5. Higher Chow Groups and Beilinson's Conjectures

5.1. **Bloch's formula.** One interesting application of Quillen's techniques involving the Q construction is the following theorem of Quillen, extending work of Bloch. This gives another hint of the close connection of algebraic K-theory and algebraic cycles.

Theorem 5.1. *Let X be a smooth algebraic variety over a field. For any $i \geq 0$, let $\mathcal{K}_i$ denote the sheaf on X (for the Zariski topology) sending an open subset $U \subset X$ to $K_i(U)$. Then there is a natural spectral sequence*

$$E_2^{p,q}(X) = \ H^p(X, \mathcal{K}_q) \Rightarrow K_{p+q}(X).$$

Moreover, there is a natural isomorphism (Bloch's formula)

$$CH^q(X) \simeq H^q(X, \mathcal{K}_q)$$

relating the Chow group of codimension q cycles on X and the Zariski cohomology of the sheaf $\mathcal{K}_q$.

In particular, for $q = 1$, Bloch's formula becomes the familiar isomorphism relating the Chow group of divisors on a smooth variety to $H^1(X, \mathcal{O}_X^*)$.

Proof. Quillen's techniques apply to $K'_*(X) = \pi_{*+1}(BQ\mathcal{M}(X))$, the K-theory of coherent $\mathcal{O}_X$-modules. When X is smooth, Quillen verifies that the natural map $K_*(X) \to K'_*(X)$ is an isomorphism.

The key result needed for the existence of a spectral sequence is Quillen's localization theorem for a not necessarily smooth scheme X: if $Y \subset X$ is closed with Zariski open complement $U = X - Y$, then there is a natural long exact sequence

$$\cdots \to K'_{q+1}(U) \to K'_q(Y) \to K'_q(X) \to K'_q(U) \to \cdots .$$

Applying this localization sequence (slightly generalized) to the filtration of the category $\mathcal{M}(X)$ of coherent $\mathcal{O}_X$-modules

$$\mathcal{M}(X) = \mathcal{M}_0(X) \supset \mathcal{M}_1(X) \supset \mathcal{M}_2(X) \supset \cdots$$

where $\mathcal{M}_p(X)$ denotes the subcategory of those coherent sheaves whose support has codimension $\geq p$, Quillen obtains long exact sequences

$$\cdots K_i(\mathcal{M}_{p+1}) \to K_i(\mathcal{M}_p(X)) \to \coprod_{x \in X^p} K_i(k(x)) \to K_{i-1}(\mathcal{M}_{p+1}) \to \cdots$$

Here, X^p denotes the set of points of X of codimension p.

These exact sequences (for varying p) determine an exact couple and thus a spectral sequence

$$E_1^{p,q} = \coprod_{x \in X^p} K_{-p-q}(k(x)) \Rightarrow K'_{-p-q}(X).$$

Following Gersten, Quillen considers the sequences

gersten

$$0 \to K_q \to \coprod_{x \in X^0} K_q(k(x)) \to \coprod_{x \in X^1} K_{q-1}(k(x)) \to \cdots \tag{5.1.1}$$

where the differential in this sequence is d_1 of the above spectral sequence, given as the composition

$$\coprod_{x \in X^p} K_i(k(x)) \to K_{i-1}(\mathcal{M}_{p+1}) \to \coprod_{x \in X^{p+1}} K_{i-1}(k(x)).$$

Gersten conjectured that this sequence should be exact if X is the spectrum of a regular local ring; Quillen proved this for local rings of the form $O_{X,x}$ where X is a smooth variety.

We now consider the sequences (5.1.1) for varying open subsets $U \subset X$ in place of X, thereby getting an exact sequence of *sheaves* on X

Gersten (5.1.2)
$$0 \to \mathcal{K}_q \to \coprod_{x\in X^0} i_{x*}K_q(k(x)) \to \coprod_{x\in X^1} i_{x*}K_{q-1}(k(x)) \to \cdots$$

Since the sheaves $i_{x*}K_q(k(x))$ are flasque, we conclude that the p-th cohomology of the sequence (5.1.2) equals $H^p(X, \mathcal{K}_q)$ which is the E_2-term of our spectral sequence.

