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CONSTRAINTS ON SUPERSYMMETRY BREAKING*

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Some non-perturbative constraints on supersymmetry breaking are derived. It is demonstrated that dynamical supersymmetry breaking does not occur in certain interesting classes of theories.

One of the most remarkable ideas in particle physics – and an idea that has not yet found its place in our understanding of nature – is supersymmetry [1]. Recently, there has been renewed interest in attempting to make a realistic model of particle physics based on supersymmetry [2–6].

If supersymmetry plays a role in nature it certainly is spontaneously broken, because we do not observe degenerate Bose-Fermi multiplets. It is therefore crucial to understand under what conditions supersymmetry is spontaneously broken.

A realistic model of particle physics based on supersymmetry might be a model in which supersymmetry is spontaneously broken at the tree level. The conditions under which supersymmetry is spontaneously broken at the tree level are well understood. On the other hand, a realistic description of particle physics might require a model in which supersymmetry is unbroken at the tree level but broken dynamically by the quantum corrections. The purpose of this paper is to derive some constraints on the conditions under which dynamical breaking of supersymmetry can occur.

Spontaneous breaking of supersymmetry has a number of special features, some of which were reviewed in ref. [2]. Supersymmetry is unbroken if and only if the energy of the vacuum is exactly zero. From this it follows that even in weakly coupled theories with small, well-defined coupling constants, it is in general difficult to decide whether supersymmetry is spontaneously broken. Even if the vacuum energy appears to be zero in some approximation, tiny corrections that have been neglected may cause the energy to be small but non-zero. Some explicit examples in which this occurs were given in ref. [2].

It is rather extraordinary that one cannot straightforwardly decide, even in the case of a weakly coupled theory, whether supersymmetry is spontaneously broken.

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Of course, it is even more difficult to decide this in the case of strongly coupled theories. In this paper, a new approach will be introduced which resolves this question in many interesting cases.

Certain quantities will be defined which can be calculated reliably in perturbation theory and which must vanish in order for supersymmetry breaking to be possible. In many interesting cases these quantities do not vanish; this establishes that in those theories supersymmetry is not spontaneously broken.

The quantities in question are, roughly speaking, topological invariants of the field theory. We have become accustomed to defining topological quantum numbers of individual classical field configurations. In supersymmetric theories, it is, as we will see, useful to define certain topological quantum numbers which are properties of the entire theory, not of any particular field configuration.

In sects. 2–4 some general constraints on supersymmetry breaking will be formulated. Simple applications to theories of spin 0 and spin $\frac{1}{2}$ fields only are discussed in sect. 5. Abelian gauge theories are the subject of sect. 6. In sects. 7–9, we study non-abelian gauge theories, and in sect. 10 we apply our techniques to the supersymmetric non-linear sigma model. Conclusions are drawn in sect. 11.

The approach in this paper was suggested in part by 't Hooft's use of periodic boundary conditions and there are some analogies with his discussion of chiral symmetry breaking in confining theories [8].

2. $\mathrm{Tr}(-1)^F$

It is very useful to consider supersymmetric theories formulated in a finite spatial volume. In a finite volume the spectrum of the hamiltonian is discrete; states in Hilbert space can be counted in a clear-cut, well-defined way; there are only a finite number of states with less than a given energy.

Since translations are part of the supersymmetry algebra, we must adopt boundary conditions that preserve translation invariance in order not to break supersymmetry explicitly. This means that we must use periodic boundary conditions – which is equivalent to taking space to be a three-dimensional torus. To preserve supersymmetry we must use the same boundary conditions for bosons as for fermions (periodic rather than antiperiodic in the spatial directions).

One could not ordinarily learn whether an internal symmetry is spontaneously broken by studying a theory formulated in a finite volume, because, ordinarily, an internal symmetry is unbroken in a finite volume whether or not it becomes broken in the infinite volume limit. Mixing between the various states usually ensures that, in a finite volume, the ground state is invariant under all internal symmetries, even though, in the infinite volume limit, the theory may break up into several sectors with the symmetry broken in each sector.

By contrast, supersymmetry can perfectly well be spontaneously broken in a finite volume. Supersymmetry breaking just means that the ground-state energy is positive, which is possible for supersymmetric theories in a finite volume or even for supersymmetric theories with only a finite number of degrees of freedom [2].

If supersymmetry is unbroken in an arbitrary finite volume V, this means that the ground-state energy $E(V)$ is zero for every V. Since the large-V limit of zero is zero, this means that the ground-state energy is zero in the infinite volume limit, and that supersymmetry is unbroken in this limit.

The converse is not true. Supersymmetry may be broken in any finite volume yet restored in the infinite volume limit. If supersymmetry is broken in a finite volume, so that $E(V)$ is positive, it may still be that $E(V)$ [or more pertinently, $E(V)/V$] vanishes as V becomes large. An example in which this occurs is described in appendix A.

In this paper, certain methods will be developed for proving that in certain classes of theories, supersymmetry is unbroken in any finite volume. As just explained, this suffices for proving that supersymmetry is unbroken in the infinite volume limit. The methods in this paper are less useful for proving that supersymmetry is broken in a finite volume, and even when this is possible it does not lead to a definite conclusion about the infinite volume theory, since, as just noted, supersymmetry may be restored in the infinite volume limit.

Given a theory defined in a volume V with a Hilbert space $\mathcal{H}$, our main concern is with the possible existence in $\mathcal{H}$ of zero-energy states. In supersymmetric theories, the energy E is equal to or greater than the magnitude of the momentum $|\boldsymbol{P}|$ for any state. Zero-energy states must therefore have $\boldsymbol{P}=0$, and we lose nothing by restricting our attention to the $\boldsymbol{P}=0$ subspace of $\mathcal{H}$.

In the subspace of states of zero momentum, the supersymmetry algebra is particularly simple. In a basis of properly normalized hermitian supersymmetry charges $Q_1, Q_2, \ldots, Q_K$ ($K=4$ for simple supersymmetry in four dimensions) the algebra is

$$Q_1^2 = Q_2^2 = \cdots = Q_K^2 = H,$$

$$Q_iQ_j + Q_jQ_i = 0, \qquad \text{for } i \neq j. \tag{1}$$

In this section it will be sufficient for us to work with any one of the Q_i, which we will simply denote as Q.

Supersymmetry maps bosons into fermions and maps fermions into bosons. Actually, in a finite volume the concept of an individual "particle" is ill-defined, but it makes sense to think of bosonic states and fermionic states in Hilbert space. A bosonic state $|\mathrm{b}\rangle$ satisfies $\exp(2\pi i J_z)|\mathrm{b}\rangle = |\mathrm{b}\rangle$; a fermionic state $|\mathrm{f}\rangle$ satisfies $\exp(2\pi i J_z)|\mathrm{f}\rangle = -|\mathrm{f}\rangle$. Of course, a bosonic state may be any state which in the infinite volume limit goes over to a configuration of, say, 92 neutrons and 148 pions.

We will often make use of the operator

$$(-1)^F = \exp(2\pi i J_z) \tag{2}$$

that distinguishes bosons from fermions. Of course, in a finite volume the infinitesimal rotation generator J_z is not well defined, but ninety degree rotations such as $\exp(\frac{1}{2}i\pi J_z)$ make sense; (2) should be understood as the fourth power of the ninety degree rotation operator.

The crucial observation is that the states of non-zero energy are paired by the action of Q. Let $|\mathrm{b}\rangle$ be any bosonic state of non-zero energy E. Define a fermionic state $|\mathrm{f}\rangle = (1/\sqrt{E})Q|\mathrm{b}\rangle$ (which is normalized if $|\mathrm{b}\rangle$ is, since $Q^2 = H$). The action of Q on $|\mathrm{b}\rangle$ and $|\mathrm{f}\rangle$ is then

$$Q|\mathrm{b}\rangle = \sqrt{E}\,|\mathrm{f}\rangle, \qquad Q|\mathrm{f}\rangle = \sqrt{E}\,|\mathrm{b}\rangle, \tag{3}$$

where the second equation is chosen to satisfy $Q^2 = H$. All states of non-zero energy are paired in two-dimensional supermultiplets with this structure. (Had we considered all the Q_i and not just one, the supermultiplets may be larger; the enlarged structure is not relevant for our purposes here.)

On the other hand, the zero-energy states are not paired in this way. With $Q^2 = H$ and Q hermitian, each state annihilated by H is also annihilated by Q. Any bosonic or fermionic state of zero energy satisfies $Q|\mathrm{b}\rangle = 0$ or $Q|\mathrm{f}\rangle = 0$. They form trivial, one-dimensional supersymmetry multiplets. In general, there may be an arbitrary number $n_{\mathrm{B}}^{E=0}$ of zero-energy bosonic states, and an arbitrary number $n_{\mathrm{F}}^{E=0}$ of zero-energy fermionic states.

The most general allowed form for the spectrum is indicated in fig. 1. There are paired states of positive energy, and there may be states, not necessarily paired, of zero energy.

What happens now as we vary the parameters of this theory? Here, the "parameters" should be understood as the volume, the mass m_i, and the coupling constants g_i.

As we vary the parameters, the states of non-zero energy move around in energy. They move, of course, in Bose-Fermi pairs. Conceivably, as the parameters are varied, one of these $E \neq 0$ pairs may move down to $E = 0$. In this case (fig. 2), $n_{\mathrm{B}}^{E=0}$ and $n_{\mathrm{F}}^{E=0}$ both increase by one.

Conceivably, as the parameters are varied, some states of zero energy may gain non-zero energy. It is not possible for a single zero-energy state to acquire a non-zero energy. As soon as it has a non-zero energy it must have a supersymmetric partner.

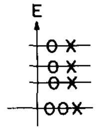

Fig. 1. The general form of the spectrum in a supersymmetric theory. A circle indicates a bosonic state. A cross indicates a fermionic state.

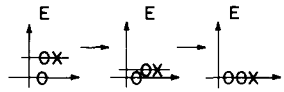

Fig. 2. A pair of states move down to zero energy as a parameter is varied.

What can occur is that a *pair* of states, one Bose, one Fermi, can move from $E=0$ to $E\neq 0$. In this case (fig. 3) $n_B^{E=0}$ and $n_F^{E=0}$ both decrease by one.

In either case, the *difference* $n_B^{E=0}-n_F^{E=0}$ does not change as one varies the parameters. This will be the basis for most arguments in this paper.

The quantity $n_B^{E=0}-n_F^{E=0}$ is useful because of two basic properties:

(i) It can be calculated reliably.

(ii) If it is not zero, supersymmetry is not spontaneously broken.

The difference $n_B^{E=0}-n_F^{E=0}$ can be calculated reliably because, since it is independent of all parameters, it can be calculated in a convenient limit, such as small volume, large bare mass, and weak coupling. As we will see in detail, almost any theory simplifies enough in this or some analogous limit to permit $n_B^{E=0}-n_F^{E=0}$ to be calculated reliably.

Of course, approximate calculations cannot in general determine how many states have *exactly* zero energy. But a valid approximate calculation does determine the *difference* $n_B^{E=0}-n_F^{E=0}$ reliably. Although corrections to the approximate calculation may give a non-zero energy to, say, a bosonic state that had zero energy in the approximation in question, if so (fig. 4), there is always a fermionic state of previously zero energy that acquires the same non-zero energy from the same corrections. So any valid approximation determines $n_B^{E=0}-n_F^{E=0}$ correctly.

As for the second point above, if $n_B^{E=0}-n_F^{E=0}\neq 0$, then obviously either $n_B^{E=0}\neq 0$ or $n_F^{E=0}\neq 0$ or both. In any case there are some states of zero energy, so supersymmetry is unbroken.

What if $n_B^{E=0}-n_F^{E=0}=0$? In this case we cannot distinguish between the following two possibilities:

(A) $n_B^{E=0}=n_F^{E=0}=0$; supersymmetry broken.

(B) $n_B^{E=0}$ and $n_F^{E=0}$ are equal but non-zero; supersymmetry is unbroken.

Despite the inability to distinguish between these two possibilities, an interesting although not quite rigorous conclusion can be drawn about theories with $n_B^{E=0}-n_F^{E=0}=0$.

We ordinarily expect the ground state of a system – the "vacuum state" – to be bosonic in a finite volume as well as in the infinite volume limit. This is a reasonable

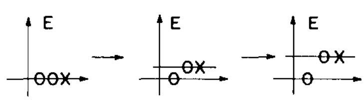

Fig. 3. A pair of states gains a non-zero energy as a parameter is varied.

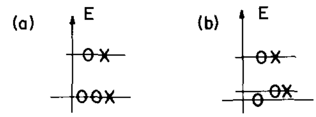

Fig. 4. Corrections to an approximate calculation (a) give an exact spectrum (b) with the same value of $\mathrm{Tr}(-1)^F$.

expectation in a theory in which (in the infinite volume limit) all fermions are massive. In such theories, any fermionic state would (in large enough volume) be expected to lie above the ground state by an amount at least equal to the mass of the lightest fermion.

What if the infinite volume theory has massless fermions?

In a finite volume one can form a normalizable state by adding a massless fermion to the vacuum in a momentum eigenstate with $\boldsymbol{P}=0$. If the volume is large, such a state is nearly degenerate with the vacuum and may be exactly degenerate.

It seems reasonable to suppose that zero-energy fermionic states that persist in arbitrarily large volume can generally be interpreted in this way – as evidence that the infinite volume theory has a massless fermion.

Returning to the options (A) and (B) above, in (A) supersymmetry is spontaneously broken and there is a massless Goldstone fermion in the infinite volume theory. In (B) supersymmetry is not broken; there is no Goldstone fermion, but there are zero-energy fermionic states which we interpret as evidence that the infinite volume theory has a massless fermion (a massless fermion is not a Goldstone fermion unless it is created from the vacuum by the supersymmetry current). In either case we conclude that if $n_{\mathrm{B}}^{E=0}-n_{\mathrm{F}}^{E=0}=0$, the infinite volume theory has a massless fermion. However, this argument is not completely rigorous.

Formally, the quantity $n_{\mathrm{B}}^{E=0}-n_{\mathrm{F}}^{E=0}$ may be regarded as the trace of the operator $(-1)^F$ introduced previously. States of non-zero energy do not contribute to the trace of $(-1)^F$ because for every bosonic state of non-zero energy that contributes $+1$ to the trace, there is a fermionic state of non-zero energy that contributes -1 and cancels the boson contribution. Therefore $\mathrm{Tr}(-1)^F$ can be evaluated among the zero-energy states only, and equals $n_{\mathrm{B}}^{E=0}-n_{\mathrm{F}}^{E=0}$.

We thus write

$$\mathrm{Tr}(-1)^F=n_{\mathrm{B}}^{E=0}-n_{\mathrm{F}}^{E=0}. \tag{4}$$

This formula should be considered as merely a useful definition, because the infinite summation over all states in Hilbert space required to define $\mathrm{Tr}(-1)^F$ is ill-defined, not being absolutely convergent. [One could regularize $\mathrm{Tr}(-1)^F$ by writing instead $\mathrm{Tr}(-1)^F\exp(-\beta H)$ for arbitrary positive β; this is actually independent of β because the states of $E\neq 0$ do not contribute. This regularization gives back eq. (4) in the limit $\beta\to 0$.]

Actually, the quantity $\mathrm{Tr}(-1)^F$ is an example of a standard mathematical concept, which plays an important role in contemporary mathematics and has had some recent applications in physics. This is the concept of the *index* of an operator [9].

We may split the Hilbert space H of our theory into bosonic and fermionic subspaces $\mathrm{H_B}$ and $\mathrm{H_F}$. Since the supersymmetry charge Q maps bosons into fermions and vice versa, it takes the following form:

$$Q=\left(\begin{array}{c|c} 0 & M^* \\ \hline M & 0 \end{array}\right), \tag{5}$$

if the states are arranged in the form

$$\left(\begin{array}{c} B \\ \hline F \end{array}\right). \tag{6}$$

Note that, because Q is hermitian the quantity designated as M^* in (5) is indeed the adjoint of M.

Now, since $H=Q^2$, the zero-energy states are precisely the states annihilated by Q. Bosonic states annihilated by Q are states ψ in $\mathrm{H_B}$ that satisfy $M\psi=0$. Fermionic states annihilated by Q are states ψ in $\mathrm{H_F}$ that satisfy $M^*\psi=0$. The quantity $n_{\mathrm{B}}^{E=0}-n_{\mathrm{F}}^{E=0}$ is therefore equal to the number of solutions of $M\psi=0$ minus the number of solutions of $M^*\psi=0$. The latter quantity is, by definition, the index of the operator M.

The fact that $\mathrm{Tr}(-1)^F$ is independent of the parameters of the theory is a special case of the fact that, in general, the index of an operator is invariant under small deformations. The argument above is a standard way of demonstrating this general fact.

Finally, let us discuss a few subtleties.

Could ultraviolet divergences invalidate the arguments above? This would not be expected because the need to cut off ultraviolet divergences only affects the highly excited states while $\mathrm{Tr}(-1)^F$ only involves the low-lying states. As long as the theory exists in the infinite cut-off limit as a supersymmetric theory, the arguments above should be valid.

A far more serious problem concerns the behavior of the potential energy for large field strengths. In a mathematical sense, the problem is the following. We have assumed that when the parameters are varied, the energy eigenvalues do not suddenly appear or disappear but move around continuously in energy. This is true as long as the changes in parameters are "gentle" enough that they can be regarded as perturbations acting within the Hilbert space already defined by the unperturbed operators.

Although ultraviolet divergences cause subtleties in defining the Hilbert space of a quantum field theory, those subtleties are not relevant here, for reasons stated above.

The serious problem involves the asymptotic behavior of the potential energy for large field strengths. A perturbation that changes this asymptotic behavior can permit new low-energy states to "move in from infinity" in field space, causing a discontinuous change of $\mathrm{Tr}(-1)^F$. For instance, consider the potential

$$V(\phi)=\left(m\phi-g\phi^2\right)^2. \tag{7}$$

At $g=0$, low-energy states correspond to ϕ near zero, but for $g\neq 0$ low-energy states may correspond to ϕ near zero or near m/g. An arbitrarily small, non-zero g cause the existence of extra low-energy states at $\phi\sim m/g$ that have no counterpart in the $g=0$ theory. In such a case $\mathrm{Tr}(-1)^F$ will have a different value at $g=0$ from its value at $g\neq 0$.

This is related to the change in asymptotic behavior of V when g is introduced. At $g=0$, $V\sim\phi^2$ for large ϕ, but for any non-zero g, $V\sim\phi^4$. The change in asymptotic behavior when g is switched on is the reason that $\mathrm{Tr}(-1)^F$ can change discontinuously at that point.

The general rule is that $\mathrm{Tr}(-1)^F$ is invariant under any change in parameters in which, in the large field regime, the hamiltonian changes by terms no bigger than the terms already present. Under such changes the energy levels of the hamiltonian change continuously; this was the crucial ingredient in showing the constancy of $\mathrm{Tr}(-1)^F$.

$\mathrm{Tr}(-1)^F$ is independent of the numerical values of the parameters in the hamiltonian as long as these are non-zero. If one wishes to set a parameter to zero or to introduce a new coupling not already present, one must make sure that this does not change the asymptotic behavior of the energy in field space and permit new states to "come in from infinity."

The concept of $\mathrm{Tr}(-1)^F$ has some generalizations, which will be discussed below when applications arise.

3. Conjugation

In the last section we have seen that in a finite volume the difference $n_{\mathrm{B}}^{E=0}-n_{\mathrm{F}}^{E=0}$ is invariant under arbitrary (reasonable) changes in the parameters of a supersymmetric theory. We will now see that actually the two numbers $n_{\mathrm{B}}^{E=0}$ and $n_{\mathrm{F}}^{E=0}$ are separately invariant under a smaller but still interesting class of changes in parameters*.

In sect. 2 we worked with a single supersymmetry charge Q. Actually, in four dimensions every supersymmetric theory contains at least four such charges, and in this section we must make use of two supersymmetry charges, say Q_1 and Q_2. If we

* This section is more technical than sect. 2, and the remainder of the paper uses primarily the results of sect. 2.

define $Q_\pm = \sqrt{\frac{1}{2}}\,(Q_1 \pm iQ_2)$, then the supersymmetry algebra, in the zero-momentum sector of Hilbert space, takes the simple form

$$Q_+^2 = Q_-^2 = 0\,, \qquad Q_+Q_- + Q_-Q_+ = H\,. \tag{8}$$

The fact that $Q_+^2 = 0$ means that the equation $Q_+\psi = 0$ has many solutions. Roughly speaking, Q_+ annihilates at least half of the states in Hilbert space. Given any state ψ, either ψ itself is annihilated by Q_+, $Q_+\psi = 0$, or its supersymmetric partner $\chi = Q_+\psi$ is annihilated by Q_+: $Q_+\chi = Q_+^2\psi = 0$.

More formally, any operator Q_+ with $Q_+^2 = 0$ can be put in Jordan canonical form by a linear transformation (not necessarily unitary). The Jordan canonical form of Q_+ would be

$$\left(\begin{array}{cc|cc|cc|ccc}
0 & 1 & & & & & & & \\
0 & 0 & & & & & & & \\
\hline
 & & 0 & 1 & & & & & \\
 & & 0 & 0 & & & & & \\
\hline
 & & & & 0 & 1 & & & \\
 & & & & 0 & 0 & & & \\
\hline
 & & & & & & 0 & & \\
 & & & & & & & 0 & \\
 & & & & & & & & 0
\end{array}\right). \tag{9}$$

There are an arbitrary number of 2×2 blocks with the structure $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. These are two-dimensional supermultiplets, consisting of two states on which Q_+ acts as a raising operator ($\psi = Q_+\chi$, $Q_+\psi = 0$). In addition, there are an arbitrary number of unpaired zeros, shown in the lower right-hand corner of (9); they are the supersymmetric zero-energy states.

