The Inversion Transformation

A non-linear transformation

The transformations of the Euclidean plane that we have studied so far have all had the property that lines have been mapped to lines. Transformations with this property are called ***linear***.

We will now investigate a specific transformation which is not linear, that is, sometimes lines are mapped to point sets which are not lines.

Inversion

Let O be a fixed circle whose center will also be called O with radius r.

For every point P other than O we define a unique point called P' on the ray OP with the property that:

$$OP \cdot OP' = r^2.$$

We refer to the mapping P → P' as ***inversion*** with respect to the circle O.

Note that if P is inside the circle then P' will be outside the circle and vice versa.

Examples

Let O be the circle with center at the origin and radius 1. Inversion with respect to this circle gives:

$P = (0,\frac{1}{2}) \rightarrow P' = (0,2)$

$P = (-3,0) \rightarrow P' = (-\frac{1}{3},0)$

$P = (0,1) \rightarrow P' = (0,1)$

$P = (3,4) \rightarrow P' = (3/25,\ 4/25)$

Let O be the circle with center at the origin and radius 2. Inversion with respect to this circle gives:

$P = (0,1) \rightarrow P' = (0,\ 4)$

$P = (2,0) \rightarrow P' = (2,0)$

$P = (2,2) \rightarrow P' = (1,1)$

Invariants & an Ideal Point

Inversion is "almost" an involution, that is, when repeated it results in the identity transformation.

A true involution pairs the points of the plane. A point could be paired with itself, such points are called ***invariant points*** of the transformation.

The problem with inversion is that the center of the circle of inversion is not paired with any point. To fix the problem, we will need to add a point to the plane, called an ***ideal point***, to pair with the center. Such a point would need to be on every line through the center … but only one point can be used, or else the pairing will not be unique.

The Circle & Lines thru Center

Every point on the circle of inversion is an invariant point of inversion.

The inverse of a line through the center of inversion is the same line.

However, only two points on such a line are invariant points … the points where the line meets the circle of inversion.

Notice that we have two situations where the set is transformed into itself, but in different ways. The circle of inversion is transformed into self in a pointwise manner (each point is invariant), while a line through the center only has some invariant points and other points are moved around.

The Circle & Lines thru Center

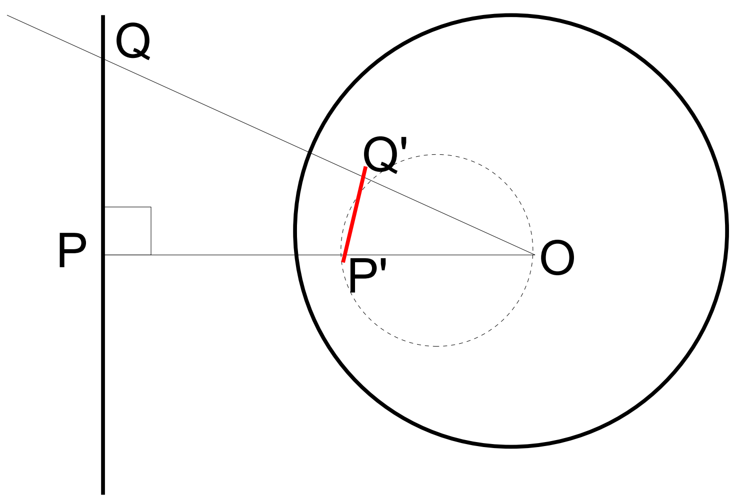
We normally would have two different terms to express these two situations.

In other situations the terms used would be ***fixed*** and ***invariant***. But as we have seen, the term invariant has been used to mean what elsewhere would be called fixed.

A possible solution to this terminology problem would be to use the terms invariant and stable.

Lines not thru the Center

Theorem 6.3: The image under inversion of a line not through the center of inversion is a circle passing through the center of inversion.



Let O be the center of inversion, OP the perpendicular from O to the given line, Q any other point on that line and P' and Q' the images of P and Q under this inversion.