Finally, Quillen verifies that the cokernel of
$$\coprod_{x\in X^{q-1}} K_1(k(x)) \to \coprod_{x\in X^q} K_0(k(x))$$
is canonically isomorphic to $CH^q(X)$ provided that X is smooth. □

5.2. **Derived categories.** In order to formulate motivic cohomology, we need to introduce the language of derived categories. Let $\mathcal{A}$ be an abelian category (e.g., the category of modules over a fixed ring) and consider the category of chain complexes $CH^\bullet(\mathcal{A})$. We shall index our chain complexes so that the differential has degree $+1$. We assume that $\mathcal{A}$ has enough injectives and projectives, so that we can construct the usual derived functors of left exact and right exact functors from $\mathcal{A}$ to another abelian category $\mathcal{B}$. For example, if $F : \mathcal{A} \to \mathcal{B}$ is right exact, then we define $L_iF(A)$ to be the i-th homology of the chain complex $F(P_\bullet)$ obtained by applying F to a projective resolution $P_\bullet \to A$ of A; similarly, if $G : \mathcal{A} \to \mathcal{B}$ is left exact, then $R^jG(A) = H^j(I^\bullet)$ where $A \to I^\bullet$ is an injective resolution of A.

The usual verification that these derived functors are well defined up to canonical isomorphism actually proves a bit more. Namely, rather take the homology of the complexes $F(P_\bullet), G(I^\bullet)$, we consider these complexes themselves and observe that they are independent *up to quasi-isomorphism* of the choice of resolutions. Recall, that a map $C^\bullet \to D^\bullet$ is a quasi-isomorphism if it induces an isomorphism on homology; only in special cases is a complex $C^\bullet$ quasi-isomorphic to its homology $H^\bullet(C^\bullet)$ viewed as a complex with trivial differential.

We define the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ as the category obtained from the category of $CH^\bullet(\mathcal{A})$ of chain complexes of $\mathcal{A}$ by inverting quasi-isomorphisms. Of course, some care must be taken to insure that such a localization of $CH^\bullet(\mathcal{A})$ is well defined. Let $Hot(CH^\bullet(\mathcal{A}))$ denote the homotopy category of chain complexes of $\mathcal{A}$: maps from the chain complex $C^\bullet$ to the chain complex $D^\bullet$ in $\mathcal{H}(CH^\bullet(\mathcal{A}))$ are chain homotopy equivalence classes of chain maps. Since chain homotopic maps induce the same map on homology, we see that $\mathcal{D}(\mathcal{A})$ can also be defined as the category obtained from $Hot(CH^\bullet(\mathcal{A}))$ by inverting quasi-isomorphisms.

The derived category $\mathcal{D}(\mathcal{A})$ of the abelian category $CH^\bullet(\mathcal{A})$ is a *triangulated category.* Namely, we have a shift operator $(-)[n]$ defined by
$$(A^\bullet[n])^j \;\equiv\; A^{n+j}.$$
This indexing is very confusing (as would be any other); we can view $A^\bullet[n]$ as $A^\bullet$ shifted "down" or "to the left". We also have *distinguished triangles*
$$A^\bullet \to B\bullet \to C^\bullet \to A^\bullet[1]$$

defined to be those "triangles" quasi-isomorphic to short exact sequences $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of chain complexes.

This notation enables us to express Ext-groups quite neatly as

$$Ext^i_{\mathcal{A}}(A,B) = H^i(Hom_{\mathcal{A}}(P_\bullet, B) = Hom_{\mathcal{D}(\mathcal{A})}(A[-i],B)$$
$$= Hom_{\mathcal{D}(\mathcal{A})}(A,B[i]) = H^i(Hom_{\mathcal{A}}(A,P^\bullet)).$$

5.3. **Bloch's Higher Chow Groups.** From our point of view, motivic cohomology should be a "cohomology theory" which bears a relationship to $K_*(X)$ analogous to the role Chow groups $CH^*(X)$ bear to $K_0(X)$ (and analogous to the relationship of $H^*_{sing}(T)$ to $K^*_{top}(T)$). In particular, motivic cohomology will be doubly indexed.

We now discuss a relatively naive construction by Spencer Bloch of "higher Chow groups" which satisfies this criterion. We shall then consider a more sophisticated version of motivic cohomology due to Suslin and Voevodsky.