Given any state χ, if $\chi = Q_+\psi$ for some ψ, then obviously $Q_+\chi$ vanishes ($Q_+\chi = Q_+^2\psi = 0$). Let us ask the converse question. Given a state χ with $Q_+\chi = 0$, can χ be written in the form $Q_+\psi$ for some ψ?

If we assume that χ is an eigenstate of the hamiltonian, $H\chi = E\chi$ for some E, the answer to the above question turns out to be the following. One can write χ in the form $\chi = Q_+\psi$ for some ψ if and only if $E \neq 0$.

The proof of this is trivial. Suppose first $E \neq 0$. Define $\psi = (1/E)Q_-\chi$. Then $Q_+\psi = (1/E)Q_+Q_-\chi = (1/E)(Q_+Q_- + Q_-Q_+)\chi = (1/E)H\chi = \chi$, so we have explicitly found a state ψ with $\chi = Q_+\psi$.

Suppose conversely that $E = 0$. It is then impossible to find a state ψ with $Q_+\psi = \chi$ for the following reason. Since Q_+ commutes with the hamiltonian, the equation $Q_+\psi = \chi$ implies that ψ has the same energy as χ. If χ has $E = 0$, ψ would also have to have zero energy. But since the Q_i annihilate all states of zero energy, if ψ has zero energy it satisfies $Q_+\psi = 0$, not $Q_+\psi = \chi$.

We arrive at the interesting conclusion that the zero-energy states are precisely the states χ such that $Q_+\chi = 0$ but $\chi \neq Q_+\psi$ for any ψ.

Let us introduce a bit of notation. Let $\ker Q_+$ be the kernel of Q_+, the space of solutions of $Q_+\chi = 0$. Let $\operatorname{im} Q_+$ be the image of Q_+, the space of all states that can be written as $Q_+\psi$ for some ψ. Let $(\ker Q_+/\operatorname{im} Q_+)$ be the quotient space (consisting of all χ in $\ker Q_+$ with χ and $\chi + Q_+\psi$ considered equivalent for any ψ). Finally, let N be the total number of zero-energy states, Bose or Fermi: $N = n_{\mathrm{B}}^{E=0} + n_{\mathrm{F}}^{E=0}$. What we have learned is that

$$N = \dim(\ker Q_+/\operatorname{im} Q_+)\,, \tag{10}$$

or, in other words, that N is equal to the dimension of the quotient space.

We wish to find conditions under which N, the total number of zero-energy states, is invariant under changes in the parameters of a supersymmetric theory. Unlike the invariance of $\mathrm{Tr}(-1)^F$ discussed in sect. 2, the invariance of N does not follow from supersymmetry alone. The following two-dimensional representation of the supersymmetry algebra [8],

$$Q_+ = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \qquad Q_- = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}, \tag{11}$$

in an explicit counterexample. Here λ is an arbitrary parameter. For $\lambda \neq 0$ no state has zero energy: the image and kernel of Q_+ are each one dimensional, and the quotient has zero dimension. But for $\lambda = 0$ there are two zero-energy states; the image of Q_+ has dimension zero, the kernel has dimension two, and the quotient consists of the two zero-energy states. At $\lambda = 0$ the number of zero-energy states jumps from zero to two.

Although in general the number of zero-energy states can change when the parameters are changed, there is a restricted but important class of changes in the parameters of a supersymmetric theory under which the total number of zero-energy states does not change. Consider the substitution from Q_+, Q_-, and H to new operators

$$\begin{aligned} \tilde{Q}_+ &= M^{-1} Q_+ M\,, \\ \tilde{Q}_- &= M^* Q_- M^{*-1}\,, \\ \tilde{H} &= \tilde{Q}_+ \tilde{Q}_- + \tilde{Q}_- \tilde{Q}_+\,, \end{aligned} \tag{12}$$

where M is an arbitrary invertible linear operator, not necessarily unitary, and M^* is the adjoint of M.

If M is unitary, $M^* = M^{-1}$, the operators $\tilde{Q}_+$, $\tilde{Q}_-$, and $\tilde{H}$ differ from Q_+, Q_-, and H merely by a change of basis in Hilbert space. They do not describe a new theory. However, if M is invertible but not unitary, the theory described by the $\tilde{Q}$ and $\tilde{H}$ is inequivalent to the one described by the Q and H.

The spectrum of non-zero-energy states of $\tilde{H}$ will differ, in general, from the spectrum of H. However, the main point here is that the number of zero-energy states for $\tilde{H}$ always equals the number of zero-energy states of H. This, indeed, follows directly from our previous observation that the total number of zero-energy states is equal to the number of linearly independent solutions of $Q_+\chi=0$ such that χ is not $Q_+\psi$ for any ψ. If χ is such a state, then $\tilde{\chi}=M^{-1}\chi$ satisfies $\tilde{Q}_+\tilde{\chi}=0$ but cannot be written as $\tilde{\chi}=\tilde{Q}_+\tilde{\psi}$ [which would imply $\chi=Q_+(M\tilde{\psi})$, contrary to the hypothesis that χ is not $Q_+\psi$ for any ψ]. Conversely if $\tilde{Q}_+\tilde{\chi}=0$ and $\tilde{\chi}$ is not $\tilde{Q}_+\tilde{\psi}$ for any ψ, then $\chi=M\tilde{\chi}$ satisfies $Q_+\chi=0$ but cannot be written as $Q_+\psi$.

In short, the mapping $\chi\leftrightarrow M^{-1}\chi$ is a one to one mapping from solutions of $Q_+\chi=0$ that cannot be written as $\chi=Q_+\psi$ to solutions of $\tilde{Q}_+\tilde{\chi}=0$ that cannot be written as $\tilde{\chi}=\tilde{Q}_+\tilde{\psi}$. Hence the number of zero-energy states of the system (Q_+, Q_-, H) is the same as for the system $(\tilde{Q}_+, \tilde{Q}_-, \tilde{H})$.

The transformation in eq. (12) from Q_+ to $\tilde{Q}_+$ is achieved by conjugation by the linear operator M. Changes in the parameters of a supersymmetric theory that can be brought about by such a transformation we will refer to as changes that can be brought about by conjugation. One might think that the operation of conjugation is too special to be of broad interest, but this is not so. Many interesting changes in the parameters of a supersymmetric theory can be brought about by conjugation.

A complete listing of the coupling constants in renormalizable supersymmetric theories is the following:

(i) the usual mass terms, scalar self-couplings, and Yukawa interactions, which are derived from the superspace potential;

(ii) abelian and non-abelian gauge interactions;

(iii) θ angles;

(iv) the Fayet-Iliopoulos D term.

We will see that the superspace potential, and so the couplings in group (i), can be changed in an arbitrary way by conjugation. Abelian gauge couplings can be changed by conjugation. Non-abelian gauge couplings can be changed by conjugation in theories in which there is no θ dependence. θ angles themselves cannot be changed by conjugation, except in the uninteresting case of theories in which the physics is θ independent. The D term cannot be changed by conjugation.

Let us now derive the above results in detail. The simplest supersymmetric model is the Wess-Zumino model. It has a single complex scalar field ϕ and a single spinor field ψ. The parameters are the mass m and the coupling g. We will see that m and g can be changed in an arbitrary way by conjugation.

Rather than a Majorana basis of hermitian supersymmetry charges, it is convenient to work with γ^0 eigenstates. Since γ^0 is imaginary in the Majorana basis, the $\gamma^0=+1$ supersymmetry charges are a complex doublet Q_α, the $\gamma^0=-1$ charges being the complex conjugates Q^*_β. The change of parameters under conjugation is brought about in the following simple way. Q_α for parameters (m_1, g_1) is related to Q_α for parameters (m_2, g_2) by the simple rule

$$Q_\alpha(m_2, g_2) = M^{-1}(m_1, g_1; m_2, g_2) Q_\alpha(m_1, g_1) M(m_1, g_1; m_2 g_2), \tag{13}$$

where

$$M(m_1, g_1; m_2, g_2) = \exp\left(2\,\mathrm{Re}\int \mathrm{d}^3x \left((m_2 - m_1)\tfrac{1}{2}\phi^2 + (g_2 - g_1)\tfrac{1}{3}\phi^3\right)\right). \tag{14}$$

M is not unitary, and the hermitian conjugate of Q_α changes oppositely under a change in parameters, $Q^*_\beta \to M Q^*_\beta M^{-1}$.

In the construction above, leading to the characterization (10) of the total number of zero-energy states, we may take Q to be any one of the Q_α. The fact that under a change in parameters Q_α changes by conjugation means that if supersymmetry is unbroken in this theory for one value of m and g, it is unbroken for all values of m and g.

To verify eqs. (13) and (14) is a straightforward matter of examining the standard definitions of Q_α and using the canonical commutation relations. The only terms in Q_α that do not commute with M are the terms involving the time derivatives $\dot\phi$ or $\dot\phi^*$ of the Bose fields. These terms are

$$Q_\alpha^{\mathrm{N.C.}} = \int \mathrm{d}^3x \left(\dot\phi^* \psi_\alpha + \dot\phi \varepsilon_{\alpha\beta}\psi^{*\beta}\right), \tag{15}$$

where ψ_α are the spinor components with $\gamma^0 = +1$; ψ_α and $\varepsilon_{\alpha\beta}\psi^{*\beta}$ both transform as spinors under rotations. With this form for the part of Q_α that does not commute with M, it is easy to see that under conjugation by M the change in Q_α is

$$\begin{aligned} Q_\alpha \to Q_\alpha + \int \mathrm{d}^3x \Big[&\left[(m_2 - m_1)\phi^* + (g_2 - g_1)\phi^{*2}\right]\psi_\alpha \\ &+ \left[(m_2 - m_1)\phi + (g_2 - g_1)\phi^2\right]\varepsilon_{\alpha\beta}\psi^{*\beta}\Big]. \end{aligned} \tag{16}$$

But a comparison with the standard formula for Q_α shows that the last term on the right-hand side is precisely the change in Q_α when (m_1, g_1) is replaced by (m_2, g_2).

This construction extends straightforwardly to arbitrary renormalizable theories of spin 0 and spin $\frac{1}{2}$ fields only. In such theories, the mass terms and the scalar and Yukawa interactions are all derived from a single function, the superspace potential W. [The scalar potential is $V = \Sigma_i |\partial W/\partial\phi_i|^2$; the Yukawa interaction is $L_{\mathrm{Yuk}} = (\partial^2 W/\partial\phi^i\,\partial\phi^j)\psi^i_{\mathrm{L}}\psi^j_{\mathrm{L}} + \mathrm{h.c.})$.] W can be changed arbitrarily by conjugation. A change from one superspace potential W_1 to another potential W_2 can be achieved by conjugation, $Q_\alpha \to M^{-1} Q_\alpha M$, where

$$M = \exp\left(2\,\mathrm{Re}\int \mathrm{d}^3x\,\left(W_2(\phi_i(x)) - W_1(\phi_i(x))\right)\right). \tag{17}$$

The verification is as before. This seemingly indicates that if supersymmetry is unbroken for one value of W it is unbroken for any value of W. That conclusion, however, is subject to a crucial restriction explained below.

As in sect. 2, one may first of all worry whether ultraviolet divergences could invalidate the above discussion. In contrast to $\mathrm{Tr}(-1)^F$, which really could not be affected by the high-energy behavior, the argument in this section must be checked carefully for its compatibility with renormalization theory. The renormalization of the operators M of eqs. (14) and (17) is quite complicated.

In theories of spin 0 and spin $\frac{1}{2}$ fields only, there is a simple way to avoid this difficulty. It is possible to regularize these theories in a supersymmetrically invariant way while preserving the fact that the coupling constants can be changed by conjugation. This can be done by a simple point-splitting method. In the interaction terms one replaces all superfields $\Phi_i(x)$ by "smeared fields" $\tilde{\Phi}_i(x, t) = \int \mathrm{d}^3y\, G(x, y)\Phi_i(y, t)$, where $G(x, y)$ is a suitable kernel. The superspace interaction, usually written as $\int \mathrm{d}^4x\, W(\Phi_i(x))$, is now written as $\int \mathrm{d}^4x\, W(\tilde{\Phi}_i(x, t))$. This eliminates all ultraviolet divergences, and W can still be changed, in essentially the same way, by conjugation. At the end of the analysis one removes the regulator, $G(x, y) \to \delta^3(x - y)$. Note that the smearing of Φ is carried out in the spatial directions only, to preserve the canonical, hamiltonian framework that is crucial for our analysis.

When gauge fields are introduced, such a simple point-splitting method is not acceptable because it conflicts with gauge invariance. While I hope that the statements below about conjugation in theories with gauge fields survive the process of renormalization, I have no proof of this.

As in the last section, a more serious problem concerns the behavior for large fields. We must check that the operator M is a well-defined operator in the Hilbert space of our theory. Acting on an energy eigenstate it must give a normalizable, finite energy state.

This imposes a crucial restriction on the change ΔW in the superspace potential that can be achieved by an allowed process of conjugation. Recall that $M = \exp(2 \int \mathrm{d}^3x\, \Delta W(\Phi(x)))$. If ΔW increases too rapidly for large ϕ, then M, acting on an energy eigenstate, will produce a nonsensical, unnormalizable state that diverges exponentially for large ϕ.

It is not difficult to see that the necessary requirement is that the change in W under a conjugation operation must grow no more rapidly than W itself for large fields.

To understand this, let us go back to the case of the Wess-Zumino model – a single complex field ϕ and fermion ψ. If W is purely quadratic, we are dealing with free field theory. The potential energy is $m^2 \int d^3x |\phi|^2$. It is useful to adopt a Schrödinger viewpoint in which the states are regarded as functionals of the fields. As in the case of the simple harmonic oscillator, in free field theory the wave functions of the energy eigenstates behave for large fields as a gaussian, $\psi \sim \exp(-m \int d^3x |\phi|^2)$. We cannot now introduce a cubic term in W by conjugation because the required M would behave for large ϕ as the exponential of $\int d^3x\, \phi^3$. Acting on one of the free field theory eigenstates this overwhelms the gaussian large-ϕ behavior and produces an unnormalizable state.

If, however, a cubic term is already present in W, the potential energy contains a quartic term $g^2 \int d^3x |\phi|^4$. We must now determine the asymptotic behavior of the wave functions in the interacting theory. The asymptotic behavior of the wave functions is now better because the potential is larger for large ϕ. The problem of determining this asymptotic behavior is not as formidable as it sounds, for the following reason. As the field becomes large the energy grows and the wave function decays sharply. To minimize the rate of decay of the wave function, the field should become large in such a way that the energy grows no more rapidly than is inevitable. While the potential energy necessarily increases when the field becomes strong, the kinetic energy can remain small if the field is strong but constant in space. The slowest decrease of the wave function thus corresponds to the field being large but constant. The rate of decrease for large fields can be determined by studying a quantum mechanics problem with a single degree of freedom (representing the constant mode of the field) and a $|\phi|^4$ potential. The wave function decays for large fields as $\Psi \sim \exp(-2g \int d^3x |\phi|^3)$.

Now suppose that we try to change g into $g + \Delta g$ by conjugation. Since $M = \exp(2\,\Delta g \operatorname{Re} \int d^3x\, \phi^3)$, M acting on Ψ gives a wave function that is still normalizable as long as Δg is smaller than g. By repeated conjugation operations g can be made arbitrarily big (or arbitrarily small, but not zero) if it is not zero to begin with.

As stated before, the generalization of this result to problems with several fields ϕ^i is that W may be changed by conjugation by any amount that increases, for large fields, no faster than W itself.

Now let us turn our attention to theories with gauge fields. We wish to determine under what conditions gauge couplings can be changed by conjugation. It is convenient to scale the fields so that the gauge coupling appears only as an overall constant multiplying the kinetic energy of the gauge field A_μ^a and its fermion partner λ_α^a,

$$\mathcal{L} = \frac{1}{e^2} \int d^4x \left(-\tfrac{1}{4} \left(F_{\mu\nu}^a \right)^2 + \bar{\lambda}^a i \not{D} \lambda^a \right). \tag{18}$$

Although additional charged fields may be present, their presence is not crucial because, with the normalization of A_μ^a and λ_α^a indicated in (18), the terms in the lagrangian containing the additional fields do not depend on the gauge coupling. To study the question of whether gauge couplings can be changed by conjugation, it is enough to study the minimal supersymmetric gauge theory (18).

The standard supersymmetry current in this theory is

$$S_\mu = \frac{1}{2e^2}\sigma_{\alpha\beta}F^{\alpha\beta^a}\gamma_\mu\lambda^a = \frac{1}{e^2}\sigma_{0i}(E_i^a + iB_i^a\gamma_5)\gamma_\mu\lambda^a, \tag{19}$$

where $E_i^a = F_{0i}^a$ and $B_i^a = \frac{1}{2}\varepsilon_{ijk}F_{jk}^a$ are the electric and magnetic fields.

In contrast to the use above of γ^0 eigenstates, we will find it convenient here to work with supersymmetry charges of definite chirality. By Q_α, $\alpha = 1,2$, we will here mean the positive chirality supersymmetry charges; the hermitian conjugates of the Q_α have negative chirality. Q_α is, of course, the integral of the positive chirality part of $S_{0\alpha}$, or, explicitly,

$$Q_\alpha = \frac{i}{e^2}\int d^3x\left(\gamma^i(E_i^a(x) + iB_i^a(x))\lambda_L^a\right)_\alpha, \tag{20}$$

where λ_L is the negative chirality part of the spinor field, $\gamma_5\lambda_L = -\lambda_L$.

Going back to the lagrangian (18), we see that, in a canonical gauge such as $A_0 = 0$, the canonical momentum conjugate to the gauge field A_i^a is $\pi_i^a = (1/e^2)E_i^a$. From a canonical point of view it is more appropriate to write Q_α in the form

$$Q_\alpha = i\int d^3x\left(\gamma^i\left(\pi_i^a(x) + \frac{i}{e^2}B_i^a(x)\right)\lambda_L^a\right)_\alpha. \tag{21}$$

In this form it is rather obvious that, if we can find an operator K with the properties

$$\begin{aligned}
&[\pi_i^a(x), K] = -iB_i^a(x),\\
&[B_i^a(x), K] = 0,\\
&[\lambda_\alpha^a(x), K] = 0,
\end{aligned} \tag{22}$$

then we can bring about a change in the gauge coupling by conjugation,

$$Q_\alpha(e_2) = (\exp(-tK))Q_\alpha(e_1)\exp tK, \tag{23}$$

with $t = (1/e_2^2 - 1/e_1^2)$.

The remarkable fact is that such an operator K exists; in fact

$$K = \tfrac{1}{2}\int d^3x\,\varepsilon_{ijk}\left(A_i^a\partial_j A_k^a - \tfrac{2}{3}f^{abc}A_i^aA_j^bA_k^c\right). \tag{24}$$

This operator has played a role in physics before; it is the operator, famous in instanton studies, that measures the "winding number" of the gauge field. Why the winding number should enter in this particular problem is not clear, but it is easy to check that K satisfies (22) and (23).

At first sight this seems to prove that gauge couplings can be changed by conjugation, but a number of points must be checked. Apart from the question of regularizing K, we must come to grips with the fact that the integrand on the right-hand side of (24) is not gauge invariant, and we must make sure that $\exp tK$ maps energy eigenstates into well-defined normalizable states.

In an abelian gauge theory, although the integrand in (24) is not gauge invariant, K itself is completely well-defined and gauge invariant. Under a gauge transformation, $A_i \to A_i + \partial_i \alpha$, the change in K is $\int d^3x\, \varepsilon_{ijk}\, \partial_i \alpha\, \partial_j A_k$, which vanishes after integration by parts (there is no surface term because of the use of periodic boundary conditions). Moreover, precisely because K is gauge invariant, K cannot in an abelian theory be extremely large without the energy being large. The wave function of an energy eigenstate therefore vanishes very rapidly in the large-K region, and $\exp tK$ is a well-defined operator acting on the energy eigenstates.

Assuming that the problem of finding a suitable regularization (or otherwise dealing with the ultraviolet divergences) can be settled, it follows that abelian gauge couplings can be changed by conjugation.

For non-abelian gauge couplings, the problem is more difficult. K is still invariant under "small" gauge transformations but it is not invariant under "large" gauge transformations that change the winding number and shift K by a constant.

The fact that K is not gauge invariant is a clear warning that conjugation by $\exp tK$ is likely to lead to trouble. Perhaps it is useful to compare $\exp tK$ to the unitary operator $\exp i\alpha K$ that has appeared in instanton studies. One would naively expect that conjugation of the hamiltonian by this unitary operator would have no physical effect, but actually the operation $H \to (\exp -(i\alpha K))H(\exp i\alpha K)$ brings about a shift $\theta \to \theta + \alpha$ in the vacuum angle θ, and this shift, in general, has physical effects. The operation $\exp tK$ is an attempt to shift θ by an imaginary amount, $\theta \to \theta - it$, and this sounds particularly dangerous.