Since $OP \cdot OP' = OQ \cdot OQ'$, $\Delta OPQ \sim \Delta OQ'P'$ (with right angle at Q'). As Q varies on the line, Q' traces a circle with diameter OP'. ∎

Circles thru the Center

The converse of this theorem is also valid, namely,

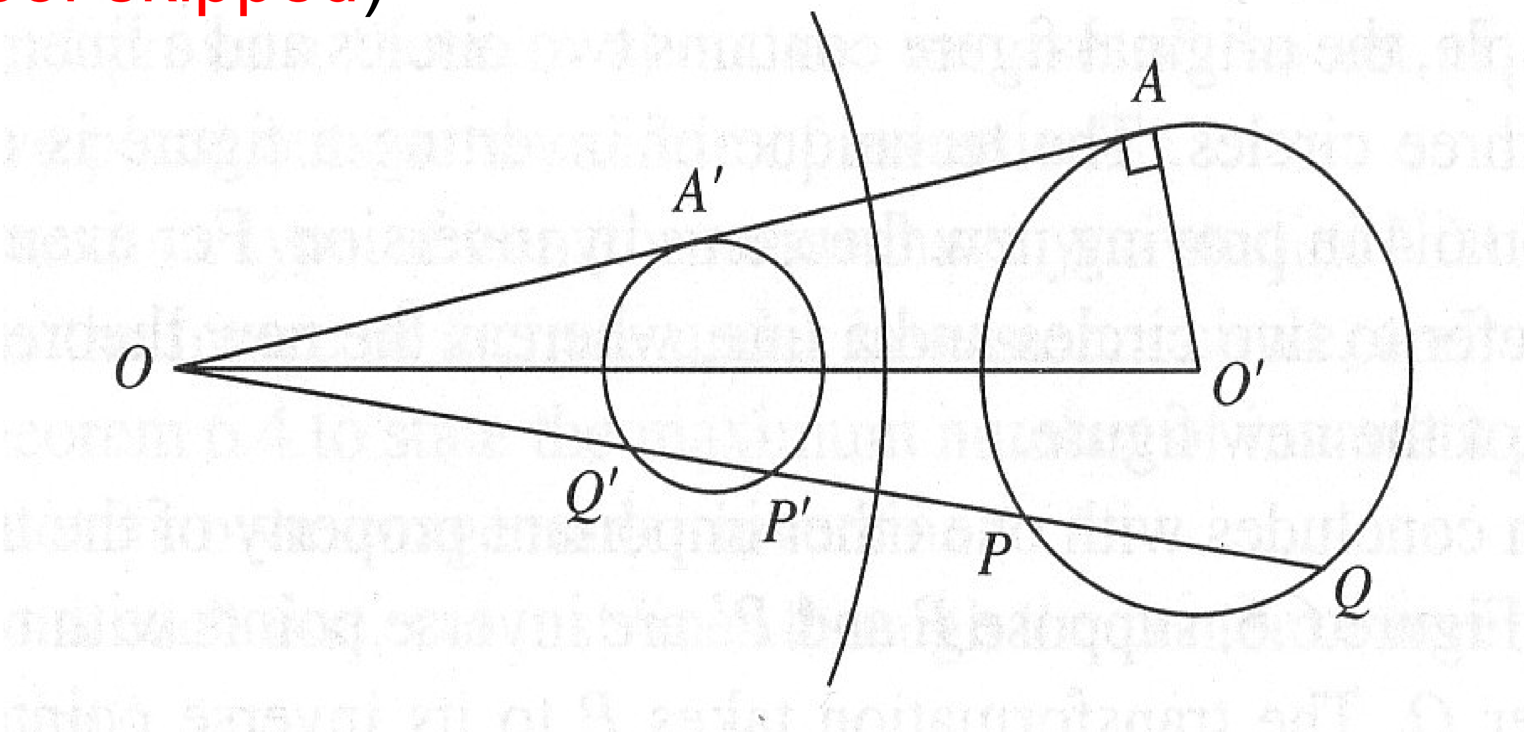
Circles through the center of inversion are mapped to lines not through the center of inversion by the inversion transformation.

The proof is essentially the reverse of the proof of the last theorem.

Circles not thru Center

Theorem 6.4: The image under inversion of a circle not passing through the center of inversion is a circle not passing through the center of inversion.

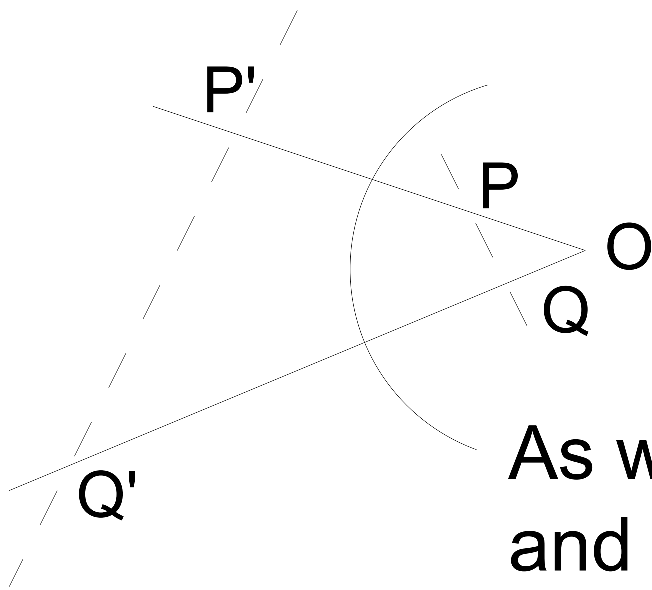
(Proof skipped)