We work over a field k and define Δ^n to be $\operatorname{Spec} k[x_0,\dots,x_n]/(\sum_i x_i - 1)$, the algebraic n-simplex. As in topology, we have face maps $\partial_i : \Delta^{n-1} \to \Delta^n$ (sending the coordinate function $x_i \in k[\Delta^n]$ to 0) and degeneracy maps $\sigma_j : \Delta^{n+1} \to \Delta^n$ (sending the coordinate function $x_j \in k[\Delta^n]$ to $x_j + x_{j+1} \in k[\Delta^{n+1}]$). More generally, a composition of face maps determines a face $F \simeq \Delta^i \to \Delta^n$. Of course, $\Delta^n \simeq \mathbb{A}^n$.

Bloch's idea is to construct a chain complex for each q which in degree n would be the codimension q-cycles on $X \times \Delta^n$. In particular, the 0-th homology of this chain complex should be the usual Chow group $CH^q(X)$ of codimension q cycles on X modulo rational equivalence. This can not be done in a completely straightforward manner, since one has no good way in general to restrict a general cycle on $X \times \Delta[n]$ via a face map ∂_i to $X \times \Delta^{n-1}$. Thus, Bloch only considers codimension q cycles on $X \times \Delta^n$ which restrict properly to all faces (i.e., to codimension q cycles on $X \times F$).

Definition 5.2. Let X be a variety over a field k. For each $p \geq 0$, we define a complex $z_p(X,*)$ which in degree n is the free abelian group on the integral closed subvarieties $Z \subset X \times \Delta^n$ with the property that for every face $F \subset \Delta^n$

$$dim_k(Z \cap (X \times F) \leq dim_k(F) + p.$$

The differential of $z_p(X,*)$ is the alternating sum of the maps induced by restricting cycles to codimension 1 faces. Define the *higher Chow homology groups* by

$$CH_p(X,n) = H_n(z_p(X,*)), \quad n,p \geq 0.$$

If X is locally equi-dimensional over k (e.g., X is smooth), let $z^q(X,n)$ be the free abelian group on the integral closed subvarieties $Z \subset X \times \Delta^n$ with the property that for every face $F \subset \Delta^n$

$$codim_{X\times F}(Z \cap (X \times F)) \geq q.$$

Define the *higher Chow cohomology groups* by

$$CH^q(X,n) = H_n(z^q(X,*), \quad n,q \geq 0,$$

where the differential of $z^q(X,*)$ is defined exactly as for $z_p(X,*)$.

Bloch, with the aid of Marc Levine, has proved many remarkable properties of these higher Chow groups.

Theorem 5.3. *Let X be a quasi-projective variety over a field. Bloch's higher Chow groups satisfy the following properties:*

- $CH_p(-,*)$ *is covariantly functorial with respect to proper maps.*
- $CH^q(-,*)$ *is contravariantly functorial on Sm_k, the category of smooth quasi-projective varieties over k.*
- $CH_p(X,0) = CH_p(X)$, *the Chow group of p-cycles modulo rational equivalence.*
- *(Homotopy invariance)* $\pi^* : CH_p(X,*) \xrightarrow{\sim} CH_{p+1}(X \times \mathbb{A}^1)$.
- *(Localization) Let $i : Y \to X$ be a closed subvariety with $j : U = X - Y \subset X$ the complement of Y. Then there is a distinguished triangle*
$$z_p(Y,*) \xrightarrow{i_*} z_p(X,*) \xrightarrow{j^*} z_p(U,*) \to z_p(Y,*)[1]$$
- *(Projective bundle formula) Let $\mathcal{E}$ be a rank n vector bundle over X. Then $CH^*(\mathbb{P}(\mathcal{E})*)$ is a free $CH^*(X,*)$-module on generators $1, \zeta, \ldots, \zeta^{n-1} \in CH^1(\mathbb{P}(\mathcal{E}),0)$.*
- *For X smooth, $K_i(X) \otimes \mathbb{Q} \simeq \oplus_q CH^q(X,i) \otimes \mathbb{Q}$ for any $i \geq 0$. Moreover, for any $q \geq 0$,*
$$(K_i(X) \otimes \mathbb{Q})^{(q)} \simeq CH^q(X,i) \otimes \mathbb{Q}.$$
- *If F is a field, the $K_n^M(F) \simeq CH^n(Spec\, F, n)$.*

The most difficult of these properties, and perhaps the most important, is localization. The proof requires a very subtle technique of moving cycles. Observe that $z_p(X,*) \to z_p(U,*)$ is not surjective because the conditions of proper intersection on an element of $z_p(U,n)$ (i.e, a cycle on $U \times \Delta^n$) might not continue to hold for the closure of that cycle in $X \times \Delta^n$.

