ON THE DIRAC EQUATION IN CURVED SPACE-TIME

M.D. Pollock

V.A. Steklov Mathematical Institute, Russian Academy of Sciences
Ulitsa Gubkina 8, Moscow 119991, Russia[†]
mdp30@cam.ac.uk

(Received March 11, 2010)

We discuss in detail the general-relativistically covariant Dirac equation derived by Fock for a particle of rest mass m and charge e in an electromagnetic potential A_i, $[i\gamma^k(\partial_k - \mathit{\Gamma}_k - ieA_k) - m]\psi = 0$. The spinorial affine connection is given in terms of the spin connection ω_{abi} and spin operator $\bar{s}_{ab}$ by the formula $\mathit{\Gamma}_i = -\omega_{abi}\bar{s}^{ab}/4$, which follows from the assumption that the curved-space gamma matrices γ_i are covariantly constant, and which we prove to be equivalent to the 'tetrad postulate' of van Nieuwenhuizen, that the tetrad $t_i{}^a$ is covariantly constant. The intermediate result that $\gamma_k \mathit{\Gamma}_i^\dagger \gamma^k = 0$ is also proven. Extension to dimensionality D is straightforward, and results in the formula $\widehat{\mathit{\Gamma}}_I = -\hat{\omega}_{abI}\widehat{\bar{s}}^{ab}/4$ for the spinorial connection. Reduction of the five-dimensional Dirac equation to four dimensions has been shown by Klein, in the approximation linear in A_i, to yield in addition an anomalous Pauli mass term $\frac{1}{2}i\sqrt{\pi G_{\mathrm{N}}}\, F_{ij}s^{ij}$, which produces a correction to the intrinsic magnetic moment of the electron by the factor $(1+\delta)$, where $\delta = -\sqrt{1/\alpha}\, m/M_{\mathrm{P}} = -4.90\times 10^{-22}$, of theoretical interest but beyond the range of current experiment. We also discuss the TCP theorem in curved space-time, with particular reference to the heterotic superstring theory of Gross *et al.*, in the expanding Friedmann Universe. Previously, we have established the interrelationship between non-invariance of the metric under T, defined with regard to comoving time by $t \to -t$, due to general relativity, and non-invariance of the superstring under P, due to the asymmetric construction of the world sheet, which contains only right-moving Majorana fermions, while TP is conserved. This motivates study of C and the dimensional fermionic existence conditions found by van Nieuwenhuizen, Chapline and Slansky, Wetterich and Gliozzi *et al.*

PACS numbers: 03.70.+k, 04.50.+h, 29.30.Kv, 75.20.–g

[†] Temporary address.

1. Introduction

It is a result due to Fock [1] that the wave function ψ for a spin-1/2 particle of rest mass m and charge e in curved space-time obeys the equation

$$\left[i\gamma^k\left(\partial_k - \Gamma_k - ieA_k\right) - m\right]\psi = 0\,. \tag{1}$$

Here, γ_i are the generalized gamma matrices defining the covariant Clifford algebra [2]

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2g_{ij}\,, \tag{2}$$

where g_{ij} is the space-time metric, whose signature is

$$\operatorname{sgn} g_{ij} = (+ - --) \tag{3}$$

(we shall also consider the opposite signature

$$\operatorname{sgn} g_{ij} = (- + ++) \tag{4}$$

below), Γ_i is the spinorial affine connection and A_i is the electromagnetic four-vector potential.

Long ago, it was shown by Ricci and Levi-Civita [3] that g_{ij} can be related at every point to a Minkowski tangent space η_{ab} via the tetrads $t_i{}^a$, which obey the orthogonality conditions

$$t_i{}^a t^j{}_a = \delta_i^j\,, \qquad t_i{}^a t^i{}_b = \delta_b^a\,, \tag{5}$$

as

$$g_{ij} = t_i{}^a t_j{}^b \eta_{ab}\,, \qquad \eta_{ab} = t^i{}_a t^j{}_b g_{ij}\,. \tag{6}$$

Application of the tetrad method to general relativity was discussed by Einstein [4] in the context of distant parallelism, and by Wigner [5], who noted the freedom to make Lorentz transformations in $t_i{}^a$.

We can now define the spin operator

$$\bar{s}_{ab} = \tfrac{1}{2}\left(\bar{\gamma}_a\bar{\gamma}_b - \bar{\gamma}_b\bar{\gamma}_a\right) \tag{7}$$

and the spin connection

$$\omega_{abj} = t_{ka}\left(\partial_j t^k{}_b + \Gamma^k_{jl} t^l{}_b\right)\,, \tag{8}$$

where barred quantities are defined in the tetrad frame, that is $\bar{\gamma}_a = t^i{}_a\gamma_i$. (It is a straightforward exercise to prove the anti-symmetry of ω_{abj} in the first two indices, $\omega_{abj} = -\omega_{baj}$.) In terms of these two quantities, the connection Γ_j can be expressed as

$$\Gamma_j = -\tfrac{1}{4}\omega_{abj}\bar{s}^{ab}\,. \tag{9}$$

The starting point for the derivation of Eq. (9) is the purely geometrical assertion that the curved-space gamma matrices are covariantly constant, in the sense that

$$\nabla'_j \gamma^k \equiv \nabla_j \gamma^k + \gamma^k \mathit{\Gamma}_j + \mathit{\Gamma}_j^\dagger \gamma^k = 0\,, \tag{10}$$

where ∇_j is the tensorial covariant derivative operator, so defined that

$$\nabla_j \gamma_k = \partial_j \gamma_k - \mathit{\Gamma}^l{}_{jk} \gamma_l\,, \tag{11}$$

and the Hermitian conjugate is $^\dagger \equiv^{\mathrm{T}*}$.

Since the $\bar\gamma_a$ are constant, Eq. (10) is precisely the 'tetrad postulate' of van Nieuwenhuizen [6] — namely, the hypothesis that the *tetrad* is covariantly constant, that is

$$\partial_j t_k{}^a - \mathit{\Gamma}^l_{jk} t_l{}^a + \omega^{ab}_j t_{kb} = 0\,. \tag{12}$$

For multiplication of Eq. (12) by the constant gamma matrix $\bar\gamma_a$, defined in the tetrad frame, leads to Eq. (10), taking into account the anti-Hermitian character of $\mathit{\Gamma}_i$, which we shall shortly prove,

$$\mathit{\Gamma}_i^\dagger = -\mathit{\Gamma}_i\,. \tag{13}$$

The resulting first two terms simply comprise the covariant derivative (11). With regard to the second two terms in Eq. (10), using Eqs. (9) and (13) we have

$$\begin{aligned}\mathit{\Delta}_{jk} &\equiv \gamma_k \mathit{\Gamma}_j + \mathit{\Gamma}_j^\dagger \gamma_k = \gamma_k \mathit{\Gamma}_j - \mathit{\Gamma}_j \gamma_k = \tfrac{1}{4}\omega_{abj}\left(\bar s^{ab}\bar\gamma_c - \bar\gamma_c \bar s^{ab}\right) t_k{}^c \\ &= \tfrac{1}{8}\omega_{abj}\left(\bar\gamma^a\bar\gamma^b\bar\gamma_c - \bar\gamma^b\bar\gamma^a\bar\gamma_c - \bar\gamma_c\bar\gamma^a\bar\gamma^b + \bar\gamma_c\bar\gamma^b\bar\gamma^a\right) t_k{}^c\,. \end{aligned} \tag{14}$$