Conformal Map

Theorem 6.5: The measure of an angle between two intersecting curves is an invariant under inversion.

Pf:

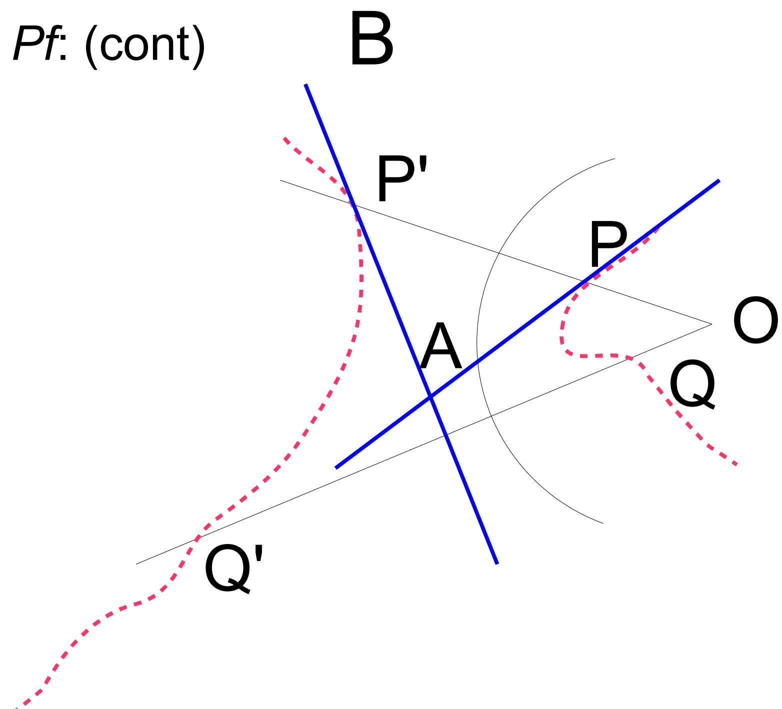


As we have seen, $\Delta OPQ \sim \Delta OQ'P'$ and so, $\angle OPQ = \angle OQ'P'$. Also, $\angle OQP = \angle OP'Q'$.

Conformal Map

Theorem 6.5: The measure of an angle between two intersecting curves is an invariant under inversion.

Pf: (cont)