5.4. **Sheaves and Grothendieck topolologies.** The motivic cohomology groups of Suslin-Voevodsky are sheaf cohomology groups, and this formulation in terms of sheaves for a suitable topology provides much more flexibility. Moreover, this fits the spirt of the Beilinson conjectures which have motivated many of the developments which relate K-theory to algebraic cycles.

Before discussing Beilinson's conjectures, let us briefly consider Grothendieck's approach to sheaf theory and introduce both the etale and Nisnevich topologies.

Grothendieck had the insight to realize that one could formulate sheaves and sheaf cohomology in a setting more general than that of topological spaces. What is essential in sheaf theory is the notion of a covering, but such a covering need not consist of open subsets.

Definition 5.4. A (Grothendieck) site is the data of a category $\mathcal{C}/X$ of schemes over a given scheme X which is closed under fiber products and a distinguished class of morphisms (e.g., Zariski open embeddings; or etale morphisms) closed under composition, base change and including all isomorphisms. A covering of an object $Y \in \mathcal{C}/X$ for this site is a family of distinguished morphisms $\{g_i : U_i \to Y\}$ with the property that $Y = \cup_i g_i(U_i)$.

The data of the site $\mathcal{C}/X$ together with its associated family of coverings is called a Grothendieck topology on X.

Example 5.5. Recall that a map $f : U \to X$ of schemes is said to be *etale* if it is flat, unramified, and locally of finite type. Thus, open immersions and covering space maps are examples of etale morphisms. If $f : U \to X$ is etale, then for each point $u \in U$ there exist affine open neighborhoods $SpecA \subset U$ of u and $SpecR \subset X$ of $f(u)$ so that A is isomorphic to $(R[t]/g(t))_h$ for some monic polynomial $g(t)$ and some h so that $g'(t) \in (R[t]/g(t))_h$ is invertible.

The (small) etale site X_{et} has objects which are etale morphisms $Y \to X$ and coverings $\{U_i \to Y\}$ consist of families of etale maps the union of whose images equals Y. The big etale site X_{ET} has objects $Y \to X$ which are locally of finite type over X and coverings $\{U_i \to Y\}$ defined as for X_{et} consisting of families of etale maps the union of whose images equals Y. If k is a field, we shall also consider the site $(Sm/k)_{et}$ which is the full subcategory of $(\operatorname{Spec} k)_{ET}$ consisting of smooth, quasi-projective varieties Y over k.

An instructive example is that of $X = SpecF$ for some field F. Then an etale map $Y \to X$ with Y connected is of the form $SpecE \to SpecF$, where E/F is a finite separable field extension.

Definition 5.6. A presheaf sets (respectively, groups, abelian groups, rings, etc) on a site $\mathcal{C}/X$ is a contravariant functor from $\mathcal{C}/X$ to $(sets)$ (resp., to groups, abelian groups, rings, etc). A presheaf $P : (\mathcal{C}/X)^{op} \to (sets)$ is said to be a sheaf if for every covering $\{U_i \to Y\}$ in $\mathcal{C}/X$ the following sequence is exact:

$$P(Y) \to \prod_i P(U_i) \rightrightarrows \prod_{i,j} P(U_i \times_X U_j).$$

(Similarly, for presheaves of groups, abelian presheaves, etc.) In other words, if for every Y, the data of a section $s \in P(Y)$ is equivalent to the data of sections $s_i \in P(U_i)$ which are compatible in the sense that the restrictions of s_i, s_j to $U_i \times_X U_j$ are equal.