Let us examine the contribution to $\mathit{\Delta}_{jk}$ for fixed $a \neq b$. When $a \neq b \neq c$, we find that $\mathit{\Delta}_{jk}\Big|_{a\neq b\neq c} = 0$, after permuting terms. Alternatively, when $c = a \neq b$, we have

$$\begin{aligned}\mathit{\Delta}_{jk}\Big|_{c=a\neq b(\not\Sigma)} &= \tfrac{1}{8}\omega_{abj}\left(\bar\gamma^a\bar\gamma^b\bar\gamma_a - \bar\gamma^b\bar\gamma^a\bar\gamma_a - \bar\gamma_a\bar\gamma^a\bar\gamma^b + \bar\gamma_a\bar\gamma^b\bar\gamma^a\right) t_k{}^a \\ &= -\tfrac{1}{2}\omega_{abj}\bar\gamma^b t_k{}^a\,, \end{aligned} \tag{15}$$

since $\bar\gamma_a\bar\gamma^a(\not\Sigma) = 1$. Similarly,

$$\mathit{\Delta}_{jk}\Big|_{a\neq b=c(\not\Sigma)} = -\tfrac{1}{2}\omega_{abj}\bar\gamma^b t_k{}^a\,. \tag{16}$$

Adding Eqs. (15) and (16) and lifting the no-summation sign, we now obtain the result

$$\mathit{\Delta}_{jk} = -\omega_{abj}\bar\gamma^b t_k{}^a\,, \tag{17}$$

substitution of which into Eq. (10) yields Eq. (12).

2. The derivation of the Dirac equation

Before deriving Eq. (1), we have to discuss the question of Hermiticity. In Lorentzian space-time, the zeroth component of the tetrad gamma matrices $\bar{\gamma}_0$ and the spatial components $\bar{\gamma}_\alpha\,(\alpha = 1, 2, 3)$ have opposite Hermiticities, the precise assignment depending upon the space-time signature. In the signature (3) assumed here, we can represent the $\bar{\gamma}_a$ for example by the standard 4×4 matrices

$$\bar{\gamma}_0 = \begin{pmatrix} 0 & -\sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \quad \bar{\gamma}_\alpha = \begin{pmatrix} 0 & \sigma_\alpha \\ -\sigma_\alpha & 0 \end{pmatrix}, \quad \gamma_5 \equiv i\bar{\gamma}_0\bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \tag{18}$$

defined in terms of the generalized 2×2 Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{19}$$

The chiral representation (18), in which γ_5 is diagonal, is obtained from the Dirac representation by interchanging $\bar{\gamma}_0$ and γ_5. We recall that the σ_i are all Hermitian, when squared all yield the unit matrix, $\sigma_i^2(\not{\Sigma}) = \mathbf{1}$, and that the spatial components σ_α obey the commutation law $\sigma_\alpha\sigma_\beta - \sigma_\beta\sigma_\alpha = 2i\epsilon_{\alpha\beta\gamma}\sigma_\gamma$, guaranteeing the Clifford algebra (2). As a consequence, $\bar{\gamma}_0$ is Hermitian, while the $\bar{\gamma}_\alpha$ are anti-Hermitian. (The opposite signature (4) is obtained by transforming the gamma matrices to $\bar{\gamma}'_a = \pm i\bar{\gamma}_a$, which reverses the Hermiticities.)

This difference in Hermiticity between $\bar{\gamma}_0$ and $\bar{\gamma}_\alpha$ renders the analysis intractable, *a priori*, due to the presence of the second term $-\Gamma^l_{jk}\gamma_l$ in the covariant derivative (11), which has the effect of mixing the components γ_k and γ_l in the definition of $\nabla_j\gamma_k$. One way of dealing with this problem is to Euclideanize the metrics η_{ab} and g_{ij} via Wick rotation of the time coordinate $x^0 \equiv t$,

$$t \to \pm it\,, \tag{20}$$

so that

$$\gamma_0 \to \mp i\gamma_0\,, \qquad g_{00} \to -g_{00}\,, \qquad A_0 \to \mp iA_0\,. \tag{21}$$

(The Euclidean formalism is used in Ref. [6].) Now all the gamma matrices have the same Hermiticity in either metric signature.

Suppose first that the γ_i are all Hermitian. Then, to maintain the Hermiticity of Eq. (10), due to the fact that $\nabla_j\gamma_k$ is Hermitian, being linear in the γ_i (note that the covariant derivative is only Hermitian in general if *all* the γ_i are Hermitian), we have to write the last two terms in Eq. (10) as

$$\Delta_{jk} = \gamma_k\Gamma_j + (\mathrm{H.c.}) = \gamma_k\Gamma_j + \Gamma_j^\dagger\gamma_k^\dagger = \gamma_k\Gamma_j + \Gamma_j^\dagger\gamma_k\,. \tag{22}$$

Alternatively, suppose that all the γ_i are anti-Hermitian. Then, analogously, to maintain the anti-Hermiticity of Eq. (10), we have to write Δ_{jk} as

$$\Delta_{jk} = \gamma_k \Gamma_j - (\mathrm{H.c.}) = \gamma_k \Gamma_j - \Gamma_j^\dagger \gamma_k^\dagger = \gamma_k \Gamma_j + \Gamma_j^\dagger \gamma_k \,, \tag{23}$$

which coincides with Eq. (22). Thus, Eq. (14) is independent of metric signature.

Premultiplication of Eq. (10) by γ_k now yields the equation

$$4\Gamma_j = -\gamma_k \Gamma_j^\dagger \gamma^k - \gamma_k \partial_j \gamma^k - \Gamma_{jl}^k \gamma_k \gamma^l \,. \tag{24}$$

Eqs. (10), (13) and (24) imply that an arbitrary anti-Hermitian function times the unit matrix can be added to Γ_i without invalidating the solution, which we write as

$$\Gamma_i' = \Gamma_i + iC_i \mathbf{1} \,, \tag{25}$$

where C_i is real, thus preserving the anti-Hermiticity of Γ_i. Comparison with Eq. (1) shows that we can in fact identify C_i with eA_i in an external field A_i. In Appendix A, we prove that the first term on the right-hand side of Eq. (24) vanishes identically, so that, after substitution from Eqs. (7) and (8), we have

$$\Gamma_j = -\tfrac{1}{4}\gamma_k \left(\partial_j \gamma^k + \Gamma_{jl}^k \gamma^l\right) = -\tfrac{1}{4}\omega_{abj}\bar{s}^{ab} \,, \tag{26}$$

thus verifying Eqs. (9) and (13), the anti-Hermiticity of Γ_j following from that of $\bar{s}^{ab}$. Finally, the Lorentzian space-time can be reinstated by reversing the transformations (20), (21) to yield Eq. (1).

3. Electromagnetism

In Minkowski space-time, the electromagnetic gauge contribution to the Dirac equation for a charged spinor has to be added as the separate entity $e\gamma^k A_k$ in Eq. (1). It is therefore of much interest that by going to a curved space-time we find a way of incorporating this term automatically into the geometrical connection Γ_i, for an arbitrary background four-vector potential A_i. That is, by changing the definition of Γ_i, we can rewrite Eq. (1) as

$$\left[i\gamma^k \left(\partial_k - \Gamma_k'\right) - m\right]\psi = 0 \,, \qquad \Gamma_k' = \Gamma_k + ieA_k \,. \tag{27}$$

Therefore, the gauge symmetry of the Dirac equation, enunciated by Weyl [7] in Minkowski space-time (see p. 331 of Ref. [7]), can be geometrized in curved space-time to invariance under the simultaneous transformations

$$\partial_k \to \partial_k - i\Gamma_k'' \,,\ \psi\left(x^i\right) \to \psi'\left(x^i\right) = e^{i\int \Gamma_k'' dx^k}\psi\left(x^i\right) \,, \tag{28}$$

where $\Gamma_k'' = eA_k\mathbf{1}$, implying that $\Gamma_k''^\dagger = \Gamma_k''$ and hence invariance of the probability current, Eq. (61) below[1].