The specific reason that acting with $\exp tK$ gives trouble in non-abelian theories is that in those theories K can be arbitrarily large while the energy remains small (since even if $F^a_{\mu\nu} = 0$, the winding number can be arbitrarily big). Because of tunneling effects, every eigenstate of the hamiltonian has an admixture of configurations with large K. In fact, the wave functions are undamped for large K because the energy does not increase as K becomes large. Acting on a state that is undamped for large K, the operator $\exp tK$ gives an exponentially divergent wave function.

Because of this, non-abelian gauge couplings cannot, in general, be changed by conjugation. However, the above argument indicates the existence of an important class of theories in which the conjugation by $\exp tK$ does make sense and can be used to change the non-abelian gauge coupling. These are theories in which an axial vector anomaly of some kind suppresses tunneling and prevents the wave functions from extending to large K.

For instance, the minimal supersymmetric gauge theory of eq. (18) has a chiral current $J_{\mu 5}=\bar{\lambda}\gamma_\mu\gamma_5\lambda$ which naively is conserved, but actually suffers from an anomaly; $\partial_\mu J^5_\mu$ does not really vanish, but is a multiple of $F_{\mu\nu}\tilde{F}^{\mu\nu}$. More elaborate theories with additional fields may or may not possess such an anomalous axial current, perhaps of a more complicated form. When such an anomalous current exists, the physics is independent of θ.

More relevant for our purposes, when there is an axial current whose conservation is spoiled only by an anomaly, the tunneling is suppressed, and properly defined eigenstates of the hamiltonian vanish very rapidly for large K. In this situation, the operator $\exp tK$ can meaningfully act on eigenstates of the hamiltonian and can change the non-abelian coupling by conjugation.

There is another way to see that the conjugation operation makes sense in theories with an anomalous axial current. Our problem, at one level, was that K was not gauge invariant. Because of the anomaly, $Q_5=\int \mathrm{d}^3x\, J_{05}$ does not commute with the hamiltonian. Rather, the hamiltonian commutes with $K-cQ_5$, where c is a constant depending on the theory. Because $[H, K-cQ_5]=0$, instead of conjugation by $\exp tK$, we may achieve the same result by conjugation by $\exp tcQ_5$. This operator is gauge invariant.

We arrive at the peculiar conclusion that non-abelian gauge couplings can be changed by conjugation in theories in which the physics is independent of θ, and only in those theories.

To complete the list of all possible interaction terms in renormalizable, supersymmetric theories, we still must consider the θ angles themselves and the Fayet-Iliopoulos D term.

It has already been mentioned that, naively, θ can be changed by conjugation by the unitary operator $\exp i\alpha K$. One would naively conclude from this that not just supersymmetry breaking, but all other physical observables, are independent of the value of θ. These naive conclusions are wrong, for reasons described in the literature on instantons. The only case in which θ can be changed by conjugation is the uninteresting case in which the physics is actually independent of θ. If the hamiltonian commutes with some operator $K-\lambda Q_5$, then θ can be changed by conjugation with the gauge invariant, genuinely unitary operator $\exp i\alpha\lambda Q_5$. In the more interesting case in which the physics depends on θ, it is perfectly possible that supersymmetry is unbroken for one value of θ and broken for the other values.

We are left with the Fayet-Iliopoulos D term. It cannot be changed by conjugation. The easiest way to realize this is to note that in the literature, there are many examples of theories in which, in perturbation theory, supersymmetry is unbroken for one value of the coefficient of the D term (usually zero) but broken for other values.

We have not explicitly discussed the question of whether a change in the volume of our finite volume theory can be brought about by conjugation. Actually, by the renormalization group, a change in the volume is equivalent to a change in the masses and coupling constants. As long as the theory is such that all masses and couplings can be changed by conjugation, a change in the volume is equivalent to a conjugation operation plus a renormalization group transformation. In this case, if supersymmetry is unbroken for some values of the parameters and of the volume, it is unbroken for all values of the parameters and of the volume, and hence also in the infinite volume limit.

Since the arguments in this section have been somewhat abstract, it may be helpful to give a tangible example.

In ref. [2], the supersymmetric problem of a particle moving in a line in one dimension was considered. The supersymmetry charges were

$$Q_1 = \tfrac{1}{2}(\sigma_1 p + \sigma_2 W(x)),$$
$$Q_2 = \tfrac{1}{2}(\sigma_2 p - \sigma_1 W(x)), \tag{25}$$

where W is an arbitrary function and $p = -i\,\mathrm{d}/\mathrm{d}x$. The hamiltonian was

$$H = \tfrac{1}{2}\left(p^2 + W^2 + \sigma_3 \frac{\mathrm{d}W}{\mathrm{d}x}\right). \tag{26}$$

We have then

$$Q_+ = \sqrt{\tfrac{1}{2}}\,(Q_1 + iQ_2) = \frac{-i}{2\sqrt{2}}(\sigma_1 + i\sigma_2)\left(\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\right). \tag{27}$$

We see immediately that W can be changed by conjugation, the relation between $Q_+(W)$ and $Q_+(\tilde{W})$ being

$$Q_+(\tilde{W}) = (\exp(-F(x)))Q_+(W)(\exp F(x)), \tag{28}$$

where $F(x)$ is a function that satisfies $\mathrm{d}F/\mathrm{d}x = \tilde{W}(x) - W(x)$.

Now, an interesting special case is $W(x) = x^2 + a^2$. For $a^2 > 0$ supersymmetry is spontaneously broken at the tree level since the classical potential energy $V(x) = W^2(x)$ is a strictly positive function. For $a^2 < 0$ supersymmetry is unbroken at the tree level and in perturbation theory, but it was shown in ref. [2] that dynamical supersymmetry breaking occurs.

The occurrence of dynamical supersymmetry breaking is related to the fact that the sign of a^2 can be changed by conjugation. In fact, one can easily see that

$$Q_+(-a^2) = \exp(2a^2x)Q_+(a^2)\exp(-2a^2x). \tag{29}$$

The total number of zero-energy states must therefore be independent of the sign of a^2.

Since for positive a^2 it is obvious that there are no zero-energy states, the number of zero-energy states must also be zero for negative a^2, even though in perturbation theory there appear to be two zero-energy states if a^2 is negative. Eq. (29) shows, without need for the explicit calculations of ref. [2], that dynamical supersymmetry breaking occurs if a^2 is negative.

Actually, the validity of this argument depends upon the fact that the operator $\exp(\pm 2a^2 x)$, acting on any eigenstate of the hamiltonian (26), gives a normalizable state. In fact, it is easily seen that if $W = x^2 + a^2$, the eigenstates of H behave for large $|x|$ as $\exp(-\frac{1}{3}|x|^3)$. On such states the operator $\exp(\pm 2a^2 x)$ is quite safe. The same reasoning shows, however, that we could not safely go by conjugation from $W = x^2 + a^2$ to, say, $W = x^2 + a^2 + \varepsilon x^3$. The required conjugation operator would be $\exp\frac{1}{4}\varepsilon x^4$, and acting on eigenstates of H that behave for large x as $\exp(-\frac{1}{3}|x|^3)$, this would give exponentially divergent wave functions. The general rule is that one may change W by conjugation as one wishes as long as one does not change its asymptotic behavior at large $|x|$.

In the previous section we learned that the difference $n_B^{E=0} - n_F^{E=0}$ is invariant under arbitrary changes in the parameters of a supersymmetric theory. In this section we have learned that under changes in parameters that can be brought about by conjugation, the sum $n_B^{E=0} + n_F^{E=0}$ is also invariant. This means, in particular, that the two numbers $n_B^{E=0}$ and $n_F^{E=0}$ are separately constants.

In practice, the new result is not as much of an improvement as it might appear. The reason for this is that, as we will see in detail, the difference $n_B^{E=0} - n_F^{E=0}$ can be calculated reliably for weak coupling in almost all theories. But it is often difficult to determine the sum $n_B^{E=0} + n_F^{E=0}$ even for very weak coupling. If one cannot evaluate the sum, the fact that it is known to be independent of the coupling constant is of little use.

There are, however, various situations, some of which will be considered later in this paper, in which knowing that the total number of zero-energy states is independent of the coupling can lead to important constraints on supersymmetry breaking.

4. Analyticity

In this section one additional general constraint on supersymmetry breaking will be considered. It is far more elementary than the constraints discussed in sects. 2, 3.

Rather than field theory, let us first consider supersymmetric quantum mechanics problems with a finite number of degrees of freedom. As long as the number of degrees of freedom is finite, the energy eigenvalues of any quantum mechanical system are analytic functions of the parameters appearing in the hamiltonian. Actually, a caveat analogous to those stated in sects. 2, 3 is necessary here. Analyticity breaks down at points in parameter space at which the asymptotic nature of the potential changes. For instance, for the one-dimensional anharmonic oscillator with $V(x) = m^2 x^2 + \lambda x^4$, the energy eigenvalues are analytic functions of m^2 and λ as long as $\lambda > 0$; $\lambda = 0$ is a singular point.

Under conditions in which such analyticity holds, it is true in particular that the ground-state energy is an analytic function of the parameters. Let us now consider supersymmetric theories. Suppose that in some theory, it is known that in some finite range of the parameters supersymmetry is unbroken and the ground-state energy is exactly zero. Since an analytic function that vanishes in a finite range of the parameters vanishes everywhere, it follows that the ground-state energy vanishes identically and that supersymmetry is unbroken for all values of the parameters.

Suppose, conversely, that in some theory it is known that in a finite range of the parameters supersymmetry is spontaneously broken and the ground-state energy is non-zero. It then follows from analyticity that there does not exist a finite range of parameters in which the vacuum energy vanishes identically. In particular, in a theory with just one coupling constant, if supersymmetry is spontaneously broken for some value of the coupling, it is restored at most for isolated values of the coupling, since an analytic function that does not vanish everywhere has at most isolated zeros. In a theory with N coupling constants, if supersymmetry is spontaneously broken at one point in coupling constant space, it is restored at most on a surface of dimension $N-1$ in the space of couplings.

A simple example of this is the quantum mechanics hamiltonian of eq. (26). Again take $W = x^2 + a^2$. The ground-state energy is an analytic function of a^2. Since it is non-vanishing for large, positive a^2, it must not vanish for negative a^2 (except possibly for isolated values of a^2). This simple argument based on analyticity thus anticipates the dynamical supersymmetry breaking found for negative a^2 in ref. [2]. Of course, the fact that the ground-state energy does not vanish even for isolated values of a^2 can be seen from the argument of sect. 3, based on conjugation, or by the explicit calculation of ref. [2].

How much of this carries over to supersymmetric field theory? To begin with, we may consider a supersymmetric field theory in a finite volume V and in the presence of an ultraviolet cut-off Λ. (We will assume that a suitable ultraviolet cut-off exists or that the ultra-violet behavior is not really crucial.) As long as the volume and the cut-off are finite, the total number of degrees of freedom is finite, so analyticity holds, along with its above-mentioned consequences. However, we are interested in the limit $V \to \infty$, $\Lambda \to \infty$. In this limit the number of degrees of freedom becomes infinite and the ground-state energy may cease to be an analytic function of the coupling constant.

Two cases must be distinguished. Suppose that it is known that for any finite V and Λ there is a non-zero range of coupling parameters in which the ground-state energy vanishes exactly. Then, since the ground-state energy is an analytic function of coupling for fixed V and Λ, it must vanish for arbitrary coupling as long as V and Λ are finite. Since the large-V, large-Λ limit of zero is zero, it follows from this that also in the limit the ground-state energy is zero and supersymmetry is unbroken.

In the opposite case, we do not reach such an interesting conclusion. If it is known that, for finite V and Λ, the ground-state energy is non-zero at least for some values of the coupling, it indeed follows that the ground-state energy is non-zero for almost all coupling, as long as V and Λ are finite. But since a non-zero energy may become zero in the large-V, large-Λ limit, and since analyticity may be lost in this limit, supersymmetry may be restored in the limit for any or all values of the coupling.

An example in which analyticity is lost in the large-V limit is described in appendix A.

How might one be able to put these considerations to use? The usual way to try to show that a symmetry is unbroken in some range of the couplings is to study the weak coupling behavior. But in the case of supersymmetry, even for weak coupling, it is hard in general to decide whether the symmetry is broken or unbroken.

A fairly simple and instructive, although exotic, situation in which the considerations of this section could play a role would be the following. Suppose that in a weak coupling calculation, in a finite volume, one finds in some theory eight states of apparently zero energy. Suppose that these are four Bose states of spin zero, and four Fermi states of spin $\frac{3}{2}$. (Of course, "spin" refers to the discrete rotation subgroup of the finite volume theory.) What conclusions can be drawn?

The first step is to ask whether the eight states in question really have exactly zero energy for sufficiently weak coupling. With the spectrum assumed, $\mathrm{Tr}(-1)^F = 0$, so the results of sect. 2 are of no help. However, we may use a line of reasoning analogous to that in sect. 2. For weak enough coupling, the eight states that in perturbation theory appear to have zero energy certainly are much lower in energy than any states whose energy does not vanish in perturbation theory. Therefore, if the supersymmetry charges do not annihilate these eight states, they must at least map them among themselves, for weak enough coupling: supersymmetry could not connect these eight states to others of higher energy.

But the supersymmetry operators transform as operators of spin one half under rotations. They cannot connect states of spin zero to states of spin $\frac{3}{2}$. So under the stated assumptions the supersymmetry charges must annihilate the eight states in question, at least for weak coupling. The ground-state energy must be zero, at least for weak coupling.

It now follows from analyticity that the ground-state energy is zero, and supersymmetry unbroken, even for strong coupling, and also in the large-volume limit.

Note that this argument would hold even if the range of validity of perturbation theory depends on the volume (as in infrared-unstable theories like QCD) as long as perturbation theory has a non-zero range of validity for any finite volume.

The argument just described is an analogue of the "lacunary principle" of Morse theory. We will have more to say on Morse theory in sect. 10.

5. Simple applications

Let us now turn to applications of the ideas developed in sects. 2–4. Most of the applications depend only on the results of sect. 2. In this section we consider the simplest applications.

The simplest supersymmetric theory is the Wess-Zumino model. There is a single complex scalar field ϕ; its supersymmetry partner is a left-handed Weyl spinor ψ. The superspace potential is $W(\phi)=\frac{1}{3}g\phi^3-(m^2/4g)\phi$, and the ordinary potential energy that describes the self-interaction of ϕ is

$$V(\phi,\phi^*)=\left|\frac{\partial W}{\partial\phi}\right|^2=g^2\left|\phi^2-\frac{m^2}{4g^2}\right|^2. \tag{30}$$

In addition, there is a Yukawa coupling

$$I_{\mathrm{Yuk}}=g\phi\psi_{\mathrm{L}}^{\alpha}\psi_{\mathrm{L}}^{\beta}\varepsilon_{\alpha\beta}+\text{h.c.}, \tag{31}$$

although it will not play a crucial role in our analysis. We will assume, temporarily, that $m\neq 0$.

Let us evaluate $\mathrm{Tr}(-1)^F$ in this theory. We will do so by studying the weak coupling behavior, in a finite volume. As we will see, no detailed calculations are necessary.

For weak coupling, perturbation theory is a good approximation. In perturbation theory there are two vacuum states, $\langle\phi\rangle=\pm m/2g$ (they are related by the discrete symmetry $\phi\to-\phi$, $\psi\to i\psi$). Expanding around either minimum of the potential, one finds that ϕ and ψ are massive for weak coupling. In fact

$$m_\phi=m_\psi=m\big(1+\mathrm{O}(g^2)\big). \tag{32}$$

The spectrum of this theory in zeroth order of perturbation theory is very simple. In each minimum of the potential (fig. 5) there is one zero-energy state, the "vacuum". The "vacuum" has spin zero and therefore is bosonic. All other states are obtained by adding ϕ and ψ quanta to the vacuum. Any state containing such quanta has an energy that is strictly positive; in fact, they have $E\gtrsim m$, since the ϕ and ψ mass is approximately m. Hence states containing ϕ and ψ quanta do not contribute to $\mathrm{Tr}(-1)^F$.

Each of the two vacuum states contributes one to $\mathrm{Tr}(-1)^F$. Since there are no other contributions, we find that in the Wess-Zumino model

$$\mathrm{Tr}(-1)^F=2. \tag{33}$$

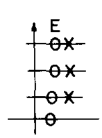

Fig. 5. The typical spectrum in a supersymmetric theory with massive particles only. The only zero-energy state is the spin zero "vacuum".

Because this is not zero, supersymmetry is not spontaneously broken in this model. Since $\mathrm{Tr}(-1)^F$ is independent of g and m, the conclusion is valid for large g as well as small g. And it is valid for $m=0$ even though the derivation assumed $m \neq 0$.

Let us now compare this to what could be learned by other methods.

For $m \neq 0$, we do not need $\mathrm{Tr}(-1)^F$ to show that supersymmetry is not spontaneously broken for small g. When supersymmetry is spontaneously broken, there is always a massless fermion, the Goldstone fermion. In this theory, for $m \neq 0$, the elementary fermion certainly has a non-zero mass if g is small enough. Moreover, for very weak coupling, a massless fermion will certainly not appear as a bound state of the massive elementary quanta. Therefore, for $m \neq 0$ and small enough g, there is no massless fermion that could possibly be a Goldstone fermion, so supersymmetry could not be spontaneously broken in this range of the parameters.

However, one might think that as g increases (for fixed m), the ψ mass might eventually decrease and go to zero. Beyond a critical value of g (fig. 6), the ψ particle might be a massless Goldstone fermion, associated with spontaneous breaking of supersymmetry. Such a phase transition occurs quite readily in the case of internal symmetries (a scalar mass goes to zero as the coupling is increased and the scalar becomes a Goldstone boson). But because $\mathrm{Tr}(-1)^F=2$, this cannot occur here; supersymmetry is unbroken even if g is large. (It may be that for large enough g, ψ is massless. But if so ψ is not a Goldstone fermion; the crucial matrix element $\langle 0|S_\mu|\psi\rangle$ vanishes, and supersymmetry is not spontaneously broken.)

One may also be interested in the $m=0$ theory. In this case, even for small g, there is a massless fermion, the ψ particle. It is possible to show [10] that even for $m=0$, the vacuum energy vanishes to all finite orders of perturbation theory. However, how do we know that, non-perturbatively, there is not a tiny, non-zero vacuum energy (and a tiny, non-zero matrix element $\langle 0|S_\mu|\psi\rangle$, making ψ a Goldstone fermion)? The fact that $\mathrm{Tr}(-1)^F \neq 0$ shows that this does not occur.

(We could have attempted to derive this result by analyticity. One is tempted to say that since the vacuum energy obviously vanishes for $m \neq 0$ and sufficiently small g, it must, by analyticity, vanish for all m and g. However, a careful attempt to think this argument through leads back to the concept of $\mathrm{Tr}(-1)^F$. Since analyticity only holds in a finite volume, we must know that the ground-state energy is exactly zero in a finite volume for $m \neq 0$ and small enough g in order to justify inferences about supersymmetry breaking that are based on analyticity. But in a finite volume the

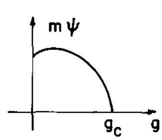

Fig. 6. What cannot happen: the fermion of the Wess-Zumino model becomes a Goldstone fermion above a critical coupling.

concept of a Goldstone fermion is ill-defined. The fact that the infinite volume theory, for $m \neq 0$ and small g, does not have a massless fermion does not prove that the ground-state energy vanishes exactly in a finite volume; in fact, a counter-example is given in appendix A. To prove that, in a suitable range of parameters, the ground-state energy of the finite volume theory is exactly zero, one needs the concept of $\mathrm{Tr}(-1)^F$, or perhaps some other non-perturbative argument that might be discovered.)

The above argument is clearly not limited to the Wess-Zumino model. Consider any supersymmetric theory in which at the tree level supersymmetry is unbroken, and in which all particles have non-zero masses at the tree level. Because the masses are non-zero, all states other than the "vacuum" states have energy greater than zero (any state but the vacuum has energy at least equal to the mass of the lightest particle). Therefore, $\mathrm{Tr}(-1)^F$ receives contributions only from the zero-energy "vacuum" states, all of which have spin zero. There are some such states, because we assumed that supersymmetry was not spontaneously broken at the tree level. So in these theories, $\mathrm{Tr}(-1)^F$ is greater than zero, and supersymmetry is not spontaneously broken, even if the coupling is strong.

The same argument applies even if there are massless particles at the tree level, as long as those massless particles could be given masses by changing the parameters in the lagrangian. (As discussed in sects. 2–4, the change in parameters must be one that does not affect the asymptotic behavior of the potential for large fields.) As long as all massless particles *could* have had mass, $\mathrm{Tr}(-1)^F$ is positive, since it is independent of the parameters and is positive if all particles are massive.

The only theories in which it may be tricky to calculate $\mathrm{Tr}(-1)^F$ are theories with massless particles that are massless (in perturbation theory) for all values of the parameters. The most important theories of this class are, of course, gauge theories; gauge theories always have massless particles in perturbation theory, unless complete breakdown of the gauge symmetry (by the Higgs mechanism) occurs at the tree level. Much of the remainder of this paper is devoted to calculating $\mathrm{Tr}(-1)^F$ in various gauge theories.