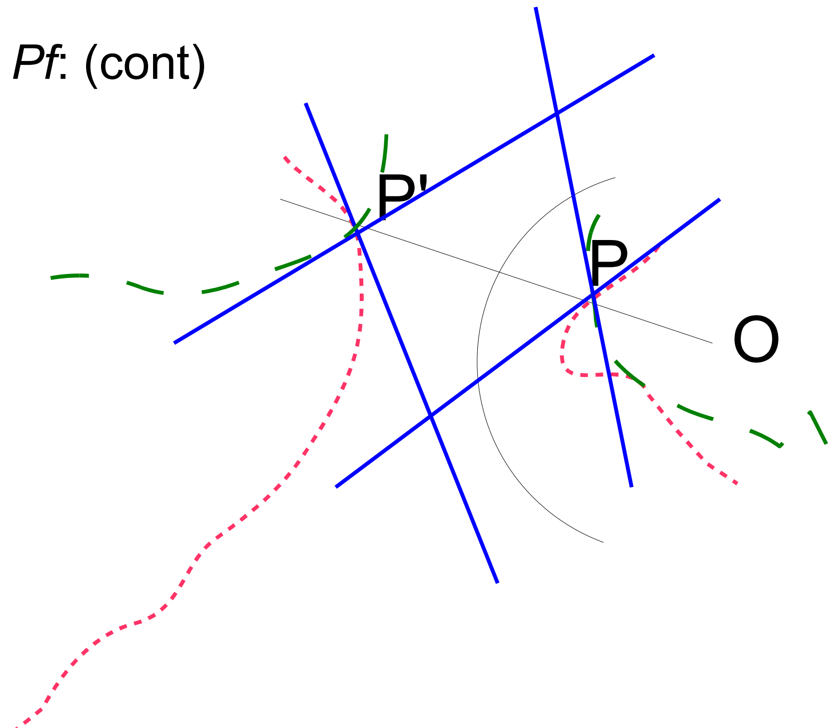


With P fixed, as Q varies along its curve approaching P, the secant PQ approaches the tangent at P, which is PA. Thus $\angle QPO$ approaches $\angle APO$. Similarly, $\angle OQ'P'$, which is always congruent to $\angle QPO$, approaches $\angle OP'B$. Thus, $\angle APO = \angle AP'O$ as supplements of congruent angles.

Conformal Map

Theorem 6.5: The measure of an angle between two intersecting curves is an invariant under inversion.

Pf: (cont)

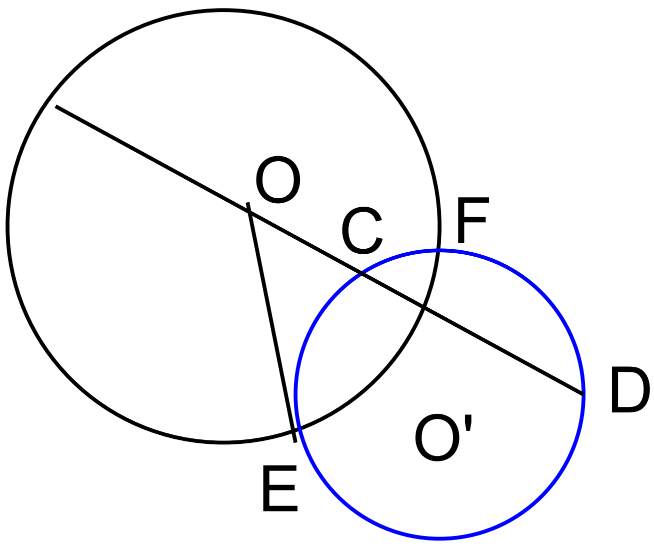


As, at the point of intersection of the two intersecting curves, P and its inversive image P', the tangent lines meet PP' in the same angles, the angle between the tangent lines must be the same at both P and P'. ∎

Orthogonal Circles

Theorem 6.6: A circle orthogonal to the circle of inversion is stabilized under the inversion transformation.

Pf:



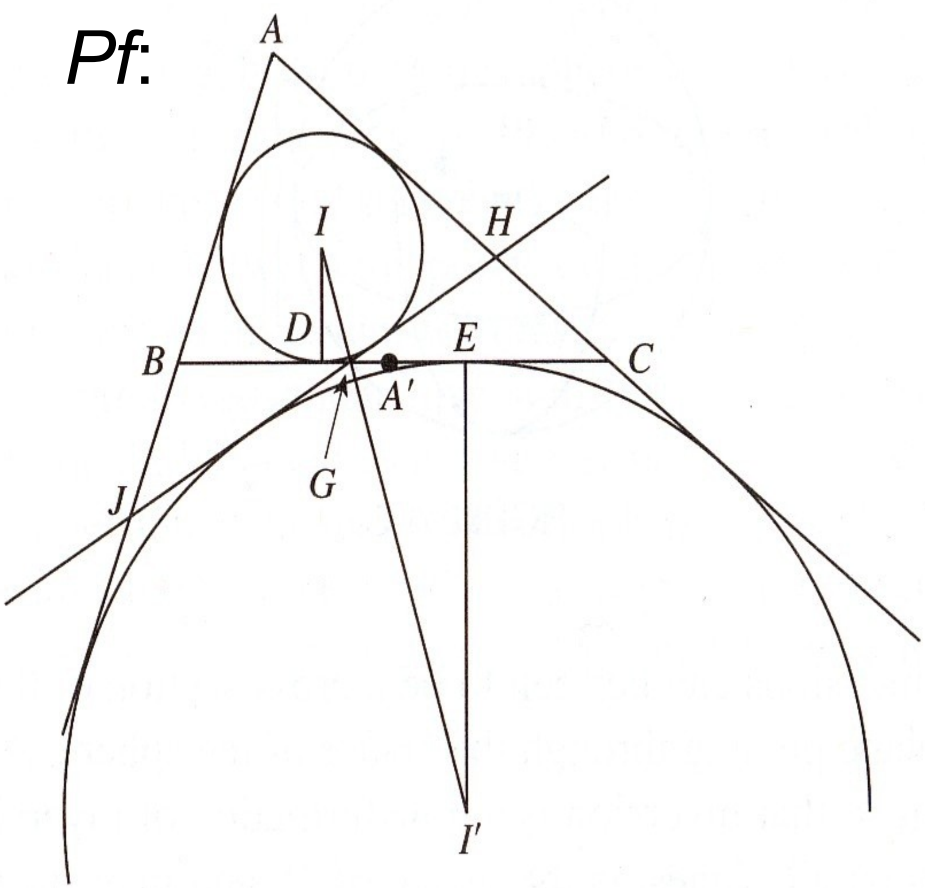
O is the circle of inversion, O' a circle orthogonal to it, meeting at points E and F. Since O and O' are orthogonal, OE is a tangent line. Thus, for any line through O meeting O' at C and D, we have:

$OC \cdot OD = (OE)^2$ so C and D are inverses with respect to circle O. The points of O' are thus stabilized by inversion with E and F invariant points.

Feuerbach's Theorem

Theorem: The nine-point circle of a triangle is tangent to the incircle and each of the three excircles of the triangle.

Pf:



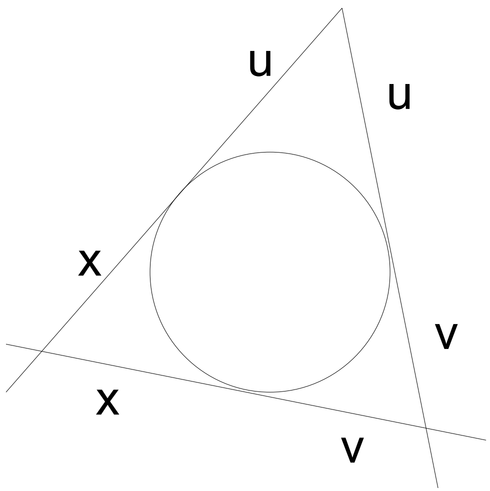
Let I be the incenter, I' an excenter, A' the midpoint of side BC. Draw radii ID and I'E which are perpendicular to BC. Note that BC is a common tangent to the incircle and this excircle.

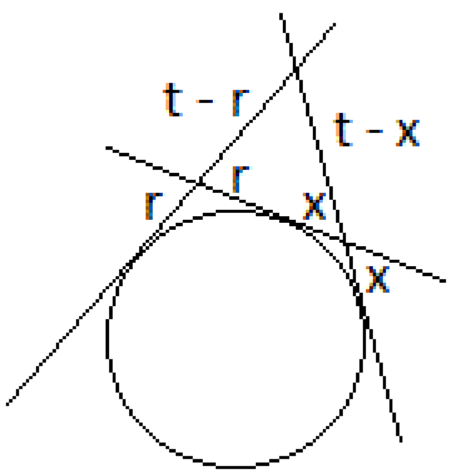
A' is also the midpoint of DE, since BD = EC.

Feuerbach's Theorem

Theorem: The nine-point circle of a triangle is tangent to the incircle and each of the three excircles of the triangle.

Pf:





Semiperimeter $s = x + u + v$
so
$BD = x = s – u – v = s - AC.$

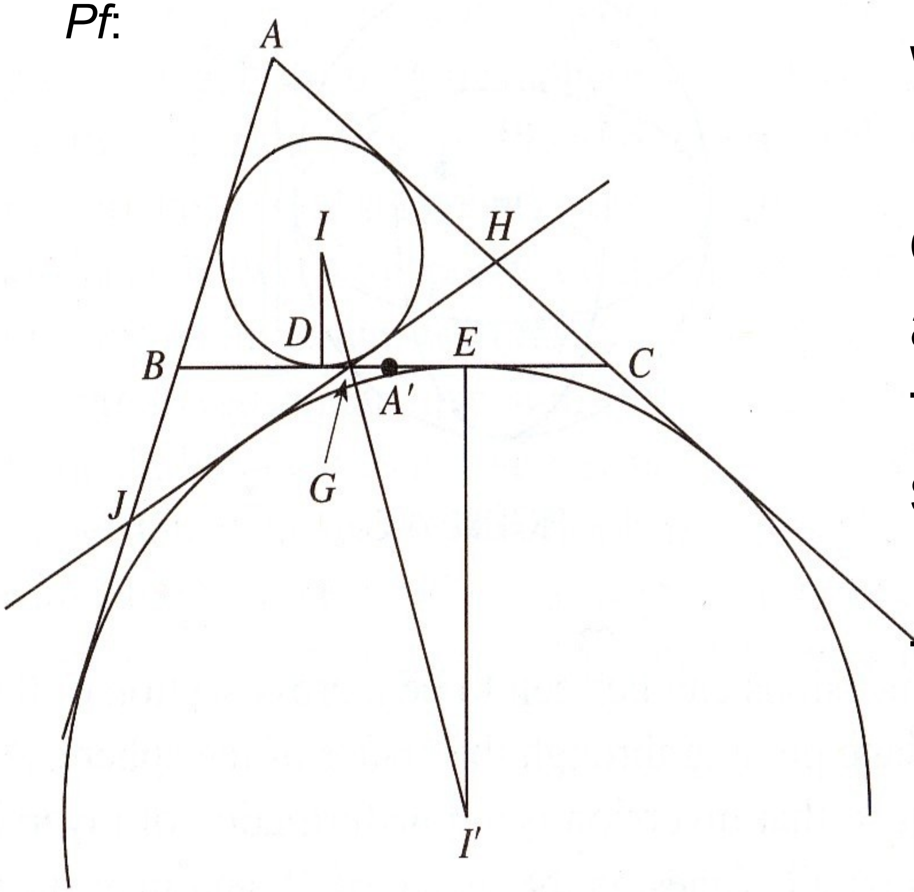
$s = ½(t – r + r + x + t – x) = t$
so
$EC = x = t – (t – x) = s – AC$
Hence, $BD = EC$.