It is also interesting, however, to analyze the result obtained by the Kaluza–Klein mechanism [8,9] applied to the five-dimensional, curved-space Dirac equation, discussed by Klein [9]. Firstly, we recall that the modified Klein–Gordon equation obtained by squaring the operator in Eq. (1) takes the form, found by Schrödinger [10],

$$\left(\frac{1}{\sqrt{-g}}D_k'\left(\sqrt{-g}g^{kl}D_l'\right) - \tfrac{1}{4}R - \tfrac{1}{2}ieF_{ij}s^{ij} + m^2\right)\psi = 0\,, \tag{29}$$

where

$$D_k' = \partial_k - \Gamma_k'\,, \tag{30}$$

R is the Ricci scalar and $F_{jk} \equiv \partial_j A_k - \partial_k A_j$ is the electromagnetic field tensor, in the Lorentz gauge $\partial_k(\sqrt{-g}A^k)/\sqrt{-g} = 0$, for example.

Eq. (29) contains the interaction term $\frac{1}{2}ieF_{ij}s^{ij}$, which survives in the Minkowski-space limit $\Gamma_i = 0$, $R = 0$, originally discovered by Dirac [11], and is essential for agreement with experiment. The curved-space Eq. (1) in the uncharged case $e = 0$ has also been discussed by Dirac [12].

The presence of the term $\frac{1}{2}ieF_{ij}s^{ij} \equiv e(\boldsymbol{\sigma}\cdot\boldsymbol{B} - i\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{E})$ reflects the fact that the electron possesses an intrinsic quantized angular momentum. This idea was first conjectured by Compton [13] to explain a number of experimental results in ferromagnetic materials, especially the phenomenon of diamagnetism, and also some X-ray crystallographic experiments [14,15], the rotation of the plane of polarization by optically active substances [16] and the helical motion revealed by particle tracks in the cloud-chamber observations by Wilson.

Subsequently, Uhlenbeck and Goudsmit [17,18] showed that the hypothesis of quantized intrinsic electronic spin results in an improved explanation of the fine structure of the alkali spectra, particularly in the X-ray levels, if this intrinsic spin is quantized in odd multiples of $\hbar/2$. It also explains the anomalous Zeeman effect if the ratio of the magnetic moment of the electron to its angular momentum is twice as large for the intrinsic as for the orbital, that is the gyromagnetic ratio is $g = 2$, so that

$$\mu = \frac{e\hbar}{2mc}\,, \tag{31}$$

as discussed by Heisenberg and Jordan [19].

---

[1] From the definition (9), the Γ_k, which are trace-free, are complex or real according as the $\bar{s}^{ab}$ are complex or real, since ω_{abj} is real. Thus, the complexity of the Γ_k is representation-dependent, and the Γ_k are only real in a Majorana representation, for which the $\bar{\gamma}^a$ are all imaginary (or, if the spinor is massless, all real), implying that the $\bar{s}^{ab}$ are all real. In this case, $\Gamma_k'' = \mathrm{Im}\,\Gamma_k' = eA_k\mathbf{1}$.

Starting instead from the five-dimensional Dirac equation for a massless, uncharged spinor $\hat{\psi}$,

$$i\hat{\gamma}^K \hat{D}_K \hat{\psi} = 0\,, \tag{32}$$

where $K = 0, 1, 2, 3, 4$ and

$$\hat{D}_K = \hat{\partial}_K - \hat{\Gamma}_K\,, \tag{33}$$

the reduction to four dimensions is accomplished via the ansatz [9]

$$\hat{g}_{ij} = g_{ij} + \beta^2 A_i A_j \hat{g}_{44}\,, \quad \hat{g}^{ij} = g^{ij}\,, \quad \frac{\hat{g}_{i4}}{\hat{g}_{44}} = \beta A_i\,, \quad \hat{g}^{i4} = -\beta A^i\,, \quad |\hat{g}_{44}| = \alpha_4\,, \tag{34}$$

in which $\alpha_4\beta^2 = 2\kappa^2$ and $\kappa^2 \equiv 8\pi G_{\rm N}$ is the four-dimensional gravitational coupling, $G_{\rm N} \equiv M_{\rm P}^{-2}$ being the Newton constant and $M_{\rm P}$ the Planck mass.

This procedure does not lead precisely to Eq. (1), however. As discussed in some detail in Ref. [9], the equations generally become rather complicated — when Eqs. (34) are substituted into the definition for $\hat{\Gamma}_i$, we obtain terms non-linear in A_i, due for example to the fact that $\hat{\gamma}_i \equiv \hat{g}_{ij}\hat{\gamma}^j$ yields the term $(g_{ij} + \beta^2 A_i A_j \hat{g}_{44})\gamma^j$. Even in the approximation analyzed in Ref. [9], retaining only terms linear in A_i, there is a difference from Eq. (1). The gauge term in the electromagnetic vector potential $e\gamma^k A_k$ can be obtained by allowing the wave function to depend upon the additional coordinate x^4 through the periodic phase factor

$$\hat{\psi}\left(x^k\right) = \psi\left(x^k\right) \exp\left(2\pi i x^4/l_4\right), \tag{35}$$

where the constant $l_4 \equiv hc\sqrt{\alpha_4}\beta/e$ defines the periodicity. The interaction term $F_{ij}s^{ij}$ now occurs not only in the Klein–Gordon equation, obtained by squaring the Dirac operator, but also in the Dirac equation itself, which reads [9]

$$\left[i\gamma^k(D_k - ieA_k) - \tfrac{1}{2} i l_0 F_{ij}s^{ij} - m + \ldots\right]\psi = 0\,, \tag{36}$$

where $l_0 = \sqrt{\alpha_4}\beta/4$.

The extra term $-\frac{1}{2} i l_0 F_{ij}s^{ij}$ in Eq. (36) is the so-called 'anomalous' interaction discussed by Pauli [20, 21], who emphasized (see p. 233 of Ref. [20]) that the requirements of relativistic invariance, gauge invariance and correspondence do not determine the Dirac equation uniquely in the Minkowski-space limit, allowing this additional possibility, derivable from the Lagrangian

$$\delta\mathcal{L} = -\tfrac{1}{2} i\sqrt{-g}\tilde{\psi} l_0 F_{ij}s^{ij}\psi\,, \tag{37}$$

where $\tilde{\psi} = \psi^\dagger\gamma^0$. Pauli [20] also emphasized, however, that the Dirac theory [11] already explains the values of the intrinsic spin and resulting magnetic moment (31) of the electron without need of modification, making the anomalous term superfluous.