The reason that massless particles cause trouble in calculating $\mathrm{Tr}(-1)^F$ is the following. In a finite volume, a normalizable state can be obtained by adding to the "vacuum" a massless particle in a momentum eigenstate with $\boldsymbol{P}=0$. In fact, several massless particles can be added in this way. Such a state will have very low energy, and may have exactly zero energy in finite volume. Such states may be bosonic or fermionic, depending on what massless quanta are added to the vacuum. It is difficult to count these states, because an arbitrary number of $\boldsymbol{P}=0$ bosons may be added to the vacuum. And it is difficult to determine how many of them have exactly zero energy. This is why, in theories that "unavoidably" have massless particles, it is frequently difficult to evaluate $\mathrm{Tr}(-1)^F$.

There is one other situation, however, in which it is easy to calculate $\mathrm{Tr}(-1)^F$. If supersymmetry is spontaneously broken at the tree level, then, in perturbation theory, there are no states of zero energy (neither bosonic nor fermionic states), so $\mathrm{Tr}(-1)^F = 0$. As explained in sect. 2, the knowledge that $\mathrm{Tr}(-1)^F = 0$ does not lead to a definite prediction about whether supersymmetry is restored for some range of the parameters. We will have nothing further to say in this paper about theories in which supersymmetry is spontaneously broken at the tree level.

6. Abelian gauge theories

In this section we will begin to come to grips with the problem of calculating $\mathrm{Tr}(-1)^F$ in theories which have massless particles because of unbroken gauge symmetries. Such massless particles make the evaluation of $\mathrm{Tr}(-1)^F$ far more difficult, for reasons explained in the last section.

We will focus here on supersymmetric QED, that is, supersymmetric theories with an unbroken U(1) gauge symmetry. However, we will limit ourselves to theories in which the charged fields of given chirality form a real representation of U(1); in other words, we will study theories in which, as in ordinary QED, the photon has vector (not axial vector) couplings to the charged fermions. This means that gauge invariant bare mass terms are possible for all charged fields. In calculating $\mathrm{Tr}(-1)^F$, we will assume such bare masses to be present; however, as $\mathrm{Tr}(-1)^F$ is independent of the bare masses, the restrictions on supersymmetry breaking derived below are valid also for the case of zero bare mass.

Charged fields in a complex representation of U(1) would make the determination of $\mathrm{Tr}(-1)^F$ more difficult. Such fields would necessarily be massless, as long as U(1) is unbroken, and massless charged fields that cannot be given bare masses would make our problem more complicated.

We will find the following generalization of the concept of $\mathrm{Tr}(-1)^F$ to be useful. Let X be any operator that commutes with the supersymmetry charges, $[X, Q_\alpha] = 0$. (Of course, X then also commutes with $H = \Sigma_\alpha Q_\alpha^2$.) Instead of calculating the trace of $(-1)^F$ in the entire Hilbert space, we may calculate the trace in the subspace of states on which X has a prescribed eigenvalue λ. In other words, letting P_λ be the projection onto the subspace with $X = \lambda$, we may calculate $\mathrm{Tr}(-1)^F P_\lambda$. The arguments of sect. 2 showing that $\mathrm{Tr}(-1)^F$ is independent of the parameters in a supersymmetric theory immediately carry over to show that, for any λ, $\mathrm{Tr}(-1)^F P_\lambda$ is independent of the parameters (but now one is restricted to considering theories in which X is conserved, along with the Q_α). Moreover, if $\mathrm{Tr}(-1)^F P_\lambda$ is non-zero for any λ, supersymmetry is definitely unbroken – for all values of the parameters.

Equivalently, introducing an arbitrary function $f(X)$ of the operator X, we may consider

$$\mathrm{Tr}(-1)^F f(X) = \sum_\lambda f(\lambda) \mathrm{Tr}(-1)^F P_\lambda . \tag{34}$$

If $\mathrm{Tr}(-1)^F f(X)$ is non-zero for some choice of $f(X)$, then supersymmetry is unbroken*.

We will find it useful to take X to be the charge conjugation operator C. In the vector-like theories we will consider the gauge interactions are invariant under an operator C that exchanges positively charged fields of given chirality with negatively charged fields of the same chirality. Under C, the gauge field A_μ and its fermionic partner ψ_α change sign. While the gauge couplings conserve C in vector-like QED, non-minimal interactions that might be added may or may not conserve C. As long as C is conserved we can use C invariance to obtain interesting information.

Because $C^2 = 1$, there are only two independent invariants to consider. They are $\mathrm{Tr}(-1)^F$ and $\mathrm{Tr}(-1)^F C$. We will find that in vector-like supersymmetric QED, $\mathrm{Tr}(-1)^F = 0$. This result, by itself, sheds no light on the possibility of dynamical supersymmetry breaking. However, we will find that in C-conserving vector-like theories, $\mathrm{Tr}(-1)^F C = 4$. The non-zero value means that in these theories, dynamical supersymmetry breaking does not occur, even for strong coupling.

Let us now attempt to derive these results. Actually, to evaluate $\mathrm{Tr}(-1)^F$ there is a very simple method, which depends on the existence of the Fayet-Iliopoulos D term. Leaving aside possible non-minimal interactions, the classical potential energy for a U(1) theory with charged scalar fields C_i of bare masses m_i and charges e_i is

$$V(C_i) = \sum m_i^2 |C_i|^2 + \left(\sum e_i |C_i|^2\right)^2. \tag{35}$$

The minimum of the potential is at $C_i = 0$. Obviously, the potential vanishes at this point, so supersymmetry is unbroken at the tree level. Whether supersymmetry is unbroken in the full quantum mechanical theory is precisely what we wish to investigate.

The theory just described can be generalized by including the D term. The modified potential is

$$V(C_i) = \sum m_i^2 |C_i|^2 + \left(\sum e_i |C_i^2| - d^2\right)^2, \tag{36}$$

where d^2 is a constant. As is easily seen, if $d^2 \neq 0$, (36) does not vanish for any value of the C_i. Thus, supersymmetry is spontaneously broken at the tree level when the D term is included (as long as all charged fields were massive). This means that $\mathrm{Tr}(-1)^F = 0$ with the D term, and therefore also without it.

The same reasoning cannot be used to calculate $\mathrm{Tr}(-1)^F C$. The reason for this is that the D term is not invariant under charge conjugation. To understand this, we must clarify how charge conjugation is defined in supersymmetric theories. The

* It does not matter, in this construction, whether X is spontaneously broken in the infinite volume limit. As long as the operator X is well-defined, and commutes with the Q_α, in the finite volume theory, $\mathrm{Tr}(-1)^F f(X)$ can be defined and gives constraints on supersymmetry breaking.

complex fields C_i in (35) and (36) are the supersymmetric partners of the left-handed Fermi fields ψ_{iL}; their complex conjugates C^{*j} are the partners of the right-handed Fermi fields $\psi_R^j = (\psi_{jL})^*$. The operation $C_i \leftrightarrow C^{*i}$ is a symmetry of (35) and (36); it actually is a CP transformation since, because of the Yukawa couplings [which are required by supersymmetry but not indicated explicitly in (35) and (36)], the left- and right-handed fermions must be exchanged at the same time.

Charge conjugation, as opposed to CP, is defined as follows. By our hypothesis, the partners C_i of the left-handed fermions form a real representation of U(1). Otherwise, the mass terms in (35) and (36) would be incompatible with gauge invariance and supersymmetry. To say that the C_i form a real representation means simply that for every C_i of charge e_i there is a C_j of charge $-e_i$. More specifically, because of gauge invariance and supersymmetry, for every C_i of charge e_i and mass m_i, there is a C_j of charge $-e_i$ and the same mass m_i. Charge conjugation is the exchange $C_i \leftrightarrow C_j$. In (35) and (36) this has the effect $e_i \leftrightarrow -e_i$, which evidently leaves (35) invariant but not (36). Thus, the D term is not invariant under charge conjugation, and the introduction of the D term cannot be used as a tool in calculating $\mathrm{Tr}(-1)^F C$.

We must therefore consider other methods. The key to a general approach to calculating $\mathrm{Tr}(-1)^F$ and $\mathrm{Tr}(-1)^F C$ is to realize that if all charged fields are massive, the charged fields can be ignored entirely. Every state involving excitations of the charged fields has energy at least equal to the mass of the lightest charged field. Very low energy states are excitations of the A_μ and its fermionic partner ψ only*. The fact that the charged fields can be ignored means that $\mathrm{Tr}(-1)^F$ and $\mathrm{Tr}(-1)^F C$ have the same values that they have in free field theory. Of course, in free field theory everything can be evaluated explicitly.

For reasons explained in sect. 5, the only thing that makes it slightly delicate to calculate $\mathrm{Tr}(-1)^F$ and $\mathrm{Tr}(-1)^F C$ is the masslessness of A_μ and ψ. The zero momentum modes of the massless particles are normalizable in a finite volume, and they carry zero energy. So they must be taken into account in evaluating $\mathrm{Tr}(-1)^F$ or $\mathrm{Tr}(-1)^F C$.

Actually, in the infinite volume theory, the zero-momentum mode of the gauge field can simply be gauged away. It is not quite as simple in the finite volume case, for reasons that we will discuss. However, we will see that in abelian gauge theories, the zero-momentum mode of the gauge field makes no contribution.

Ignoring the zero-momentum mode of the gauge field for a moment, let us evaluate the contribution of the zero-momentum mode of the fermions. Because of

* We assume that neutral fields other than A_μ and ψ, if present in the theory, are massive and have vacuum expectation values that are uniquely determined, at the tree level, by the classical potential. The statements below require modification otherwise. Massless fields that cannot be given masses without changing the asymptotic behavior of the potential would as always cause complications. If the potential has several minimima, the contribution of each minimum to $\mathrm{Tr}(-1)^F$ and $\mathrm{Tr}(-1)^F C$ must be included.

Fermi statistics, only a finite number of zero-momentum fermions can fit in the box. This gives a finite number of states, which can be straightforwardly counted. As sketched in fig. 7, our finite volume box may contain no zero-momentum fermions, a single zero-momentum fermion, which may have spin up or spin down, or two zero-momentum fermions, one of spin up and one of spin down. These four states have zero energy at least in perturbation theory (the zero-momentum fermions carry zero energy, and the zero-point energy of the other modes cancels, in the usual way, between bosons and fermions). Of the four states, two are bosonic (the states with zero or two fermions present) and two are fermionic (the states with one fermion present). So $\mathrm{Tr}(-1)^F = 2-2=0$, in agreement with our previous results.

Turning now to $\mathrm{Tr}(-1)^F C$, we must determine how the four states in fig. 7 transform under C. Since there is no other physical principle that fixes the overall sign of the operator C, we may define the zero fermion state to have $C=+1$. Then, since the elementary Fermi field ψ is odd under C (like its supersymmetric partner, the gauge field), the states containing one fermion have $C=-1$, and the state with two fermions has $C=+1$. Thus, of the four states in fig. 7, the bosonic states have $C=+1$ and the fermionic states have $C=-1$. This means that $\mathrm{Tr}(-1)^F C = 4$.

To justify these statements, we must show that the zero-momentum mode of the gauge field can in fact be ignored. In the infinite volume theory, the zero-momentum mode can simply be gauged away. Under a gauge transformation, we have $A_\mu \to A_\mu + \partial_\mu \varepsilon$. If we choose

$$\varepsilon = c_\mu x^\mu , \tag{37}$$

where c_μ is a constant, then $A_\mu \to A_\mu + c_\mu$ and this can be used to eliminate the zero-momentum component of the field.

In a finite volume, there is no difficulty in using (37) to eliminate the zero-momentum component of A_0. In fact, A_0 can be eliminated altogether by a gauge transformation, and we will do so henceforth. However, we must be careful in trying to eliminate the zero-momentum components of A by means of (37), because $\varepsilon = \boldsymbol{c}\cdot\boldsymbol{x}$ is not periodic.

In the absence of charged fields, this does not matter. Even though ε is not periodic, the operation $A_i \to A_i + \partial_i \varepsilon$ preserves the periodicity of A_i, and this is the only requirement a gauge transformation must satisfy. Thus, in free field theory, one can eliminate the zero-momentum mode of A_i by means of a gauge transformation and forget about it. $\mathrm{Tr}(-1)^F$ and $\mathrm{Tr}(-1)^F C$ can then be evaluated in terms of the fermions only, as we have already done.

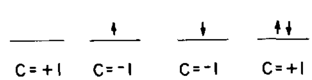

Fig. 7. The spectrum of zero-energy states in vector-like supersymmetric QED.

Technically, "eliminating" the zero-momentum mode of the gauge field means the following. As the operator that implements a gauge transformation by $\varepsilon = \boldsymbol{c} \cdot \boldsymbol{x}$ commutes with the hamiltonian, we can require the states to be annihilated by this operator. In this restricted Hilbert space the zero-momentum mode of A_μ is absent, and our previous determination of $\mathrm{Tr}(-1)^F$ and $\mathrm{Tr}(-1)^F C$ is valid.

When charged fields are present, we must be more careful. Under the gauge transformation discussed above, a field N of charge e transforms as

$$N \to \exp(ie\boldsymbol{c} \cdot \boldsymbol{x})N. \tag{38}$$

We will assume that all electric charges are integral multiples of some basic charge* e. In that case, all fields remain periodic under the action of our gauge transformation if and only if

$$c_i = \frac{2\pi}{eL} n_i, \tag{39}$$

where L is the length of the box and the n_i are arbitrary integers.

Because of the restriction (39), we cannot simply eliminate the zero-momentum mode, but we can shift it by a constant. Defining the zero-momentum mode as

$$h_i = \frac{1}{V} \int \mathrm{d}^3 x \, A_i, \tag{40}$$

we can shift $h_i \to h_i + (2\pi/eL)n_i$ by a gauge transformation. The h_i are periodic variables, the period being $2\pi/eL$.

We must now quantize the h_i. The term in the action involving h_i is easily evaluated; in fact

$$L = -\tfrac{1}{4} \int \mathrm{d}^3 x \, F_{\mu\nu}^2 = \tfrac{1}{2} L^3 \sum_i \left(\frac{\mathrm{d}h_i}{\mathrm{d}t} \right)^2 + \cdots, \tag{41}$$

where the terms omitted are independent of the h_i. Quantizing this lagrangian, the relevant part of the hamiltonian is

$$H = \frac{1}{2L^3} \sum \pi_i^2, \tag{42}$$

where $\pi_i = -i\,\partial/\partial h_i$. The spectrum of this operator is well known. There is a unique,

* This is true if the ratios of the charges of the various fields are rational. Nothing is essentially different if the ratios are irrational since an irrational number can be approximated arbitrarily well by a rational number.

zero-energy ground state, with wave function $\Psi = 1$. It is normalizable (as are all eigenstates of H) since the h_i are periodic variables. Since the period is $2\pi/eL$, the lowest excited state has energy $e^2/2L$.

Placing the gauge field in its unique ground state and re-introducing the fermions, we obtain the previously stated results. Allowing for states that contain zero, one, or two zero-momentum fermions, we get $\mathrm{Tr}(-1)^F = 0$, $\mathrm{Tr}(-1)^F C = 4$.

A technicality in the above derivation should be clarified. In defining $\mathrm{Tr}(-1)^F$ in sect. 2, we assumed that, in a finite volume, the spectrum of the hamiltonian is discrete. Otherwise, the counting of states used in sect. 2 is invalid. $\mathrm{Tr}(-1)^F$ is ill-defined, or could be made well-defined only through sophisticated arguments, if the hamiltonian even in a finite volume has a continuous spectrum.

While commonly the spectrum in a finite volume is discrete, this may be untrue in theories with massless particles because the zero momentum modes of the massless particles may have a continuous spectrum. It is absolutely crucial that in our problem the h_i can be regarded as periodic variables. Otherwise, the hamiltonian (42) would have a continuous spectrum, and $\mathrm{Tr}(-1)^F$ and $\mathrm{Tr}(-1)^F C$ would be ill-defined. It is only in a restricted Hilbert space in which the states are required to be invariant under $A_i \to A_i + (2\pi/eL)h_i$ that $\mathrm{Tr}(-1)^F$ and $\mathrm{Tr}(-1)^F C$ are well-defined.

Our result $\mathrm{Tr}(-1)^F C = 4$ shows that in charge-conjugation invariant, vector-like QED, dynamical breaking of supersymmetry does not occur. The same is true if C is violated only by terms in the lagrangian that could be eliminated by an allowed conjugation operation.

It should be noted that we have not assumed, or proved, that dynamical breaking of C invariance does not occur. Because $\mathrm{Tr}(-1)^F C = 4$, there are at least four zero-energy states for any value of the coupling constant and of the volume. This means that the ground-state energy vanishes also in the infinite volume theory, and supersymmetry is unbroken. It does not prove that, in the infinite volume limit, the proper vacuum states – the ones in which cluster decomposition holds – are C eigenstates.

A final comment, analogous to remarks in sect. 5 about the Wess-Zumino model, should be made. For weak coupling, a far more trivial argument shows that dynamical supersymmetry breaking does not occur in vector-like QED with massive charged particles and C invariance. For weak coupling, the only possible massless Goldstone fermion would be the fermionic partner ψ of the photon. It is odd under C; the supersymmetry current S_μ is even under C. Hence, the crucial matrix element $\langle 0 | S_\mu | \psi \rangle$ vanishes as long as C is conserved. But in theories such as the one with potential energy (35), spontaneous breaking of C invariance certainly does not occur if the coupling is weak enough.

The virtue of the argument based on $\mathrm{Tr}(-1)^F C$ is that it shows that, in the class of theories considered here, supersymmetry remains unbroken even if the coupling is strong, and even if one removes the bare mass terms of some of the charged fields.

7. Non-abelian gauge theories

Let us now turn to the interesting question of dynamical supersymmetry breaking in non-abelian gauge theories. In this and the next section, we will consider the minimal supersymmetric non-abelian gauge theory. The only fields are the gauge field A_μ^a and its supersymmetric partner ψ_α^a. The lagrangian takes the minimally coupled form,

$$\mathcal{L} = -\frac{1}{4g^2}F_{\mu\nu}^a F_{\mu\nu}^a + \tfrac{1}{2}\bar{\psi}^a i \not{D} \psi^a , \tag{43}$$

and is supersymmetric because, as may easily be demonstrated, the supersymmetry current

$$S_{\mu\alpha} = \left(\sigma_{\gamma\delta} F^{\gamma\delta a} \gamma_\mu \psi^a\right)_\alpha \tag{44}$$

is conserved.

Because of asymptotic freedom and the corresponding infrared instability, it is reasonable to guess that, as in conventional QCD, there are many non-perturbative phenomena that occur in this theory. We may expect confinement, dynamical mass generation, and perhaps other phenomena as well. It is natural to suspect that, although supersymmetry is unbroken in perturbation theory in this model, dynamical breaking of supersymmetry may occur, presumably with the binding of a color singlet Goldstone fermion. We will be able to answer this question by calculating $\mathrm{Tr}(-1)^F$. As we will see, for a simple Lie group of rank r, $\mathrm{Tr}(-1)^F = r+1$. The non-zero value means that dynamical supersymmetry breaking does not occur.

As in our previous discussions, it is only the zero-momentum modes of the massless particles that make the evaluation of $\mathrm{Tr}(-1)^F$ difficult. Massless fermions of zero momentum can be straightforwardly counted, as we did in the last section. But massless bosons are more troublesome. Any number of zero-momentum massless bosons can be placed in our finite volume box. Because of the non-linear interactions, it is difficult to count these states and to determine how many have zero energy.

To calculate $\mathrm{Tr}(-1)^F$, it is sufficient to evaluate the spectrum of the theory in perturbation theory. However, the existence of the zero-momentum mode of the gauge field, which carries zero energy and cannot be gauged away, means that, in a finite volume, it is not straightforward to formulate perturbation theory for the Yang-Mills field.

It turns out that, for the gauge group $\mathrm{SU}(N)$, it is possible to avoid this problem, by formulating the theory in such a way that the zero-momentum mode is absent. We will follow that approach here. In sect. 8, a more general approach, valid for arbitrary gauge groups, is described.

In a finite volume, we cannot gauge away the zero-momentum mode. As we have discussed in the abelian case, the required gauge transformation would not be periodic. We will return to this point in sect. 8. However, if the gauge group in SU(N), we can eliminate the zero-momentum mode in another way. We can choose boundary conditions that it does not satisfy. This can be done by introducing 't Hooft's "twisted boundary conditions".

In a box of length L, the usual periodic boundary conditions are that A_μ and ψ are periodic functions in the usual sense, for example

$$\begin{aligned} A_\mu(x, y, z) &= A_\mu(x+L, y, z) = A_\mu(x, y+L, z) \\ &= A_\mu(x, y, z+L) . \end{aligned} \tag{45}$$

't Hooft has shown [7] that for some purposes, it is useful to consider a generalization of (45). In a special case that is general enough for our purposes, the modified boundary conditions take the form

$$\begin{aligned} A_\mu(x, y, z) &= P A_\mu(x+L, y, z) P^{-1} \\ &= Q A_\mu(x, y+L, z) Q^{-1} \\ &= A_\mu(x, y, z+L) . \end{aligned} \tag{46}$$

Here P and Q are constant matrices which, for the gauge group SU(N), are chosen to satisfy

$$PQ = QP \exp(2\pi i/N) . \tag{47}$$

For instance, these matrices may be taken to be

$$P = \alpha \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & \ddots & 1 \\ 1 & & & & & 0 \end{bmatrix} ,$$

$$Q = \beta \begin{pmatrix} 1 & & & & \\ & e^{2\pi i/N} & & & \\ & & e^{4\pi i/N} & & \\ & & & \ddots & \\ & & & & e^{2\pi i(N-1)/N} \end{pmatrix} , \tag{48}$$

where α and β are phases chosen so $\det P = \det Q = 1$. Of course, if A_μ is required to satisfy (46), then supersymmetry holds only if ψ is required to satisfy (46) also.