The category of abelian sheaves on a Grothendieck site $\mathcal{C}/X$ is an abelian category with enough injectives, so that we can define sheaf cohomology in the usual way. If $F : \mathcal{C}/X)^{op} \to (Ab)$ is an abelian sheaf, then we define

$$H^i(X_{\mathcal{C}/X}, F) = R^i\Gamma(X, F).$$

Etale cohomology has various important properties. We mention two in the following theorem.

Theorem 5.7. *Let X be a quasi-projective, complex variety. Then the etale cohomology of X with coefficients in (constant) sheaf $\mathbb{Z}/n$, $H^*(X_{et}, \mathbb{Z}/n)$, is naturally isomorphic to the singular cohomology of X^{an},*

$$H^*(X_{et}, \mathbb{Z}/n) \;\simeq\; H^*_{sing}(X^{an}, \mathbb{Z}/n).$$

Let $X = Speck$, the spectrum of a field. Then an abelian sheaf on X for the etale topology is in natural 1-1 correspondence with a (continuous) Galois module for the Galois group $Gal(\overline{k}/k)$. Moreover, the etale cohomology of X with coefficients in such a sheaf F is equivalent to the Galois cohomology of the associated Galois module,

$$H^*(k_{et}, F) \;\simeq\; H^*(Gal(\overline{F}/F), F(k)).$$

From the point of view of sheaf theory, the essence of a continuous map $g : S \to T$ of topological spaces is a mapping from the category of open subsets of T to the open subsets of S. We shall consider a map of sites $g : \mathcal{C}/X \to \mathcal{D}/Y$ to a functor from

$\mathcal{C}/Y$ to cC/X which takes distinguished morphisms to distinguished morphisms. For example, one of Beilinson's conjectures involves the map of sites

$$\pi : X_{et} \to X_{Zar}, \quad (U \subset X) \mapsto U \to X.$$

Such a map of sites induces a map on sheaf cohomology: if $F : (\mathcal{D}/Y)^{op} \to (Ab)$ is an abelian sheaf on $\mathcal{C}/Y$, then we obtain a map

$$H^*(Y_{\mathcal{D}/Y}, F) \to H^*(X_{\mathcal{C}/X}, g^*F).$$

5.5. **Beilinson's Conjectures.** We give below a list of conjectures due to Beilinson which relate motivic cohomology and K-theory. Bloch's higher Chow groups go some way toward providing a theory which satisfies these conjectures. Namely, Beilinson conjectures the existence of complexes of sheaves $\Gamma_{Zar}(r)$ whose cohomology (in the Zariski topology) $H^p(X, \Gamma_{Zar}(r))$ one could call "motivic cohomology". If we set

$$H^p(X, \Gamma_{Zar}(r)) = CH^r(X, 2r-p),$$

then many of the cohomological conjectures Beilinson makes for his conjectured complexes are satisfied by Bloch's higher Chow groups $CH^\bullet(X, *)$.

Conjecture 5.8. *Let X be a smooth variety over a field k. Then there should exist complexes of sheaves $\Gamma_{Zar}(r)$ of abelian groups on X with the Zariski topology, well defined in $\mathcal{D}(AbSh(X_{Zar}))$, functorial in X, and equipped with a graded product, which satisfy the following properties:*

(1) $\Gamma_{Zar}(1) = \mathbb{Z}$; $\Gamma_{Zar}(1) \simeq G_m[-1]$.
(2) $H^{2n}(X, \Gamma_{zar}(n)) = A^n(X)$.
(3) $H^i(Spec\, k, \Gamma_{Zar}(i)) = \mathcal{K}_i^M k$, *Milnor K-theory.*
(4) *(Motivic spectral sequence) There is a spectral sequence of the form*

$$E_2^{p,q} = H^{p-q}(X, \Gamma_{Zar}(q)) \Rightarrow K_{-p-q}(X)$$

which degenerates after tensoring with $\mathbb{Q}$. Moreover, for each prime ℓ, there is a mod-ℓ version of this spectral sequence

$$E_2^{p,q} = H^{p-q}(X, \Gamma_{Zar}(q) \otimes^L \mathbb{Z}/\ell) \Rightarrow K_{-p-q}(X, \mathbb{Z}/\ell)$$