Regarding the question of the electromagnetic interaction energy, the anomalous contribution can be understood directly, for the addition of expression (37) to the uncharged Lagrangian density $\mathcal{L}$ implies a change in the spinor Hamiltonian density, defined (see Schweber [22], for example) as

$$\mathcal{H} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} \partial_0 \psi - \mathcal{L} = \sqrt{-g} \left[i\tilde{\psi}\gamma^\alpha \partial_\alpha \psi + \left(m + \tfrac{1}{2} i l_0 F_{ij} s^{ij}\right) \tilde{\psi}\psi \right] , \tag{38}$$

by the increment

$$\delta\mathcal{H} = \tfrac{1}{2} i l_0 \sqrt{-g} \tilde{\psi} F_{ij} s^{ij} \psi \, . \tag{39}$$

In other words, the effective mass of the spinor is

$$m'_{\text{eff}} = m + \tfrac{1}{2} i l_0 F_{ij} s^{ij} = m + \boldsymbol{\mu}_{\text{anomalous}} \cdot \boldsymbol{B} - i l_0 \gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{E} \, , \tag{40}$$

where the electric and magnetic three-vector fields are defined by[2]

$$E_\alpha = -F_{0\alpha} \qquad \text{and} \qquad B^\alpha = \tfrac{1}{2} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma} \, , \tag{41}$$

respectively, while the spatial components of the spin operator can be written as

$$s_{\alpha\beta} = -i \epsilon_{\alpha\beta\gamma} \sigma^\gamma \, , \tag{42}$$

which follows from the commutator of the Pauli matrices σ_α.

Ignoring the electric field, Eq. (40) shows how the effective mass of an uncharged fermion is increased by the amount $\boldsymbol{\mu}_{\text{anomalous}} \cdot \boldsymbol{B}$ in an external magnetic field $\boldsymbol{B}$, where the anomalous magnetic moment is defined by

$$\boldsymbol{\mu}_{\text{anomalous}} = l_0 \boldsymbol{\sigma} \, . \tag{43}$$

The Lagrangian of the original Dirac theory [11], on the other hand, does not contain a magnetic-moment term *per se*. Due to the quantum-theoretical nature of the wave equation (1) for ψ, this term appears explicitly only upon squaring the Dirac operator. (The reason for this is that the quantum-mechanical operator replacement $-i\hbar\partial_k$ for the momentum p_k does not commute with the gauge vector potential A_k.) The effective mass-squared in the resulting Klein–Gordon equation is [11]

$$m^2_{\text{eff}} = m^2 - \tfrac{1}{2} i e F_{ij} s^{ij} \approx m^2 - e \sigma_\alpha B^\alpha \, . \tag{44}$$

[2] The three-vector field identifications (41) are obtained from the identity ${}^*F^{ij}{}_{;j} \equiv 0$, where ${}^*F^{ij} \equiv \epsilon^{ijkl} F_{kl}$ is the dual field tensor and $\epsilon^{ijkl} = \delta^{ijkl}/\sqrt{-g}$. The $(^\alpha)$ component can be written in curved space-time as $\epsilon^{\alpha\beta\gamma}(\sqrt{h}\partial_\beta F_{0\gamma}) - \frac{1}{2}\partial_0(\epsilon^{\alpha\beta\gamma}\sqrt{h}F_{\beta\gamma}) = 0$, or in three-vector notation as $\operatorname{curl}\boldsymbol{E} + \partial_0(\sqrt{h}\boldsymbol{B})/\sqrt{h} = 0$, where $\epsilon^{\alpha\beta\gamma} = \delta^{\alpha\beta\gamma}/\sqrt{h}$, $h = \det h_{\alpha\beta} = -g/g_{00}$ and $h_{\alpha\beta} \equiv -g_{\alpha\beta} + g_{0\alpha}g_{0\beta}/g_{00}$ is the physical three-metric. The scalar and three-vector potentials are defined as $\varphi = -A_0$, $(\boldsymbol{A})_\alpha = A_\alpha$, so that $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \partial_0 \boldsymbol{A}$ and $\boldsymbol{B} = \operatorname{curl}\boldsymbol{A}$.

In the non-relativistic limit $|e\sigma_\alpha B^\alpha| \ll m^2$, Eq. (44) implies a change in the effective mass of the electron to the level

$$m_{\text{eff}} \approx m - \frac{e\sigma_\alpha B^\alpha}{2m}\,, \tag{45}$$

enabling us to identify the intrinsic magnetic moment of the electron with expression (31).

To recapitulate, the mass parameter m of the uncharged theory appears on the same footing in the Dirac Eq. (1) and the Klein–Gordon Eq. (29). It is only the intrinsic magnetic-moment term which is present in Eq. (29), yet absent in Eq. (1), this difference being due solely to the *operator* nature of the equation, independently, in particular, of any specific solution for the wave function ψ. Note also that the gamma matrices γ^i occur linearly in Eq. (1), which is thus representation-dependent, but quadratically in Eq. (29), where they combine to form the metric g_{ij}. In this sense, it therefore seems that we need Eq. (29) to give complete expression to the ideas of general relativistic covariance and the interaction with an external electromagnetic field.

If the wave equation for an electrically charged spinor also includes the 'anomalous' term, then in Minkowski space-time, from Eq. (36) we have

$$\left[i\gamma^k(\partial_k - ieA_k) - m'_{\text{eff}}\right]\psi = 0\,, \tag{46}$$

where

$$m'_{\text{eff}} = m + \tfrac{1}{2} i l_0 F_{ij} s^{ij}\,. \tag{47}$$

To obtain the net magnetic moment, it is necessary to square the operator of Eq. (46), which yields

$$\left[\partial'_k \partial'^k + m^2_{\text{eff}} - \tfrac{1}{2} l_0 \gamma^k \partial_k \left(F_{ij} s^{ij}\right)\right]\psi = 0\,, \tag{48}$$

where

$$\partial'_k = \partial_k - ieA_k \tag{49}$$

and

$$m^2_{\text{eff}} = m'^2_{\text{eff}} - \tfrac{1}{2} ie F_{ij} s^{ij} = m^2 - \tfrac{1}{2} ie \left(1 - \sqrt{\frac{1}{\alpha}}\frac{m}{M_{\text{P}}}\right) F_{ij} s^{ij} - \tfrac{1}{4} l_0^2 \left(F_{ij} s^{ij}\right)^2. \tag{50}$$

In deriving Eq. (48), we have used the fact that $\gamma_i \gamma_j = g_{ij} + s_{ij}$ and have introduced the fine structure constant

$$\alpha = \frac{e^2}{4\pi\hbar c \epsilon_0} = \frac{e^2}{4\pi}\,, \tag{51}$$

in units such that $c = \hbar = \epsilon_0 = 1$, where ϵ_0 is the permittivity of free space.

From Eq. (50), we see, ignoring the term quadratic in $F_{ij}s^{ij}$, that the anomalous electromagnetic interaction produces a correction to the intrinsic magnetic moment of the electron by the factor $(1+\delta)$, where

$$\delta = -\sqrt{\frac{1}{\alpha}}\frac{m}{M_{\mathrm{P}}} \approx -4.90\times 10^{-22}\,. \tag{52}$$

Quantum electrodynamics, on the other hand, yields a magnetic-moment anomaly given, to within $\sim 0.2\%$, by the one-loop term calculated by Schwinger [23],

$$\delta_{\mathrm{QED}}\Big|_{\mathrm{one-loop}} = \frac{\alpha}{2\pi} \approx 1.16\times 10^{-3}\,, \tag{53}$$

the expansion to four loops agreeing with experiment to approximately one part in 10^9 (see Ref. [22] and Amsler *et al.* [24]).