With $BC = a$, $AC = b$ and s the semiperimeter $= ½(a+b+c)$ we have
$DE = a – 2(s – b) = b – c.$

Feuerbach's Theorem

Theorem: The nine-point circle of a triangle is tangent to the incircle and each of the three excircles of the triangle.

Pf:



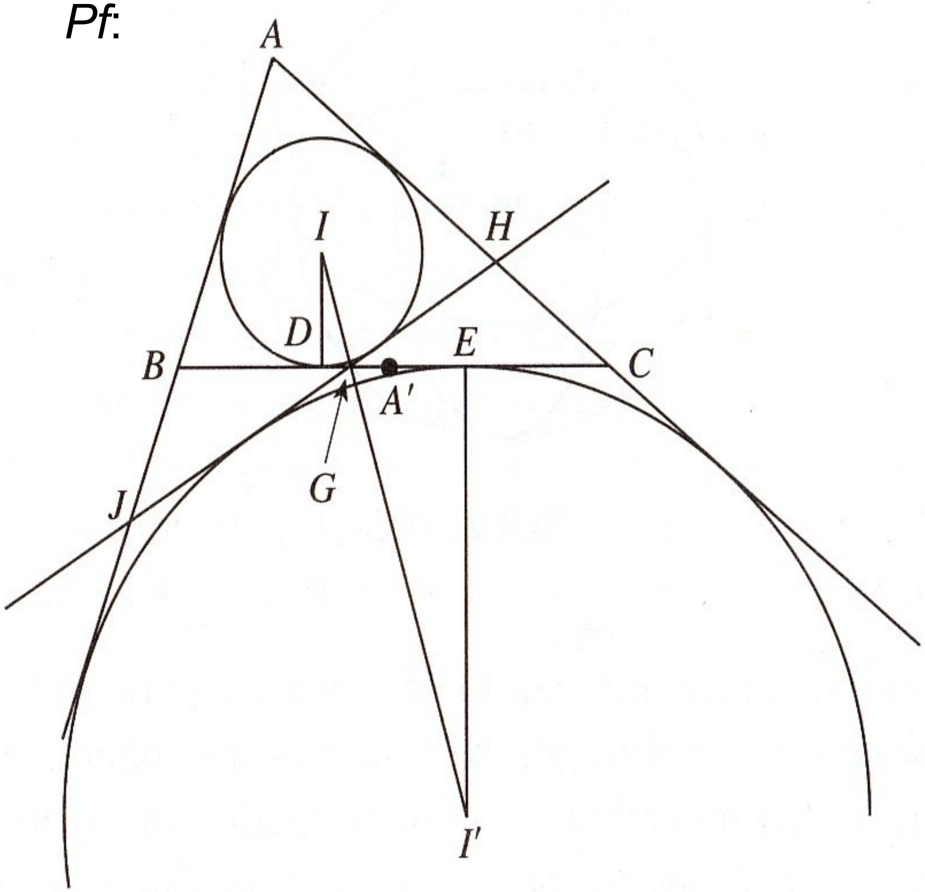
We will take A' as the center of inversion with circle having diameter DE. As both the incircle and excircle are orthogonal to the circle of inversion, they are stabilized by inversion.