We are entitled to consider (46) instead of (45) because the large-volume limit of the theory is expected to be independent of the boundary conditions. If for *any* choice of boundary conditions we can evaluate $\mathrm{Tr}(-1)^F$ and show it to be non-zero, this shows that supersymmetry is unbroken in the infinite volume limit, with this choice of boundary conditions and therefore with any other choice.

According to 't Hooft, the boundary conditions (46) specify a theory with one unit of magnetic flux in the z direction. The reason for this terminology is irrelevant for our purposes. What is relevant is that the zero-momentum mode does not obey (46). Indeed, the zero-momentum mode C of any field satisfying (46) would have to obey $C = PCP^{-1} = QCQ^{-1}$. For C in the Lie algebra of $\mathrm{SU}(N)$, this requires $C = 0$. (To commute with Q, C would have to be diagonal; a traceless, diagonal matrix that commutes with P must vanish.)

With the zero-momentum mode eliminated by the boundary conditions, it is straightforward to set up perturbation theory and calculate $\mathrm{Tr}(-1)^F$. Expanding around $A_\mu = 0$, all modes of A_μ and ψ (except for modes of A_μ that can be gauged away) have a non-zero momentum and therefore a positive energy. Just as if the massless particles did not exist at all, one finds in perturbation theory a spectrum like that in fig. 8. There is a single zero-energy ground state, of spin zero; and there are Bose-Fermi pairs for positive energy. The expansion around $A_\mu = 0$ would thus give the value one for $\mathrm{Tr}(-1)^F$.

However, perturbation theory is not merely an expansion around $A_\mu = 0$. We must expand around any configuration which, classically, has zero energy (and thus is degenerate with $A_\mu = 0$). We will show below that, with twisted boundary conditions, any configuration that classically has zero energy may be written as $A_\mu = -i(\partial_\mu U)U^{-1}$ for some U. If U can be continuously deformed to the identity, it can be simply gauged away. But if U cannot be so deformed, the expansion around $A_\mu = -i(\partial_\mu U)U^{-1}$ gives rise, in perturbation theory, to a new sector of Hilbert space. Since the new sector differs from the old one only by the gauge transformation U, it likewise contributes one to $\mathrm{Tr}(-1)^F$. The numerical value of $\mathrm{Tr}(-1)^F$ is obtained, as in the Wess-Zumino model, by summing over the various sectors. As we will see, in a Hilbert space specified by a definite value of the vacuum angle θ, there are N sectors that must be included. Hence, altogether, $\mathrm{Tr}(-1)^F = N$.

The numerical value of $\mathrm{Tr}(-1)^F$, which depends on some technicalities, is not so important. What is important is that, with the zero-momentum modes eliminated, only the "vacuum" states, which always have spin zero, contribute to $\mathrm{Tr}(-1)^F$.

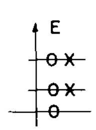

Fig. 8. The spectrum of supersymmetric QCD with twisted boundary conditions expanding near $A_\mu = 0$.

Hence, $\mathrm{Tr}(-1)^F$ is positive, just as if there were no massless particles at all. From this it follows that supersymmetry is not spontaneously broken.

The remainder of this section is more technical, and perhaps of more specialized interest. In the above analysis, we made the assertion that, with twisted boundary conditions, any configuration that classically has zero energy may be written as a pure gauge, $A_\mu = -i(\partial_\mu U)U^{-1}$ for some U. This assertion is untrue with ordinary periodic boundary conditions, because of the zero-momentum mode. Since twisted boundary conditions eliminate the zero-momentum mode, it is plausible that the assertion is true, if twisted boundary conditions are used. But we must demonstrate it. We also must justify the above claim concerning the numerical value of $\mathrm{Tr}(-1)^F$.

If A satisfies the twisted boundary conditions (46), what boundary conditions* should be imposed on a gauge transformation U? The most obvious choice would be to require

$$\begin{aligned} U(x,y,z) &= PU(x+L,y,z)P^{-1} \\ &= QU(x,y+L,z)Q^{-1} \\ &= U(x,y,z+L). \end{aligned} \tag{49}$$

If U satisfies (49), then the transformation $A_\mu \to UA_\mu U^{-1} - i(\partial_\mu U)U^{-1}$ is consistent with (46). However, following 't Hooft, we may equally well permit U to satisfy

$$\begin{aligned} U(x,y,z) &= \exp(2\pi i k_x/N)PU(x+L,y,z)P^{-1} \\ &= \exp(2\pi i k_y/N)QU(x,y+L,z)Q^{-1} \\ &= \exp(2\pi i k_z/N)U(x,y,z+L), \end{aligned} \tag{50}$$

where k_x, k_y, and k_z are any integers (modulo N). The action of U on the gauge field still preserves the condition (46).

If we choose standard gauge transformations T_x, T_y, and T_z, which satisfy (50) with $\boldsymbol{k} = (1,0,0)$, $(0,1,0)$, and $(0,0,1)$, respectively, then any gauge transformation U which satisfies (50) may be written as

$$U = (T_x)^{k_x}(T_y)^{k_y}(T_z)^{k_z}\tilde{U}, \tag{51}$$

where $\tilde{U}$ satisfies (49). Remarkably enough, we may choose T_x to be a constant, $T_x(x,y,z) = Q$. Likewise, we may choose $T_y(x,y,z) = P^{-1}$. No particularly simple choice is possible for T_z, and we will not make a particular choice.

* In this discussion we will go into more detail than actually needed, in order to clarify some points about twisted boundary conditions.

Obviously, with the above choices, $(T_x)^N=(T_y)^N=1$. What about $(T_z)^N$? One might guess that T_z can be chosen so that $(T_z)^N=1$, but this is not so.

$(T_z)^N$ is a gauge transformation that satisfies the simpler boundary condition (49). What is the topological classification of gauge transformations that satisfy (49)? By using the fact that SU(N) is simply connected and that $\pi_2(\mathrm{SU}(N))$ is trivial, it is easy to see that any gauge transformation U which satisfies (49) can be continuously deformed so that $U=1$ if x, y, or z equals zero or L. Once this is done, the behavior of U for $0\leqslant \boldsymbol{x}\leqslant L$ determines an element of $\pi_3(\mathrm{SU}(N))$, corresponding to the winding number of instanton theory.

It turns out that, no matter how T_z is defined, $(T_z)^N$ has a non-zero winding number. In particular, it is impossible to choose T_z so that $(T_z)^N=1$. However, if we choose a gauge transformation T that satisfies (49) and has winding number one, then T_z can be chosen so that $(T_z)^N=T$.

(This fact, which will not really be needed in what follows, is closely related to 't Hooft's observations on θ dependence in the presence of twisted boundary conditions, and to his discovery of instanton solutions of Pontryagin number $1/N$. The relationship $(T_z)^N=T$ is also analogous to some facts pointed out in the last few paragraphs of ref. [11] and in ref. [12].)

Let us now count the sectors of pure gauge configurations, and explain why, numerically, $\mathrm{Tr}(-1)^F=N$. Since T_x and T_y are constant gauge transformations (by Q and P^{-1}, respectively) they give nothing new when acting on $A_\mu=0$. They map $A_\mu=0$, and the entire Hilbert space obtained by expanding around $A_\mu=0$, into itself.

Acting on $A_\mu=0$ with T, which shifts the winding number, we do get something new. However, this merely corresponds to the fact that the theory contains an arbitrary vacuum angle θ. Because of the massless fermions and the axial anomaly, all physical quantities, including $\mathrm{Tr}(-1)^F$, are independent of θ. We may therefore also calculate $\mathrm{Tr}(-1)^F$ in a Hilbert space of states with a definite value of θ. Making this choice gives all states a prescribed behavior under the action of T (they are multiplied by $\mathrm{e}^{i\theta}$), and we may forget about T.

Having chosen θ, the only way to obtain from $A_\mu=0$ a new sector of Hilbert space is to act with T_z. T_z may be applied $0,1,2,\ldots,N-1$ times. Applying T_z N times we would just multiply the states by $\mathrm{e}^{i\theta}$. This action of T_z gives N sectors of Hilbert space, in perturbation theory. Each sector, as described above, contributes one to $\mathrm{Tr}(-1)^F$, so this, finally, explains why $\mathrm{Tr}(-1)^F$ is equal to N*.

We are now almost ready to prove that with twisted boundary conditions, a configuration of zero energy can always be written as $A_\mu=-i(\partial_\mu U)U^{-1}$. However, a simple algebraic lemma is necessary first. In eq. (48), we wrote two explicit matrices

* According to 't Hooft, the eigenvalue of T_z is the electric flux in the z direction. Since T_z commutes with supersymmetry, we may calculate the value of $\mathrm{Tr}(-1)^F$ in a subspace labeled by the value of the electric flux (as we did for charge conjugation in sect. 6). Since T_z permutes the N sectors of Hilbert space, we find $\mathrm{Tr}(-1)^F=1$ in any one of these N subspaces.

P and Q that satisfy $PQ=QP\exp(2\pi i/N)$. The same relationship, $P'Q'=Q'P'\exp(2\pi i/N)$, is obviously satisfied by the matrices

$$P'=XPX^{-1}, \qquad Q'=XQX^{-1}, \tag{52}$$

if X is any unitary matrix. We will find it useful to know that the converse is also true. Any two SU(N) matrices P' and Q' which satisfy $P'Q'=Q'P'\exp(2\pi i/N)$ can be written in the form (52), for some X. In fact, the relationship $P'Q'=Q'P'\exp(2\pi i/N)$ means that the action of P' multiplies the eigenvalues of Q' by a factor $\exp(2\pi i/N)$. This, and the fact that $\det Q'=1$, means that Q' has the same eigenvalues as Q. So $Q'=XQX^{-1}$, for some X. Once Q' is diagonalized, the relationship $P'Q'=Q'P'\exp(2\pi i/N)$ determines P' uniquely, except for some phases, which can be eliminated by a unitary transformation that commutes with Q'.

Incidentally, twisted boundary conditions can be introduced for groups other than SU(N), as long as the center of the group is non-trivial. However, the analogue of the uniqueness theorem just stated does not hold. It is for this reason only that, for groups other than SU(N), twisted boundary conditions are not useful in calculating $\mathrm{Tr}(-1)^F$.

Considering now a configuration that classically has zero energy, we wish to show that $A_\mu=-i(\partial_\mu U)U^{-1}$ for some U. Actually A_0 can always be gauged away; let us assume this has been done. For the energy to vanish classically, it must be that $F_{ij}=\partial_i A_j-\partial_j A_i-i[A_i, A_j]$ vanishes. We must show that, under this condition, we can write $A_i=-i(\partial_i U)U^{-1}$.

Of course, with $F_{ij}=0$, we always can write $A_i=-i(\partial_i U)U^{-1}$ locally. In fact, this expression is not unique. If $A_i=-i(\partial_i U)U^{-1}$, then also $A_i=-i(\partial_i \tilde{U})\tilde{U}^{-1}$, if $\tilde{U}=UK$, K being any constant matrix. This ambiguity will play a role in what follows.

As a first try in attempting to satisfy $A_i=-i(\partial_i U)U^{-1}$, let us write

$$U(x, y, z)=\mathrm{T}\exp\left(i\int_{(0,0,0)}^{(x,y,z)} A_i \,\mathrm{d}x^i\right). \tag{53}$$

The path of integration does not matter, since $F_{ij}=0$. The only problem with (53) is that it may not satisfy the boundary condition (50). In fact, from the boundary condition (46) satisfied by A_μ, and the definition (54), it is easy to see that U satisfies

$$\begin{aligned} U(x, y, z) &= PU(x+L, y, z)K_x P^{-1} \\ &= QU(x, y+L, z)K_y Q^{-1} \\ &= U(x, y, z+L)K_z, \end{aligned} \tag{54}$$

where

$$K_x = \mathrm{T}\exp\left(i\int_{(L,0,0)}^{(0,0,0)} A_i\,\mathrm{d}x^i\right),$$

$$K_y = \mathrm{T}\exp\left(i\int_{(0,L,0)}^{(0,0,0)} A_i\,\mathrm{d}x^i\right),$$

$$K_z = \mathrm{T}\exp\left(i\int_{(0,0,L)}^{(0,0,0)} A_i\,\mathrm{d}x^i\right). \tag{55}$$

By applying (55) twice we can learn that

$$\begin{aligned} U(x,y,z) &= PU(x+L,y,z)K_xP^{-1} \\ &= PQU(x+L,y+L,z)K_yQ^{-1}K_xP^{-1}, \end{aligned} \tag{56}$$

and also that

$$\begin{aligned} U(x,y,z) &= QU(x,y+L,z)K_yQ^{-1} \\ &= QPU(x+L,y+L,z)K_xP^{-1}K_yQ^{-1}. \end{aligned} \tag{57}$$

Combining these relations, and recalling that $PQ = QP\exp(2\pi i/N)$, we learn

$$\left(PK_x^{-1}\right)\left(QK_y^{-1}\right) = \left(QK_y^{-1}\right)\left(PK_x^{-1}\right)\exp(2\pi i/N). \tag{58}$$

In view of the uniqueness theorem that we discussed surrounding eq. (52), it follows from this that

$$PK_x^{-1} = XPX^{-1}, \qquad QK_y^{-1} = XQX^{-1}, \tag{59}$$

for some X in SU(N).

By manipulations similar to those that gave (56)–(58), but working now in the (xz) or (yz) planes, two more consistency conditions can be derived. They are

$$K_z\left(K_xP^{-1}\right) = \left(K_xP^{-1}\right)K_z, \qquad K_z\left(K_yQ^{-1}\right) = \left(K_yQ^{-1}\right)K_z. \tag{60}$$

From these relations it follows that K_z is an element of the center of SU(N):

$$K_z = \exp(2\pi ik/N), \tag{61}$$

for some k. (It is obvious from (48) that a matrix that commutes with P^{-1} and Q^{-1}

must be a multiple of the identity. This is likewise true for a matrix which commutes with $K_x P^{-1}$ and $K_y Q^{-1}$, which differ from P^{-1} and Q^{-1} only by conjugation.)

Eqs. (59) and (61) are the key to our problem. It follows readily from these equations that if we define

$$\tilde{U}(x, y, z) = U(x, y, z) X, \tag{62}$$

then $\tilde{U}$ satisfies the boundary condition (50), with $(k_x, k_y, k_z) = (0, 0, k)$. Moreover, $A_i = -i(\partial_i U)U^{-1} = -i(\partial_i \tilde{U})\tilde{U}^{-1}$. Thus, we have finally proved that with twisted boundary conditions, any classical configuration of zero energy is a pure gauge. This completes our analysis of $\mathrm{Tr}(-1)^F$ with twisted boundary conditions.

8. More on non-abelian gauge theories

In the last section, we calculated $\mathrm{Tr}(-1)^F$ in the minimal supersymmetric $\mathrm{SU}(N)$ gauge theory, using twisted boundary conditions. We found $\mathrm{Tr}(-1)^F = N$.

In this section, we will calculate $\mathrm{Tr}(-1)^F$ with ordinary (untwisted) boundary conditions, for the minimal supersymmetric theory based on an arbitrary Lie group. We will find that, for any simple, non-abelian Lie group of rank r, $\mathrm{Tr}(-1)^F = r + 1$.

We will work in the gauge $A_0 = 0$. As we have discussed, the whole difficulty arises from the zero-momentum mode of the gauge field A_i. Suppose that the A_i are independent of x, y, and z and are equal to constant matrices C_i. As long as the C_i commute with each other, the energy vanishes because $F_{ij} = \partial_i A_i - \partial_j A_i - [A_i, A_j] = 0$.

If for simplicity we consider first the gauge group SU(2), then the C_i, to commute with each other, must be proportional to some one generator of the group. After making a global gauge transformation, if necessary, it can be assumed that this generator is (say) T^3. So in this gauge

$$A_i^a = c_i \delta^{a3}, \tag{63}$$

where the c_i are constants.

The easiest way to establish that the c_i cannot be gauged away is to consider (fig. 9) a Wilson loop running from one end to the other of our box. The trace of this Wilson loop is invariant under all periodic gauge transformations. If, for example, the Wilson loop runs in the x direction, one may readily calculate

$$\mathrm{Tr}\,\mathrm{T}\exp\left(i\oint A i\, \mathrm{d}x^i \right) = 2\cos c_x L, \tag{64}$$

where L is the length of the box. The non-trivial dependence on c_x of this gauge-invariant expression shows that the c_i cannot be gauged away.

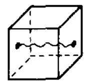

Fig. 9. A Wilson loop running through the box.

While the c_i cannot be gauged away, every configuration which has $F_{ij}=0$ can be put in the form (63) by a gauge transformation. This hopefully plausible statement will be proved at the end of this section.

In a background gauge field of the type (63), the Dirac equation has zero-energy solutions. The zero modes take the form

$$\psi_\alpha^a = \varepsilon_\alpha \delta^{a3}, \tag{65}$$

where ε_α is an arbitrary constant spinor. Note that to make a zero-energy solution, ψ must be parallel to A in the SU(2) algebra, because of the $[A_\mu, \psi]$ term in the Dirac equation.

Our strategy will be to quantize the degrees of freedom (63) and (65) while ignoring the rest. The rationale for this is that while (63) and (65) carry zero energy classically, all other degrees of freedom (except for modes of the gauge field that can be gauged away) carry a momentum and energy of at least $2\pi/L$. As in the conventional Born-Oppenheimer approximation in molecular physics, the very low lying energy levels – which are all that we need to calculate $\mathrm{Tr}(-1)^F$ – can be calculated by quantizing the degrees of freedom that have zero energy classically. As in the abelian case [see the remarks following eq. (42)], the quantization of (63) and (65) gives a spectrum with an energy scale $g^2 2\pi/L$, g being the gauge coupling. If $g^2 \ll 1$, these are the low-lying states.

(One might wonder whether there are other sectors of Hilbert space in addition to that constructed in expanding around (63). Actually, with standard, untwisted periodic boundary conditions, the only sectors are those associated with the instanton winding number. These can be ignored if we agree to work in a Hilbert space labeled by a definite value of the vacuum angle θ.)

Let us now discuss some further properties of the c_i. A look at (64) suggests that, while the c_i cannot be eliminated by a gauge transformation, it may be possible by a gauge transformation to shift the c_i by a multiple of $2\pi/L$. This is indeed true. The gauge functions

$$\begin{aligned} U_x &= \exp\frac{2\pi i x}{L}\sigma_3, \\ U_y &= \exp\frac{2\pi i y}{L}\sigma_3, \\ U_z &= \exp\frac{2\pi i z}{L}\sigma_3 \end{aligned} \tag{66}$$

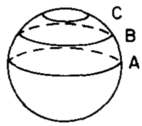

Fig. 10. A gauge transformation is deformed to a constant (A → B → C).

are well defined and periodic. They generate gauge transformations which shift, respectively, c_x, c_y, and c_z by $2\pi/L$. Requiring physical states to be invariant under the action of U_x, U_y, and U_z means that the c_i are periodic variables with period $2\pi/L$.

This may seem reminiscent of the discussion of the abelian case in sect. 6. There, too, we defined a gauge transformation [see eqs. (37)–(39)] that shifted the zero-momentum mode of the gauge field by a constant*. But there are some fundamental differences. In the abelian case, we had the *option* of requiring states to be invariant under the gauge transformation that shifts the gauge field by a constant. For reasons explained in sect. 6, it was advantageous to make use of this option. In the non-abelian case, however, we *must* require the states to be invariant under action of (66). This follows from the constraint of Gauss' law. The gauge transformations (66), which wrap around the "equator" in group space, can be continuously deformed to the identity, much as a lasso can be slipped off of a sphere (see fig. 10). Gauss' law therefore requires that physical states be invariant under (66). In the non-abelian case, the c_i *must* be regarded as periodic variables.

A related difference between the abelian and non-abelian situations is the following. In the abelian case, a well-defined gauge transformation that shifts the zero-momentum mode by a constant exists only if one assumes that electric charge is quantized, in some units. In the non-abelian case, the charges are automatically quantized, because of the non-linear commutation relations in the Lie algebra. Correspondingly, the operator (66) always exists in the non-abelian theory.

As we have mentioned, and will demonstrate at the end of this section, any gauge field with $F_{ij}=0$ can be put in the form (63) by a gauge transformation. Actually, this can be done in two ways. The constant gauge transformation

$$G=i\sigma_2 \tag{67}$$

preserves the form of (63), but it reverses the sign of the zero-momentum modes of A_i and ψ_α. It brings about the transformation

$$c_i \leftrightarrow -c_i, \qquad \varepsilon_\alpha \leftrightarrow -\varepsilon_\alpha. \tag{68}$$

* A slightly different normalization was used in sect. 6, however.

Obviously, the gauge invariant Wilson loop [eq. (64)] is invariant under this transformation.

Again, G has an analogue in the abelian theory. It is analogous to the charge-conjugation operator C which we utilized in sect. 6. Like G, C commuted with supersymmetry and reversed the sign of the zero-momentum modes. However, there are again some fundamental differences. In the abelian case, C was a global symmetry, not part of the gauge group. We had the liberty to consider theories that were or were not C invariant. In the non-abelian case, G is a gauge transformation, and so is a symmetry as long as the theory is gauge invariant.