The result (52), of gravitational origin and thus smaller by a factor $\sim -4.2\times 10^{-19}$, is beyond the reach of current technology, and consequently not susceptible to experimental test. It is nevertheless of interest that the initially five-dimensional Kaluza–Klein theory differs in its predictions from the four-dimensional theory by the presence of this additional anomalous term.

Essentially the same reasoning applies to the other fermionic elementary particles obeying the Dirac equation, both leptons (the μ and τ) and quarks. The precise numerical values can be obtained from Ref. [24].

4. The construction of the charge-conjugate spinor

The Hermitian conjugate of Eq. (1) for the wave function ψ of a massive, electrically charged spinor in the curved space-time with Lorentzian signature (3) is

$$\left[i\left(\partial_k\psi^\dagger - \psi^\dagger\Gamma_k^\dagger\right) - eA_k\psi^\dagger\right]\left(\gamma^k\right)^\dagger + m\psi^\dagger = 0\,, \tag{54}$$

assuming that m and A_k are both real. In order to obtain the Hermitian adjoint $\tilde{\psi}$, we have to postmultiply the conjugate wave function $\psi^\dagger$ by a quantity that reduces to $\bar{\gamma}^0$ in the Minkowski-space limit

$$t^i_{\ a} = \delta^i_{\ a}\,, \tag{55}$$

for which we require the flat-space, Hermitian-adjoint wave function

$$\tilde{\psi} = \psi^\dagger\bar{\gamma}^0\,. \tag{56}$$

The probability current is then

$$j^k = \tilde{\psi}\gamma^k\psi\,, \tag{57}$$

which is conserved upon application of the Dirac equation for ψ, yielding

$$\partial_k j^k = 0\,. \tag{58}$$

The zero component of j^k is the probability density which is positive semi-definite,

$$\bar{\rho} \equiv \bar{j}^0 = \tilde{\psi}\bar{\gamma}^0\psi = \psi^\dagger\psi \geq 0\,. \tag{59}$$

In curved space-time a problem arises, because the corresponding matrix quantity γ^0 is generally coordinate dependent, and therefore, in particular, the first term in Eq. (54), upon postmultiplication by γ^0, would yield

$$\left(\partial_k\psi^\dagger\right)\gamma^0 = \partial_k\left(\psi^\dagger\gamma^0\right) - \psi^\dagger\partial_k\gamma^0\,. \tag{60}$$

The presence of the second term on the right-hand side of Eq. (60) means that the quantity $\psi^\dagger\gamma^0$ will not exactly obey the Dirac equation, and to resolve this difficulty Bargmann [25] argued that one should define the Hermitian adjoint $\tilde{\psi}$ by Eq. (56), even when $\bar{\gamma}^0 \neq \gamma^0$. The resulting probability current

$$j^k = \psi^\dagger\bar{\gamma}^0\gamma^k\psi \tag{61}$$

is conserved in the sense [25] (see Eq. (41) of Ref. [25])

$$\nabla_k j^k = 0\,, \tag{62}$$

which follows by construction of the sum $[\,\tilde{\psi} \times$ Eq. (1) + Eq. (69) $\times\, \psi\,]$, remembering Eqs. (10) and (13).

This formalism was discussed further by Parker [26,27], who pointed out that the probability density could be made positive semi-definite at each point by choosing a locally inertial coordinate system in which $\gamma^k(x^l) = \bar{\gamma}^k$, so that $\rho = \bar{\rho}$, given by Eq. (59). (He also applied the formulae to calculate the gravitational energy change for the hydrogen atom, obtaining the result that a shift in the lowest energy levels (non-relativistic 1S, 2S and 2P and relativistic $1\mathrm{S}_{1/2}$, $2\mathrm{S}_{1/2}$ and $2\mathrm{P}_{1/2}$) comparable to the theoretical and experimental value of 4.4×10^{-6} eV for the Lamb shift would require a characteristic space-time curvature of $\lesssim 10^{-3}$ cm.)

To proceed, let us postmultiply Eq. (54) by $\bar{\gamma}^0$, remembering that

$$\left(\bar{\gamma}^0\right)^\dagger = \bar{\gamma}^0\,, \qquad (\bar{\gamma}^\alpha)^\dagger = -\bar{\gamma}^\alpha \quad \text{and} \quad \bar{\gamma}^0\bar{\gamma}^\alpha = -\bar{\gamma}^\alpha\bar{\gamma}^0\,, \tag{63}$$

and consequently that

$$\left(\gamma^k\right)^\dagger\bar{\gamma}^0 \equiv t^k_{\ a}\,(\bar{\gamma}^a)^\dagger\,\bar{\gamma}^0 = \bar{\gamma}^0\gamma^k\,. \tag{64}$$

The second term in Eq. (54), times $\bar{\gamma}^0$, can now be written as

$$-i\psi^\dagger \Gamma_k^\dagger \left(\gamma^k\right)^\dagger \bar{\gamma}^0 = -i\tilde{\psi}\Gamma_k^{'''\dagger}\gamma^k\,, \tag{65}$$

in which we have inserted a factor of $\left(\bar{\gamma}^0\right)^2 \equiv 1$ after $\psi^\dagger$ and introduced the transformed spinorial affine connection $\Gamma_k^{'''}$, defined by

$$\Gamma_k^{'''} = \bar{\gamma}_0 \Gamma_k \bar{\gamma}_0\,. \tag{66}$$

From Eq. (54) times $\bar{\gamma}_0$, Eq. (64) and Eq. (66), we now obtain the equation for the Hermitian adjoint (56),

$$\left[i\left(\partial_k\tilde{\psi} - \tilde{\psi}\Gamma_k^{'''\dagger}\right) - eA_k\tilde{\psi}\right]\gamma^k + m\tilde{\psi} = 0\,. \tag{67}$$

The quantity $\Gamma_k^{'''\dagger}$ occurring in the second term in Eq. (67) can be expanded, from the definition (9) of Γ_k and the Hermiticity and commutivity relations (63), as

$$\begin{aligned}\Gamma_k^{'''\dagger} = -\tfrac{1}{4}\omega_{abk}\bar{\gamma}^0\left(\bar{s}^{ab}\right)^\dagger\bar{\gamma}^0 &= -\tfrac{1}{2}\omega_{0\alpha k}\bar{\gamma}^0\bar{s}^{0\alpha}\bar{\gamma}^0 + \tfrac{1}{4}\omega_{\alpha\beta k}\bar{\gamma}^0\bar{s}^{\alpha\beta}\bar{\gamma}^0\\ &= \tfrac{1}{2}\omega_{0\alpha k}\bar{s}^{0\alpha} + \tfrac{1}{4}\omega_{\alpha\beta k}\bar{s}^{\alpha\beta}\\ &= -\Gamma_k\,,\end{aligned} \tag{68}$$

enabling us to rewrite Eq. (67) as

$$\left[i\left(\partial_k\tilde{\psi} + \tilde{\psi}\Gamma_k\right) - eA_k\tilde{\psi}\right]\gamma^k + m\tilde{\psi} = 0\,. \tag{69}$$

Following Ref. [22], we take the transpose $(^{\mathrm{T}})$ of Eq. (69) and make the substitution

$$\left(\gamma^k\right)^{\mathrm{T}} = -C^{-1}\gamma^k C\,, \tag{70}$$

where C is the charge-conjugation operator discussed by Kramers [28] and in Ref. [21]. The existence and constancy of C can be verified by writing γ^i in terms of the $\bar{\gamma}^a$, since the tetrads $t^i{}_a$ are invariant under the transposition operator. This implies the equation

$$\left(\bar{\gamma}^a\right)^{\mathrm{T}} = -C^{-1}\bar{\gamma}^a C\,, \tag{71}$$