(5) $gr^r_\gamma(K_j(X) \otimes \mathbb{Q} \simeq \mathbb{H}^{2r-i}(X_{Zar}, \Gamma_{Zar}(r))_{\mathbb{Q}}$.
(6) *(Beilinson-Lichtenbaum Conjecture) $\Gamma_{Zar} \otimes^L \mathbb{Z}/\ell \simeq \tau_{\leq r}\mathbf{R}\pi_*(\mu_\ell^{\otimes} r)$ in the derived category $\mathcal{D}(AbSh(X_{Zar}))$ provided that ℓ is invertible in $\mathcal{O}_X$, where $\pi : X_{et} \to X_{Zar}$ is the change of topology morphism.*
(7) *(Vanishing Conjecture) $\Gamma_{Zar}(r)$ is acyclic outside $[1, r]$ for $r \geq 1$.*

These conjectures require considerable explanation, of course. Essentially, Beilinson conjectures that algebraic K-theory can be computed using a spectral sequence of Atiyah-Hirzebruch type (4) using "motivic complexes" $\Gamma_{Zar}(r)$ whose cohomology plays the role of singular cohomology in the Atiyah-Hirzebruch spectral sequence for topological K-theory. I have indexed the spectral sequence as Beilinson suggests, but we could equally index it in the Atiyah-Hirzebruch way and write (by simply re-indexing)

$$E_2^{p,q} = H^p(X, \mathbf{\Gamma}_{Zar}(-q/2)) \Rightarrow K_{-p-q}(X).$$

where $\Gamma_{Zar}(-q/2) = 0$ if $-q$ is not an even non-positive integer and $\Gamma_{Zar}(-q/2) = \Gamma_{Zar}(i)$ is $-q = 2i \geq 0$.

(1) and (2) just "normalize" our complexes, assuring us that they extend usual Chow groups and what is known in codimensions 0 and 1. Note that (1) and (2) are compatible in the sense that

$$H^2(X, \Gamma_{Zar}(1)) = H^2(X, \mathcal{O}_X^*[-1]) = H^1(X, \mathcal{O}_X^*) = Pic(X).$$

(3) asserts that for a field k, the n-th cohomology of $\Gamma_{Zar}(n)$ – the part of highest weight with respect to the action of Adams operations – should be Milnor K-theory. This has been verified for Bloch's higher Chow groups by Suslin-Nesternko and Totaro.

The (integral) spectral sequence of (4) has been established thanks to the work of many authors. This spectral sequence "collapses" at the E_2-level when tensored with $\mathbb{Q}$, so that $E_2 \otimes \mathbb{Q} = E_\infty \otimes \mathbb{Q}$. (5) asserts that this collapsing can be verified by using Adams operations, interpreted using the γ-filtration.

The vanishing conjecture of (7) is the most problematic, and there is no consensus on whether it is likely to be valid. However, (6) incorporates the mod-ℓ version of the vanishing conjecture.

(6) asserts that if we consider the complexes $\Gamma_{Zar}(r)$ modulo ℓ (in the sense of the derived category), then the result has cohomology closely related to etale cohomology with $\mu_\ell^{\otimes r}$ coefficients, where μ_ℓ is the etale sheaf of ℓ-th roots of unity (isomorphic to $\mathbb{Z}/\ell$ if all ℓ-th roots of unity are in k. If the terms in the mod-ℓ spectral sequence were simply etale cohomology, then we would get etale K-theory which would violate the vanishing conjectured in (7) (and which would imply periodicity in low degrees which we know to be false). So Beilinson conjectures that the terms modulo ℓ should be the cohomology of complexes which involve a truncation.

More precisely, $\mathbf{R}\pi_*F$ is a complex of sheaves for the Zariski topology (given by applying π_* to an injective resolution $F \to I^\bullet$) with the property that $H^*_{Zar}(X, \mathbf{R}\pi_*F) = H^*_{et}(X, F)$. Now, the n-th truncation of $\mathbf{R}\pi_*F$, $\tau_{\leq n}\mathbf{R}\pi_*F$, is the truncation of this complex of sheaves in such a way that its cohomology sheaves are the same as those of $\mathbf{R}\pi_*F$ in degrees $\leq n$ and are 0 in degrees greater than n. (We do this by retaining coboundaries in degree $n+1$ and setting all higher degrees equal to 0.)