In the abelian case, assuming C to be a symmetry, we could label the states as even or odd under C. Corresponding to this, we had the two invariants $\mathrm{Tr}(-1)^F$ and $\mathrm{Tr}(-1)^F C$. In the non-abelian case, the Gauss' law constraint requires that physical states should be invariant under G. This is so because G, like any other constant gauge transformation, can be continuously deformed to the identity. Therefore, in the non-abelian case, there is only one invariant, $\mathrm{Tr}(-1)^F$, to be calculated, but it should be calculated in the space of states that are even under G.

With the machinery in place, the actual evaluation of $\mathrm{Tr}(-1)^F$ for an SU(2) gauge theory is an almost trivial repetition of the abelian case, except for a few difficult points that we will not come to grips with fully. As has been explained, the strategy will be to quantize the zero momentum modes, ignoring the modes of non-zero momentum. We thus express the lagrangian

$$L=\int \mathrm{d}^3x\left(-\frac{1}{4g^2}\left(F^a_{\mu\nu}\right)^2+\tfrac{1}{2}\bar{\psi}^a i\not{D}\psi^a\right),$$

in terms of c_i and ε_α. The terms that depend on c_i and ε_α are

$$L_{\text{eff}}=V\left(\frac{1}{g^2}\sum_i\left(\dot{c}_i\right)^2+\bar{\varepsilon}i\gamma^0\frac{\partial\varepsilon}{\partial t}\right), \tag{69}$$

where $V=L^3$ is the volume of the box. After quantization, the hamiltonian is simply

$$H_{\text{eff}}=\frac{g^2}{V}\sum\pi_i^2, \tag{70}$$

where π_i is the momentum canonically conjugate to c_i*. Note that the spinor ε has disappeared, because of the linear dependence of (69) on $\partial\varepsilon/\partial t$.

As in the abelian case, the hamiltonian (70) (if one ignores the fermions) has a unique zero-energy ground state, the wave function being a constant. Since the c_i are periodic with period $2\pi/L$, the excited states have energies of order g^2/L. For small

* One really should study the Faddeev-Popov determinant to obtain the proper measure for the c_i. We will not attempt that here. $\mathrm{Tr}(-1)^F$ is expected to be independent of the measure, just as it is independent of other parameters.

g^2, this is much less than the energy scale, $1/L$, of the modes of non-zero momentum. It is this that justifies ignoring the modes of non-zero momentum in determining the spectrum of low-lying states.

When the fermions are included, we have, as in the abelian case, four zero-energy states in this approximation. The zero-momentum mode of the fermions may be empty, once occupied, or twice occupied (fig. 7). However, of these states, the physical states are those of $G = +1$.

As the overall sign of the operator G is difficult to determine, let us assume for the moment (fig. 11) that the state with no zero-momentum fermions has $G = +1$. The once occupied states then have $G = -1$. since the fermion zero mode ε is odd under G. The twice occupied state has $G = +1$. We see that in the physical subspace of $G = +1$, we have $\mathrm{Tr}(-1)^F = 2$.

If instead we assumed the empty state to have $G = -1$, then all G assignments would be reversed. We would conclude that among states of $G = 1$, $\mathrm{Tr}(-1)^F = -2$.

In the abelian case, the overall sign of the charge-conjugation operator C is a matter of arbitrary choice. In the non-abelian case, the sign of G is in principle fixed, since G is part of the gauge group. Unfortunately, I do not know of a simple, direct argument determining the overall sign of G is the space of states considered here. The existence of fermions makes it rather difficult to work out the overall sign.

An indirect argument indicating that the choice in fig. 11 is correct will be mentioned below.

While this point should be cleared up, the main conclusion is the same in either case. Whether the G assignments are as in fig. 11 or reversed, $\mathrm{Tr}(-1)^F$ is non-zero, and the ground-state energy of the system is exactly zero, for any volume.

Now let us apply these considerations to an arbitrary simple, non-abelian Lie group.

Given any Lie group G, the rank r is defined as the maximum number of commuting generators. Choosing r commuting generators $T^1, \ldots, T^r$, we obtain an abelian subalgebra of the Lie algebra of G known as the Cartan subalgebra.

As stated above, my classical configuration of zero energy can, by a gauge transformation, be put in the form $A_i = C_i$, the C_i being constant matrices which commute with one another. The fact that the C_i commute with one another means that, by a global gauge rotation, they can be taken to lie in the Cartan subalgebra. In this gauge

$$A_i = \sum_\sigma c_i^\sigma T^\sigma , \tag{71}$$

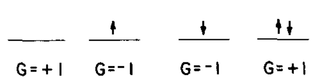

Fig. 11. The spectrum of zero-energy states in supersymmetric QCD. A certain assumption has been made concerning the G assignments.

where $\sigma = 1,\ldots,r$ runs over the Cartan subalgebra (*not* over the whole Lie algebra), and the c_i^σ are now simply numbers.

What will play the role played by G invariance in the SU(2) example? There is more than one way to put A_i in the form (71). In every Lie group G, there is a discrete subgroup, known as the Weyl group W, consisting of elements of G do not commute with the generators of the Cartan subalgebra, but which map this subalgebra into itself.

For example, in the case of SU(N), the rank is $r = N-1$. The Cartan subalgebra consists of the diagonal matrices. The Weyl group is P_N, the permutation group of N objects, and it acts by permuting the N eigenvalues of a diagonal matrix. In the case of SU(2) the Weyl group consists only of two elements, G and the identity.

The action of the Weyl group preserves the form of (71) but rearranges the c_i^σ. Physical states must be invariant under the Weyl group, because the Weyl transformations are global gauge transformations which can be reached continuously from the identity.

In a background gauge field of the form (71), the Dirac equation has zero-energy solutions. If we arrange the Dirac field as a matrix $\psi_\alpha = \Sigma \psi_\alpha^a T^a$ (with a running over the entire Lie algebra), the zero modes take the simple form

$$\psi_\alpha = \sum \varepsilon_\alpha^\sigma T^\sigma , \tag{72}$$

where the $\varepsilon_\alpha^\sigma$ are constants and σ runs over the Cartan subalgebra. (The fermion zero modes must lie in the Cartan subalgebra because of the $[A_i, \psi]$ term in the Dirac equation.)

As before, the spectrum of low-lying states can be computed by quantizing the c_i^σ and $\varepsilon_\alpha^\sigma$, ignoring the other modes. The hamiltonian is again trivial, depending only on the momenta π_i^σ conjugate to the c_i^σ:

$$H = \frac{g^2}{V} \sum (\pi_i^\sigma)^2 . \tag{73}$$

This hamiltonian by itself would give a single zero-energy state, with constant wave function.

We may put the boson degrees of freedom c_i^σ in their ground state and forget about them. However, we must take account of the fermions.

The fermions have two spin states $\alpha = 1,2$ for each value of the internal symmetry index σ. We may describe the fermions by means of creation and annihilation operators $a_\alpha^{*\sigma}$ and a_α^σ ($\alpha = 1,2$; $\sigma = 1,\ldots,r$). In the fermion Hilbert space there is a state $|\Omega\rangle$ that is annihilated by the annihilation operators. The other states can be obtained by acting with creation operators, $a_\alpha^{*\sigma}|\Omega\rangle$, $a_\alpha^{*\sigma} a_\beta^{*\tau}|\Omega\rangle$, etc.

Obviously, $|\Omega\rangle$ transforms as a one-dimensional representation of the Weyl group. However, here we meet an ambiguity similar to the sign ambiguity in the case of SU(2). The Weyl group always has two one-dimensional representations. In addition to the trivial representation in which each element is represented by the identity, there is a non-trivial one-dimensional representation in which each element is represented by plus or minus one*. States transforming in the trivial and non-trivial one-dimensional representations of the Weyl group may be described as true invariants and pseudo-invariants, respectively. Gauss' law requires that physical states be true invariants.

It is not obvious whether $|\Omega\rangle$ is a true invariant or a pseudo-invariant. As we will not come to grips with this question, we will have to consider both possibilities.

What invariant and pseudo-invariant states can be made by acting on $|\Omega\rangle$ with creation operators? It is equivalent to ask what invariant operators can be formed from the creation operators $a_\alpha^{*\sigma}$. Since $a_\alpha^{*\sigma}$ transforms in the fundamental r-dimensional representation of the Weyl group, we must ask what invariants can be formed in this representation.

The Weyl group is a discrete subgroup of the orthogonal group $O(r)$. $O(r)$ admits one invariant, the Kronecker delta $\delta_{\sigma\tau}$, and one pseudo-invariant, the completely antisymmetric tensor $\varepsilon_{\sigma_1\sigma_2\cdots\sigma_r}$. These are certainly invariant under the Weyl subgroup. So we can form a Weyl invariant operator $U=a_\alpha^{*\sigma}a_\beta^{*\sigma}\varepsilon^{\alpha\beta}$. U has spin zero; the spin index must be contracted antisymmetrically, by Fermi statistics. We can also form the pseudo-invariant $V_{\alpha_1\cdots\alpha_r}=a_{\alpha_1}^{*\sigma_1}a_{\alpha_2}^{*\sigma_2}\cdots a_{\alpha_r}^{*\sigma_r}\varepsilon_{\sigma_1\sigma_2\cdots\sigma_r}$. Because of Fermi statistics, V is symmetric in its indices; it has spin $\frac{1}{2}r$, and $r+1$ components.

Of course, the Weyl group has other invariants that are not invariants of the full orthogonal group $O(r)$. However, because of Fermi statistics and the fact that Pauli spinors have only two components, the Weyl invariants that are not $O(r)$ invariants cannot be combined non-trivially with the $a_\alpha^{*\sigma}$. For instance, in the case of $SU(N)$, the Weyl group admits a completely symmetric third-rank invariant** $d_{\sigma\tau\mu}$. However, the operator $d_{\sigma\tau\mu}a_\alpha^{*\sigma}a_\beta^{*\tau}a_\gamma^{*\mu}$ is identically zero, by Fermi statistics (it would have to be antisymmetric in α, β, and γ, which only take two values).

Moreover, by Fermi statistics we have the following relations: $U^{r+1}=0$; $UV=VU=0$ (for any component of V); and V^2 can be expressed in terms of U^r.

It is now trivial, except for a twofold ambiguity, to identify the invariant states and calculate $\mathrm{Tr}(-1)^F$. If we assume that $|\Omega\rangle$ is a true invariant, then the true invariants are the $r+1$ states $|\Omega\rangle, U|\Omega\rangle, U^2|\Omega\rangle,\ldots, U^r|\Omega\rangle$. They are all bosonic, so $\mathrm{Tr}(-1)^F=r+1$.

If, instead, we assume that $|\Omega\rangle$ is a pseudo-invariant, then the only way to form true invariants is to act on $|\Omega\rangle$ with the pseudo-invariant V. The states $V_{\alpha_1\alpha_2\cdots\alpha_r}|\Omega\rangle$ have spin $\frac{1}{2}r$, so under this assumption $\mathrm{Tr}(-1)^F=(-1)^{r+1}(r+1)$.

* The non-trivial representation can be described as follows. The commuting generators T^σ transform in the fundamental r-dimensional representation of the Weyl group. In this representation, the determinant of each element of the Weyl group is ± 1. Representing each element by its determinant, we get a non-trivial one-dimensional representation of the Weyl group.

** This corresponds to the fact that the permutation group of N objects $x_1, x_2\cdots x_N$ leaves invariant the cubic polynomial $x_1^3+x_2^3+\cdots+x_N^3$.

Even without attempting to resolve the sign ambiguity, we see that $\mathrm{Tr}(-1)^F$ is non-zero (equal to $r+1$ in absolute value). Therefore, dynamical supersymmetry breaking does not occur in these theories. It is also possible to make a few interesting, but not rigorous, comments about some physical questions other than supersymmetry breaking.

For $SU(N)$, we have proved that in a finite volume, there are at least $N=r+1$ degenerate ground states. How can we interpret this degeneracy?

This theory possesses, classically, a U(1) chiral symmetry $\psi \to e^{i\alpha\gamma_5}\psi$. Because of instantons and the Adler-Bell-Jackiw anomaly, only a discrete subgroup of this U(1) symmetry really commutes with the hamiltonian. In the case of $SU(N)$, this is a $2N$-fold discrete symmetry, consisting of the transformations

$$\psi \to e^{\pi i k \gamma_5/N}\psi, \qquad k=0,1,2,\ldots,2N-1. \tag{74}$$

Note that for $k=N$, this is $\psi \to -\psi$, which, come what may, will presumably not be spontaneously broken (it is equivalent to a 2π rotation).

By analogy with conventional QCD, one may guess that in the supersymmetric $SU(N)$ gauge theory, the operator $\mathrm{Tr}\,\bar{\psi}\psi$ gets a non-zero vacuum expectation value. If so, this spontaneously breaks the $2N$-fold discrete chiral symmetry down to a twofold symmetry, the unbroken symmetries being simply $\psi \to \pm\psi$. There would be N vacuum states.

It is quite plausible that the N zero-energy states whose existence we have proved go over, in the infinite volume limit, to the N vacuum states of spontaneously broken chiral symmetry. This interpretation is lent some credence by the following considerations. If we choose the creation and annihilation operators a^* and a employed previously to have chirality one and minus one, respectively, then the operator $U=a_\alpha^{*\sigma}a_\beta^{*\sigma}\varepsilon^{\alpha\beta}$ has chirality two. Normalizing the chiral charge so that our state $|\Omega\rangle$ has chirality zero*, the states $|\Omega\rangle, U|\Omega\rangle, U^2|\Omega\rangle,\ldots,U^{N-1}|\Omega\rangle$ have chirality $0,2,4,6,\ldots,2N-2$. These quantum numbers are suitable so that appropriate linear combinations of those states (the combinations $(1/\sqrt{N})\Sigma_{p=0}^{N-1}\exp(2\pi ip/N)U^p|\Omega\rangle)$ could be the N vacua of spontaneously broken chiral symmetry.

There also are some soluble examples in which the type of reasoning just given yields the correct answer. These are the supersymmetric S^N and CP^N non-linear sigma models in 1 + 1 dimensions, which we will discuss in sect. 10. In those models, the pattern of zero-energy states predicted by $\mathrm{Tr}(-1)^F$ coincides with the vacuum structure found in the $1/N$ expansion.

What do we find if we apply this reasoning to four-dimensional $Sp(2N)$ and $O(N)$ gauge theories?

* It is being assumed here that $|\Omega\rangle$ is a true invariant.

In the Sp($2N$) case, the results are suggestive of chiral symmetry breaking. In this case the discrete chiral symmetry is Z_{2N+2}. If it is spontaneously broken to $\psi \to \pm\psi$, there should be $N+1$ vacuum states. Indeed, $\mathrm{Tr}(-1)^F = N+1$, and the $N+1$ zero-energy states found by our analysis may go over, in the large-volume limit, to the $N+1$ vacuum states of spontaneously broken chiral symmetry.

For O(N), with $N > 6$, this type of counting does not work. The chiral symmetry is Z_{2N-4}, so if $\langle \mathrm{Tr}\,\bar{\psi}\psi \rangle \neq 0$ there are $N-2$ vacuum states. But $\mathrm{Tr}(-1)^F$ equals $\frac{1}{2}(N+1)$ or $\frac{1}{2}(N+2)$ depending on whether N is odd or even. Perhaps in this case the chiral symmetry is unbroken. The $\frac{1}{2}(N+1)$ or $\frac{1}{2}(N+2)$ zero-energy states may be related, not to chiral symmetry breaking, but to the zero-momentum modes of physical color singlet bound states.

It should be evident that these arguments are, at best, only suggestive.

Another question on which some interesting but inconclusive remarks can be made is the question of confinement, in 't Hooft's sense, of electric and magnetic flux.

We have here so far only considered gauge transformations that are strictly periodic. However, as in sect. 7, we may consider gauge transformations that are periodic up to an element of the center of the gauge group. According to 't Hooft, these operators measure the electric flux trapped in the box. Since the flux commutes with the supersymmetry operators, we may calculate $\mathrm{Tr}(-1)^F$ in a sector of Hilbert space labeled by the value of the electric flux in the x, y, and z directions.

All of the zero-energy states found in our above discussion are in the sector of zero electric flux. In fact, the operators that measure electric flux can be chosen so that they shift the c_i^σ by constants; our zero-energy wave functions were independent of the c_i^σ and so are invariant under such shifts. So in the sector of zero electric flux, $\mathrm{Tr}(-1)^F = r+1$, and the ground-state energy is zero, but in other sectors, $\mathrm{Tr}(-1)^F = 0$. This is consistent with the possibility that, in the large volume limit, a large energy is associated with electric flux.

On the other hand, for the group SU(N), we showed in sect. 7 that the ground-state energy is zero in the presence of non-zero magnetic flux. There is no confinement of magnetic flux in these theories.

One can also ask whether the ground-state energy vanishes in the presence of electric as well as magnetic flux. This can be answered on the basis of the results in sect. 7, where we studied the theory with magnetic flux in the z direction. The operators T_x, T_y, and T_z defined in sect. 7 are precisely the operators that measure electric flux. The zero-energy states found in sect. 7 were invariant under T_x and T_y, so $\mathrm{Tr}(-1)^F = 0$ in any sector with electric flux not parallel to the magnetic flux. What if the electric flux is in the z direction? The zero-energy states found in sect. 7 were of the form $|\Phi\rangle, T_z|\Phi\rangle, \ldots, T_z^{N-1}|\Phi\rangle$. From linear combinations of these, eigenstates of T_z can be formed with every possible eigenvalue. So $\mathrm{Tr}(-1)^F = 1$ for any value of the electric flux parallel to the magnetic flux, as long as there is no electric flux in the transverse directions. In these sectors, the ground-state energy vanishes.

We have found, in the case of SU(N), that $\mathrm{Tr}(-1)^F = N$ both with normal periodic boundary conditions and with twisted ones. Was there any *a priori* reason to expect this equality?

In general, $\mathrm{Tr}(-1)^F$ depends on the boundary conditions. It is only in the infinite volume limit that physics becomes independent of the boundary conditions. In a finite volume, the same theory may have zero ground-state energy with one set of boundary conditions, and non-zero ground-state energy with another set of boundary conditions. In fact, examples are easily given. Consider any theory with a global symmetry that commutes with supersymmetry and is spontaneously broken. If one imposes as a boundary condition that the fields should be periodic up to a global symmetry operation, then with such boundary conditions, $\mathrm{Tr}(-1)^F = 0$, because in the case of a spontaneously broken global symmetry, a non-zero energy is associated with a twist in the fields. This is so even if the theory would have $\mathrm{Tr}(-1)^F \neq 0$ if formulated with normal periodic boundary conditions.

However, in the particular case of an SU(N) gauge theory, the theories with and without magnetic flux can be related to each other by duality arguments that were introduced by 't Hooft. In this way, one can in principle derive consistency conditions relating our results with and without magnetic flux. We will not delve into this matter here, except to note that it is not too hard to prove that consistency is possible only if the state $|\Omega\rangle$ is a true Weyl invariant.

Finally, to conclude this section, we must prove an assertion that was made at the beginning of this section and used heavily. With untwisted boundary conditions, consider any gauge field A_i with $F_{ij} = 0$. We must show that, by a gauge transformation, we can set the A_i equal to constant matrices* C_i.

The first step is to define

$$U(x, y, z) = \mathrm{T}\exp\left(i\int_{(0,0,0)}^{(x,y,z)} A_i \,\mathrm{d}x^i \right). \tag{75}$$

The integral is independent of the path since $F_{ij} = 0$. If U were periodic, a gauge transformation by U could be used to set A_i to zero. However, U is in general not periodic. Instead, let us define

$$\begin{aligned} U_x &= U(L,0,0), \\ U_y &= U(0, L, 0), \\ U_z &= U(0,0, L). \end{aligned} \tag{76}$$

* In a non-abelian gauge theory, one may regard the gauge fields A_i as matrices in any representation of the gauge group that is of interest. In the statements that follow, the A_i should be regarded as matrices in a representation of the gauge group in which the center acts faithfully. In the case of SU(N), ordinary $N \times N$ matrices form a suitable representation.

From the fact that $F_{ij}=0$, it follows that the U_i commute with each other $U_iU_j=U_jU_i$.

Because the U_i commute with one another, they can be put by a global gauge transformation in the Cartan subgroup – the maximal abelian subgroup of the gauge group. With this done, we can write $U_i=\exp(-iC_i)$, where now the C_i are matrices in the Cartan subalgebra.

We next define

$$V(x,y,z)=U(x,y,z)\exp\frac{i}{L}\left(\sum x_iC_i\right). \tag{77}$$

Now, V is perfectly periodic, and a gauge transformation by V sets $A_i=C_i$, as was desired.

9. Inclusion of matter

What happens in supersymmetric non-abelian gauge theories when one includes matter multiplets – charged fields other than the gauge field and its supersymmetric partner?

If the matter multiplets are massive, they do not affect the determination of $\mathrm{Tr}(-1)^F$. The trace is still non-zero, so supersymmetry is unbroken.

Now suppose that some of the matter fields are massless, but that they lie in a real representation of the gauge group and therefore *could* have had bare masses. Turning on the bare masses, our previous calculation of $\mathrm{Tr}(-1)^F$ is valid. So the ground-state energy is zero for any non-zero value of the bare mass, and hence also in the limit $m\to 0$. The only assumption needed here is that the $m\to 0$ limit of the massive theory does exist, and defines the massless theory.