(assuming that $t^i{}_a \neq 0$), to which there are solutions for C in all representations[3] of $\bar{\gamma}^a$. Thus,

$$C^{-1}\gamma^k C\left[i\left(\partial_k\tilde{\psi}^{\mathrm{T}} + \Gamma_k^{\mathrm{T}}\tilde{\psi}^{\mathrm{T}}\right) - eA_k\tilde{\psi}^{\mathrm{T}}\right] - m\tilde{\psi}^{\mathrm{T}} = 0\,, \tag{72}$$

[3] In the chiral representation (18), we find that $C = i\begin{pmatrix}-\sigma_2 & 0\\ 0 & \sigma_2\end{pmatrix}$.

premultiplication of which by C yields the equation

$$\left[i\gamma^k\left(\partial_k + C\Gamma_k^{\mathrm{T}}C^{-1} + ieA_k\right) - m\right]\psi^{\mathrm{c}} = 0 \tag{73}$$

for the charge-conjugate wave function

$$\psi^{\mathrm{c}} = C\tilde{\psi}^{\mathrm{T}}\,. \tag{74}$$

The operator for the second term in Eq. (73) is

$$i\gamma^k C\Gamma_k^{\mathrm{T}}C^{-1} = \frac{1}{8}i\gamma^k\omega_{abk}C\left[(\bar{\gamma}^a)^{\mathrm{T}}\left(\bar{\gamma}^b\right)^{\mathrm{T}} - \left(\bar{\gamma}^b\right)^{\mathrm{T}}(\bar{\gamma}^a)^{\mathrm{T}}\right]C^{-1} = -\gamma^k\Gamma_k\,, \tag{75}$$

and therefore Eq. (73) reads

$$\left[i\gamma^k\left(\partial_k - \Gamma_k + ieA_k\right) - m\right]\psi^{\mathrm{c}} = 0\,. \tag{76}$$

Eq. (76) is obtained from Eq. (1) for ψ by the charge reversal

$$e \to -e\,, \tag{77}$$

the two equations coinciding when $e = 0$, which thus proves the existence of uncharged Majorana fermions in an arbitrary, four-dimensional curved space-time.

Let us now study the Majorana representation in which the $\bar{\gamma}^a$ are all imaginary, so that

$$(\bar{\gamma}^a)^* = -\bar{\gamma}^a\,. \tag{78}$$

Taking the complex conjugate of Eq. (1), we have

$$\left[-i\left(\gamma^k\right)^*\left(\partial_k - \Gamma_k^* + ieA_k\right) - m\right]\psi^* = 0\,. \tag{79}$$

The reality of the space-time and tangent-space metrics g_{ij} and η_{ab} implies that the tetrad components and hence, from Eq. (8), the spin connection, are also real, as is the spin operator,

$$\left(t^i{}_a\right)^* = t^i{}_a\,, \qquad \omega_{abk}^* = \omega_{abk} \quad \text{and} \quad (\bar{s}_{ab})^* = \bar{s}_{ab}\,. \tag{80}$$

Consequently, the γ_i reverse sign under complex conjugation, while the spinorial affine connection (9) is invariant,

$$(\gamma_i)^* = -\gamma_i\,, \qquad \Gamma_i^* = \Gamma_i\,. \tag{81}$$

Substitution of the relationships (81) into Eq. (79) yields

$$\left[i\gamma^k\left(\partial_k - \Gamma_k + ieA_k\right) - m\right]\psi^* = 0\,, \tag{82}$$

which is precisely Eq. (76).

This is the special representation originally discovered and investigated by Majorana [29] in the Minkowski-space limit $\Gamma_i = 0$, with a view to elucidating the nature of the particle–anti-particle symmetry, for which the charge conjugate spinor (74) is defined by

$$\psi^{\mathrm{c}} \equiv C\tilde{\psi}^{\mathrm{T}} = \psi^*\,. \tag{83}$$

The wave function of the anti-particle is simply the complex conjugate of that of the particle, as discussed further by Racah [30].

For a massless, uncharged particle, Eq. (83) implies that the wave function is real,

$$\psi^* = \psi\,, \tag{84}$$

and in this case a Majorana representation is possible in which the γ_k are all imaginary or alternatively all real.

5. Discussion

In combination with the operators of time reversal T and parity, or spatial reflection, P, the charge-conjugation operator C gives rise to the theorem expressing conservation of the product TCP for a local theory defined by a Hermitian Lagrangian that is invariant under proper Lorentzian transformations and obeys the spin-statistics theorem (see Ref. [22] and references therein).

Previously [31], we have explored the application of this theorem to the superstring theory, initially formulated in a curved-space background, and it is therefore interesting to extend the foregoing analysis to arbitrary dimensionalities. See also Ref. [32] for quantum-theoretical aspects of TP, and hence TCP, in curved space-time.

Let us, then, consider the expanding Friedmann Universe, after reduction to four dimensions, the line element for which is

$$ds^2 = dt^2 - a^2(t)d\boldsymbol{x}^2\,, \tag{85}$$

where t is the comoving time coordinate and $a(t)$ is the radius function of the three-space $d\boldsymbol{x}^2$, and assume the heterotic superstring theory of Gross *et al.* [33–35]. A prime concern of Ref. [31] was to establish the interrelationship between non-invariance of the metric (85) under T, defined by

$$t \to -t\,, \tag{86}$$

due to general relativity, and non-invariance of the superstring under P, due to the asymmetric construction of the string world-sheet, which contains only right-moving Majorana fermions, whilst maintaining conservation of the product TP. The link between T and P results from the fact that far from the Planck era, the cosmological time coordinate t can also be identified with the world-sheet time coordinate τ.

When the space-time is generalized from the four-dimensional metric g_{ij} to a D-dimensional metric $\hat{g}_{AB}$, the analysis becomes modified in two respects. Analogously to Eq. (1), we start from the D-dimensional Dirac equation for the wave function $\hat{\psi}$,

$$\left[i\hat{\gamma}^K\left(\hat{\partial}_K - \hat{\Gamma}_K - ie\hat{A}_K\right) - m\right]\hat{\psi} = 0\,. \tag{87}$$

To derive the formula for $\hat{\Gamma}_I$, we assume the gamma matrices $\hat{\gamma}_I$ to be covariantly constant, and as in Eq. (10) write

$$\hat{\nabla}'_J\hat{\gamma}^K \equiv \hat{\nabla}_J\hat{\gamma}^K + \hat{\gamma}^K\hat{\Gamma}_J + \hat{\Gamma}^\dagger_J\hat{\gamma}^K = 0\,. \tag{88}$$

In place of Eq. (24), premultiplication of Eq. (88) by $\hat{\gamma}_K$ yields

$$D\hat{\Gamma}_J = -\hat{\gamma}_K\hat{\Gamma}^\dagger_J\hat{\gamma}^K - \hat{\gamma}_K\hat{\partial}_J\hat{\gamma}^K - \hat{\Gamma}^K_{JL}\hat{\gamma}_K\hat{\gamma}^L\,, \tag{89}$$

resulting in the solution analogous to (25),

$$\hat{\Gamma}'_J = \hat{\Gamma}_J + i\hat{C}_J\hat{\mathbf{1}}\,, \tag{90}$$

where

$$\hat{\Gamma}_J = -\frac{1}{4}\hat{\omega}_{ABJ}\hat{\hat{s}}^{AB}\,, \tag{91}$$

since $\hat{\gamma}_K\hat{\Gamma}_J\hat{\gamma}^K = (D-4)\hat{\Gamma}_J$ (see Appendix A).

