverse of P.



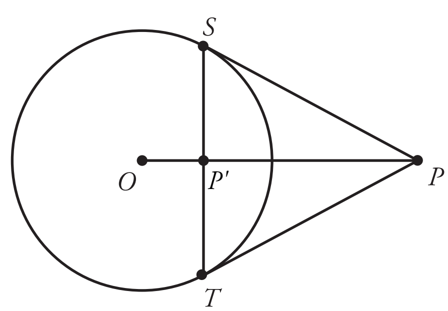
Proof. The isosceles triangles OQP and $OP'Q$ are similar (by **AA**) so

$$\frac{OP}{OQ} = \frac{OQ}{OP'}$$

and so $$OP \cdot OP' = OQ^2 = r^2.$$ □

The tangent method (Ogilvy p. 27, Eves p. 122)

Another construction for finding the inverse, this time with a compass and straightedge, is as follows (several construction lines are omitted):



Here we are given the circle $\mathcal{C}(O, r)$ and the point P outside the circle.

Draw the segment OP, and construct the tangents PS and PT to the circle with S and T being the points of tangency.

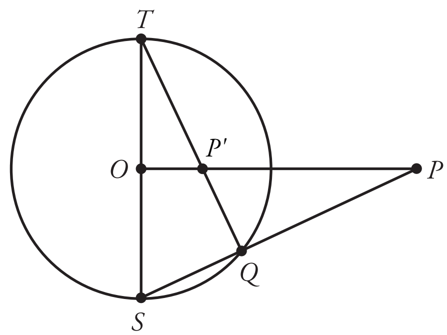
Let $P' = ST \cap OP$. Then P' is the inverse of P.

We leave the proof as an exercise.

Note that an easy modification works to find P' when P is inside the circle. (Draw the line through P perpendicular to OP cutting the circle at S and T. Draw the tangents at S and T meeting at P'.)

The perpendicular diameter method

Another method that works when P is inside or outside the circle is as follows:



1. Draw ST, the diameter perpendicular to OP.
2. Let Q be the point where the line SP meets the circle.
3. Let P' be the point where the line TQ meets OP.

Then P' is the inverse of P.

The proof is left as an exercise.

The inversive plane

Given a circle $\mathcal{C}$, every point P in the plane has an inverse with respect to $\mathcal{C}$ *except* the centre of the circle, O. The point O has no inverse and is itself not the inverse of any point. As far as inversion is concerned, the point O may as well not exist.

In order to overcome this blemish we append a single ***point at infinity***, I, to the plane so that the inversion maps O to I and vice-versa. The point I is also called the ***ideal point***, and it is considered to be on every line in the plane. The Euclidean plane together with this single ideal point is called the ***inversive plane***.

When we want to exclude the ideal point from the discussion, we refer to the non-ideal points as being ***ordinary points***.

In the inversive plane, all lines pass through the ideal point. Two lines that meet at a single point in the Euclidean plane meet at two points in the inversive plane. Two lines that are parallel in the Euclidean plane meet only at the ideal point in the inversive plane. Technically speaking, there are no parallel lines in the inversive plane, however continue to use the terminology "parallel lines" to mean that the lines meet only at the ideal point. And, as is the case in the Euclidean plane, lines that coincide are also said to be parallel.

The following facts are immediate consequences of the definitions of the inversion in $\mathcal{C}(O, r)$:

Theorem 1.2.

1. *The point P' is the inverse of the point P if and only if P is the inverse of P'.*
2. *If $|OP| = kr$, then $|OP| = \frac{1}{k}r$.*
3. *The inversion maps every point outside the circle to some point inside the circle and vice-versa. Each point on the circle of inversion is mapped onto itself.*

Problem 1.3. *Suppose that P and Q are points on the ray $\overrightarrow{OP}$. Let p' and Q' be the respective inverses. Show that if $OQ = k \cdot OP$ the $OP' = k \cdot OQ'$.*

Solution. Let r be the radius of inversion. Then $OP \cdot OP' = r^2$ and $OQ \cdot OQ' = r^2$. Multiplying both sides of the equation $OQ = k \cdot OP$ by $OP' \cdot OQ'$ we get

$$
\begin{aligned}
& OP' \cdot OQ' \cdot OQ = OP' \cdot OQ' \cdot k \cdot OP \\
\Longrightarrow \qquad & OP' \cdot r^2 = OQ' \cdot k \cdot r^2 \\
\text{that is,} \qquad & OP' = k \cdot OQ'.
\end{aligned}
$$

□