There remains the difficult case in which there are charged fields in a complex representation of the gauge group. Such fields cannot be given gauge-invariant masses. We will not attempt to calculate $\mathrm{Tr}(-1)^F$ in such theories.

Finally, let us discuss the implications of a very important phenomenon that frequently arises in supersymmetric theories when there are scalar fields of zero bare mass. It often happens [13] that the potential energy is identically zero in some directions in field space.

Before discussing the implications of this, let us consider a few examples. If for simplicity we assume the superspace potential to be zero, then the potential energy of the scalar fields receives a contribution only from the gauge interactions. In this case the scalar potential is simply

$$\sum_a\left(e_a(\Phi^*,T_a\Phi)\right)^2, \tag{78}$$

where Φ are the scalar fields (scalar partners of left-handed fermions), T_a are the generators of the gauge group, and e_a are the gauge couplings.

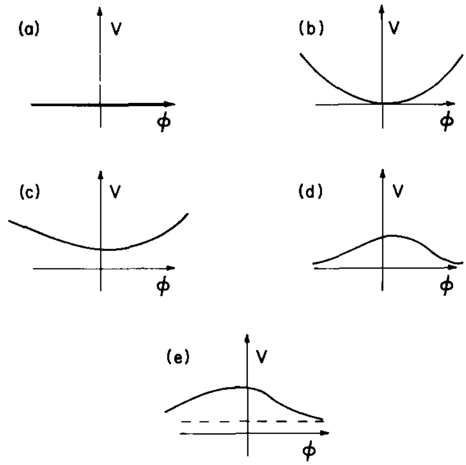

Fig. 12. The possible behavior of a theory in which a field ϕ is undetermined in perturbation theory. This field may remain undetermined, be uniquely determined, or be sent to infinity by the non-perturbative corrections.

There may be certain directions in field space in which (78) vanishes. For instance, if one considers an SU(N) gauge theory with an N and $\bar{N}$, A^i and B_j, then (78) vanishes as long as $B_i = (A^i)^*$. B and A can be arbitrarily large as long as they are related in this way. If one adds a complex field A^i_j in the adjoint representation of SU(N), then (78) vanishes as long as $[A, A^*] = 0$.

An example in which such a degeneracy does *not* occur is an SU(5) theory with one 10 and one $\bar{5}$. With two 10's and two $\bar{5}$'s, however, there is such a degeneracy. Denoting the 10's as C^{ij} and D^{ij} (antisymmetric in i and j) and the $\bar{5}$'s as G_i and H_i, it is easy to see that (78) vanishes if $C^{12} = G_1 = H_2$ and all other components vanish.

Degeneracies like this can also occur when supersymmetry is spontaneously broken at the tree level [6]. However, we will here consider theories with unbroken supersymmetry at the tree level in which minimizing the potential does not determine the fields uniquely. It is known that with supersymmetry unbroken, whatever degeneracy exists classically persists to all orders of perturbation theory.

Let us denote as ϕ a field that is undetermined classically. Thus, the effective potential $V(\phi)$ (with all other fields set equal to zero), vanishes identically in perturbation theory. In this situation, if perturbation theory can be believed, one may postulate an arbitrary vacuum expectation value for ϕ. As the masses and couplings of the elementary particles will (in general) depend on ϕ, every choice of ϕ leads to a physically different theory.

When non-perturbative effects are included, at least five qualitatively different possibilities can be imagined. The true quantum effective potential may be exactly zero for all ϕ, as appears to be the case in perturbation theory (fig. 12a). In this case, the theory really does have a continuous infinity of vacuum states.

Or the non-perturbative effects may give a non-trivial $V(\phi)$. This function may have a minimum at some value of ϕ, say $\phi=\phi_0$. In this case the vacuum is uniquely determined. Supersymmetry may remain unbroken, if $V(\phi_0)=0$ (fig. 12b), or be dynamically broken, if $V(\phi_0)>0$ (fig. 12c).

Finally, it may be that the function $V(\phi)$, with non-perturbative effects included, has no absolute minimum for any finite ϕ. It may vanish for large ϕ (fig. 12d) or approach a positive limit as $\phi\to\infty$ (fig. 12e). Such a theory would have no vacuum state at all, but might make sense in cosmology. A theory which exhibits such strange behavior in the classical approximation was pointed out in ref. [6].

Can we use the techniques of this paper to learn something about what option a given theory chooses? Unfortunately, there are severe difficulties in doing so. One serious problem is that $\mathrm{Tr}(-1)^F$ is likely to be mathematically ill-defined whenever there is a field ϕ which classically can be arbitrarily large at no cost in energy.

As always, the problem comes from the zero-momentum mode of the ϕ field. If ϕ can be arbitrarily large at no cost in energy, then, in a finite volume, the quantization of the zero-momentum mode gives rise to a continuous spectrum starting at zero energy. (Since the ϕ field can be arbitrarily large, the quantum mechanics of the zero mode resembles the motion of a free particle that can escape to infinity.) But we know that $\mathrm{Tr}(-1)^F$ is mathematically ill-defined when the hamiltonian has a continuous spectrum.

If the non-perturbative effects prevent ϕ from escaping to infinity, $\mathrm{Tr}(-1)^F$ may actually be mathematically well-defined. But this is of little help. We wanted to use $\mathrm{Tr}(-1)^F$ to learn about non-perturbative effects, not the other way around!

In some cases, some information can be gained by adding a perturbation that lifts the classical degeneracy. For instance, in the SU(N) theory with an N and $\overline{N}$, A^i and B_i, we may add a bare mass for A and B. This uniquely determines $A=B=0$ at the tree level. In the massive theory, $\mathrm{Tr}(-1)^F\neq 0$, and therefore the ground-state energy is zero. Taking the limit as the bare mass goes to zero, we learn, as discussed earlier, that the massless theory has a zero-energy ground state.

This reasoning has the following limitation. The massless theory has, according to perturbation theory, a one-parameter family of inequivalent vacuum states, with $B_i=(A^i)^*$, for any value of $|A|$. Taking the $m\to 0$ limit of the massive theory, we obtain the state at $A=B=0$, and we learn that this state really has zero energy. We do not learn about the states at non-zero A and B, which cannot be reached as the limit of the massive theory. So we do not learn whether the theory is of type (a) or type (b) in fig. 12.

More seriously, of course, regarding the massless theory as the limit of a massive theory is not possible when there are matter fields in a complex representation.

One final comment is in order. In an asymptotically free, infrared unstable theory, the question of dynamical symmetry breaking is usually regarded as a strong coupling problem. However, in the many interesting cases in which there is a field ϕ that classically can be arbitrarily big, this is not entirely true. If one could learn the large ϕ behavior of $V(\phi)$, one could learn a great deal. Is $V(\phi)$ identically zero for very large ϕ? If not, does it increase or decrease with ϕ? The answers to these questions would rule out many of the possibilities in fig. 12. But, given asymptotic freedom, the large ϕ behavior of $V(\phi)$ is a problem in the weak coupling domain! Unfortunately, even for weak coupling it is hard to determine whether dynamical supersymmetry breaking occurs, or equivalently, whether $V(\phi)$ is identically zero.

10. The non-linear sigma model

In this section we will evaluate $\mathrm{Tr}(-1)^F$ in the supersymmetric non-linear sigma model [14].

Although the non-linear sigma model (with or without supersymmetry) is unrenormalizable in four dimensions, it can arise as a low-energy approximation to a renormalizable theory. For instance, low-energy pion dynamics in QCD can be well-described by a non-linear sigma model. It is conceivable that, in the future, the results in this section could have applications to renormalizable supersymmetric theories, with the non-linear sigma model arising as an approximation, valid for computing low-energy quantities such as $\mathrm{Tr}(-1)^F$.

Apart from the speculative possibility of eventual applications, the non-linear σ model will be considered here for its mathematical interest.

The discussion will be formulated in a $1+1$ dimensional language, because in $1+1$ dimensions the non-linear sigma model is renormalizable. The results, however, apply equally well in $3+1$ dimensions.

The non-linear sigma model is, in general, a theory in which the scalar field takes its values in some riemannian manifold M. Simple choices for M such as S^N and CP^N have been widely studied. However, we may just as well imagine other choices of M – for example, the two-dimensional surface with two handles pictured in fig. 13. (On this space, with its negative curvature, the non-linear sigma model would not be asymptotically free. But asymptotic freedom will play no role in our discussion.)

Let ϕ^i be coordinates for the manifold M, and let $\gamma_{ij}(\phi^k)$ be the metric tensor of M. The non-linear sigma model is defined by introducing superfields $\Phi^i = \begin{pmatrix} \phi^i \\ \psi^i \end{pmatrix}$,

Fig. 13. A surface with two handles.

and a superspace lagrangian

$$\mathcal{L}=\tfrac{1}{2}\int \mathrm{d}^2x\,\mathrm{d}^2\theta\,\gamma_{ij}(\Phi^k)D_\alpha\Phi^i D_\beta\Phi^j\varepsilon^{\alpha\beta}. \tag{79}$$

In terms of ordinary component fields, this is

$$\mathcal{L}=\int \mathrm{d}^2x\left(\tfrac{1}{2}\gamma_{ij}\,\partial_\mu\phi^i\,\partial_\mu\phi^j+\tfrac{1}{2}\bar{\psi}_i i\gamma^k D_k\psi_i+\tfrac{1}{8}R_{ijkl}(\phi)\bar{\psi}^i\psi^k\bar{\psi}^j\psi^l\right), \tag{80}$$

where R_{ijkl} is the curvature tensor of the manifold M, and D_k is a covariant derivative, the details of which need not concern us.

The difficulty in calculating $\mathrm{Tr}(-1)^F$ is that the classical vacuum state is not unique, and possesses a continuous degeneracy. Any configuration $\phi^i(x,t)=\phi^i$, the ϕ^i being any constants, has zero energy at the classical level. As in all of our discussions of the zero-momentum modes of massless particles, this introduces some subtlety in the determination of $\mathrm{Tr}(-1)^F$.

We will discuss two methods to deal with the difficulty. The first and simpler approach is to switch on a perturbation that lifts the degeneracy. The second approach, by means of which we will be able to calculate also some generalizations of $\mathrm{Tr}(-1)^F$, is to carry out a Born-Oppenheimer quantization of the zero-momentum modes, ignoring the modes of non-zero momentum.

To lift the degeneracy we will apply a "magnetic field" in a supersymmetrically invariant way. We introduce an arbitrary function $h(\phi^i)$ defined on the manifold M. We add to the superspace potential a new term

$$\Delta\mathcal{L}=\tfrac{1}{2}\int \mathrm{d}^2x\,\mathrm{d}^2\theta\,h(\Phi). \tag{81}$$

This addition cannot change $\mathrm{Tr}(-1)^F$, according to our usual arguments. In terms of components the addition to the lagrangian is

$$\Delta\mathcal{L}=\int \mathrm{d}^2x\left(-\tfrac{1}{2}\gamma^{ij}\frac{\partial h}{\partial\phi^i}\frac{\partial h}{\partial\phi^j}-\frac{1}{2}\frac{\partial^2 h}{\partial\phi^i\,\partial\phi^j}\bar{\psi}^i\psi^j\right). \tag{82}$$

Thus, the scalar potential and the fermion mass matrix are, respectively,

$$V(\phi^i)=\tfrac{1}{2}|\nabla h|^2,\qquad m^2_{ij}=\frac{\partial^2 h}{\partial\phi^i\,\partial\phi^j}. \tag{83}$$

The potential energy $V(\phi^i)$ removes the troublesome degeneracy. The energy now vanishes, at the classical level, only if ϕ^i is such that $\partial h/\partial\phi^i=0$.

We now assume that h is a generic function, chosen so that $\partial h/\partial\phi^i=0$ only at isolated points on the manifold M. For instance, in fig. 14 we consider again the surface with two handles; h is chosen as the "height" function, and there are exactly 6 points at which $\partial h/\partial\phi^i = 0$.

Let us denote as p^a, $a = 1,\dots,k$, the points at which $\partial h/\partial\phi^i = 0$. We will assume that h is chosen generically so that at each of the p^a, the matrix $\partial^2 h/\partial\phi^i\,\partial\phi^j$ has no zero eigenvalues.

With the degeneracy lifted in this way, it is trivial to construct perturbation theory. We may expand around any of the p^a. Regardless of which of the p^a we expand around – which vacuum state we choose – supersymmetry is unbroken in perturbation theory, because the potential $V = \frac{1}{2}|\nabla h|^2$ vanishes at each of the p^a. There are no massless particles in perturbation theory, since we have assumed the mass matrix $\partial^2 h/\partial\phi^i\,\partial\phi^j$ to have no zero eigenvalues. This means that in perturbation theory, expanding around any of the p^a, there is precisely one zero-energy state, the "vacuum", and all other states have energy at least equal to the mass of the lightest particle. For example, in the problem illustrated in fig. 14, there are altogether precisely six zero-energy states in perturbation theory.

The only difficulty is that we must determine which of the zero-energy states are bosonic and which are fermionic. This is not as trivial as one might at first expect.

In three dimensions, $(-1)^F$ can be defined as $\exp(2\pi i J_z)$. Bosons can be distinguished from fermions by their angular momentum. In particular, when one finds precisely one zero-energy state in expanding around a given minimum of the potential, it is necessarily bosonic, because fermionic representations of the angular momentum algebra have dimension two or higher.

In one dimension, there is no angular momentum, so a definition such as $(-1)^F = \exp(2\pi i J_z)$ is not available. One must define the operator $(-1)^F$ more abstractly, as being the operator which commutes with all elementary Bose fields and anticommutes with all elementary Fermi fields. In other words, $(-1)^F$ is defined by requiring it to obey

$$(-1)^F\phi = \phi(-1)^F, \qquad (-1)^F\psi = -\psi(-1)^F, \tag{84}$$

where ϕ and ψ are any elementary Bose and Fermi fields.

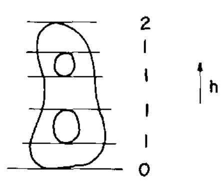

Fig. 14. A surface with two handles; a "magnetic field", indicated by the "height function" h, has been introduced to lift the degeneracy that exists in the supersymmetric non-linear sigma model. This function h has six stationary points. The planes tangent to the surface are drawn at those six points. Each stationary point is labeled by the number of negative eigenvalues of $\partial^2 h/\partial\phi^i\,\partial\phi^j$ at that point (0, 1, or 2).

This definition does not fix the overall sign of the operator $(-1)^F$. If $(-1)^F$ satisfies (84), then obviously $-(-1)^F$ satisfies (84) equally well. Conventionally, one removes this ambiguity by *defining* the vacuum $|\Omega\rangle$ to be bosonic, $(-1)^F|\Omega\rangle = +|\Omega\rangle$.

If there is only one vacuum state, that is the end of the story. However, in our problem, we have k "vacuum" states $|\Omega^a\rangle$, obtained by expanding about the various points p^a at which $\partial h/\partial\phi^i = 0$. Moreover, as we are working in a finite volume, tunneling processes communicate between these states; they are all part of the same Hilbert space. We may define any one of our k states to be, say, bosonic, but we must then *determine* which of the others are bosonic and which are fermionic*.

The answer to this question is perhaps rather surprising. It can be explained in the following simple and somewhat heuristic way**. Consider first the theory of a free Majorana fermion with mass term $m\bar{\psi}\psi$:

$$\mathcal{L} = \tfrac{1}{2}\int d^2x\,(\bar{\psi} i \not{\partial}\psi - m\bar{\psi}\psi)\,. \tag{85}$$

Of course, either sign of m may be considered; the physical mass of the fermion is $|m|$, and by chiral symmetry the content of the theory does not depend on the sign of m.

Let us define the zero momentum modes of the Fermi field ψ:

$$\sigma_1 = \frac{1}{\sqrt{L}}\int dx\,\psi_1(x)\,, \qquad \sigma_2 = \frac{1}{\sqrt{L}}\int dx\,\psi_2(x)\,.$$

The names σ_1 and σ_2 are motivated by the fact that, after quantization, these operators satisfy the sigma matrix algebra $\sigma_1^2 = \sigma_2^2 = 1$, $\sigma_1\sigma_2 + \sigma_2\sigma_1 = 0$. The hamiltonian for the zero-momentum mode is very simple. As the zero momentum mode has no kinetic energy, it receives a contribution only from the mass terms. This contribution is simply $H = -im\sigma_1\sigma_2$ or equivalently,

$$H = m\sigma_3\,, \tag{86}$$

where $\sigma_3 = -i\sigma_1\sigma_2$ is the "number operator" of the zero-momentum mode.

* From this description, it may seem that in (3 + 1) dimensions, the relation $(-1)^F = \exp(2\pi i J_z)$ could be used to prove that the $|\Omega^a\rangle$ are all bosonic. This is not true, for the following reason. For technical reasons, switching on the magnetic field h in 3 + 1 dimensions can preserve supersymmetry (or rather, a sufficiently large portion of the supersymmetry algebra for our purposes) only if it is done in a way that ruins rotation invariance. One can still use the perturbation h to lift the degeneracy and facilitate calculating $\mathrm{Tr}(-1)^F$; and setting h to zero at the end of the calculation restores rotation invariance without changing $\mathrm{Tr}(-1)^F$. But rotation invariance cannot be used, in the presence of a non-zero h, to determine whether $|\Omega^a\rangle$ are bosonic or fermionic. In fact, the argument given in the text yields the correct answer in 3 + 1 as well as in 1 + 1 dimensions.

** What follows is somewhat analogous with recent work by Goldstone and Wilczek [15].

Now, (86) shows that, in the ground state, the zero-momentum mode is empty or filled according to whether m is positive or negative. When m is changed in sign, the ground state gains a fermion it did not have, or loses one it had. The ground state goes from being bosonic to being fermionic when m is changed in sign*.

Let us now consider a theory with N Majorana fermions of masses $m_1, m_2, \dots, m_N$. If we normalize $(-1)^F$ so that the vacuum is considered a boson if all m_i are positive, then in general the vacuum is bosonic or fermionic depending on whether the number of negative m_i is even or odd. The number of negative m_i is the number of additional fermions that the vacuum contains, relative to the number it would have if the m_i were all positive.

Now let us apply this to our problem. We have seen that the fermion mass matrix is $\partial^2 h/\partial\phi^i\,\partial\phi^j$. Let n^a be the number of negative eigenvalues of this matrix, evaluated at the ath minimum of the potential p^a. Then the ath vacuum state $|\Omega^a\rangle$ is bosonic or fermionic depending on whether n^a is even or odd, and it contributes $(-1)^{n^a}$ to $\mathrm{Tr}(-1)^F$. Adding the various contributions, we conclude

$$\mathrm{Tr}(-1)^F = \sum_a (-1)^{n^a}. \tag{87}$$

This is our desired result. For example, for the surface with two handles of fig. 14, of the six stationary points of h, one is the absolute minimum with $n^a = 0$, one is the absolute maximum with $n^a = 2$, and four are saddle points with $n^a = 1$. Altogether $\mathrm{Tr}(-1)^F = 1 + 1 - 4 = -2$ for this space.

Something about eq. (87) may seem puzzling. The left-hand side, $\mathrm{Tr}(-1)^F$, is a property of the non-linear sigma model only. But the right-hand of (87) appears to depend on the particular choice of the function h. However, it is a theorem in topology that the right-hand side of (87) is actually independent of h. It is one of the basic theorems of Morse theory** that $\Sigma_a(-1)^{n^a}$ is equal to the Euler characteristic $\chi(\mathrm{M})$ of the manifold M. This is so for any choice of h.

Before discussing special choices of M, let us first state the simplest generalization of (87). Suppose that our manifold M possesses an isometry $k: \mathrm{M} \to \mathrm{M}$. In the quantum field theory, there is then a conserved operator K that commutes with the supersymmetry charges and with the hamiltonian: $[K, Q_\alpha] = [K, H] = 0$. As in our discussion of charge conjugation invariance in sect. 6, we may then calculate the additional invariant quantity $\mathrm{Tr}(-1)^F K$.

It is not difficult to use the methods just described to evaluate $\mathrm{Tr}(-1)^F K$. One must choose the function h to be invariant under k. One finds $\mathrm{Tr}(-1)^F K = \mathrm{Lef}(k)$, the Lefschetz number of the mapping k.

* Modes of non-zero momentum can be ignored here because their energy (equal to $\sqrt{k^2+m^2}$) never goes to zero as a function of m.

** A readable introduction to Morse theory has been given by Bott [16].

Let us now consider some particular examples. A widely studied special case is the choice $M = S^N$, the N-dimensional sphere. An interesting choice for k is in this case the "isotopic parity" operator which reflects one of the coordinates (say $x_N \leftrightarrow -x_N$ if the sphere is defined by $x_0^2 + x_1^2 + \cdots + x_N^2 = 1$). In this case, by standard results in topology (or by a convenient choice of h, such as $h = x_0$) we have

$$\mathrm{Tr}(-1)^F = 1 + (-1)^N, \qquad \mathrm{Tr}(-1)^F K = 1 - (-1)^N. \tag{88}$$

For any value of N, one or the other of these expressions is non-zero, so supersymmetry is unbroken for any N.