If $X = Speck$, then $H^p(Speck, \tau_{\leq n}\mathbf{R}\pi_*\mu_\ell^{\otimes n})$ equals $H^p_{et}(Speck, \mu_\ell^{\otimes n})$ for $p \leq n$ and is 0 otherwise. For a positive dimensional variety, this truncation has a somewhat mystifying effect on cohomology.

It is worth emphasizing that one of the most important aspects of Beilinson's conjectures is its explicit nature: Beilinson conjectures precise values for algebraic K-groups, rather than the conjectures which preceded Beilinson which required the degree to be large or certain torsion to be ignored. Such a precise conjecture should be much more amenable to proof.

Now, the Suslin-Voevodsky motivic complexes $\mathbb{Z}(n)$ are complexes of sheaves for the *Nisnevich topology*, a Grothendieck topology with finer than the Zariski topology but less fine than the etale topology.

Definition 5.9. A *Nisnevich covering* $\{U_i \to X\}$ of a scheme X is an etale covering with the property that every for every point $x \in X$ there exists some $u_x \in U_i$ mapping to x inducing an isomorphism $k(x) \to k(u_x)$ of residue fields.

We define $(Sm/k)_{Nis}$ to be the Grothendieck site whose objects are smooth quasi-projective varieties over a field k and whose distinguished class of morphisms consists of etale morphisms $g : U \to X$ with the property that for every point $x = g(u)$ there exists some point $u_x \in U$ with $k(x) \simeq k(u_x)$.

The Nisnevich topology is particularly useful for the study of blow-ups a smooth variety along a smooth subvariety. Although many of the proofs of Suslin-Voevodsky require use of this topology rather than the Zariski topology, there are various theorems to the effect that cohomology with respect to the Nisnevich topology agrees with that with respect to the Zariski topology for special types of coefficient sheaves.

It is instructive to observe that the "localization" of a scheme X at a point $x \in X$ is the spectrum of the local ring $\mathcal{O}_{X,x}$ if we consider the Zariski topology, is the spectrum of the *henselization* of $\mathcal{O}_{X,x}$ if we consider the Nisnevich topology, and is the the spectrum the *strict henselization* of $\mathcal{O}_{X,x}$ if we consider the etale topology.

References

A [1] M. Atiyah, K-thoery, Benjamin 1967.

AH [2] M. Atiyah and F. Hirzebruch, Vector bundles and homogeneous spaces, Proc. Symp in Pure Math, vol 3, A.M.S. (1961).

BMS [3] H. Bass, J. Milnor, and J.-P. Serre, Solution of the congruence subgroup problem for $SL_n(n \geq 3)$ and $Sp_{2n}(n \geq 2)$, Publ. Math. I.H.E.S. [bf 33, 1967.

Bei [4] A. Beilinson, Height Paring between algebraic cycles. K-theory , arithmetic, and geometry. Lecture Notes in Mathematics #1289, springer (1986).

Ber [5] J. Berrick, An Approach to Algebraic K-theory, Pitman 1982.

Bl [6] S. Bloch, Algebraic cycles and higher K-theory. Advances in Math **61** (1986), 267-304.

B [7] R. Bott. Math Review MR0116022 (22 # 6817) of Le théorème de Riemann-Roch by Armand Borel and Jean-Pierre Serre.

FL [8] E. Friedlander and H.B. Lawson, Moving algebraic cycles of bounded degree. Inventiones Math. **132** (1998), 91-119.

Fu [9] W. Fulton, *Intersection Theory*, Springer 1984.

G [10] D. Grayson, *Higher algebraic K-theory. II (after Daniel Quillen)* L.N.M. # 551, pp 217-240.

H [11] R. Hartshorne, *Algebraic Geometry*, Springer 1977.

M [12] J. Milnor, *Introduction to Algebraic K-theory*, Annals of Math Studies 72, Princeton University press 1971.

Q [13] D. Quillen, Higher algebraic K-theory, I. L.N.M # 341, pp. 85 – 147.

W [14] C. Weibel, *Algebraic K-theory.* http://math.rutgers.edu/ weibel/Kbook.html

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