Thus, we note, firstly, that Eq. (91) generalizes Eqs. (9) or (26), with the same numerical coefficient $-1/4$ on the right-hand side.

Secondly, concerning the charge-conjugate wave function, defined by

$$\hat{\psi}^{\mathrm{c}} = \hat{C}\hat{\bar{\psi}}^{\mathrm{T}}\,, \tag{92}$$

the derivation proceeds exactly as in Section 4 for the four-dimensional spinor ψ^{c}. In particular, Eq. (70) for the transformation taking γ^k to $\left(\gamma^k\right)^{\mathrm{T}}$ is replaced by

$$\left(\hat{\gamma}^K\right)^{\mathrm{T}} = -\hat{C}^{-1}\hat{\gamma}^K\hat{C}\,, \tag{93}$$

which, as previously, implies the corresponding equation relating the gamma matrices $\hat{\hat{\gamma}}^A$ and $\left(\hat{\hat{\gamma}}^A\right)^{\mathrm{T}}$ in the D-bein frame,

$$\left(\hat{\bar{\gamma}}^{A}\right)^{\mathrm{T}} = -\hat{C}^{-1}\hat{\bar{\gamma}}^{A}\hat{C}\,, \tag{94}$$

where

$$\hat{\bar{\gamma}}_{A}\hat{\bar{\gamma}}_{B} + \hat{\bar{\gamma}}_{B}\hat{\bar{\gamma}}_{A} = 2\hat{\eta}_{AB}\,. \tag{95}$$

Eq. (94) has the important consequence that the existence of the charge-conjugate spinor $\hat{\psi}^{\mathrm{c}}$ — and hence, taking the zero-charge limit, of Majorana spinors $\hat{\psi}_{\mathrm{M}}$ — depends solely upon the existence of the matrix $\hat{C}$ in the D-dimensional Minkowski space-time. The condition that this imposes upon the values of D has been analyzed in detail by van Nieuwenhuizen [36], Chapline and Slansky [37] and Wetterich [38], who found that Majorana spinors, for which $\hat{\psi}^{\mathrm{c}} = \hat{\psi}$, assuming a real representation, exist only in dimensionalities

$$D = 0, 1, 2, 3, 4 \bmod 8\,. \tag{96}$$

This result can be traced back to the fundamental geometrical fact that the metric $\hat{g}_{IJ}$ is both real and symmetric, as a consequence of which the Clifford algebra is equivalently stated in terms of the Dirac matrices $\hat{\gamma}_I$, their complex conjugates $\hat{\gamma}_I^*$ and their transposes $\hat{\gamma}_I^{\mathrm{T}}$, *viz*

$$\{\hat{\gamma}_I, \hat{\gamma}_J\} = \{\hat{\gamma}_I^*, \hat{\gamma}_J^*\} = \left\{\hat{\gamma}_I^{\mathrm{T}}, \hat{\gamma}_J^{\mathrm{T}}\right\} = 2\hat{g}_{IJ}\,. \tag{97}$$

Pauli [21, 39, 40] and Kramers [28] then showed that there must exist non-singular matrices $\hat{B}$ and $\hat{C}$, that are unitary to preserve the property of Hermiticity, the first relating $\hat{\gamma}_I$ and $\hat{\gamma}_I^*$, which we write, using Schur's lemma [41], the representation of the $\hat{\gamma}_I$ being of minimal dimension $2^{D/2}$, as [38]

$$\hat{\gamma}_I^* = \pm\hat{B}\hat{\gamma}_I\hat{B}^{-1}\,, \tag{98}$$

and where the charge-conjugation operator $\hat{C}$ is defined by Eq. (93). Note that Eq. (93) is invariant under the change of metric signature from (3) to (4), which implies that

$$\hat{\gamma}_I \to \pm i\hat{\gamma}_I\,, \tag{99}$$

while Eq. (98) reverses sign. The minus sign in Eq. (98) is chosen to yield an imaginary $\hat{\gamma}_I$ when $\hat{B} \propto \hat{\mathbf{1}}$, allowing massless or massive Majorana spinors, while the plus sign would yield a real $\hat{\gamma}_I$, when $\hat{B} \propto \hat{\mathbf{1}}$, allowing only massless Majorana spinors.

Analogously to Eqs. (93) and (94), Eq. (98) implies the corresponding relationship between the D-bein components $\hat{\bar{\gamma}}_A$ and $\hat{\bar{\gamma}}_A^*$, namely

$$\hat{\bar{\gamma}}_A^* = \pm\hat{B}\hat{\bar{\gamma}}_A\hat{B}^{-1}\,. \tag{100}$$

Eq. (96) follows from the requirement that the transformation properties of $\hat{B}$ and $\hat{C}$ are completely consistent with one another, and is therefore equally valid in curved space-time. In fact $\hat{B}$ is set equal to unity, so that the wave function does indeed reduce to the D-dimensional analogue of Eq. (84),

$$\hat{\psi}^* \equiv \hat{B}\hat{\psi} = \hat{\psi} \tag{101}$$

(see Refs. [36–38]).

Also, Weyl spinors exist in all even dimensionalities D, for which the $(D+1)$-th gamma-matrix product

$$\hat{\hat{\gamma}}_{D+1} = (-1)^{(D-2)/4}\hat{\hat{\gamma}}_0\hat{\hat{\gamma}}_1 \ldots \hat{\hat{\gamma}}_{(D-1)} \tag{102}$$

can be so defined in the D-bein frame that

$$\hat{\hat{\gamma}}^2_{D+1} = 1\,, \tag{103}$$

enabling us to construct the projection operators $(1 \pm \hat{\hat{\gamma}}_{D+1})/2$, and hence the left- and right-handed Weyl spinors

$$\hat{\psi}_\pm = \tfrac{1}{2}\left(1 \pm \hat{\hat{\gamma}}_{D+1}\right)\hat{\psi}\,. \tag{104}$$

Setting $\hat{B} = 1$, so that the $\hat{\gamma}_I$ are purely imaginary or purely real and $\hat{\psi}$ is real, the even dimensionalities (96) are

$$D = 0, 2, 4, \mod 8\,. \tag{105}$$

If these Majorana spinors are also to satisfy the Weyl condition, however, the eigenvalues of $\hat{\hat{\gamma}}_{D+1}$ have to be real, which means from Eq. (102) that

$$\hat{\hat{\gamma}}_{D+1} = \pm 1\,, \qquad D = 2 \bmod 4\,. \tag{106}$$

Consequently, Majorana–Weyl spinors exist for the subset of the dimensionalities (96) discovered earlier in D-dimensional Minkowski space by Gliozzi *et al.* [42], that is

$$D = 2 \bmod 8\,, \tag{107}$$

for which values it is possible to find an imaginary, chiral Majorana representation of the Dirac matrices.

The paper was written at the University of Cambridge, Cambridge, England.