There is a second common tangent, JH. Let G be the intersection of JH and BC. The point G lies on the angle bisector at A, which is the line II'.

Feuerbach's Theorem

Theorem: The nine-point circle of a triangle is tangent to the incircle and each of the three excircles of the triangle.

Pf:



This follows since the reflection with axis II' stabilizes the two circles and interchanges the common tangents BC and HJ. It follows that AC = AJ and AH = AB.

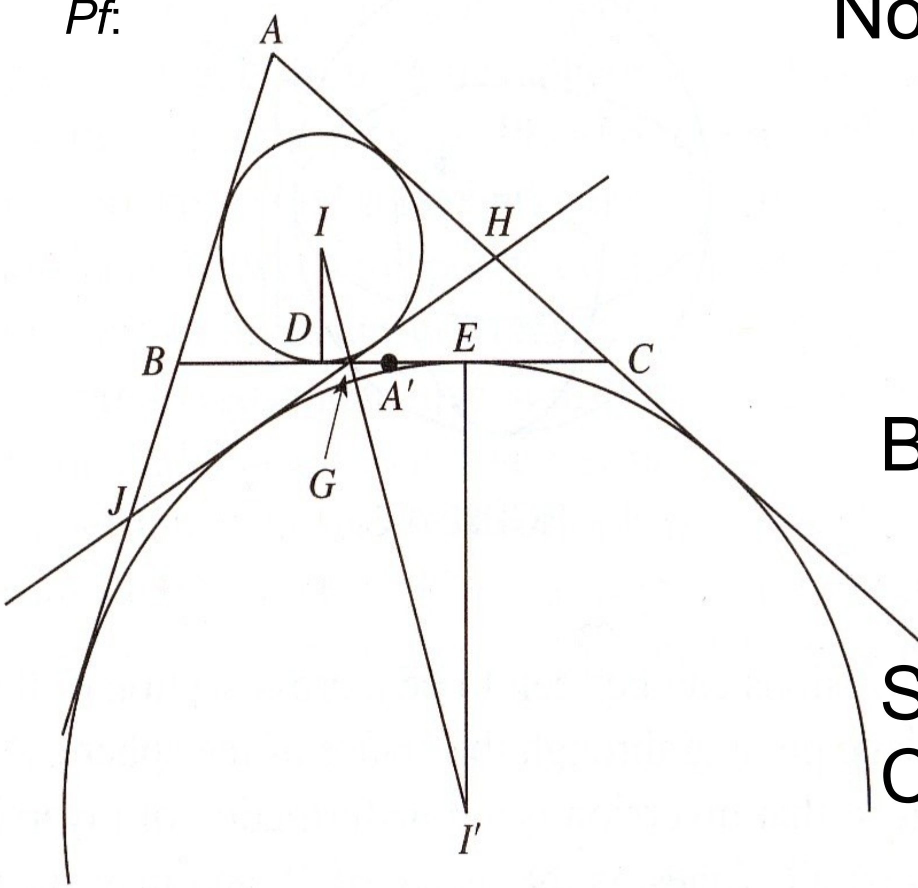
Since G is on the angle bisector at A in ∆ABC, the segments GB and GC are in the ratio of the sides b/c, so

$$GC = \frac{ab}{b+c} \text{ and } GB = \frac{ac}{b+c}.$$

Feuerbach's Theorem

Theorem: The nine-point circle of a triangle is tangent to the incircle and each of the three excircles of the triangle.

Pf:



Now, BG + GA' = ½a, so we get:

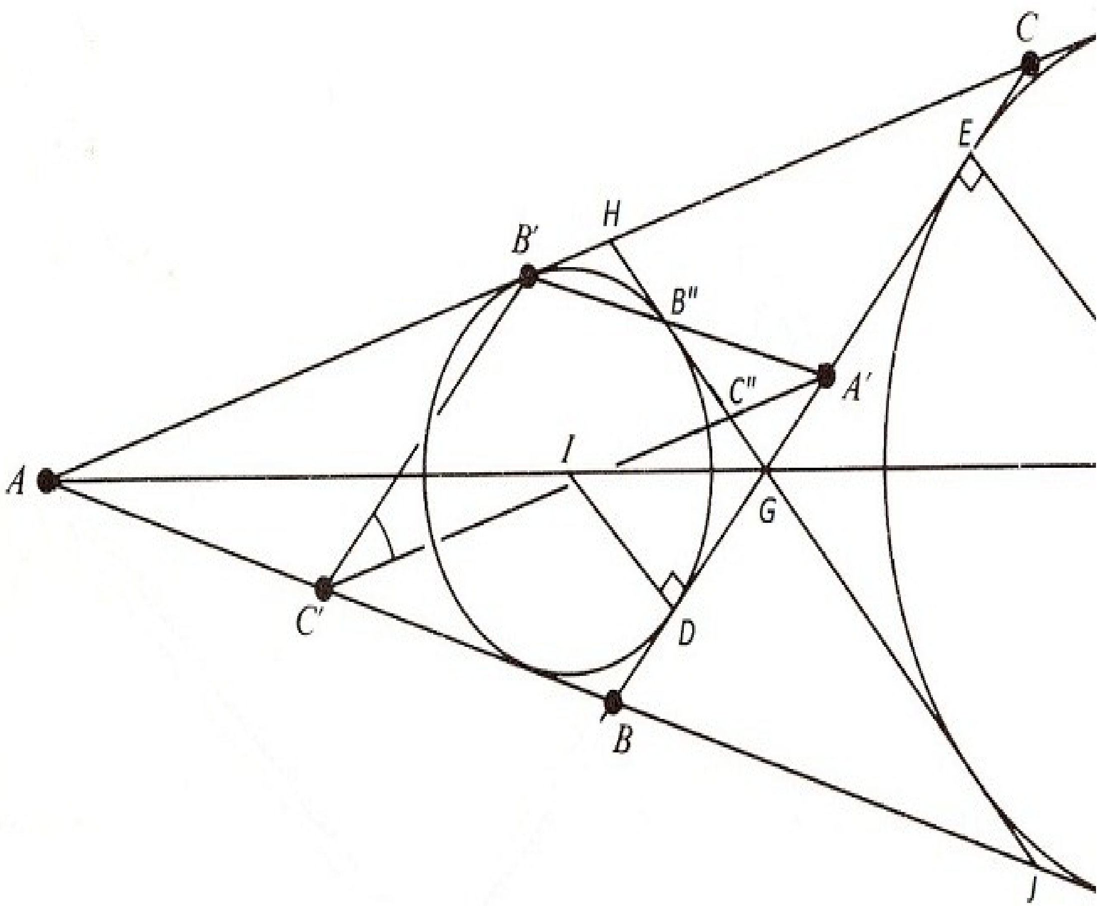
$$GA' = \frac{a(b-c)}{2(b+c)}.$$

BJ = AJ – AB = AJ – c = AC – c
so BJ = b – c.

Similarly,
CH = AC – AH = b – AH = b – AB
and HC = b - c.

Feuerbach's Theorem

Theorem: The nine-point circle of a triangle is tangent to the incircle and each of the three excircles of the triangle.



Let B'' = A'B' ∩ HJ and C'' = A'C' ∩ HJ

ΔGA'B'' ~ ΔGBJ and

ΔGA'C'' ~ ΔGCH.

$$\frac{A'B''}{BJ} = \frac{GA'}{GB} \text{ and}$$

$$\frac{A'C''}{CH} = \frac{GA'}{GC}$$

Feuerbach's Theorem

Theorem: The nine-point circle of a triangle is tangent to the incircle and each of the three excircles of the triangle.

Pf:

We can now calculate:

$$A'B'' = \frac{(b-c)^2}{2\mathrm{c}} \quad \text{and}$$

$$A'C'' = \frac{(b-c)^2}{2\mathrm{b}}.$$

Which gives us:

$$A'B' \times A'B'' = \left(\frac{c}{2}\right)\left(\frac{(b-c)^2}{2\mathrm{c}}\right) = \left(\frac{b-c}{2}\right)^2$$

$$\text{and} \quad A'C' \times A'C'' = \left(\frac{b}{2}\right)\left(\frac{(b-c)^2}{2\mathrm{b}}\right) = \left(\frac{b-c}{2}\right)^2.$$

Feuerbach's Theorem

Theorem: The nine-point circle of a triangle is tangent to the incircle and each of the three excircles of the triangle.

Pf:

Thus, B' and B'' as well as C' and C'' are inverse images with respect to our inversion transformation. Since B' and C' are on the 9-points circle, and the 9-pts circle passes through the center of inversion (A'), it is mapped to the line containing B'' and C'', which is HJ.

Since HJ is a common tangent to the incircle and this excircle, applying inversion again gives us that the 9-points circle is tangent to the incircle and this excircle since they are stabilized by inversion.

The argument can be repeated for the other excircles. ∎