Another widely studied case is the supersymmetric CP^N model. The Euler characteristic of CP^N is $N+1$, so for CP^N

$$\mathrm{Tr}(-1)^F = N + 1. \tag{89}$$

This is positive for all N, so supersymmetry is not spontaneously broken.

These results are in agreement with what has been found in explicit calculation in the $1/N$ expansion [17]. It has been found that, at least for large N, dynamical supersymmetry breaking does not occur.

Moreover, the S^N and CP^N models both possess a discrete chiral symmetry. In the S^N model there is a twofold chiral symmetry. In the CP^N case there is an apparent continuous chiral symmetry, broken by an anomaly to an $N+1$-fold discrete symmetry. According to the $1/N$ expansion, all of these discrete symmetries are spontaneously broken, so that the S^N (or CP^N) models possess two (or $N+1$) vacuum states, respectively.

This is in striking agreement with a plausible inference that might be drawn from (88) and (89). According to (88), the S^N model has at least two degenerate ground states, for any finite volume. According to (89), there are at least $N+1$ such states in the CP^N model. Apparently, the degeneracy predicted by (88) and (89) goes over, in the large-volume limit, to the degeneracy among the vacuum states of spontaneously broken chiral symmetry.

Now let us discuss another approach to the calculation of $\mathrm{Tr}(-1)^F$ in these models. As the modes of non-zero momentum carry a non-zero energy even at the classical level, we may ignore them, and determine the spectrum of low-lying states by quantizing the zero-momentum modes, which carry zero energy classically.

This amounts to saying that, prior to quantization, we should drop the spatial dependence of the fields. Assuming ϕ^i and ψ^i to be functions of time only, we integrate (80) over space and get the lagrangian

$$\mathcal{L} = \tfrac{1}{2} L \int \mathrm{d}t \left(\gamma_{ij}(\phi^k) \frac{\partial \phi^i}{\partial t} \frac{\partial \phi^j}{\partial t} + \bar{\psi}^i i \gamma^0 \frac{\mathrm{D}}{\mathrm{D}t} \psi_i + \tfrac{1}{4} R_{ijkl} \bar{\psi}^i \psi^k \bar{\psi}^j \psi^l \right). \tag{90}$$

This theory coincides with the original one for all states of energy much less than $1/L$, so it can be used to calculate $\mathrm{Tr}(-1)^F$, and analogous quantities.

Since the spatial dependence has been eliminated, (90) describes, after quantization, a supersymmetric quantum mechanics problem with only a finite number of degrees of freedom. It is plausible to suppose that such a problem could be analyzed. But it comes as a surprise to realize that (90) is equivalent to one of the most famous of all problems in mathematics!

In a convenient basis, $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and a Majorana spinor is of the form $\begin{pmatrix} \psi \\ \psi^* \end{pmatrix}$ (the lower component is the hermitian conjugate of the upper one). In this basis, the supersymmetry algebra is $Q^2 = Q^{*2} = 0$, $QQ^* + Q^*Q = H$.

The spinors ψ_i and their hermitian conjugate ψ_j^* satisfy, after quantization, the algebra

$$\{\psi_i, \psi_j\} = \{\psi_i^*, \psi_j^*\} = 0, \qquad \{\psi_i, \psi_j^*\} = \gamma_{ij}(\phi^k). \tag{91}$$

Thus, we may regard the ψ_j^* and the ψ_i as creation and annihilation operators, respectively.

After canonical quantization, the supersymmetry charges are

$$Q = i\sum \psi_i^* p_i, \qquad Q^* = -i\sum \psi_i p_i, \tag{92}$$

where $p_i = -i\mathrm{D}/\mathrm{D}\phi^i$ is the appropriate covariant derivative – the momentum conjugate to ϕ^i.

How may one describe the quantum mechanical states of this system? It is first of all possible to consider the states in which the fermion Hilbert space is completely empty, states that are annihilated by the ψ_i. The amplitude for such a state is an arbitrary function $A(\phi^k)$ of the scalar coordinates ϕ^k.

We may act on such states with a fermion creation operator ψ_i^*, to get a state containing a single fermion, of type i. The wave function of such a state is, for any i, an arbitrary function of the ϕ^k. So we describe these states by a wave function $A_i(\phi^k)$, which has a single index i tangent to the manifold.

States with two fermions, i and j, would be described by a wave function $A_{ij}(\phi^k)$, which, by Fermi statistics, must be antisymmetric in the two indices i and j labeling the fermions.

Continuing in this way, we see that a state of this system is described by specifying a function $A(\phi^k)$, a vector field $A_i(\phi^k)$, an antisymmetric second-rank tensor field $A_{ij}(\phi^k)$, an antisymmetric third-rank tensor field $A_{ijm}(\phi^k)$, and so on until we reach (if M has dimension N) an Nth rank antisymmetric tensor $A_{i_1 i_2 \cdots i_N}(\phi^k)$ which describes the completely filled Fermi sea. The Nth rank antisymmetric tensor must be proportional to $\varepsilon_{i_1 i_2 \cdots i_N}$, so it is equivalent again to a scalar function.

This is precisely the description of the de Rham complex of a manifold – the space of all p-forms, with $p = 0, 1, 2, \ldots, N$.

Now looking back at eq. (92), how does $Q=\Sigma\psi_i^* p_i$ act on a state $A_{i_1 i_2\cdots i_q}(\phi^k)$ with, say, q fermions? Q adds a new fermion, producing a new wave function that should have $q+1$ indices. At the same time Q differentiates the old wave function. The derivative has an index i which is the index that is added to produce the wave function. The new wave function is

$$\tilde{A}_{i,i_1,i_2\cdots i_q}=\left(\frac{\mathrm{D}}{\mathrm{D}\phi^i}A_{i_1 i_2\cdots i_q}\pm\cdots\right). \tag{93}$$

A cyclic permutation of indices, needed to ensure that $\tilde{A}$ is antisymmetric in the indices $(i, i_1,\ldots,i_q)$, as required by Fermi statistics, has not been written explicitly on the right-hand side of (93). We see, in short, that acting with Q takes the curl of the wave function – differentiate, and antisymmetrize with respect to all indices. Q is the exterior derivative d of the de Rham theory.

What about Q^*? Q^* removes a fermion, producing a wave function with one index less. Of course, one can only remove a fermion that is present. Q^* removes a fermion of type i while differentiating in the i direction. Acting on a wave function $A_{i_1 i_2\cdots i_q}(\phi^k)$, Q^* produces the new wave function

$$A_{i_1\cdots i_{q-1}}(\phi^k)=\frac{\mathrm{D}}{\mathrm{D}\phi^{i_q}}A_{i_1 i_2\cdots i_{q-1}}{}^{i_q}. \tag{94}$$

Q^* is the divergence operator – the adjoint operator d^* of the Rham theory.

The hamiltonian $H=QQ^*+Q^*Q=dd^*+d^*d$ is usually referred to as the laplacian acting on forms. The number of zero-energy states with q indices (zero-energy q forms) is known as the qth Betti number of the manifold, B_q. We have interpreted q as the number of fermions present, so q forms are to be regarded as bosonic or fermionic depending on whether q is even or odd. Therefore

$$\mathrm{Tr}(-1)^F=\sum_q(-1)^q B_q. \tag{95}$$

The right-hand side of (95) is equal, by the de Rham theory, to the Euler characteristic of M, so we regain our previous result.

Once one has interpreted the low-energy theory (90) as the de Rham theory on M, it follows more or less by definition that $\mathrm{Tr}(-1)^F K$ is equal to the Lefschetz number of k, as stated earlier.

We can use this framework to derive some new results. The non-linear sigma model in $1+1$ dimensions has a discrete chiral symmetry $\psi\to\gamma_5\psi$. Let Q_5 be the operator that implements this symmetry. In terms of supersymmetry operators $Q_\pm$ of definite chirality, Q_5 satisfies the relation $Q_5 Q_\pm=\pm Q_\pm Q_5$. We will have no further use for Q_+, but the relation

$$Q_5 Q_- = -Q_- Q_5 \tag{96}$$

is of interest.

This relation is analogous to the fact that $(-1)^F Q_\alpha = -Q_\alpha(-1)^F$. We may use (96) to repeat the analysis of sect. 2, but with Q_5 playing the role that previously was played by $(-1)^F$. Working in the sector of Hilbert space with $p=0$, only states of zero energy are annihilated by Q_-. Among states of non-zero energy, for every state $|\psi\rangle$ of $Q_5=+1$, $Q_-\,|\psi\rangle$ is a state of the same energy with $Q_5=-1$. However, among the zero-energy states, the number of states of $Q_5=+1$ is not necessarily equal to the number of states of $Q_5=-1$.

We can thus define a quantity $\operatorname{Tr} Q_5$, analogous to $\operatorname{Tr}(-1)^F$, which is saturated entirely by the zero-energy states, and cannot change when the parameters of the theory are changed. If $\operatorname{Tr} Q_5 \neq 0$, supersymmetry is unbroken, for any value of the parameters. Given an isometry $k: \mathrm{M} \to \mathrm{M}$, we can define a further invariant quantity $\operatorname{Tr} Q_5 K$, analogous to $\operatorname{Tr}(-1)^F K$.

How can we evaluate $\operatorname{Tr} Q_5$ and $\operatorname{Tr} Q_5 K$? We cannot now make use of the "magnetic field" h that made possible a simple evaluation of $\operatorname{Tr}(-1)^F$. The problem is that in the presence of such a perturbation, Q_5 is not conserved.

However, we can evaluate $\operatorname{Tr} Q_5$ and $\operatorname{Tr} Q_5 K$ by using the fact that the $0+1$ dimensional theory of eq. (90) is valid as a low-energy approximation to the $1+1$ dimensional theory of interest. $\operatorname{Tr} Q_5$ and $\operatorname{Tr} Q_5 K$ have the same values in $0+1$ as in $1+1$ dimensions.

How can we interpret γ_5 invariance in the de Rham theory? Recall that the fermion creation and annihilation operators ψ_i^* and ψ_i were defined as γ_0 eigenstates. Since $\gamma_5\gamma_0 = -\gamma_0\gamma_5$, Q_5 exchanges operators of $\gamma_0=+1$ with operators of $\gamma_0=-1$. Thus Q_5 exchanges the creation operators ψ_i^* with the annihilation operators ψ_i.

Q_5, therefore, is the operator that fills empty fermion states and empties filled ones. Q_5 is the symmetry between the empty Fermi sea and the filled sea. It exchanges q forms with $N-q$ forms. Q_5 is what is known in mathematics as the operation of Poincaré duality.

Once this is realized, it follows, more or less by definition, that $\operatorname{Tr} Q_5$ equals the Hirzebruch signature of the manifold M, while $\operatorname{Tr} Q_5 K$ equals the signature of the isometry k.

We may now make a list of correspondences between invariant quantities in the non-linear sigma model and topological invariants of the underlying manifold:

$$\begin{aligned} \operatorname{Tr}(-1)^F &= \chi(\mathrm{M}), \\ \operatorname{Tr}(-1)^F K &= \operatorname{Lef}(k), \\ \operatorname{Tr} Q_5 &= \operatorname{sign}(\mathrm{M}), \\ \operatorname{Tr} Q_5 K &= \operatorname{sign}(k). \end{aligned} \tag{97}$$

Here χ is the Euler characteristic, Lef is the Lefschetz number, and sign refers to the signature of M or k respectively.

If any quantity in (97) is non-zero, supersymmetry is not spontaneously broken. However, it is an open question whether, for M such that all quantities in (97) vanish (for each possible choice of k), there are cases where dynamical breaking of supersymmetry does occur.

11. Conclusion

We have shown that dynamical breaking of supersymmetry does not occur in certain interesting classes of theories.

One must be cautious about drawing conclusions. For certain classes of theories – notably, the gauge theories with complex matter representations – our methods do not apply, at least not without significant extension. It is not clear whether this is a serious or purely technical loophole.

One should also bear in mind that the incorporation of gravity will require significant change in the framework of this paper, and may ultimately give a very different flavor to the subject of dynamical breaking of supersymmetry.

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Appendix A

In this appendix, a simple example will be given of a theory in which supersymmetry is spontaneously broken in any finite volume, but restored in the infinite volume limit.

The theory will be a $1+1$ dimensional theory with a real scalar field ϕ and a Majorana fermion ψ. The lagrangian

$$\mathcal{L}=\int d^2x\left(\tfrac{1}{2}(\partial_\mu\phi)^2+\tfrac{1}{2}\bar{\psi}i\not{\partial}\psi-\tfrac{1}{2}\lambda^2(\phi^2+a^2)^2-\tfrac{1}{2}\lambda\phi\bar{\psi}\psi\right) \tag{98}$$

describes a supersymmetric version of ϕ^4 theory. The quantum mechanics model discussed in ref. [2] and in sect. 3 can be obtained from this model by dimensional reduction to $0+1$ dimensions.

For a^2 large and positive, supersymmetry is spontaneously broken in this theory, the ground-state energy being approximately $\frac{1}{2}\lambda^2a^4$. As long as we work in a finite volume, the sign of a^2 can be changed by conjugation, rather as in the discussion in sect. 3. Hence, in a finite volume, supersymmetry is spontaneously broken for arbitrary negative a^2 as well as positive a^2.

In the infinite volume limit, this is not so. For negative a^2, ϕ has a vacuum expectation value, and therefore ψ has a non-zero mass. There being no massless fermion that could become a Goldstone fermion, supersymmetry is unbroken if a^2 is large and negative.

We thus conclude that, for negative a^2, supersymmetry is spontaneously broken in a finite volume but restored in the infinite volume limit. How does this come about? For negative a^2, there are two vacuum states, with $\phi = \pm\sqrt{-a^2}$. As long as we work in a finite volume, tunneling between the two states is possible. The relevant instanton solutions are independent of space, and functions of time only. The action for such an instanton solution is proportional to the volume. As long as we work in a finite volume, instantons that tunnel between the two vacua trigger spontaneous supersymmetry breaking, exactly as in the quantum mechanics problem in one dimension less. In the infinite volume limit, the tunneling between the two vacua is suppressed, and supersymmetry is restored.

This also means that in the infinite volume limit, the ground-state energy per unit volume is not an analytic function of a^2. It vanishes identically for negative a^2 but not for positive a^2.

In this theory, $\mathrm{Tr}(-1)^F = 0$ whether a^2 is positive or negative. For positive a^2 this is obvious, there being no zero-energy states in perturbation theory. For negative a^2 there are two zero-energy states in perturbation theory. One is bosonic and one is fermionic; this can be seen by analogy with the discussion of the non-linear sigma model in sect. 10. Since $\mathrm{Tr}(-1)^F = 0$, it is possible for tunneling processes to trigger supersymmetry breaking.

Appendix B

In sect. 3 we introduced the concept of changing a coupling constant "by conjugation". We showed that if supersymmetry is unbroken for one value of a coupling constant that can be changed by conjugation, it is unbroken for any value.

That is a non-perturbative statement, but it has interesting implications for perturbation theory. Consider a supersymmetric theory which depends on a coupling, λ, which can be changed by conjugation, and on some other couplings α_i. As in sect. 3, we work in the $\boldsymbol{P} = 0$ sector of Hilbert space, and we consider any two hermitian supersymmetry charges Q_1 and Q_2. Equivalently, we may work with $Q_\pm = \sqrt{\frac{1}{2}}(Q_1 \pm iQ_2)$.

The statement that λ can be changed by conjugation means that

$$Q_+(\lambda_1; \alpha_i) = M^{-1}(\lambda_1, \lambda_2) Q_+(\lambda_2; \alpha_i) M(\lambda_1, \lambda_2). \tag{99}$$

(M may depend on the α_i, but this is irrelevant for our purposes.) In this appendix, we will not work in all generality but will specialize to the situations that actually arise in practice. In all practical cases M has the form

$$M(\lambda_1, \lambda_2) = \exp(\lambda_1 - \lambda_2)K, \tag{100}$$

where K is a hermitian matrix which obeys

$$[[Q_\alpha, K], K] = 0. \tag{101}$$

Suppose that we know that at $\lambda = 0$, supersymmetry is unbroken. We would like to know what happens at $\lambda \neq 0$. As discussed in sect. 3, the relation

$$Q_+(\lambda, \alpha_i) = \exp(-\lambda K) Q_+(0, \alpha_i) \exp \lambda K \tag{102}$$

implies that supersymmetry is also unbroken at $\lambda \neq 0$ provided that the operator $\exp \lambda K$ has a sensible behavior when acting on the energy eigenstates of the $\lambda = 0$ hamiltonian. However, in many interesting situations that is not the case.

If $\exp \lambda K$ is not a "good" operator, we can still learn something interesting by expanding (102) in perturbation theory. Returning to the hermitian basis of supersymmetry charges Q_1 and Q_2, relations (101) and (102) are equivalent (after some simple manipulations) to

$$\begin{aligned} Q_1(\lambda, \alpha_i) &= Q_1(0, \alpha_i) + i\lambda[Q_2(0, \alpha_i), K], \\ Q_2(\lambda, \alpha_i) &= Q_2(0, \alpha_i) - i\lambda[Q_1(0, \alpha_i), K]. \end{aligned} \tag{103}$$

We will now show that eq. (103) implies that, if the vacuum energy vanishes at $\lambda = 0$, it also vanishes to all finite orders in perturbation theory in λ. (This result would hold non-perturbatively if $\exp \lambda K$ were a "good" operator, but we will not assume this.)

Let $|\Omega_\sigma\rangle$, $\sigma = 1, \ldots, k$, be the states that have zero energy at $\lambda = 0$. These states are annihilated, therefore, by $Q_1(0, \alpha_i)$ and $Q_2(0, \alpha_i)$. Let us now diagonalize the operator $Q_1(\lambda, \alpha_i)$ perturbatively in λ. We will see that in perturbation theory $Q_1(\lambda, \alpha_i)$ has k zero eigenvalues.

The first step in degenerate perturbation theory is to diagonalize the matrix

$$\begin{aligned} M^{(1)}_{\sigma\tau} &= \langle \Omega_\sigma | \delta Q_1 | \Omega_\tau \rangle \\ &= i\lambda \langle \Omega_\sigma | [Q_2(0, \alpha_i), K] | \Omega_\tau \rangle. \end{aligned} \tag{104}$$

This vanishes because $Q_2(0, \alpha_i)$ annihilates the $|\Omega_\sigma\rangle$.

In second-order degenerate perturbation theory we must diagonalize the matrix

$$M^{(2)}_{\sigma\tau} = -\langle \Omega_\sigma | \delta Q_1 P \frac{1}{Q_1(0, \alpha_i)} P \delta Q_1 | \Omega_\tau \rangle. \tag{105}$$

Here P is the projection operator that annihilates the $|\Omega_\sigma\rangle$. Although Q_1 is not invertible, the operator $P(1/Q_1)P$ is well defined.

Since $\delta Q_1 = i\lambda[Q_2(0,\alpha_i), K]$, and since $Q_2(0,\alpha_i)$ annihilates the $|\Omega_\sigma\rangle$, (105) can be rewritten

$$M^2_{\sigma\tau} = -\lambda^2 \langle \Omega_\sigma | K Q_2(0,\alpha_i) P \frac{1}{Q_1(0,\alpha_i)} P Q_2(0,\alpha_i) K | \Omega_\tau \rangle . \tag{106}$$

Because Q_2 annihilates precisely those states that Q_1 annihilates, $Q_2 P = P Q_2$. We may therefore rewrite (106) in the form

$$M^{(2)}_{\sigma\tau} = -\lambda^2 \langle \Omega_\sigma | K P Q_2 \frac{1}{Q_1} Q_2 P K | \Omega_\tau \rangle . \tag{107}$$

But

$$\begin{aligned} P Q_2 \frac{1}{Q_1} Q_2 P &= -P Q_2^2 \frac{1}{Q_1} P \\ &= -P Q_1^2 \frac{1}{Q_1} P \\ &= -P Q_1 P , \end{aligned} \tag{108}$$

where the relations $Q_1^2 = Q_2^2 = H$, and $\{Q_1, Q_2\} = 0$ have been used. Also $PQ_1P = Q_1$, since the states annihilated by P are annihilated by Q_1 anyway. So (107) becomes

$$\begin{aligned} M^{(2)}_{\sigma\tau} &= \lambda^2 \langle \Omega_\sigma | K Q_1 K | \Omega_\tau \rangle \\ &= \tfrac{1}{2}\lambda^2 \langle \Omega_\sigma | Q_1 K^2 + K^2 Q_1 | \Omega_\tau \rangle \\ &\quad + \tfrac{1}{2}\lambda^2 \langle \Omega_0 | [K, [Q_1, K]] | \Omega_\tau \rangle . \end{aligned} \tag{109}$$

But (109) vanishes, because of eq. (101) and because Q_1 annihilates the $|\Omega_\sigma\rangle$.

We have shown that up to second order of perturbation theory, Q_1 has the same number of zero eigenvalues as at $\lambda = 0$. It is not difficult to show, by means of the same tricks, that this result holds to all finite orders of perturbation theory.

The result of this section may be regarded as a generalization of (one aspect of) the "non-renormalization" theorems of supersymmetric theories. It has been proved previously that if supersymmetry is unbroken at the tree level, then it is unbroken to all finite orders of perturbation theory. This usual result refers to conventional perturbation theory with free field theory as the starting point. The result of this section is more general. The theory at $\lambda = 0$, around which we perturb, may be any theory in which supersymmetry is not spontaneously broken; the other couplings α_i may have arbitrary values.

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