Appendix A

The evaluation of the terms $\gamma_k \mathit{\Gamma}_j \gamma^k$ and $\hat{\gamma}_K \hat{\mathit{\Gamma}}_J \hat{\gamma}^K$

The derivation of formula (26) for the spinorial affine connection,

$$\mathit{\Gamma}_j = -\tfrac{1}{4}\omega_{abj}\bar{s}^{ab}\,, \tag{A.1}$$

presupposes the vanishing of the first term on the right-hand side of Eq. (24), which, since Eq. (A.1) implies the anti-Hermiticity of $\mathit{\Gamma}_j$, can be written, using the orthogonality (5) of the tetrads, as

$$X_j \equiv \gamma_k \mathit{\Gamma}_j \gamma^k \equiv \bar{\gamma}_a \mathit{\Gamma}_j \bar{\gamma}^a = 0\,. \tag{A.2}$$

We now prove Eq. (A.2) by induction, starting from the solution Eq. (A.1), substitution of which into X_j yields

$$X_j = -\tfrac{1}{4}\omega_{abj}\bar{\gamma}_c \bar{s}^{ab}\bar{\gamma}^c\,. \tag{A.3}$$

In fact it is easy to see that each contribution on the right-hand side of expression (A.3) vanishes separately for all $a, b \neq a$, due to the anti-symmetry of the spin connection in the tetrad indices, $\omega_{abj} = -\omega_{baj}$, and the properties of the gamma-matrix bilinears $\bar{\gamma}^a\bar{\gamma}^b$. Thus,

$$\begin{aligned} X_j\Big|_{a,b\neq a} &\equiv -\tfrac{1}{4}\omega_{abj}\bar{\gamma}_c\bar{\gamma}^a\bar{\gamma}^b\bar{\gamma}^c \not{\Sigma}_{a,b} \\ &= -\tfrac{1}{4}\omega_{abj}\left(\bar{\gamma}_a\bar{\gamma}^a\bar{\gamma}^b\bar{\gamma}^a + \bar{\gamma}_b\bar{\gamma}^a\bar{\gamma}^b\bar{\gamma}^b + 2\bar{\gamma}_c\bar{\gamma}^a\bar{\gamma}^b\bar{\gamma}^c\right)\not{\Sigma}_{a,b,c\neq a,b} \\ &= -\tfrac{1}{4}\omega_{abj}\left(\bar{\gamma}^b\bar{\gamma}^a + \bar{\gamma}^b\bar{\gamma}^a + 2\bar{\gamma}^a\bar{\gamma}^b\right)\not{\Sigma}_{a,b} = 0\,. \end{aligned} \tag{A.4}$$

Therefore,

$$X_j \equiv \sum_{a,b\neq a} X_j\Big|_{a,b\neq a} = 0\,. \tag{A.5}$$

This completes the proof.

Analogously, when $D \neq 4$, starting from the assumption that

$$\hat{\mathit{\Gamma}}_J = -\left(\frac{1}{Z}\right)\hat{\omega}_{abJ}\hat{\bar{s}}^{ab}\,, \tag{A.6}$$

we obtain the result

$$\hat{X}_J \equiv \hat{\gamma}_K \hat{\mathit{\Gamma}}_J \hat{\gamma}^K = (D-4)\hat{\mathit{\Gamma}}_J\,, \tag{A.7}$$

substitution of which into Eq. (89) yields $Z = 4$ and thus Eq. (91).

REFERENCES

[1] V. Fock, *Z. Phys.* **57**, 261 (1929).

[2] H. Tetrode, *Z. Phys.* **50**, 336 (1928).

[3] M.M.G. Ricci, T. Levi-Civita, *Math. Ann.* **54**, 125 (1900); 608(E) (1901).

[4] A. Einstein, *Sitz. Preuss. Akad. Wiss. Berlin* 217 (1928).

[5] E. Wigner, *Z. Phys.* **53**, 592 (1929).

[6] P. van Nieuwenhuizen, *Phys. Rep.* **68**, 189 (1981).

[7] H. Weyl, *Z. Phys.* **56**, 330 (1929).

[8] Th. Kaluza, *Sitz. Preuss. Akad. Wiss. Berlin* 966 (1921).

[9] O. Klein, *Ark. Mat. Astr. Fys.* **34A**, No.1 (1947).

[10] E. Schrödinger, *Sitz. Preuss. Akad. Wiss. Berlin* 105 (1932).

[11] P.A.M. Dirac, *Proc. R. Soc. London* **A117**, 610 (1928).

[12] P.A.M. Dirac, in *Max-Planck-Festschrift* 1958, ed. W. Frank, Deutscher Verlag der Wissenschaft, Berlin 1959, p. 339.

[13] A.H. Compton, *J. Franklin Inst.* **192**, 145 (1921).

[14] K.T. Compton, E.A. Trousdale, *Phys. Rev.* **5**, 315 (1915).

[15] A.H. Compton, O. Rognley, *Phys. Rev.* **16**, 464 (1920).

[16] H.S. Allen, *Phil. Mag.* **40**, 426 (1920).

[17] G.E. Uhlenbeck, S. Goudsmit, *Naturwissenschaften* **47**, 953 (1925).

[18] G.E. Uhlenbeck, S. Goudsmit, *Nature* **117**, 264 (1926).

[19] W. Heisenberg, P. Jordan, *Z. Phys.* **37**, 263 (1926).

[20] W. Pauli, in *Handbuch der Physik*, Vol. 24/1 *Quantentheorie*, Springer Verlag, Berlin 1933, pp. 83–272.

[21] W. Pauli, *Rev. Mod. Phys.* **13**, 203 (1941).

[22] S.S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, Row, Peterson and Company, Elmsford, New York 1961.

[23] J. Schwinger, *Phys. Rev.* **73**, 416 (1948).

[24] C. Amsler *et al.* [Particle Data Group], *Phys. Lett.* **B667**, 1 (2008).

[25] V. Bargmann, *Sitz. Preuss. Akad. Wiss. Berlin* 346 (1932).

[26] L. Parker, *Phys. Rev. Lett.* **44**, 1559 (1980).

[27] L. Parker, *Phys. Rev.* **D22**, 1922 (1980).

[28] H.A. Kramers, *Kon. Akad. Wet. Amsterdam* **40**, 814 (1937).

[29] E. Majorana, *Nuovo Cimento* **14**, 171 (1937).

[30] G. Racah, *Nuovo Cimento* **14**, 322 (1937).

[31] M.D. Pollock, *Mod. Phys. Lett.* **A23**, 2665 (2008).

[32] M.D. Pollock, *Mod. Phys. Lett.* **A23**, 797 (2008).

[33] D.J. Gross, J.A. Harvey, E. Martinec, R. Rohm, *Phys. Rev. Lett.* **54**, 502 (1985).

[34] D.J. Gross, J.A. Harvey, E. Martinec, R. Rohm, *Nucl. Phys.* **B256**, 253 (1985).

[35] D.J. Gross, J.A. Harvey, E. Martinec, R. Rohm, *Nucl. Phys.* **B267**, 75 (1986).

[36] P. van Nieuwenhuizen, in *Supergravity '81*, ed. S. Ferrara, J.G. Taylor, Cambridge University Press, Cambridge 1982, p. 151.

[37] G. Chapline, R. Slansky, *Nucl. Phys.* **B209**, 461 (1982).

[38] C. Wetterich, *Nucl. Phys.* **B211**, 177 (1983).

[39] W. Pauli, *Ann. Phys.* **18**, 337 (1933).

[40] W. Pauli, *Ann. Inst. H. Poincaré* **6**, 109 (1936).

[41] I. Schur, *Sitz. Preuss. Akad. Wiss. Berlin* 406 (1905).

[42] F. Gliozzi, J. Scherk, D. Olive, *Nucl. Phys.* **B122**, 253 (1977).